# Competitively orientable complete multipartite graphs 

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#### Abstract

We say that a digraph $D$ is competitive if any pair of vertices has a common out-neighbor in $D$ and that a graph $G$ is competitively orientable if there exists a competitive orientation of $G$. The notion of competitive digraphs arose while studying digraph whose competition graphs are complete. We derive some useful properties of competitively orientable graphs and show that a complete graph of order $n$ is competitively orientable if and only if $n \geq 7$. Then we completely characterize a competitively orientable complete multipartite graph in terms of the sizes of its partite sets. Moreover, we present a way to build a competitive multipartite tournament in each of competitively orientable cases.


Keywords. competitive digraph; competitively orientable graph; complete multipartite graph; competitive multipartite tournament; competition graph.
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## 1 Introduction

In this paper, for graph-theoretical terminology and notations not defined, we follow [1]. We consider finite simple graphs. For a digraph $D$, the underlying graph of $D$ is the graph $G$ such that $V(G)=V(D)$ and $E(G)=\{u v \mid(u, v) \in A(D)\}$. An orientation of a graph $G$ is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is $G$.

We say that two vertices compete in a digraph $D$ if they have a common out-neighbor in $D$ and that a digraph $D$ is competitive if any pair of vertices competes in $D$. A graph $G$ is said to be competitively orientable if there exists a competitive orientation of $G$. For example, the complete graph $K_{7}$ is completely orientable as shown in Figure 1. By the way, we deduce Theorem 2.6 which guarantees $K_{n}$ being competitively orientable for $n \geq 7$ from $K_{7}$ being competitively orientable. Yet, $K_{n}$ is not competitively orientable for any integer $2 \leq n \leq 6$ by Theorem 2.4(3).


Figure 1: A competitive orientation of $K_{7}$

The notions of competitive digraph and competitive orientation arose during a research on competition graphs of complete multipartite graphs. The competition graph of a digraph $D$ is defined as the graph with the vertex set $V(D)$ and an edge $u v$ if and only if $u$ and $v$ compete in $D$. Competition graphs arose in connection with an application in ecology (see [4]) and also have applications in coding, radio transmission, and modeling of complex economic systems. Early literature of the study on competition graphs is summarized in the survey papers by Kim [15] and Lundgren [17]. The competition graphs of tournaments and those of bipartite tournaments have been actively studied (see [2], [3], [5], 6], [7], 10], [11], and [16] for papers related to this topic).

By the definition of competition graph, it is easy to see that a digraph is competitive if and only if its competition graph is a complete graph. On the other hand, the competition graph of a digraph $D$ being complete may be rephrased as: The adjacency matrix of a digraph $D$ is "scrambling". A matrix $A$ is said to be scrambling if for any pair of indices $i, j$, there exists $k$ such that $A_{i k} \neq 0$ and $A_{j k} \neq 0$. Scrambling matrices were first defined in [14] to study weak ergodicity of inhomogeneous Markov chains.

Kim and Lee [19] studied acyclic digraphs whose competition graphs consist of only complete components.

In this paper, we completely characterize a competitively orientable complete multipartite graph in terms of the sizes of its partite sets. We first show that there is no competitively orientable complete bipartite graph (Corollary 2.3). Then we show that for each integer $k \geq 7$, any $k$-partite complete graph is competitively orientable (Proposition (3.1). Next we characterize competitively orientable complete 6-partite graphs as follows.

Theorem 1. Let $n_{1}, \ldots, n_{6}$ be positive integers such that $n_{1} \geq \cdots \geq n_{6}$. Then a complete 6-partitie graph $K_{n_{1}, n_{2}, \ldots, n_{6}}$ is competitively orientable if and only if one of the following holds: (a) $n_{1} \geq 5$ and $n_{2}=1$; (b) $n_{1} \geq 3, n_{2} \geq 2$, and $n_{3}=1$; (c) $n_{3} \geq 2$.

The remaining cases are also completely taken care of in the following manner.
Theorem 2. Let $n_{1}, n_{2}$, and $n_{3}$ be positive integers such that $n_{1} \geq n_{2} \geq n_{3}$. Then a complete tripartite graph $K_{n_{1}, n_{2}, n_{3}}$ is competitively orientable if and only if $n_{1} \geq 5$ and
$n_{3} \geq 4$.
Theorem 3. Let $n_{1}, \ldots, n_{4}$ be positive integers such that $n_{1} \geq \cdots \geq n_{4}$. Then a complete 4-partite graph $K_{n_{1}, n_{2}, n_{3}, n_{4}}$ is competitively orientable if and only if one of the following holds: (a) $n_{1} \geq 4, n_{3} \geq 3$, and $n_{4}=1$; (b) $n_{1} \geq 4, n_{3}=2$, and $n_{4}=2$; (c) $n_{3} \geq 3$ and $n_{4} \geq 2$.

Theorem 4. Let $n_{1}, \ldots, n_{5}$ be positive integers such that $n_{1} \geq n_{2} \geq \cdots \geq n_{5}$. Then a complete 5-partite graph $K_{n_{1}, n_{2}, \ldots, n_{5}}$ is competitively orientable if and only if one of the following holds: (a) $n_{1}=3, n_{2}=3$, $n_{3} \geq 2$, and $n_{4}=1$; (b) $n_{1} \geq 4, n_{3} \geq 2$, and $n_{4}=1$; (c) $n_{4} \geq 2$.

A tournament is an orientation of a complete graph. A $k$-partite tournament is an orientation of a complete $k$-partite graph for some positive integer $k \geq 2$. If a digraph is a $k$-partite tournament for some integer $k \geq 2$, then it is called a multipartite tournament. Multipartite tournaments have been actively studied by graph theorists (see [8], [9], [12], [13], and a survey paper [18]).

We make a useful observation that any complete multipartite graph containing a competitively orientable complete multipartite graph as a subgraph is competitively orientable (Corollary [2.7). Thanks to this observation, showing a complete multipartite graph $K_{n_{1}, n_{2}, \ldots, n_{k}}$ is competitively orientable becomes much simpler: once we present a deliberately designed concrete competitive multipartite tournament as a base, the proposition guarantees that it will expand to a competitive multipartite tournament with partite sets of sizes $n_{1}, n_{2}, \ldots, n_{k}$.

In Section 2, we derive some properties of competitively orientable graphs which are useful in proving our main results. In Section 3.1, we study structure of competitive 6 -partite tournament to prove Theorem 1. In Section 3.2, we deal with competitive tripartite tournaments and prove Theorem 2, In Section 3.3, we prove Theorems 3 and 4 . In Section 4, we summarize the main results in the aspect of vertices of the competition graph of a competitive multipartite tournament (Theorem 4.1).

## 2 Preliminaries

In this section, we derive properties of competitive digraphs and competitively orientable graphs, and develop tools to prove our mains results.

### 2.1 Competitively orientable graphs

Given a digraph $D$, we denote by $N_{D}^{+}(x)$ the set of out-neighbors of a vertex $x$ in $D$ and by $N_{D}^{-}(x)$ the set of in-neighbors of a vertex $x$ in a digraph $D$. If no confusion is likely, we omit the subscript $D$.

Given a digraph $D$ and a vertex $u$ of $D$, we add a new vertex $v$ and arcs to $D$ including the arcs in the set $\{(v, w) \mid(u, w) \in A(D)\}$. We call the resulting digraph a digraph competitively expanded from $D$ via $u$ by $v$.

Proposition 2.1. Given a nontrivial competitive digraph $D$ and a vertex $u$ in $D$, each digraph competitively expanded from $D$ via $u$ by a new vertex $v$ is competitive.

Proof. Let $D^{\prime}$ be a digraph competitively expanded from $D$ via $u$ by a new vertex $v$. By the hypothesis, $A(D) \subset A\left(D^{\prime}\right)$ and $N_{D}^{+}(u) \subseteq N_{D^{\prime}}^{+}(v)$. Take two vertices $x$ and $y$ in $D^{\prime}$. If $x \neq v$ and $y \neq v$, then $x$ and $y$ compete in $D$ and so in $D^{\prime}$. By symmetry, now we suppose $x=v$. Then $y \neq v$, so $y \in V(D)$. Since $D$ is a nontrivial competitive digraph, $N_{D}^{+}(y) \cap N_{D}^{+}(u) \neq \emptyset$. Then $N_{D^{\prime}}^{+}(x) \cap N_{D^{\prime}}^{+}(y)=N_{D^{\prime}}^{+}(v) \cap N_{D^{\prime}}^{+}(y) \supseteq N_{D}^{+}(u) \cap N_{D}^{+}(y) \neq \emptyset$ and so $x$ and $y$ compete in $D^{\prime}$.

Proposition 2.2. Let $D$ be a competitive digraph. Then the following are true:
(1) if $D$ is a nontrivial digraph and has a vertex $v$ of indegree at most 1 , then $D-v$ is competitive;
(2) if $D_{v}$ is the subdigraph of $D$ induced by $N^{+}(v)$ for $a$ vertex $v$ in $D$, then each vertex has outdegree at least one in $D_{v}$;
(3) each vertex in $D$ has outdegree at least 3, especially, if a vertex $u$ has outdegree 3 in $D$, then its out-neighbors form a directed cycle;
(4) there exist at least $\max \{4|V(D)|-|A(D)|, 0\}$ vertices of outdegree 3 in $D$.

Proof. The statement (1) is obviously true.
To show the statement (2), take a vertex $v$ in $D$. Let $D_{v}$ is the subdigraph of $D$ induced by $N^{+}(v)$. If there exists an out-neighbor $w$ of $v$ which has out-degree 0 in $D_{v}$, then $v$ and $w$ cannot compete in $D$, a contradiction. The statement (3) is an immediate consequence of the statement (2).

Let $l$ be the number of vertices of outdegree 3 . Since each vertex in $D$ has outdegree at least 3 by the statement (3),

$$
4(|V(D)|-l)+3 l \leq|A(D)| .
$$

Therefore $4|V(D)|-|A(D)| \leq l$. Thus the statement (4) is true.
The following is an immediate consequence of Proposition 2.2(2).
Corollary 2.3. There is no competitive bipartite tournament.
Proposition 2.2 may be rephrased as graph version in the following.

Theorem 2.4. Let $G$ be a competitively orientable graph. Then the following are true:
(1) each vertex in $G$ has at least three neighbors, especially, if a vertex has exactly three neighbors, then its neighbors form a clique;
(2) if $G$ is nontrivial and has a vertex $v$ of degree at most 4 , then $G-v$ is a competitively orientable graph;
(3) $|V(G)| \geq 7$ and $|E(G)| \geq 3|V(G)|$.

Proof. Let $D$ be a competitive orientation of $G$. Then the statement (1) is immediately true by Proposition 2.2(3).

To show the statement (2), suppose there exists a vertex $v$ of degree at most 4 . Then, by Proposition [2.2(3), $v$ has indegree at most 1 . Therefore $D-v$ is competitive by Proposition [2.2(1). Thus $G-v$ is competitively orientable and so the statement (2) is true.

Since each vertex in $D$ has outdegree at least 3 by Proposition [2.2(3), $|A(D)| \geq$ $3|V(D)|$ and so $|E(G)| \geq 3|V(G)|$. By the way, since $G$ is simple, $|E(G)| \leq\binom{|V(G)|}{2}$. Therefore

$$
3|V(G)| \leq \frac{|V(G)|(|V(G)|-1)}{2}
$$

Thus $|V(G)| \geq 7$.
Remark 2.5. The inequality $|V(G)| \geq 7$ and $|E(G)| \geq 3|V(G)|$ given in Theorem [2.4 is tight. By the way, for each integer $m \geq 7$, there exists a competitively orientable graph of order $m$ with $3 m$ edges.

Proof. Take an integer $m \geq 7$ and a digraph $D$ given in Figure 1, which is a competitive orientation of $K_{7}$. We note that each vertex in $D$ has outdegree 3. Since $\left|E\left(K_{7}\right)\right|=$ $21=3\left|V\left(K_{7}\right)\right|, K_{7}$ is the desired one for $m=7$. Now we assume $m \geq 8$. We begin with $D$ to construct a desired digraph. Take a vertex $u$ in $D$. Then $\left|N_{D}^{+}(u)\right|=3$. Inductively, we identify $D_{0}$ with $D_{0}$ and competitively expand $D_{i}$ from $D_{i-1}$ via $u$ by a new vertex $v_{i}$ so that $N_{D_{i}}^{+}\left(v_{i}\right)=N_{D}^{+}(u)$ and $N_{D_{i}}^{-}\left(v_{i}\right)=\emptyset$ for each $1 \leq i \leq m-7$. Then $\left|A\left(D_{i}\right)\right|=|A(D)|+3 i=3(7+i)=3\left|V\left(D_{i}\right)\right|$ and $D_{i}$ is competitive for each $1 \leq i \leq m-7$ by Proposition 2.1. Therefore the underlying graph of $D_{m-7}$ is the desired one.

We make a useful observation as follows.
Theorem 2.6. Let $G$ be a competitively orientable graph and $G^{\prime}$ be a supergraph of $G$ such that for each vertex $v$ in $G^{\prime}$, there exists a vertex $u$ in $G$ satisfying $N_{G}(u) \subset N_{G^{\prime}}(v)$. Then $G^{\prime}$ is also competitively orientable.

Proof. Suppose that $D$ is a competitive orientation of $G$. If $V\left(G^{\prime}\right)=V(G)$, then each orientation $D^{\prime}$ of $G^{\prime}$ obtained by orienting edges in $E\left(G^{\prime}\right) \backslash E(G)$ arbitrarily so that $A(D) \subset A\left(D^{\prime}\right)$ is competitive.

Suppose $V\left(G^{\prime}\right) \neq V(G)$. Then $V\left(G^{\prime}\right) \backslash V(G)=\left\{v_{1}, \ldots, v_{k}\right\}$ for a positive integer $k$. By the hypothesis, there exists a vertex $u_{i}$ in $G$ such that $N_{G}\left(u_{i}\right) \subset N_{G^{\prime}}\left(v_{i}\right)$ for each $1 \leq i \leq k$. Let $G_{0}=G, D_{0}=D$, and $G_{i}=G^{\prime}\left[V(G) \cup\left\{v_{1}, \ldots, v_{i}\right\}\right]$ for each $1 \leq i \leq k$. Then the orientation $D_{i}$ of $G_{i}$ obtained by orienting edges in $E\left(G_{i}\right) \backslash E\left(G_{i-1}\right)$ arbitrarily as long as $A\left(D_{i-1}\right) \subset A\left(D_{i}\right)$ and $N_{D_{i-1}}^{+}\left(u_{i}\right) \subset N_{D_{i}}^{+}\left(v_{i}\right)$ is competitive for each $1 \leq i \leq k$ by Proposition 2.1. Therefore $D_{k}$ is a competitive orientation of $G^{\prime}$.

The following are immediate consequences of Theorem 2.6. Especially, Corollary 2.7 plays a key role throughout this paper.

Corollary 2.7. Let $k$ and $l$ be positive integers with $l \geq k \geq 3 ; n_{1}, \ldots, n_{k}$ be positive integers such that $n_{1} \geq \cdots \geq n_{k} ; n_{1}^{\prime}, \ldots, n_{l}^{\prime}$ be positive integers such that $n_{1}^{\prime} \geq \cdots \geq n_{l}^{\prime}$, $n_{1}^{\prime} \geq n_{1}, n_{2}^{\prime} \geq n_{2}, \ldots$, and $n_{k}^{\prime} \geq n_{k}$. If $K_{n_{1}, \ldots, n_{k}}$ is competitively orientable, then $K_{n_{1}^{\prime}, \ldots, n_{l}^{\prime}}$ is also competitively orientable.

Corollary 2.8. Let $k$ be a positive integer with $k \geq 3 ; n_{1}, \ldots, n_{k}, n_{k}^{\prime}, n_{k+1}^{\prime}$ be positive integers such that $n_{k}=n_{k}^{\prime}+n_{k+1}^{\prime}$. If $K_{n_{1}, \ldots, n_{k-1}, n_{k}}$ is competitively orientable, then $K_{n_{1}, \ldots, n_{k-1}, n_{k}^{\prime}, n_{k+1}^{\prime}}$ is also competitively orientable.

### 2.2 Competitive multipartite tournaments

Proposition 2.9. Suppose that $D$ is a competitive multipartite tournament. If the outneighbors of a vertex $v$ are included in exactly two partite sets $U$ and $V$ of $D$, then $\left|N^{+}(v) \cap U\right| \geq 2$ and $\left|N^{+}(v) \cap V\right| \geq 2$.

Proof. Suppose that there exists a vertex $v$ whose out-neighbors are included in exactly two partite sets $U$ and $V$ of $D$. If $N^{+}(v) \cap U=\{u\}$ for some vertex $u$ in $D$, then $u$ is a common out-neighbor of each vertex in $N^{+}(v) \cap V$ and $v$, and so $u$ has no out-neighbor in $N^{+}(v)$, which contradicts Proposition $2.2(2)$. Therefore $\left|N^{+}(v) \cap U\right| \geq 2$. By symmetry, $\left|N^{+}(v) \cap V\right| \geq 2$. Thus the statement is true.

Lemma 2.10. For $k \in\{5,6\}$, if a competitive $k$-partite tournament $D$ of order 8 has at least two vertices of outdegree at least 4 , then $k=6$ and $D$ is an orientation of $K_{2,2,1,1,1,1}$ in which there exist exactly two vertices of outdegree at least 4.

Proof. Suppose that a competitive $k$-partite $D$ has 8 vertices at least two of which have outdegree at least 4 for some $k \in\{5,6\}$. Suppose $k=5$. It is easy to check that the numbers of arcs in $K_{4,1,1,1,1}, K_{3,2,1,1,1}$, and $K_{2,2,2,1,1}$ are 22,24 , and 25 , respectively. Therefore $|A(D)|$ becomes maximum when $D$ is an orientation of $K_{2,2,2,1,1}$, so $|A(D)| \leq 25$.


Figure 2: The subdigraph $\tilde{D}$ obtained in the proof of Theorem 2.11

By Proposition [2.2(4), there exist at least $\max \{4|V(D)|-|A(D)|, 0\}$ vertices of outdegree 3 in $D$, so at least 7 vertices have outdegree 3 in $D$. Therefore there exists at most one vertex of outdegree at least 4 , which is a contradiction to the hypothesis. Thus $k=6$ and $D$ is an orientation of $K_{3,1,1,1,1,1}$ or $K_{2,2,1,1,1,1}$. If $D$ is an orientation of $K_{3,1,1,1,1,1}$, then $|A(D)|=25$ and so, by the same reason, we reach a contradiction. Thus $D$ is an orientation of $K_{2,2,1,1,1,1}$. By the way, $D$ has 8 vertices and 26 arcs, so $4|V(D)|-|A(D)|=6$. Then there exist at least 6 vertices of outdegree 3 by Proposition 2.2(4). Therefore $D$ has exactly two vertices of outdegree at least 4.

Theorem 2.11. Suppose that $D$ is a competitive $k$-partite tournament for some integer $k \in\{4,5,6\}$ which has a vertex u of outdegree 3 . Then $D$ contains a subdigraph isomorphic to the digraph $\tilde{D}$ in Figure 圆 and $|V(D)| \geq 9$. In particular, if $k=4$, then $|V(D)| \geq 10$.

Proof. Each pair of vertices has a common out-neighbor in $D$ since $D$ is competitive. Let $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\}$. Without loss of generality, we may assume that $C:=v_{1} \rightarrow$ $v_{2} \rightarrow v_{3} \rightarrow v_{1}$ is a directed cycle of $D$ by Proposition [2.2(3). Let $w_{i}$ be a common outneighbor of $v_{i}$ and $v_{i+1}$ for each $1 \leq i \leq 3$ (identify $v_{4}$ with $v_{1}$ ). If $w_{j}=w_{k}$ for some distinct $j, k \in\{1,2,3\}$, then $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\} \subseteq N^{-}\left(w_{j}\right)$ and so $w_{j}$ does not share a common out-neighbor with $u$, which is a contradiction. Therefore $w_{1}, w_{2}$, and $w_{3}$ are all distinct. Moreover, since $u$ and $w_{i}$ share a common out-neighbor for each $1 \leq i \leq 3$, $\left\{\left(w_{1}, v_{3}\right),\left(w_{2}, v_{1}\right),\left(w_{3}, v_{2}\right)\right\} \subset A(D)$. Thus, so far, we have a subdigraph $\tilde{D}$ of $D$ with the vertex set $\left\{u, v_{1}, v_{2}, v_{3}, w_{1}, w_{2}, w_{3}\right\}$ given in Figure 2,

If $|V(D)|=7$, then the underlying graph of $D$ must have 21 edges by Theorem [2.4(3) and so $D$ is a 7-partite tournament, which is a contradiction. Thus $|V(D)| \geq 8$. To reach a contradiction, suppose that $|V(D)|=8$. Then $V(D)=V(\tilde{D}) \cup\{x\}$ for some vertex $x$


Figure 3: The subdigraphs $D_{1}$ and $D_{2}$ considered in the proof of Theorem 2.11
in $D$ and

$$
\begin{equation*}
\left|N^{+}\left(v_{i}\right)\right|=3 \text { or } 4 \tag{1}
\end{equation*}
$$

for each $1 \leq i \leq 3$. Since $x$ and $u$ must compete and $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\}$, one of $v_{1}, v_{2}$, $v_{3}$ is a common out-neighbor of $u$ and $x$. Without loss of generality, we may assume $v_{1}$ is a common out-neighbor of $x$ and $u$. Then

$$
N^{+}\left(v_{1}\right)=\left\{v_{2}, w_{1}, w_{3}\right\} \text { and }\left(x, v_{1}\right) \in A(D)
$$

By Proposition2.2(3), the out-neighbors of $v_{1}$ form a directed cycle. Therefore $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{3}\right\}$ forms a 5 -tournament in $D$, so

$$
k \geq 5
$$

By the way, since $\left(w_{3}, v_{2}\right)$ and $\left(v_{2}, w_{1}\right)$ are arcs of $D$,

$$
\left(w_{1}, w_{3}\right) \in A(D)
$$

(see the digraph $D_{2}$ given in Figure 3 for an illustration). Since $v_{1}$ and $w_{2}$ compete and $N^{+}\left(v_{1}\right)=\left\{v_{2}, w_{1}, w_{3}\right\}$,

$$
\begin{equation*}
N^{+}\left(w_{2}\right) \cap\left\{w_{1}, w_{3}\right\} \neq \emptyset \tag{2}
\end{equation*}
$$

We first claim that $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{2}, w_{3}\right\}$ forms a tournament in $D$. Since $D_{2}$ is a subgraph of $D$, we need to show that $\left\{w_{1}, w_{2}, w_{3}\right\}$ forms a tournament in $D$. As we have shown that $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{3}\right\}$ is a tournament in $D$, it remains to show that $w_{2}$ is adjacent to $w_{1}$ and $w_{3}$ in $D$.

Suppose, to the contrary, that there is no arc between $w_{1}$ and $w_{2}$. Then $\left(w_{2}, w_{3}\right) \in$ $A(D)$ by (2). Then the vertices $v_{1}, w_{2}, w_{3}$ cannot form a directed cycle. Yet, $v_{1}, w_{2}$, $w_{3}$ are out-neighbors of $v_{3}$, so $\left|N^{+}\left(v_{3}\right)\right|=4$ by (1) and Proposition [2.2(3). Since $x$
is the only possible new out-neighbor of $v_{3}$ in $D, N^{+}\left(v_{3}\right)=\left\{v_{1}, w_{2}, w_{3}, x\right\}$. Since $x$ is the only possible common out-neighbor of $w_{2}$ and $v_{2}, N^{+}\left(w_{2}\right) \cap N^{+}\left(v_{2}\right)=\{x\}$. Thus $N^{+}\left(v_{2}\right)=\left\{x, v_{3}, w_{1}, w_{2}\right\}$ and $\left\{v_{1}, w_{3}, x\right\} \subseteq N^{+}\left(w_{2}\right)$. Since $v_{2}$ and $v_{3}$ have outdegree $4, D$ is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Then $\left\{w_{1}, w_{2}\right\}$ forms a partite set of $D$. Since $u$ has outdegree 3 in $D,\left(w_{2}, u\right) \in A(D)$ and so $\left\{u, v_{1}, w_{3}, x\right\} \subseteq N^{+}\left(w_{2}\right)$. Then $v_{2}$, $v_{3}$, and $w_{2}$ have outdegree at least 4 , which contradicts Lemma 2.10. Thus there is an arc between $w_{1}$ and $w_{2}$.

Now we suppose, to the contrary, that there is no arc between $w_{2}$ and $w_{3}$. Then $v_{1}$, $w_{2}, w_{3}$ cannot form a directed cycle. Since they are out-neighbors of $v_{3},\left|N^{+}\left(v_{3}\right)\right|=4$ by (11) and Proposition [2.2(3) and so $N^{+}\left(v_{3}\right)=\left\{v_{1}, w_{2}, w_{3}, x\right\}$. Since there is no arc between $w_{2}$ and $w_{3}$, there is an $\operatorname{arc}\left(w_{2}, w_{1}\right)$ in $D$ by (2). For the same reason, $x$ is the only possible common out-neighbor of $w_{3}$ and $v_{2}$, so $N^{+}\left(w_{3}\right) \cap N^{+}\left(v_{2}\right)=\{x\}$. Thus $v_{2}$ has outdegree 4 by (11). Since $v_{3}$ also has outdegree $4, D$ is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Thus $\left\{w_{2}, w_{3}\right\}$ is a partite set of $D$. Since $N^{+}\left(v_{1}\right)=\left\{v_{2}, w_{1}, w_{3}\right\}$ and $N^{+}\left(w_{3}\right) \cap N^{+}\left(v_{2}\right)=\{x\},\left(x, w_{1}\right)$ must be an arc of $D$ in order for $v_{1}$ and $x$ to compete. Since $u$ is the only possible common out-neighbor of $x$ and $w_{1}$, there exist $\operatorname{arcs}(x, u)$ and $\left(w_{1}, u\right)$ in $D$. Then $\left\{x, u, v_{1}, v_{2}, v_{3}, w_{1}\right\}$ forms a tournament and we reach a contradiction to the fact that $D$ is an orientation of $K_{2,2,1,1,1,1}$ with $\left\{w_{2}, w_{3}\right\}$ as a partite set of $D$. Therefore $\left\{v_{1}, v_{2}, v_{3}, w_{1}, w_{2}, w_{3}\right\}$ forms a tournament as we claimed. Thus $k=6$ and each of $u$ and $x$ belongs to a partite set of size at least 2 . Furthermore, since $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\}, u$ cannot form a partite set with $v_{1}, v_{2}$, or $v_{3}$ and so $u$ and exactly one of $w_{1}, w_{2}$, and $w_{3}$ belong to the same partite set.

Suppose, to the contrary, that $\left(w_{3}, w_{2}\right) \in A(D)$. Then $\left(w_{2}, w_{1}\right) \in A(D)$ by (2). Therefore $w_{1} \rightarrow w_{3} \rightarrow w_{2} \rightarrow w_{1}$ forms a directed cycle. Then, for each pair of $w_{1}, w_{2}$, and $w_{3}, x$ and $u$ are its only possible common out-neighbors. Since $u$ and one of $w_{1}, w_{2}$, and $w_{3}$ belong to the same partite set, exactly one pair of $w_{1}, w_{2}$, and $w_{3}$ can prey on $u$. Then the other two pair of $w_{1}, w_{2}$, and $w_{3}$ prey on $x$. Therefore $\left\{w_{1}, w_{2}, w_{3}\right\} \subseteq N^{-}(x)$. Thus $x$ and exactly one of $v_{2}$ and $v_{3}$ belong to the same partite set (recall that we assumed $\left.\left(x, v_{1}\right) \in A(D)\right)$. Let $v_{j+1}$ be the vertex $j \in\{1,2\}$ belonging to the same partite set with $x$. Then, since $\left\{u, v_{j-1}\right\} \subseteq N^{-}\left(v_{j}\right)$ (identify $v_{0}$ with $v_{3}$ ) and $\left\{w_{1}, w_{2}, w_{3}\right\} \subseteq N^{-}(x), x$ and $v_{j}$ have no common out-neighbor in $D$, which is a contradiction. Therefore ( $w_{3}, w_{2}$ ) $\notin A(D)$ and so

$$
\left(w_{2}, w_{3}\right) \in A(D)
$$

Thus $N^{+}\left(w_{3}\right) \subseteq\left\{u, v_{2}, x\right\}$ and so, by Proposition [2.2(3), $N^{+}\left(w_{3}\right)=\left\{u, v_{2}, x\right\}$. Then $x$ is the only possible common out-neighbor of each pair of $v_{3}$ and $w_{3}$, and $v_{2}$ and $w_{3}$. Therefore $N^{+}\left(v_{3}\right)=\left\{v_{1}, w_{2}, w_{3}, x\right\}$ and $N^{+}\left(v_{2}\right)=\left\{v_{3}, w_{1}, w_{2}, x\right\}$ by (11). Thus $D$ is an orientation of $K_{2,2,1,1,1,1}$ by Lemma 2.10. Moreover, since $N^{+}\left(v_{1}\right)=\left\{v_{2}, w_{3}, w_{1}\right\}$, $w_{1}$ is the only possible common out-neighbor of $x$ and $v_{1}$ and so $\left(x, w_{1}\right) \in A(D)$. Then $u$ must be a common out-neighbor of $w_{1}$ and $w_{3}$. Therefore $\left\{w_{2}, u\right\}$ is a partite sets of size 2 in $D$. Then, since $\left\{\left(w_{3}, x\right),\left(v_{2}, x\right),\left(v_{2}, x\right),\left(x, v_{1}\right),\left(x, w_{1}\right)\right\} \subset A(D),\{x\}$ should be a partite set
of $D$ and so $k \geq 7$, which is a contradiction. Therefore we have shown that $|V(D)| \neq 8$ and so $|V(D)| \geq 9$.

To show the "particular" part, suppose $k=4$. Let $V_{1}, V_{2}, V_{3}$, and $V_{4}$ be the partite sets of $D$. By Proposition 2.2(3), $u, v_{1}, v_{2}$, and $v_{3}$ belong to distinct partite sets. Without loss of generality, we may assume that $u \in V_{1}, v_{1} \in V_{2}, v_{2} \in V_{3}$, and $v_{3} \in V_{4}$. Let $y_{i}$ be a common out-neighbor of $v_{i}$ and $w_{i}$ in $D$ for each $1 \leq i \leq 3$. If $y_{1}=y_{2}=y_{3}$, then $\left\{v_{1}, v_{2}, v_{3}\right\} \subseteq N^{-}\left(y_{1}\right)$, which implies that $u$ and $y_{1}$ do not share a common out-neighbor, and we reach a contradiction. Therefore at least two of $y_{1}, y_{2}$, and $y_{3}$ are distinct. In addition, we may see from a subdigraph $\tilde{D}$ given in Figure 2, that $\left\{u, w_{1}, w_{2}, w_{3}\right\} \subseteq V_{1}$. Therefore $y_{i}$ cannot be $w_{j}$ for each $1 \leq i, j \leq 3$. Suppose, to the contrary, that $|V(D)|=9$. Then exactly two of $y_{1}, y_{2}$, and $y_{3}$ are the same. Without loss of generality, we may assume $y_{1}=y_{2}$ and $y_{1} \neq y_{3}$. Neither $v_{1}$ nor $v_{2}$ is a common out-neighbor of $y_{1}$ and $u$. Thus $v_{3}$ must be a common out-neighbor of $y_{1}$ and $u$. Yet, $y_{1}$ is a common out-neighbor of $v_{1}, v_{2}$, $w_{1}$, and $w_{2}$, so $y_{1} \in V_{4}$ and we reach a contradiction. Thus $|V(D)| \geq 10$.

## 3 Proofs

### 3.1 A proof of Theorem 1

In this subsection, we characterize complete $k$-partite graphs which are competitively orientable for an integer $k \geq 6$.

Since $K_{1,1,1,1,1,1,1} \cong K_{7}$ and a competitive orientation of $K_{7}$ is given in Figure 1, the following proposition is immediately true by Corollary 2.7.

Proposition 3.1. For each integer $k \geq 7$, any $k$-partite complete graph is competitively orientable.

We now have completely characterized sizes of the partite sets of a competitive $k$ partite tournament for $k=2$ or $k \geq 7$ by Corollary 2.3 and Proposition 3.1. Accordingly, it remains to study competitive $k$-partite tournaments for $3 \leq k \leq 6$. Especially, in the rest of this section, we characterize competitively orientable complete 6-partite graphs.

Proposition 3.2. Let $D$ be a competitive $k$-partite tournament for some positive integer $k \geq 3$ with the partite sets $V_{1}, \ldots, V_{k}$. Then there exists a competitive $k$-partite tournament $D^{*}$ with the partite sets $V_{1}, \ldots, V_{k}$ such that each vertex in $D^{*}$ has indegree at least 2 .

Proof. Suppose that there exists a vertex $v_{1}$ of indegree at most 1 in $V_{t_{1}}$ for some $t_{1} \in\{1, \ldots, k\}$. Let $D_{1}=D-v_{1}$. Then $D_{1}$ is competitive by Proposition [2.2(1). By Corollary 2.3, $D_{1}$ is not a bipartite tournament. Suppose that there exists a vertex $v_{2}$ of indegree at most 1 in $V_{t_{2}}$ for some $t_{2} \in\{1, \ldots, k\}$ in $D_{1}$. Let $D_{2}=D_{1}-v_{2}$. Therefore $D_{2}$ is competitive by Proposition 2.2(1) and so, by Corollary 2.3, $D_{2}$ is not a bipartite
tournament. We keep repeating this process. Since $D$ has a finite number of vertices, this process terminates to produce digraphs $D_{1}, D_{2}, \ldots, D_{l}$ each of which is competitive and none of which is a bipartite tournament. Since $D_{l}$ is competitive, the number of partite sets in $D_{l}$ is at least 3. The fact that the process ended with $D_{l}$ implies that each vertex in $D_{l}$ has indegree at least 2 . As some of partite sets of $D_{l}$ are proper subsets of corresponding partite sets of $D$, we need to add vertices to obtain a desired $k$-partite tournament. Let $X$ be the partite set of $D_{l-1}$ to which $v_{l}$ belongs. Then $X \subseteq V_{t_{l}}$. In the following, we construct a multipartite tournament $D_{l-1}^{*}$ from $D_{l}$ such that $V\left(D_{l-1}\right)=V\left(D_{l-1}^{*}\right), D_{l-1}$ and $D_{l-1}^{*}$ have the identical partite sets, and $D_{l-1}^{*}$ is competitive. We consider two cases for $X$.

Case 1. $X=\left\{v_{l}\right\}$. We take a vertex $v^{\prime}$ in $D_{l}$. Then $v^{\prime}$ has indegree at least 2. Now we add $v_{l}$ to $D_{l}$ so that $\left\{v_{l}\right\}$ is a partite set of $D_{l-1}^{*}, v_{l}$ takes the out-neighbors and the in-neighbors of $v^{\prime}$ as its out-neighbors and in-neighbors, respectively, and the remaining out-neighbors and in-neighbors of $v_{l}$ are arbitrarily taken. Then the indegree of $v_{l}$ in $D_{l-1}^{*}$ is at least 2. Moreover,

$$
V\left(D_{l}\right) \cup\left\{v_{l}\right\}=V\left(D_{l-1}^{*}\right), \quad A\left(D_{l}\right) \subset A\left(D_{l-1}^{*}\right), \quad \text { and } \quad N_{D_{l}}^{+}\left(v^{\prime}\right) \subset N_{D_{l-1}^{*}}^{+}\left(v_{l}\right)
$$

Case 2. $\quad\left\{v_{l}\right\} \nsubseteq X$. Then there exists a vertex $v^{\prime}$ distinct from $v_{l}$ in $X$. Since $D_{l}=D_{l-1}-v_{l}, v^{\prime}$ is a vertex of $D_{l}$. Now we add $v_{l}$ to the partite set of $D_{l}$ where $v^{\prime}$ belongs so that $\left\{v_{l}, v^{\prime}\right\}$ is involved in a partite set of $D_{l-1}^{*}, v_{l}$ takes the out-neighbors and the in-neighbors of $v^{\prime}$ as its out-neighbors and in-neighbors, respectively. Then the indegree of $v_{l}$ in $D_{l-1}^{*}$ is at least 2 since the indegree of $v^{\prime}$ is at least 2 in $D_{l}$. Moreover,

$$
V\left(D_{l}\right) \cup\left\{v_{l}\right\}=V\left(D_{l-1}^{*}\right), \quad A\left(D_{l}\right) \subset A\left(D_{l-1}^{*}\right), \quad \text { and } \quad N_{D_{l}}^{+}\left(v^{\prime}\right) \subseteq N_{D_{l-1}^{*}}^{+}\left(v_{l}\right)
$$

In both cases, $D_{l-1}^{*}$ is competitive by Proposition 2.1,
Now we add $v_{l-1}$ to $D_{l-1}^{*}$ and apply an argument similar to the above one to obtain competitive multipartite tournament $D_{l-2}^{*}$ each vertex in which has indegree at least 2 . We may repeat this process until we obtain a competitive $k$-partite tournament $D_{0}^{*}$ each vertex of which has indegree at least 2 . Since we added $v_{i}$ to the partite set of $D_{i}^{*}$ which is included in $V_{t_{i}}$ for each $1 \leq i \leq l$, it is true that the partite sets of $D_{0}^{*}$ are the same as $D$. Thus $D_{0}^{*}$ is a desired $k$-partite tournament.

Proposition 3.3. The complete 6 -partite graph $K_{4,1,1,1,1,1}$ is not competitively orientable.
Proof. Suppose, to the contrary, that there exists a competitive orientation of $K_{4,1,1,1,1,1}$. Then, by Proposition 3.2, there exists a competitive orientation $D$ of $K_{4,1,1,1,1,1}$ each vertex of which has indegree at least 2 . Let $V_{1}, \ldots, V_{6}$ be the partite sets of $D$ with $\left|V_{1}\right|=4$. By Proposition 2.2(3), each vertex has outdegree at least 3 in $D$. Then, since each vertex has indegree at least 2 in $D$,

$$
\begin{equation*}
\left|N^{+}(v)\right|=3 \quad \text { and } \quad\left|N^{-}(v)\right|=2 \tag{3}
\end{equation*}
$$

for each vertex $v$ in $V_{1}$. By Proposition [2.2(4), there exist at least max $\{4|V(D)|-$ $|A(D)|, 0\}$ vertices of outdegree 3 in $D$. Since $4|V(D)|-|A(D)|=6$, there exist at least 6 vertices of outdegree 3. Thus at least two vertices of outdegree 3 do not belong to $V_{1}$. Let $u$ be a vertex of outdegree 3 which is not in $V_{1}$. Without loss of generality, we may assume $u \in V_{2}$.

Let $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\}$. By Proposition 2.2(3), $N^{+}(u)$ forms a directed cycle in $D$ and we may assume $v_{1} \rightarrow v_{2} \rightarrow v_{3} \rightarrow v_{1}$. Since each out-neighbor of $u$ has indegree at least 3 by Theorem 2.11, $N^{+}(u) \cap V_{1}=\emptyset$ by (3). Therefore we may assume that $V_{3}=\left\{v_{1}\right\}, V_{4}=\left\{v_{2}\right\}, V_{5}=\left\{v_{3}\right\}, V_{6}=\{x\}$, and $v_{1}$ is a common out-neighbor of $x$ and $u$. Then $\left\{u, v_{3}, x\right\} \subseteq N^{-}\left(v_{1}\right)$. Let $w_{1}$ be a common out-neighbor of $v_{1}$ and $v_{2}$. Then $w_{1} \in V_{1}$. Therefore, by (3), $N^{-}\left(w_{1}\right)=\left\{v_{1}, v_{2}\right\}$ and $N^{+}\left(w_{1}\right)=\left\{u, v_{3}, x\right\}$. Thus $N^{+}\left(w_{1}\right) \subseteq N^{-}\left(v_{1}\right)$ and so $w_{1}$ and $v_{1}$ have no common out-neighbor, which is a contradiction.

Let $D$ be a digraph with the vertex set $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and $A=\left(a_{i j}\right)$ be the adjacency matrix of $D$ such that

$$
a_{i j}= \begin{cases}1 & \text { if there is an arc }\left(v_{i}, v_{j}\right) \text { in } D \\ 0 & \text { otherwise }\end{cases}
$$

Now we are ready to prove Theorem 1 .
Proof of Theorem 1. To show the "only if" part, suppose that there exists a competitive orientation $D$ of $K_{n_{1}, n_{2}, \ldots, n_{6}}$. We suppose $n_{3}=1$.

Case 1. $n_{2}=1$. If $n_{1} \leq 4$, then there exists a competitive orientation of $K_{4,1,1,1,1,1}$ by Corollary [2.7, which contradicts Proposition 3.3. Therefore $n_{1} \geq 5$.

Case 2. $n_{2} \geq 2$. Then $n_{1} \geq 2$. Suppose, to the contrary, that $n_{1}=2$. Then $n_{2}=2$, so $D$ is an orientation of $K_{2,2,1,1,1,1}$. Therefore $4|V(D)|-|A(D)|=6$. By Proposition [2.2(4), there exists a vertex of outdegree 3 in $D$. Therefore $|V(D)| \geq 9$ by Theorem 2.11, which is a contradiction. Thus $n_{1} \geq 3$. Hence the "only if" part is true.

Now we show the "if" part. Let $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$ be the digraphs whose adjacency matrix are $A_{1}, A_{2}$, and $A_{3}$, respectively, given in Figure 4. It is easy to check that the inner product of each pair of rows in each matrix is nonzero, so $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$ are competitive. By applying Corollary 2.7 to $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$, we may obtain competitive orientations $D_{\alpha}^{\prime}, D_{\beta}^{\prime}$, and $D_{\gamma}^{\prime}$ of $K_{n_{1}, n_{2}, n_{3}, n_{4}, n_{5}, n_{6}}$ for (a) $n_{1} \geq 5$ and $n_{2}=1$; (b) $n_{1} \geq 3$, $n_{2} \geq 2$; (c) $n_{3} \geq 2$, respectively. Therefore we have shown that the "if" part is true.

### 3.2 A proof of Theorem 2

In this subsection, we characterize complete tripartite graphs which are competitively orientable.

$$
\begin{aligned}
& A_{1}=\left(\begin{array}{llllllllll}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0
\end{array}\right) \\
& A_{2}=\left(\begin{array}{lllllllll}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right) \\
& A_{3}=\left(\begin{array}{lllllllll}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right)
\end{aligned}
$$

Figure 4: The adjacency matrices $A_{1}, A_{2}$, and $A_{3}$ which are orientations of $K_{5,1,1,1,1,1}$, $K_{3,2,1,1,1,1}$, and $K_{2,2,2,1,1,1}$, respectively, in the proof of Theorem 1

Proposition 3.4. Let $D$ be a competitive orientation of $K_{n_{1}, n_{2}, n_{3}}$ for some positive integers $n_{1}, n_{2}$, and $n_{3}$. Then $n_{i} \geq 4$ for each $1 \leq i \leq 3$.

Proof. Let $V_{1}, V_{2}$, and $V_{3}$ be the partite sets of $D$ with $\left|V_{i}\right|=n_{i}$ for each $1 \leq i \leq 3$. Suppose, to the contrary, that $n_{j} \leq 3$ for some $j \in\{1,2,3\}$. Without loss of generality, we may assume that $n_{1} \leq 3$. Take $v_{1} \in V_{1}$. By Proposition $2.2(2)$, the out-neighbors of each vertex in $D$ are included in at least two partite sets. Then, since $D$ is tripartite tournament, the out-neighbors of each vertex in $D$ are included in exactly two partite sets. Thus, by Proposition [2.9, there are four vertices $u_{1}, u_{2}, w_{1}$, and $w_{2}$ such that $\left\{u_{1}, u_{2}\right\} \subseteq$ $N^{+}\left(v_{1}\right) \cap V_{2}$ and $\left\{w_{1}, w_{2}\right\} \subseteq N^{+}\left(v_{1}\right) \cap V_{3}$. By the same proposition, there are two vertices $v_{2}$ and $v_{3}$ in $V_{1}$ such that $\left\{v_{2}, v_{3}\right\} \subseteq N^{+}\left(u_{1}\right) \cap V_{1}$. Since $\left(v_{1}, u_{1}\right) \in A(D), v_{2}$ and $v_{3}$ are distinct from $v_{1}$. Since $n_{1} \leq 3, n_{1}=3$. Then $V_{1}=\left\{v_{1}, v_{2}, v_{3}\right\}$. Therefore, by Proposition [2.9, $N^{+}(v) \cap V_{1}=\left\{v_{2}, v_{3}\right\}$ for each vertex $v$ in $N^{+}\left(v_{1}\right)$. This implies that each out-neighbor of $v_{1}$ has $v_{2}$ as its out-neighbor. Therefore $v_{1}$ and $v_{2}$ cannot compete, which is a contradiction.
Lemma 3.5. If $D$ is a competitive orientation of $K_{4,4,4}$ with the partite sets $V_{1}, V_{2}$, and $V_{3}$, then, for distinct $i, j \in\{1,2,3\}$ and each $u \in V_{i},\left|N^{+}(u) \cap V_{j}\right|=\left|N^{-}(u) \cap V_{j}\right|=2$.

Proof. Suppose that there exists a competitive orientation $D$ of $K_{4,4,4}$ with partite sets $V_{1}, V_{2}$, and $V_{3}$. Take distinct $i$ and $j$ in $\{1,2,3\}$. Then there are exactly 16 arcs between $V_{i}$ and $V_{j}$. On the other hand, by Proposition 2.9, for each $u \in V_{i}$ and $v \in V_{j}$,

$$
\left|N^{+}(u) \cap V_{j}\right| \geq 2 \quad \text { and } \quad\left|N^{+}(v) \cap V_{i}\right| \geq 2
$$

Therefore

$$
16=\sum_{u \in V_{i}}\left|N^{+}(u) \cap V_{j}\right|+\sum_{v \in V_{j}}\left|N^{+}(v) \cap V_{i}\right| \geq 16 .
$$

and so $\left|N^{+}(u) \cap V_{j}\right|=\left|N^{+}(v) \cap V_{i}\right|=2$ for each $u \in V_{i}$ and $v \in V_{j}$. Hence $\left|N^{+}(u) \cap V_{j}\right|=$ $\left|N^{-}(u) \cap V_{j}\right|=2$ for each $u \in V_{i}$.
Lemma 3.6. If $D$ is a competitive orientation of $K_{4,4,4}$ with the partite sets $V_{1}, V_{2}$, and $V_{3}$, then, for some distinct $i$ and $j$ in $\{1,2,3\}$, there is a pair of vertices $x$ and $y$ in $V_{i}$ such that $N^{+}(x) \cap V_{j}=N^{+}(y) \cap V_{j}$.

Proof. Suppose that there exists a competitive orientation $D$ of $K_{4,4,4}$ with the partite sets $V_{1}, V_{2}$, and $V_{3}$. Suppose, to the contrary, that, for distinct $i, j \in\{1,2,3\}$,

$$
\begin{equation*}
N^{+}(u) \cap V_{j} \neq N^{+}(v) \cap V_{j} \tag{4}
\end{equation*}
$$

for any pair of vertices $u$ and $v$ in $V_{i}$. Fix $i \in\{1,2,3\}$ and $u, v \in V_{i}$. Let $w$ and $z$ be the remaining vertices in $V_{i}$. Since $D$ is competitive, $u$ and $v$ have a common out-neighbor in $V_{j}$ for some $j \in\{1,2,3\} \backslash\{i\}$. By Lemma 3.5] and (4), $N^{+}(u) \cap V_{j}=\left\{v_{1}, v_{2}\right\}$ and

$$
\begin{equation*}
N^{+}(v) \cap V_{j}=\left\{v_{1}, v_{3}\right\} \tag{5}
\end{equation*}
$$

for distinct vertices $v_{1}, v_{2}$, and $v_{3}$ in $V_{j}$. Then, by Lemma 3.5, $N^{+}\left(v_{1}\right) \cap V_{i}=\{w, z\}$. Let $v_{4}$ be the remaining vertex in $V_{j}$. Then, by Lemma 3.5 again, $N^{-}(u) \cap V_{j}=\left\{v_{3}, v_{4}\right\}$ and $N^{-}(v) \cap V_{j}=\left\{v_{2}, v_{4}\right\}$. Therefore $N^{-}\left(v_{4}\right) \cap V_{i}=\{w, z\}$ by the same lemma and so $N^{+}\left(v_{4}\right) \cap V_{i}=\{u, v\}$. Thus $v_{1}$ and $v_{4}$ cannot have a common out-neighbor in $V_{i}$. Hence they have a common out-neighbor in $V_{k}$ for $k \in\{1,2,3\} \backslash\{i, j\}$. By Lemma 3.5 and (4) again, $N^{+}\left(v_{1}\right) \cap V_{k}=\left\{w_{1}, w_{2}\right\}$ and $N^{+}\left(v_{4}\right) \cap V_{k}=\left\{w_{1}, w_{3}\right\}$ for distinct vertices $w_{1}, w_{2}$, and $w_{3}$ in $V_{k}$. Let $w_{4}$ be the remaining vertex in $V_{k}$. Then, by Lemma 3.5,

$$
N^{-}\left(v_{1}\right) \cap V_{k}=\left\{w_{3}, w_{4}\right\} \quad \text { and } \quad N^{-}\left(v_{4}\right) \cap V_{k}=\left\{w_{2}, w_{4}\right\} .
$$

Meanwhile we note that $w_{2}$ and $u$ have a common out-neighbor in $V_{j}$. Since $N^{+}(u) \cap V_{j}=$ $\left\{v_{1}, v_{2}\right\}$ and $N^{+}\left(v_{1}\right) \cap V_{k}=\left\{w_{1}, w_{2}\right\}, v_{2}$ is a common out-neighbor of $w_{2}$ and $u$. Then $N^{+}\left(w_{2}\right) \cap V_{j}=\left\{v_{2}, v_{4}\right\}$ by Lemma 3.5. Thus $w_{2}$ and $v$ cannot have a common out-neighbor in $V_{j}$ by (5). Since $w_{2}$ and $v$ belong to $V_{k}$ and $V_{i}$, respectively, they cannot compete in $D$ and we reach a contradiction.

Theorem 3.7. The complete tripartite graph $K_{4,4,4}$ is not competitively orientable.
Proof. Suppose, to the contrary, that there exists a competitive orientation $D$ of $K_{4,4,4}$. Let $V_{1}, V_{2}$, and $V_{3}$ be the partite sets of $D$. Then, by Lemmas 3.5 and 3.6, for some distinct $i$ and $j$ in $\{1,2,3\}$, there is a pair of vertices $u_{1}$ and $u_{2}$ in $V_{i}$ such that $N^{+}\left(u_{1}\right) \cap V_{j}=$ $N^{+}\left(u_{2}\right) \cap V_{j}=\left\{v_{1}, v_{2}\right\}$ for some vertices $v_{1}$ and $v_{2}$ in $V_{j}$. Without loss of generality, we may assume that $i=1$ and $j=2$. Let $u_{3}$ and $u_{4}$ (resp. $v_{3}$ and $v_{4}$ ) be the remaining vertices in $V_{1}$ (resp. $V_{2}$ ). Then, by Lemma 3.5,

$$
\begin{aligned}
& N^{+}\left(u_{3}\right) \cap V_{2}=N^{+}\left(u_{4}\right) \cap V_{2}=\left\{v_{3}, v_{4}\right\}, \\
& N^{+}\left(v_{1}\right) \cap V_{1}=N^{+}\left(v_{2}\right) \cap V_{1}=\left\{u_{3}, u_{4}\right\}, \\
& N^{+}\left(v_{3}\right) \cap V_{1}=N^{+}\left(v_{4}\right) \cap V_{1}=\left\{u_{1}, u_{2}\right\}
\end{aligned}
$$

(see Figure 5 for an illustration). Therefore each of the following pairs does not have a common out-neighbor in $V_{2}:\left\{u_{1}, u_{3}\right\} ;\left\{u_{1}, u_{4}\right\} ;\left\{u_{2}, u_{3}\right\} ;\left\{u_{2}, u_{4}\right\}$. In addition, each of the following pairs does not have a common out-neighbor in $V_{1}:\left\{v_{1}, v_{3}\right\} ;\left\{v_{1}, v_{4}\right\} ;\left\{v_{2}, v_{3}\right\}$; $\left\{v_{2}, v_{4}\right\}$. Then each of these pairs has a common out-neighbor in $V_{3}$. Let $w_{1}, w_{2}, w_{3}$, and $w_{4}$ be the common out-neighbors of $\left\{u_{1}, u_{3}\right\},\left\{u_{1}, u_{4}\right\},\left\{u_{2}, u_{3}\right\}$, and $\left\{u_{2}, u_{4}\right\}$, respectively. Then, by Lemma 3.5, $w_{i} \neq w_{j}$ for distinct $i, j \in\{1,2,3,4\}$ and so $V_{3}=\left\{w_{1}, w_{2}, w_{3}, w_{4}\right\}$. Without loss of generality, we may assume that $w_{1}, w_{2}, w_{3}$, and $w_{4}$ are the common outneighbors of $\left\{v_{1}, v_{3}\right\},\left\{v_{1}, v_{4}\right\},\left\{v_{2}, v_{3}\right\}$, and $\left\{v_{2}, v_{4}\right\}$, respectively. Then, by Lemma 3.5, $w_{1}$ and $w_{2}$ are out-neighbors of $u_{1}$ and $w_{3}$ and $w_{4}$ are out-neighbors of $v_{2}$ in $V_{3}$, so $u_{1}$ and $v_{2}$ do not compete in $D$, which is a contradiction.

Now we are ready to prove Theorem 2.


Figure 5: The arcs between $V_{1}$ and $V_{2}$

Proof of Theorem 园. To show the "only if" part, suppose that $D$ is a competitive orientation of $K_{n_{1}, n_{2}, n_{3}}$. Then, by Proposition 3.4, $n_{i} \geq 4$ for each $1 \leq i \leq 3$. If $n_{1}=4$, then $n_{2}=n_{3}=4$, which contradicts Theorem 3.7. Therefore $n_{1} \geq 5$ and so the "only if" part is true.

Now we show the "if" part. Let $D_{\alpha}$ be the digraph whose adjacency matrix is $A_{4}$ given in Figure 6. It is easy to check that $D_{\alpha}$ is an orientation of $K_{5,4,4}$ and the inner product of each pair of rows in each matrix is nonzero, so $D_{\alpha}$ is competitive. If $n_{1} \geq 5$ and $n_{3} \geq 4$, then, by applying Corollary 2.7 to $D_{\alpha}$, we obtain a competitive orientation $D_{\alpha}^{\prime}$ of $K_{n_{1}, n_{2}, n_{3}}$.

### 3.3 Proofs of Theorems 3 and 4]

In this subsection, we characterize complete $k$-partite graphs which are competitively orientable for the cases $k=4$ and $k=5$.

Proposition 3.8. Any competitive $k$-partite tournament for $k \in\{4,5\}$ has at most $k-3$ singleton partite sets.

Proof. Let $V_{1}, V_{2}, \ldots, V_{k}$ be the partite sets of a competitive $k$-partite tournament $D$ for some $k \in\{4.5\}$. We may assume that $x \in V_{1}, y \in V_{2}$, and $z \in V_{3}$.

We suppose $k=4$. To reach a contradiction, suppose that there are at least 2 partite sets of size 1. Without loss of generality, we may assume $\left|V_{1}\right|=\left|V_{2}\right|=1$. Then $V_{1}=\{x\}$ and $V_{2}=\{y\}$. Without loss of generality, we may assume $z$ is a common out-neighbor of $x$ and $y$. Then $N^{+}(z) \subseteq V_{4}$, which contradicts Proposition 2.2(2). Therefore $D$ has at most 1 partite set of size 1 .

Suppose $k=5$. To reach a contradiction, suppose that there are at least 3 partite sets of size 1. Without loss of generality, we may assume $\left|V_{1}\right|=\left|V_{2}\right|=\left|V_{3}\right|=1$. Then

$$
A_{4}=\left(\begin{array}{lllllllllllll}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Figure 6: The adjacency matrix $A_{4}$ which is an orientation of $K_{5,4,4}$ in the proof of Theorem 2
$V_{1}=\{x\}, V_{2}=\{y\}$, and $V_{3}=\{z\}$. Suppose that $x$ and $y$ have a common out-neighbor $w$ in $V_{4} \cup V_{5}$. Without loss of generality, we may assume $w \in V_{4}$. Then $N^{+}(w) \subseteq V_{3} \cup V_{5}$. By Proposition[2.2(2), $N^{+}(w) \cap V_{3} \neq \emptyset$ and $N^{+}(w) \cap V_{5} \neq \emptyset$. However, $N^{+}(w) \cap V_{3}=\{z\}$, which contradicts Proposition [2.9, Thus $w \notin V_{4} \cup V_{5}$ and so $w=z$. By symmetry, the only possible common out-neighbor of $y$ and $z$ is $x$. Since $z \in N^{+}(x), x$ cannot be an out-neighbor of $z$ and we reach a contradiction. Therefore $D$ has at most 2 partite sets having size 1.

By Theorem 2.11, the out-neighbors of a vertex of outdegree 3 in a competitive $k$ partite tournament for some $k \in\{4,5\}$ form a directed cycle and we have the following lemma.

Lemma 3.9. Let $D$ be a competitive $k$-partite tournament for some $4 \leq k \leq 5$. Suppose that a vertex u has outdegree 3. If $N^{+}(u) \subseteq U \cup V \cup W$ for distinct partite sets $U, V$, and $W$ of $D$, then $|U|+|V|+|W| \leq|V(D)|-4$.

Proof. Suppose that $N^{+}(u) \subseteq U \cup V \cup W$ for distinct partite sets $U, V$, and $W$ of $D$. Since $u$ has outdegree 3, by Theorem 2.11, $D$ contains a subdigraph isomorphic to $\tilde{D}$ given in Figure 2, We may assume that the subdigraph is $D_{1}$ itself including labels. We may assume $v_{1} \in U, v_{2} \in V$, and $v_{3} \in W$. Then $\left\{u, w_{1}, w_{2}, w_{3}\right\} \cap(U \cup V \cup W)=\emptyset$. Thus $|V(D) \backslash(U \cup V \cup W)| \geq 4$ and so $|U|+|V|+|W|=|U \cup V \cup W| \leq|V(D)|-4$.

Corollary 3.10. Neither $K_{3,3,2,2}$ nor $K_{3,3,3,1}$ is competitively orientable.

Proof. Suppose, to the contrary, that there exists a competitive orientation $D$ of $K_{3,3,2,2}$ or $K_{3,3,3,1}$. Then $|A(D)|<40$. If each vertex in $D$ has outdegree at least 4 , then $|A(D)| \geq 40$, which is a contradiction. Therefore there exists a vertex $u$ of outdegree 3, then, the outneighbors of $u$ belong to three distinct partite sets $U, V$, and $W$ by Proposition 2.2(3) and, by Lemma [3.9, $|U|+|V|+|W| \leq|V(D)|-4=6$, which is impossible.

Lemma 3.11. Let $n_{1}, n_{2}$, and $n_{3}$ be positive integers such that $n_{1} \geq n_{2} \geq n_{3}$. If $K_{n_{1}, n_{2}, n_{3}, 1}$ is competitively orientable, then $n_{3} \geq 3$.

Proof. Suppose that there exists a competitive orientation $D$ of $K_{n_{1}, n_{2}, n_{3}, 1}$. Then $n_{3} \geq 2$ by Proposition 3.8. Suppose, to the contrary, that $n_{3}=2$. Let $V_{1}, \ldots, V_{4}$ be the partite sets of $D$ satisfying $\left|V_{1}\right|=n_{1},\left|V_{2}\right|=n_{2},\left|V_{3}\right|=2$, and $\left|V_{4}\right|=1$. Let $V_{3}=\left\{x_{1}, x_{2}\right\}$ and $y$ be a common out-neighbor of $x_{1}$ and $x_{2}$. Then $V_{3} \cap N^{+}(y)=\emptyset$, so, by Proposition 2.2(2), $N^{+}(y)$ is included in exactly two partite sets. If $y \in V_{1} \cup V_{2}$, then $\left|N^{+}(y) \cap V_{4}\right|=1$, which contradicts Proposition 2.9. Therefore $y \notin V_{1} \cup V_{2}$ and so $y \in V_{4}$. Thus $V_{4}=\{y\}$. Hence $N^{+}(y) \subseteq V_{1} \cup V_{2}$. Take a vertex $u$ in $N^{+}(y)$. Then $N^{+}(u) \subseteq V_{1} \cup V_{3}$ or $V_{2} \cup V_{3}$. Therefore $N^{+}(u)$ is included in exactly two partite sets by Proposition 2.2(2). Since $\left|V_{3}\right|=2$, $N^{+}(u) \cap V_{3}=V_{3}$, that is, $u$ is a out-neighbor of neither $x_{1}$ nor $x_{2}$, by Proposition 2.9, Since $u$ was arbitrarily chosen in $N^{+}(y)$, any out-neighbor of $y$ is a out-neighbor of neither $x_{1}$ nor $x_{2}$. Thus $x_{1}$ and $y$ have no common out-neighbor in $D$, which is a contradiction. Hence $n_{3} \neq 2$ and so $n_{3} \geq 3$.

Now we are ready to show Theorem 3.
Proof of Theorem 3. To show the "only if" part, suppose that $D$ is a competitive orientation of $K_{n_{1}, n_{2}, n_{3}, n_{4}}$.

Case 1. $n_{4}=1$. Then $n_{3} \geq 3$ by Lemma 3.11, so $n_{1} \geq 3$. If $n_{1}=3$, then $n_{1}=n_{2}=$ $n_{3}=3$ and so $D$ is an orientation of $K_{3,3,3,1}$, which contradicts Corollary 3.10. Therefore $n_{1} \geq 4$.

Case 2. $n_{4} \geq 2$. Then $n_{3} \geq 2$ and so (c) holds. Suppose $n_{3}=2$. Then $n_{4}=2$. If $n_{1}=3$, then, by applying Corollary 2.7 to $D$, we obtain a competitive orientation $D^{*}$ of $K_{3,3,2,2}$, which contradicts Corollary 3.10. Therefore $n_{1} \geq 4$. Thus the "only if" part is true.

Now we show the "if" part. Let $D_{\alpha}, D_{\beta}, D_{\gamma}$ be the digraphs whose adjacency matrices are $A_{5}, A_{6}$, and $A_{7}$, respectively, given in Figure 7. It is easy to check that $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$ are orientations of $K_{4,3,3,1}, K_{4,2,2,2}$, and $K_{3,3,3,2}$, respectively, and the inner product of each pair of rows in each matrix is nonzero, so $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$ are competitive. By applying Corollary 2.7 to $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$, we may obtain orientations $D_{\alpha}^{\prime}, D_{\beta}^{\prime}$, and $D_{\gamma}^{\prime}$ of $K_{n_{1}, n_{2}, n_{3}, n_{4}}$ each of which are competitive for (a) $n_{1} \geq 4, n_{3} \geq 3$, and $n_{4} \geq 1$; (b) $n_{1} \geq 4$, $n_{3}=2$, and $n_{4}=2$; (c) $n_{3} \geq 3$ and $n_{4} \geq 2$, respectively. Therefore we have shown that the "if" part is true.

$$
\begin{aligned}
& A_{5}=\left(\begin{array}{lllllllllll}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}\right) \\
& A_{6}=\left(\begin{array}{llllllllll}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right) \\
& A_{7}=\left(\begin{array}{lllllllllll}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0
\end{array}\right)
\end{aligned}
$$

Figure 7: The adjacency matrices $A_{5}, A_{6}, A_{7}$ which are orientations of $K_{4,3,3,1}, K_{4,2,2,2}$, and $K_{3,3,3,2}$ in the proof of Theorem 3,

In the following, we study 5-partite tournaments which are competitive.
Theorem 3.12. The complete 5-partite graph $K_{3,2,2,1,1}$ is not competitively orientable.
Proof. Suppose, to the contrary, that there exists a competitive orientation $D$ of $K_{3,2,2,1,1}$. Let $V_{1}, \ldots, V_{5}$ be the partite sets of $D$ with $\left|V_{1}\right|=3,\left|V_{2}\right|=\left|V_{3}\right|=2$, and $\left|V_{4}\right|=\left|V_{5}\right|=1$. Since $4|V(D)|-|A(D)|=5$,
$(\dagger)$ there exist at least 5 vertices of outdegree 3 in $D$
by Proposition [2.2(4). Take a vertex $u$ of outdegree 3. Then $D$ contains a subdigraph containing $u$ isomorphic to $\tilde{D}$ given in Figure 2 by Theorem 2.11. We may assume that the subdigraph is $D_{1}$ itself including labels. For each $i=1,2,3$, since $w_{i}$ is adjacent to each of $v_{1}, v_{2}$ and $v_{3}$ in $D$,
(§) $w_{i}$ cannot belong to a partite set containing an out-neighbor of $u$.
By Proposition 2.2(3), the out-neighbors $v_{1}, v_{2}$, and $v_{3}$ of $u$ belong to three distinct partite sets $U, V$, and $W$. By Lemma [3.9, $|U|+|V|+|W| \leq|V(D)|-4=5$. Therefore

$$
\begin{equation*}
\left|N^{+}(u) \cap V_{1}\right|=\left|N^{+}(u) \cap V_{4}\right|=\left|N^{+}(u) \cap V_{5}\right|=1 \tag{6}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|N^{+}(u) \cap V_{i}\right|=\left|N^{+}(u) \cap V_{j}\right|=\left|N^{+}(u) \cap V_{k}\right|=1 \tag{7}
\end{equation*}
$$

for $2 \leq i<j<k \leq 5$. We first show that each vertex in $V_{4} \cup V_{5}$ has outdegree at least 4 .
Suppose, to the contrary, that $V_{4} \cup V_{5}$ contains a vertex of outdegree at most 3. Then, by Proposition [2.2(3), the vertex has outdegree 3. We may regard it as $u$ since $u$ is a vertex of outdegree 3 arbitrarily chosen. Without loss of generality, we may assume $u \in V_{5}$. Then $N^{+}(u) \cap V_{5}=\emptyset$. Therefore (6) cannot happen and so (7) holds. Thus, without loss of generality, we may assume that $v_{1} \in V_{2}, v_{2} \in V_{3}$, and $v_{3} \in V_{4}$. By (§), $V_{1}=\left\{w_{1}, w_{2}, w_{3}\right\}$. Let $V_{2}=\left\{v_{1}, x_{1}\right\}$ and $V_{3}=\left\{v_{2}, x_{2}\right\}$. Then

$$
N^{-}(u)=\left\{w_{1}, w_{2}, w_{3}, x_{1}, x_{2}\right\}
$$

Since $x_{1}$ is the only possible common out-neighbor of each of pairs $\left\{v_{2}, w_{2}\right\}$ and $\left\{v_{2}, w_{3}\right\}$,

$$
\begin{equation*}
\left\{v_{2}, w_{2}, w_{3}\right\} \subseteq N^{-}\left(x_{1}\right) \tag{8}
\end{equation*}
$$

In addition, $x_{2}$ is the only possible common out-neighbor of each of pairs $\left\{v_{1}, w_{1}\right\}$ and $\left\{v_{1}, w_{2}\right\}$. Therefore $\left\{v_{1}, w_{1}, w_{2}\right\} \subseteq N^{-}\left(x_{2}\right)$. By the way, $x_{1}$ and $x_{2}$ are the only possible common out-neighbors of $v_{3}$ and $w_{3}$. If $x_{2}$ is a common out-neighbor of $v_{3}$ and $w_{3}$, then $\left\{v_{1}, v_{3}, w_{1}, w_{2}, w_{3}\right\} \subseteq N^{-}\left(x_{2}\right)$ and so $N^{+}\left(x_{2}\right) \subseteq\left\{u, x_{1}\right\}$, which contradicts Proposition 2.2(3). Therefore $x_{1}$ is a common out-neighbor of $v_{3}$ and $w_{3}$. Then $\left\{v_{2}, v_{3}, w_{2}, w_{3}\right\} \subseteq$
$N^{-}\left(x_{1}\right)$ by (8), so $N^{+}\left(x_{1}\right) \subseteq\left\{u, w_{1}, x_{2}\right\}$. Thus $N^{+}\left(x_{1}\right)=\left\{u, w_{1}, x_{2}\right\}$ by Proposition[2.2(3). However, since $\left\{w_{1}, x_{2}\right\} \subset N^{-}(u), N^{+}\left(x_{1}\right)$ cannot form a directed cycle, which contradicts Proposition 2.2(3). Hence $u \notin V_{4} \cup V_{5}$ and we reach a contradiction. Therefore

$$
\begin{equation*}
\left|N^{+}(v)\right| \geq 4 \tag{9}
\end{equation*}
$$

for each vertex $v$ in $V_{4} \cup V_{5}$.
Now we show that each of $V_{2}$ and $V_{3}$ has exactly one vertex of outdegree 3, which implies that each vertex of $V_{1}$ has outdegree 3 . Since $D$ has at least 5 vertices of outdegree 3 by $(\dagger), V_{2} \cup V_{3}$ has at least 2 vertices of outdegree 3 by (9). Take a vertex of outdegree 3 in $V_{2} \cup V_{3}$. Then we may regard it as $u$. Take a vertex $v$ of outdegree 3 distinct from $u$ in $V_{2} \cup V_{3}$. Then, $N^{+}(x) \cap V_{2}=\emptyset$ or $N^{+}(x) \cap V_{3}=\emptyset$ for each vertex $x$ in $\{u, v\}$, so, by (6) and (7),

$$
\left|N^{+}(u) \cap V_{i}\right|=\left|N^{+}(u) \cap V_{4}\right|=\left|N^{+}(u) \cap V_{5}\right|=1
$$

for some $i \in\{1,2,3\}$ and

$$
\left|N^{+}(v) \cap V_{j}\right|=\left|N^{+}(v) \cap V_{4}\right|=\left|N^{+}(v) \cap V_{5}\right|=1
$$

for some $j \in\{1,2,3\}$. Thus, since $\left|V_{4}\right|=\left|V_{5}\right|=1$, the vertices in $V_{4} \cup V_{5}$ are common out-neighbors of $u$ and $v$ and we may assume that

$$
N^{+}(v)=\left\{v_{1}^{\prime}, v_{2}, v_{3}\right\}
$$

for some vertex $v_{1}^{\prime}$ in $D, V_{4}=\left\{v_{2}\right\}$, and $V_{5}=\left\{v_{3}\right\}$ by symmetry. Since $\left(v_{2}, v_{3}\right) \in A(D)$,

$$
\left(v_{1}^{\prime}, v_{2}\right) \in A(D)
$$

by Proposition 2.2(3). Therefore

$$
\begin{equation*}
\left\{v, v_{1}^{\prime}\right\} \subseteq N^{-}\left(v_{2}\right) \tag{10}
\end{equation*}
$$

To reach a contradiction, we suppose that $u$ and $v$ are contained in the same partite set. Without loss of generality, we may assume $\{u, v\} \subseteq V_{2}$, Then $V_{2}=\{u, v\}$. Suppose $v_{1}=v_{1}^{\prime}$. Then $N^{+}(u)=N^{+}(v)$ and $N^{-}(u)=N^{-}(v)$. Therefore any pair of vertices having $v$ as a common out-neighbor has $u$ as a common out-neighbor. Then, since $D$ is competitive, $D-u$ is competitive. However, $D-u$ is an orientation of $K_{3,1,2,1,1}$, which contradicts Proposition 3.8. Therefore $v_{1} \neq v_{1}^{\prime}$. Thus

$$
N^{+}(u) \cap N^{+}(v)=\left\{v_{2}, v_{3}\right\} .
$$

If $v_{1} \in V_{1}$, then $N^{+}(u) \subset V_{1} \cup V_{4} \cup V_{5}$ and so, by (§), $v=w_{i}$ for some $i \in\{1,2,3\}$, which contradicts $\left\{v_{2}, v_{3}\right\} \subseteq N^{+}(v)$. Therefore $v_{1} \notin V_{1}$ and so $v_{1} \in V_{3}$. Thus $N^{+}(u) \subset$ $V_{3} \cup V_{4} \cup V_{5}$ and so $V_{1}=\left\{w_{1}, w_{2}, w_{3}\right\}$ by (§). We may show that, by applying the same
argument to $v_{1}^{\prime}, v_{1}^{\prime} \notin V_{1}$. Then $v_{1}^{\prime} \in V_{3}$, so $\left\{v_{1}, v_{1}^{\prime}\right\} \subseteq V_{3}$. Therefore $V_{3}=\left\{v_{1}, v_{1}^{\prime}\right\}$. We know from $D_{1}$ that $\left\{u, v_{1}, w_{3}\right\} \subseteq N^{-}\left(v_{2}\right)$. Moreover, $\left\{v, v_{1}^{\prime}\right\} \subseteq N^{-}\left(v_{2}\right)$ by (10). Thus $\left\{u, v, v_{1}, v_{1}^{\prime}, w_{3}\right\} \subseteq N^{-}\left(v_{2}\right)$ and so $N^{+}\left(v_{2}\right) \subseteq\left\{v_{3}, w_{1}, w_{2}\right\}$. Hence $N^{+}\left(v_{2}\right)=\left\{v_{3}, w_{1}, w_{2}\right\}$ by Proposition [2.2(3). However, $w_{1}$ and $w_{2}$ belong to the same partite set $V_{1}$, which contradicts Proposition [2.2(3). Therefore $u$ and $v$ belong to the distinct partite sets. Since $u$ and $v$ were vertices of outdegree 3 arbitrarily chosen, each of $V_{2}$ and $V_{3}$ has at most one vertex of outdegree 3 . By the way, $V_{2} \cup V_{3}$ has at least 2 vertices of outdegree 3, so we may conclude that each of $V_{2}$ and $V_{3}$ has exactly one vertex of outdegree 3. Thus each vertex of $V_{1}$ has outdegree 3 by $(\dagger)$.

Without loss of generality, we may assume that $u \in V_{2}, v \in V_{3}$, and

$$
(u, v) \in A(D)
$$

Then $v_{1}=v$. Therefore $\left|N^{+}(u) \cap V_{3}\right|=\left|N^{+}(u) \cap V_{4}\right|=\left|N^{+}(u) \cap V_{5}\right|=1$ by (7). If $w_{2}$, which is a common out-neighbor of $v_{2}$ and $v_{3}$, is contained in $V_{1}$, then $N^{+}\left(w_{2}\right) \subseteq V_{2} \cup V_{3}$ and so, by Proposition [2.9, $w_{2}$ has outdegree at least 4 , which is a contradiction to the fact that each vertex of $V_{1}$ has outdegree 3 . Therefore $w_{2} \in V_{2}$ by (§). Then

$$
V_{2}=\left\{u, w_{2}\right\}
$$

Thus $\left\{w_{1}, w_{3}\right\} \subset V_{1}$ by (§) and so each of $w_{1}$ and $w_{3}$ has outdegree 3 . Let

$$
V_{1}=\left\{w_{1}, w_{3}, z\right\} \quad \text { and } \quad V_{3}=\left\{v_{1}, y\right\}
$$

for some vertices $y$ and $z$ in $D$. We know from $\tilde{D}$ given in Figure 2 that $N^{+}\left(w_{1}\right) \cap$ $\left\{v_{1}, v_{2}, v_{3}\right\}=\left\{v_{3}\right\}$ and $N^{+}\left(w_{3}\right) \cap\left\{v_{1}, v_{2}, v_{3}\right\}=\left\{v_{2}\right\}$. Since each of $w_{1}$ and $w_{3}$ has outdegree 3 , the out-neighbors of $w_{i}$ belong to distinct partite sets for $i=1,3$ by Proposition 2.2(3). By recalling that $N^{+}(u)=\left\{v_{1}, v_{2}, v_{3}\right\}$, we may conclude that $N^{+}\left(w_{1}\right)=\left\{u, v_{3}, y\right\}$ and $N^{+}\left(w_{3}\right)=\left\{u, v_{2}, y\right\}$. Since $\left(u, v_{2}\right) \in A(D)$ and $\left(u, v_{3}\right) \in A(D),\left(v_{2}, y\right) \in A(D)$ and $\left(v_{3}, y\right) \in A(D)$ by the same lemma. Therefore $\left\{v_{2}, v_{3}, w_{1}, w_{3}\right\} \subseteq N^{-}(y)$ and so $N^{+}(y) \subseteq\left\{u, w_{2}, z\right\}$. Thus $N^{+}(y)=\left\{u, w_{2}, z\right\}$ by Proposition 2.2(3). However, there is no arc between $u$ and $w_{2}$ and so $N^{+}(y)$ cannot form a directed cycle, which contradicts Proposition 2.2(3).

Now we are ready to prove Theorem (4)
Proof of Theorem 4. To show the "only if" part, suppose that there exists a competitive orientation $D$ of $K_{n_{1}, n_{2}, \ldots, n_{5}}$. By Proposition 3.8,

$$
n_{3} \geq 2
$$

If $n_{4} \geq 2$, then (c) holds. Now suppose $n_{4}=1$. Then $n_{5}=1$. Suppose, to the contrary, that $n_{1}=2$. Then $D$ is an orientation of $K_{2,2,2,1,1}$. Since $4|V(D)|-|A(D)|=7>0, D$ has

$$
\begin{aligned}
A_{8} & =\left(\begin{array}{llllllllll}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}\right) \\
A_{9} & =\left(\begin{array}{lllllllll}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}\right)
\end{aligned}
$$

Figure 8: The adjacency matrices $A_{8}$ and $A_{9}$ which are orientations of $K_{3,3,2,1,1}, K_{2,2,2,2,1}$ respectively, in the proof of Theorem 4.
a vertex of outdegree 3 by Proposition [2.2(4). Therefore $|V(D)| \geq 9$ by Theorem 2.11, which is impossible. Thus

$$
n_{1} \geq 3
$$

If $n_{1} \geq 4$, then (b) holds. Now suppose $n_{1}=3$. Then $n_{2} \leq 3$. If $n_{2}=2$, then $n_{3}=2$ and so $D$ is an orientation of $K_{3,2,2,1,1}$, which contradicts Theorem 3.12. Therefore $n_{2}=1$ or 3. Then, since $n_{2} \geq n_{3} \geq 2, n_{2}=3$ and (a) holds. Thus we have shown the "only if" part.

Now we show the "if" part. Let $D_{\alpha}, D_{\gamma}$ be the digraphs whose adjacency matrix are $A_{8}$ and $A_{9}$, respectively, given in Figure 8. Then $D_{\alpha}$ and $D_{\gamma}$ are orientations of $K_{3,3,2,1,1}$ and $K_{2,2,2,2,1}$, respectively. Let $D_{\beta}^{*}$ be the digraph whose adjacency matrix is $A_{6}$ given in Figure 7. It is easy to check that the inner product of each pair of rows in each matrix is nonzero, so $D_{\alpha}, D_{\gamma}$, and $D_{\beta}^{*}$ are competitive. By Lemma 2.8, we obtain a competitive orientation $D_{\beta}$ of $K_{4,2,2,1,1}$ from $D_{\beta}^{*}$. By applying Corollary 2.7 to $D_{\alpha}, D_{\beta}$, and $D_{\gamma}$, we may obtain orientations $D_{\alpha}^{\prime}, D_{\beta}^{\prime}$, and $D_{\gamma}^{\prime}$ of $K_{n_{1}, n_{2}, \ldots, n_{5}}$ each of which are competitive for (a) $n_{1}=3, n_{2}=3, n_{3} \geq 2, n_{4}=1$, and $n_{5}=1$; (b) $n_{1} \geq 4, n_{2} \geq n_{3} \geq 2, n_{4}=1$, and $n_{5}=1$; (c) $n_{4} \geq 2$, respectively. Therefore we have shown that the "if" part is true.

## 4 Closing remarks

By Corollary 2.3 , there is no complete graph that is the competition graph of a bipartite tournament. For an integer $k \geq 3$, Proposition 3.1, and Theorems 2, 3, 4, 1 may be summarized in the aspect of the number of vertices of a complete graph which is the competition graph of a $k$-partite tournament as follows.

Theorem 4.1. A complete graph $K_{n}$ is the competition graph of a $k$-partite tournament for some integer $k \geq 3$ if and only if

$$
\begin{cases}n \geq 13 & \text { if } k=3 \\ n \geq 10 & \text { if } k=4 \\ n \geq 9 & \text { if } k \in\{5,6\} \\ n \geq k & \text { if } k \geq 7\end{cases}
$$

Proof. For an integer $k \geq 3$, suppose that a complete graph $K_{n}$ is the competition graph of a $k$-partite tournament which is an orientation of $K_{n_{1}, n_{2}, \ldots, n_{k}}$. Then it is competitive and $n_{1}+n_{2}+\cdots+n_{k}=n$. If $k=3$, then $\sum_{i=1}^{3} n_{i} \geq 13$ by Theorem 2, If $k=4$, then $\sum_{i=1}^{4} n_{i} \geq 10$ by Theorem 3. If $k=5$, then $\sum_{i=1}^{5} n_{i} \geq 9$ by Theorem 4. If $k=6$, then $\sum_{i=1}^{6} n_{i} \geq 9$ by Theorem 1. If $k \geq 7$, then $\sum_{i=1}^{k} n_{i} \geq k$ by Proposition 3.1. Each of the above theorems also guarantees the existence of a competitive $k$-partite tournament for the corresponding $k$, so the "if" part is true.

As we mentioned previously, there is no graph of order $n$ which is competitively orientable for any integer $3 \leq n \leq 6$. For $n \geq 7$, if a graph $G$ of order $n$ is competitively orientable, then $G$ must have at least $3 n$ edges by Theorem 2.4(3). Furthermore, we showed that for each $m \geq 7$, there is a competitively orientable graph of order $m$ with exactly $3 m$ edges in Remark [2.5. However, for a complete multipartite graph, we doubt that there is a proper spanning subgraph which is competitively orientable because the matrices which we adapted to construct competitive orientations of complete multipartite graphs seem to represent minimal competitive digraphs.

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