Planar graphs without normally adjacent short cycles

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Abstract

Let $\mathscr{G}$ be the class of plane graphs without triangles normally adjacent to 8^{-}-cycles, without 4-cycles normally adjacent to 6^{-}-cycles, and without normally adjacent 5-cycles. In this paper, it is shown that every graph in $\mathscr{G}$ is 3-choosable. Instead of proving this result, we directly prove a stronger result in the form of "weakly" DP-3-coloring. The main theorem improves the results in [J. Combin. Theory Ser. B 129 (2018) 38–54; European J. Combin. 82 (2019) 102995]. Consequently, every planar graph without 4-, 6-, 8-cycles is 3-choosable, and every planar graph without 4-, 5-, 7-, 8-cycles is 3-choosable. In the third section, using almost the same technique, we prove that the vertex set of every graph in $\mathscr{G}$ can be partitioned into an independent set and a set that induces a forest, which strengthens the result in [Discrete Appl. Math. 284 (2020) 626–630]. In the final section, tightness is discussed.

Keywords: DP-coloring; Near-bipartite; IF-coloring; List coloring

1 Introduction

All graphs in this paper are finite, undirected and simple. For a graph G, a **list-assignment** L assigns to each vertex v a set $L(v)$ of colors available at v. An **L-coloring** of G is a proper coloring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. A **list k-assignment** L is a list-assignment such that $|L(v)| \geq k$ for all $v \in V(G)$. A graph G is **k-choosable** or **list k-colorable** if it has an L-coloring for any list k-assignment L. The **list chromatic number** or **choice number** $\chi_\ell(G)$ is the least integer k such that G is k-choosable.

The Four Color Theorem states that every planar graph is 4-colorable. Grötzsch [6] showed that every planar graph without triangles is 3-colorable. Much more sufficient conditions for 3-colorability and 3-choosability are extensively studied. Thomassen [21] showed that every planar graph with girth at least five is 3-choosable. Borodin [1] conjectured that every planar graph without cycles of length 4 to 8 is 3-choosable.

A widely used technique in ordinary vertex coloring is the identification of vertices, but this is not feasible in general for list-coloring because different vertices may have different lists. To overcome this difficulty, Dvořák and Postle [5] introduced DP-coloring, also called correspondence coloring, as a generalization of list-coloring.

Definition 1. Let G be a simple graph and L be a list-assignment for G. For each vertex $v \in V(G)$, let $L_v = \{v\} \times L(v)$; for each edge $uv \in E(G)$, let $\mathscr{M}_{uv}$ be a matching between the sets L_u and L_v, and let $\mathscr{M} := \bigcup_{uv \in E(G)} \mathscr{M}_{uv}$. We call $\mathscr{M}$ a **matching assignment**. The matching assignment is called a **k-matching assignment** if $L(v) = [k]$ for each $v \in V(G)$. A **cover** of G is a graph $H_{L,\mathscr{M}}$ (simply write H) satisfying the following two conditions:

(C1) the vertex set of H is the disjoint union of L_v for all $v \in V(G)$;

(C2) the edge set of H is the matching assignment $\mathscr{M}$.

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Note that the matching $\mathscr{M}_{uv}$ is not required to be a perfect matching between the sets L_u and L_v, and possibly it is empty. The induced subgraph $H[L_v]$ is an independent set for each vertex $v \in V(G)$.

Definition 2. Let G be a simple graph and H be a cover of G. An $\mathscr{M}$**-coloring** of G is an independent set $\mathcal{I}$ in H such that $|\mathcal{I} \cap L_v| = 1$ for each vertex $v \in V(G)$. The graph G is **DP-k-colorable** if for any list-assignment $L(v) \supseteq [k]$ and any matching assignment $\mathscr{M}$, it has an $\mathscr{M}$-coloring. The **DP-chromatic number** $\chi_{\mathrm{DP}}(G)$ of G is the least integer k such that G is DP-k-colorable.

DP-coloring is quite different from list-coloring, for example each even cycle is 2-choosable but it is not DP-2-colorable. Dvořák and Postle gave a relation between DP-coloring and list-coloring.

Let $W = w_1 w_2 \ldots w_m$ with $w_m = w_1$ be a closed walk of length $m - 1$ in G, a matching assignment is **inconsistent** on W, if there exist $c_1, \ldots, c_m$ such that $c_i \in L(w_i)$ for $i \in [m]$ and $(w_i, c_i)(w_{i+1}, c_{i+1})$ is an edge in $\mathscr{M}_{w_i w_{i+1}}$ for $i \in [m-1]$ and $c_1 \neq c_m$. Otherwise, the matching assignment is **consistent** on W. We say that a matching assignment is **consistent** if it is consistent on every closed walk in G.

Theorem 1.1 (Dvořák and Postle [5]). A graph G is k-choosable if and only if G is $\mathscr{M}$-colorable for every consistent k-matching assignment $\mathscr{M}$.

With the aid of DP-coloring, Dvořák and Postle [5] solved the longstanding conjecture by Borodin.

Theorem 1.2 (Dvořák and Postle [5]). Every planar graph without cycles of length 4 to 8 is 3-choosable.

An edge uv in G is **straight** in a k-matching assignment $\mathscr{M}$ if $(u, c_1)(v, c_2) \in \mathscr{M}_{uv}$ satisfies $c_1 = c_2$. An edge uv in G is **full** in a k-matching assignment $\mathscr{M}$ if $\mathscr{M}_{uv}$ is a perfect matching.

Lemma 1.1 (Dvořák and Postle [5]). Let G be a graph with a k-matching assignment $\mathscr{M}$, and let K be a subgraph of G. If for every cycle $\mathcal{Q}$ in K, the assignment $\mathscr{M}$ is consistent on $\mathcal{Q}$ and all edges of $\mathcal{Q}$ are full, then we may rename $L(u)$ for $u \in V(K)$ to obtain a k-matching assignment $\mathscr{M}'$ for G such that all edges of K are straight in $\mathscr{M}'$.

In order to prove Theorem 1.2, they showed a stronger result as the following.

Theorem 1.3 (Dvořák and Postle [5]). Let G be a plane graph without cycles of length 4 to 8. Let S be a set of vertices of G such that $|S| \leq 1$ or S consists of all vertices on a face of G. Let $\mathscr{M}$ be a 3-matching assignment for G such that $\mathscr{M}$ is consistent on every closed walk of length three in G. If $|S| \leq 12$, then every $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring φ of G.

Two cycles are **adjacent** if they have at least one common edge. An ℓ_1-cycle and an ℓ_2-cycle are **normally adjacent** if they form an $(\ell_1 + \ell_2 - 2)$-cycle with exactly one chord. In other words, two cycles are normally adjacent if their intersection is K_2. Recently, Liu and Li [13] improved Theorem 1.3 to the following result by allowing cycles of length 4 to 8 but forbidding adjacent cycles of length at most 8.

Theorem 1.4 (Liu and Li [13]). Let G be a plane graph without adjacent cycles of length at most 8. Let S be a set of vertices of G such that $|S| \leq 1$ or S consists of all vertices on a face of G. Let $\mathscr{M}$ be a 3-matching assignment for G such that $\mathscr{M}$ is consistent on every closed walk of length three in G. If $|S| \leq 12$, then every $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring φ of G.

This implies the 3-choosability of planar graphs without adjacent cycles of length at most 8.

Theorem 1.5 (Liu and Li [13]). Every planar graph without adjacent cycles of length at most 8 is 3-choosable.

The first goal of this paper is to further improve Theorem 1.4 to the following result by allowing adjacent cycles of length 6 to 8 and changing the condition on precolored vertices from faces to cycles. But before we state the main theorem, it's necessary to give a new concept. A cycle is **abnormal** if it is the 11- or 12-cycle in a subgraph isomorphic to a configuration in Fig. 1. A cycle is **normal** if it is not an abnormal cycle. A d-vertex, d^+-vertex or d^--vertex is a vertex of degree d, at least d, or at most d respectively. Similar definitions can be applied to faces and cycles.

Let $\mathscr{G}$ be the class of plane graphs without triangles normally adjacent to 8^--cycles, without 4-cycles normally adjacent to 6^--cycles, and without normally adjacent 5-cycles.
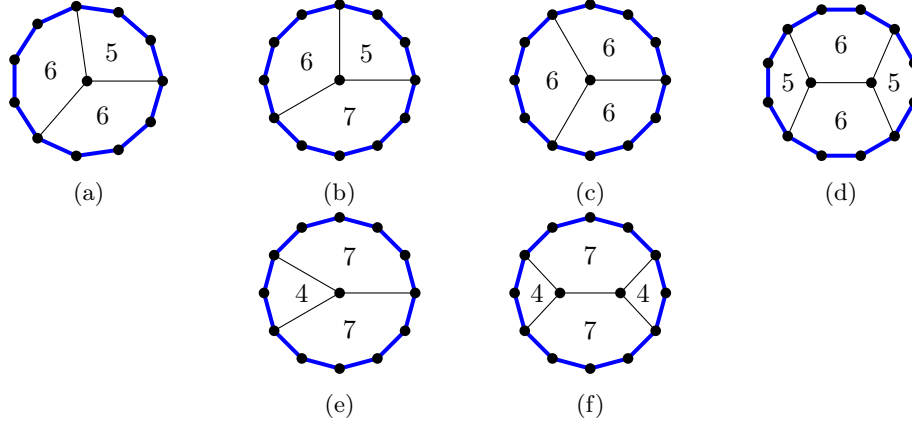
Fig. 1: The abnormal 12^--cycles in blue.

Theorem 1.6. Let G be a graph in $\mathscr{G}$. Let S be a set of vertices of G such that $|S| \leq 1$ or S consists of all vertices on a normal cycle of G. Let $\mathscr{M}$ be a 3-matching assignment for G such that $\mathscr{M}$ is consistent on every closed walk of length three in G. If $|S| \leq 12$, then every $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring of G.

Remark 1. The graphs in Fig. 1 are in the class $\mathscr{G}$. It is observed that not every $\mathscr{M}$-coloring of the 11- or 12-cycle can be extended to the whole graph. Thus, we require the condition that S consists of all vertices on a "normal" cycle. On the other hand, we find all the non-extendable 12^--cycles.

Observation: Every 10^--cycle is normal. For a normal 12^--cycle $\mathcal{O}$, every vertex not on $\mathcal{O}$ has at most two neighbors on $\mathcal{O}$. Every edge on an abnormal 12^--cycle is contained in a 4-, 5-, 6- or 7-cycle.

The following result is a direct consequence of Theorem 1.6, and it extends Theorem 1.5.

Theorem 1.7. Every graph in $\mathscr{G}$ is 3-choosable.

The following three results are immediate consequences of Theorem 1.7. The first one generalizes the 3-colorability of such graphs described by Luo, Chen and Wang [19], and the second one generalizes the 3-colorability of such graphs described by Wang and Chen [22].

Corollary 1.8. Every planar graph without 4-, 6-, 8-cycles is 3-choosable.

Corollary 1.9. Every planar graph without 4-, 5-, 7-, 8-cycles is 3-choosable.

Remark 2. Theorem 1.2, Theorem 1.5 and Theorem 1.7 are only for 3-choosable, but not for DP-3-colorable. Since we require the "consistency" on every closed walk of length three, the current arguments cannot guarantee the DP-3-colorability of the graphs in Theorem 1.3, Theorem 1.4 and Theorem 1.6. So it is interesting to know whether such graphs are DP-3-colorable.

Note that we only require the "consistency" on every closed walk of length three, if triangles are forbidden in a graph, then we can obtain the following results on DP-3-coloring.

Corollary 1.10 (Liu et al. [16])**.** Every planar graph without 3-, 5-, 6-cycles is DP-3-colorable.

Corollary 1.11 (Liu et al. [16])**.** Every planar graph without 3-, 6-, 7-, 8-cycles is DP-3-colorable.

There are some other sufficient conditions for planar graphs to be DP-3-colorable which extend the 3-choosability of such graphs. We refer the reader to [15, 16, 24]. DP-4-colorable planar or toroidal graphs can be found in [2, 11, 12, 14]. Thomassen [20] showed that every planar graph is 5-choosable. Dvořák and Postle [5] observed that every planar graph is DP-5-colorable. Recently, Li and Wang [10] extended these results to K_5-minor-free graphs.

In addition, we prove a result on the vertex partition using almost the same technique as that in Theorem 1.6. An **IF-coloring** of a graph G is a mapping $\phi : V(G) \to \{I, F\}$ such that the subgraph induced by all the vertices $\phi^{-1}(I)$ is an independent set, and the subgraph induced by all the vertices $\phi^{-1}(F)$ is a forest. In other words, an **IF-coloring** of a graph G is a partition of $V(G)$ into two parts I and F, such that $G[I]$ is an independent set and $G[F]$ is a forest.

Kawarabayashi and Thomassen [8] proved that every planar graph with girth at least five has an IF-coloring. Wang and Chen [23] showed that every planar graph without 4-, 6- and 8-cycles is 3-colorable. Very recently, Liu and Yu [17] proved that every planar graph without 4-, 6- and 8-cycles has an IF-coloring. We prove that every graph in $\mathscr{G}$ has an IF-coloring.

Theorem 1.12. The vertex set of every graph in $\mathscr{G}$ can be partitioned into an independent set and a set that induces a forest.

This paper is organized as follows. In the remainder of this section, we introduce some notations and results utilized in the proof of Theorem 1.6. In section 2, we present a proof of Theorem 1.6. In section 3, we prove a slightly stronger result than Theorem 1.12. Finally, we conclude with some open questions in section 4.

Let G be a plane graph. The edges and vertices divide the plane into a number of **faces**. The unbounded face is called the **outer face**, and the other faces are called **inner faces**. An **internal vertex** is a vertex that is not incident with the outer face. An **internal face** is a face having no common vertices with the outer cycle. Let $\mathcal{O}$ be a cycle of a plane graph G, the cycle $\mathcal{O}$ divides the plane into two regions, the subgraph induced by all the vertices in the unbounded region is denoted by $\text{ext}(\mathcal{O})$, and the subgraph induced by all the vertices in the other region is denoted by $\text{int}(\mathcal{O})$. If both $\text{int}(\mathcal{O})$ and $\text{ext}(\mathcal{O})$ contain at least one vertex, then we call the cycle $\mathcal{O}$ a **separating cycle** of G. The subgraph obtained from G by deleting all the vertices in $\text{ext}(\mathcal{O})$ is denoted by $\text{Int}(\mathcal{O})$, and the subgraph obtained from G by deleting all the vertices in $\text{int}(\mathcal{O})$ is denoted by $\text{Ext}(\mathcal{O})$. Let $\mathcal{N}$ be the set of inner faces having at least one common vertex with the outer face.

We need the following two special covers of cycles.

- The **circular ladder** Γ_n is the Cartesian product of the cycle C_n and an independent set with two vertices.

- The **Möbius ladder** M_n is the graph with vertex set $\{ (i, j) \mid i \in [n], j \in [2] \}$, in which two vertices (i, j) and (i', j') are adjacent if and only if either

    — $i' = i + 1$ and $j = j'$ for $1 \leq i \leq n - 1$, or
    — $i = n$, $i' = 1$ and $j \neq j'$.

The following result can be derived from a theorem in [9, 18].

Theorem 1.13. Let C be a cycle, and let H be a cover with a 2-list assignment. If the cover is not a circular ladder or a Möbius ladder, then H has an $\mathscr{M}$-coloring.

2 Proof of Theorem 1.6

In this section, we prove the following main result.

Theorem 1.6. Let G be a graph in $\mathscr{G}$. Let S be a set of vertices of G such that $|S| \leq 1$ or S consists of all vertices on a normal cycle of G. Let $\mathscr{M}$ be a 3-matching assignment for G such that $\mathscr{M}$ is consistent on every closed walk of length three in G. If $|S| \leq 12$, then every $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring of G.

Proof. Suppose that G is a minimal counterexample to Theorem 1.6. That is, there exists an $\mathscr{M}$-coloring of $G[S]$ that cannot be extended to an $\mathscr{M}$-coloring of G such that

$$|V(G)| + |E(G)| - |S| \text{ is minimized.} \tag{1}$$

Subject to (1),

$$\text{the number of edges in the 3-matching assignment } \mathscr{M} \text{ is maximized.} \qquad (2)$$

By the condition (2), each edge that is not in a triangle is full in the matching assignment $\mathscr{M}$. By the structure of G, we immediately have the following result on 8^--cycles.

Lemma 2.1. Every 8^--cycle has no chords.

Next, we give some structural results on G. Some of the lemmas are almost the same as the ones in [5] and [13], but for completeness we give detailed proofs here.

Lemma 2.2.

(a) $S \neq V(G)$.

(b) G is 2-connected, and the boundary of every face is a cycle.

(c) Each vertex not in S has degree at least three.

(d) Either $|S| = 1$ or $G[S]$ is an induced cycle of G.

(e) There are no separating normal k-cycles for $3 \leq k \leq 12$. Thus, every edge on an abnormal cycle is incident with a 4-, 5-, 6- or 7-face.

(f) $G[S]$ is an induced cycle of G. For convenience, we can redraw the graph G such that $G[S]$ is the outer cycle $\mathcal{C}$ of G. Let D be the outer face which is bounded by $G[S]$.

(g) Every 5-face ($\neq$ outer face) is incident with at most one 2-vertex.

Proof. (a) Suppose to the contrary that $S = V(G)$. Every $\mathscr{M}$-coloring of $G[S]$ is an $\mathscr{M}$-coloring of G, a contradiction.

(b) By the condition (1), G is connected. Suppose to the contrary that G has a cut-vertex w. We may assume that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{w\}$. By the assumption of the set S, either $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$. We may assume that $S \subseteq V(G_1)$. By the condition (1), the $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring ϕ_1 of G_1, and $\phi_1(w)$ can be extended to an $\mathscr{M}$-coloring ϕ_2 of G_2. These two colorings ϕ_1 and ϕ_2 together give an $\mathscr{M}$-coloring of G whose restriction on $G[S]$ is ϕ, a contradiction.

(c) Suppose that there exists a vertex w not in S having degree two. By the condition (1), the $\mathscr{M}$-coloring of $G[S]$ can be extended to an $\mathscr{M}$-coloring of $G - w$. Since w has degree two, there are at most two forbidden colors for w, thus we can extend the $\mathscr{M}$-coloring of $G - w$ to an $\mathscr{M}$-coloring of G, a contradiction.

(d) If $S = \emptyset$, then we put any vertex into S to make $|S| = 1$. Suppose that $S = V(\mathcal{Q})$ and $\mathcal{Q}$ is a cycle with a chord uv. It is observed that the $\mathscr{M}$-coloring of $G[S]$ is also an $\mathscr{M}$-coloring of the induced subgraph in $G - uv$. By the condition (1), the $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring of $G - uv$, and hence it is also an $\mathscr{M}$-coloring of G, a contradiction.

(e) We first show that $G[S]$ cannot be a separating cycle. Suppose to the contrary that $G[S]$ is a separating (normal) cycle $\mathcal{O}$. By the condition (1), the $\mathscr{M}$-coloring ϕ of $\mathcal{O}$ can be extended to an $\mathscr{M}$-coloring ϕ_1 of $\text{Int}(\mathcal{O})$, and another $\mathscr{M}$-coloring ϕ_2 of $\text{Ext}(\mathcal{O})$. These two colorings ϕ_1 and ϕ_2 together give an $\mathscr{M}$-coloring of G whose restriction on $G[S]$ is ϕ, a contradiction.

Thus, either $|S| = 1$ or S consists of all vertices on a face of G. Let $\mathcal{Q}$ be a separating normal k-cycle with $3 \leq k \leq 12$. Thus, we may assume that $S \subseteq \text{Ext}(\mathcal{Q})$. By the condition (1), the $\mathscr{M}$-coloring ϕ of $G[S]$ can be extended to an $\mathscr{M}$-coloring φ_1 of $\text{Ext}(\mathcal{Q})$. Similarly, the restriction of φ_1 on Q can be extended to an $\mathscr{M}$-coloring φ_2 of $\text{Int}(\mathcal{Q})$. These two colorings φ_1 and φ_2 together give an $\mathscr{M}$-coloring of G whose restriction on $G[S]$ is ϕ, a contradiction.

(f) According to (d), suppose to the contrary that $S = \{w\}$. We first assume that w is on a 10^--cycle $\mathcal{Q}$. Without loss of generality, we may assume that $\mathcal{Q}$ is a shortest cycle containing w. Then $\mathcal{Q}$ is an induced cycle. By (e), we may assume that $\text{ext}(\mathcal{Q}) = \emptyset$ and $\mathcal{Q}$ is the outer cycle. By (c) and $\mathcal{Q}$ is an induced cycle, every vertex on $\mathcal{Q}$ other than w has a neighbor in $\text{int}(\mathcal{Q})$, which implies that $\text{int}(\mathcal{Q}) \neq \emptyset$.

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By the condition (1), the $\mathscr{M}$-coloring ϕ of $\{w\}$ can be extended to an $\mathscr{M}$-coloring ϕ_1 of $\mathcal{Q}$. By the condition (1), the $\mathscr{M}$-coloring ϕ_1 of $\mathcal{Q}$ can be further extended to an $\mathscr{M}$-coloring of G, a contradiction.

So we may assume that every cycle containing w has length at least 11. Let w be incident with a face $[w_1ww_2\dots w_1]$. Let G' be obtained from G by adding a chord w_1w_2 in the face, let $S' = \{w, w_1, w_2\}$ and let the 3-matching assignment $\mathscr{M}'$ for G' be obtained from $\mathscr{M}$ by setting the matching corresponding to w_1w_2 is edgeless. We can easily check that G' is a plane graph satisfying the assumption of Theorem 1.6. By the condition (1), the $\mathscr{M}$-coloring ϕ of $\{w\}$ can be extended to an $\mathscr{M}'$-coloring ϕ_1 of $G'[S']$. By the condition (1), the $\mathscr{M}'$-coloring ϕ_1 of $G'[S']$ can be further extended to an $\mathscr{M}'$-coloring φ of G'. It is observed that φ is an $\mathscr{M}$-coloring of G, a contradiction.

(g) Note that every 2-vertex and its two neighbors are all on the outer cycle. Suppose to the contrary that $f = [x_1x_2x_3x_4x_5]$ is a 5-face which is incident with two 2-vertices. Note that the two 2-vertices must be adjacent on the outer cycle, say x_2 and x_3. It follows that x_1 and x_4 are on the outer cycle $\mathcal{C}$. If x_5 has three neighbors on the outer cycle $\mathcal{C}$, then $\mathcal{C}$ is abnormal (see Fig. 1a and Fig. 1b), a contradiction. Thus, x_5 has a neighbor not on the outer cycle $\mathcal{C}$, and $\mathcal{C}' = (\mathcal{C} - \{x_2, x_3\}) \cup \{x_1x_5, x_4x_5\}$ is a separating 11^--cycle. By (e), $\mathcal{C}'$ is an abnormal 11-cycle (see Fig. 1a). It follows that $\mathcal{C}$ is an abnormal 12-cycle (see Fig. 1d), a contradiction. $\qquad\square$

Lemma 2.3. There are no 3-faces adjacent to 8^--faces, no 4-faces adjacent to 6^--faces, and no adjacent 5-faces.

Proof. Recall that every face is bounded by a cycle. Assume that f is an 8^--face and it is adjacent to a 5^--face g. By Lemma 2.1, it suffices to consider that g is a 4- or 5-face.

Suppose that $f = [w_1w_2\dots w_k]$ is a 6^--face and $g = [uvw_3w_2]$ is a 4-face. Since there are no 4-cycles normally adjacent to 6^--cycles, we have that either u or v is on f. By symmetry, we may assume that u is on f. Recall that every 6^--face is bounded by a cycle and this cycle has no chord, so $u = w_1$ and v is not on f. It is observed that w_2 is a 2-vertex and it must be on the outer cycle $\mathcal{C}$. It follows that either f or g is the outer face. If f is the outer face, then v is an internal vertex and it has a neighbor not on the outer cycle $\mathcal{C}$ (since $[w_1vw_3\dots w_k]$ has no chords by Lemma 2.1), thus there is a separating 6^--cycle $[w_1vw_3\dots w_k]$, a contradiction. Similarly, if g is the outer face, then there is an internal vertex on f having a neighbor not on the outer cycle $\mathcal{C}$, thus there is a separating 6^--cycle containing w_1vw_3, a contradiction.

Suppose that f and g are 5-faces. Since every 8^--cycle has no chord, there are only four cases (up to symmetry) for the local structures (see Fig. 2). Since there are no normally adjacent 5-cycles, the first case will not appear. For the other three cases, we first assume that x is an internal vertex. Since every internal vertex has degree at least three, x has a neighbor x' other than w_2 and y. It is observed that x is on a 6^--cycle $\mathcal{O}_x$ not containing w_3. If x' is on $\mathcal{O}_x$, then xx' is a chord of $\mathcal{O}_x$, but this contradicts Lemma 2.1; if x' is not on $\mathcal{O}_x$, then $\mathcal{O}_x$ is a separating 6^--cycle, this contradicts Lemma 2.2(e). So we may assume that x is on the outer cycle. In the second and third cases, w_3 is a 2-vertex, so it is on the outer cycle, and g must be the outer face. In the fourth case, by the symmetry of x and z, we have that z is on the outer cycle, and g is the outer face. Therefore, g is the outer face in the last three cases, and there is an internal vertex on f having a neighbor not on g, and then there is a separating 6^--cycle containing w_2xy, a contradiction. $\qquad\square$

Lemma 2.4. If $[x_1x_2x_3]$ is a triangle and x_2, x_3 are internal 3-vertices, then all the edges in the triangle are full.

Proof. Suppose to the contrary that at least one of x_1x_3, x_2x_3 and x_1x_2 is not full. By applying Lemma 1.1 to $\{x_1x_3, x_2x_3\}$, we may assume that x_1x_3 and x_2x_3 are straight in $\mathscr{M}$. Let $\mathscr{M}'$ be a new 3-matching assignment for G by setting $\mathscr{M}'_e = \mathscr{M}_e$ for each $e \notin \{x_1x_3, x_2x_3, x_1x_2\}$ and all edges in $\{x_1x_3, x_2x_3, x_1x_2\}$ are straight and full. Note that x_1x_3 and x_2x_3 are straight in $\mathscr{M}$, thus $\mathscr{M}_{x_1x_3} \subseteq \mathscr{M}'_{x_1x_3}$ and $\mathscr{M}_{x_2x_3} \subseteq \mathscr{M}'_{x_2x_3}$. Since all the edges in $\{x_1x_3, x_2x_3, x_1x_2\}$ are full in $\mathscr{M}'$ but not in $\mathscr{M}$, the number of edges in $\mathscr{M}'$ is greater than that in $\mathscr{M}$. Since there are no adjacent triangles, every closed walk of length three is consistent in $\mathscr{M}'$. By the condition (2), the $\mathscr{M}$-coloring ϕ (also $\mathscr{M}'$-coloring) of the outer cycle $\mathcal{C}$ can be extended to an $\mathscr{M}'$-coloring ϕ' of G, but ϕ' is not an $\mathscr{M}$-coloring of G by our
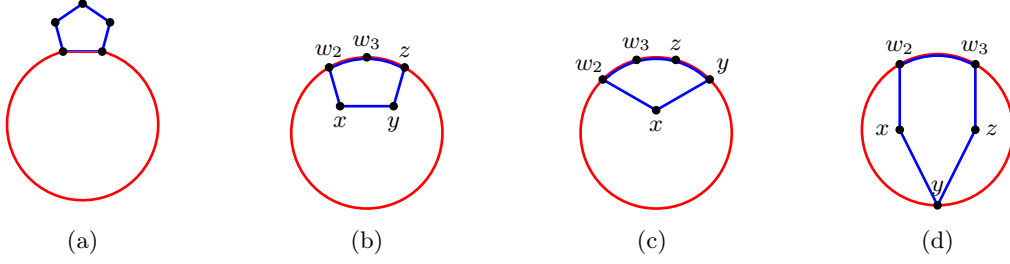
Fig. 2: A 5-face is adjacent to an 8^--face, where the blue cycle bounds a 5-face and the red cycle bounds an 8^--face.
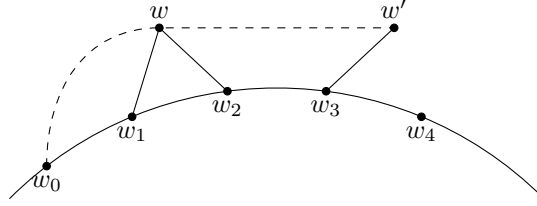

Fig. 3: A case in Lemma 2.5.

assumption. Note that $\mathscr{M}_e \subseteq \mathscr{M}'_e$ for any $e \neq x_1 x_2$, so we may assume that $\phi'(x_1) = 1$, $\phi'(x_2) = 2$ and $(x_1, 1)(x_2, 2) \in \mathscr{M}_{x_1 x_2}$. If $(x_1, 1)$ has an incident edge in $\mathscr{M}_{x_1 x_3}$ and $(x_2, 2)$ has an incident edge in $\mathscr{M}_{x_2 x_3}$, then the closed walk $x_3 x_1 x_2$ is not consistent in $\mathscr{M}$, a contradiction. If $(x_1, 1)$ has no incident edge in $\mathscr{M}_{x_1 x_3}$, then we can modify ϕ' to obtain an $\mathscr{M}$-coloring of G by recoloring x_2 and x_3 in order, a contradiction. So we may assume that $(x_1, 1)$ has an incident edge in $\mathscr{M}_{x_1 x_3}$ and $(x_2, 2)$ has no incident edge in $\mathscr{M}_{x_2 x_3}$. Since $x_1 x_3$ is straight in $\mathscr{M}$, we have that $(x_1, 1)(x_3, 1) \in \mathscr{M}_{x_1 x_3}$. Furthermore, we may assume that $(x_3, 1)$ has no incident edge in $\mathscr{M}_{x_2 x_3}$, otherwise the closed walk $x_2 x_3 x_1$ is not consistent in $\mathscr{M}$. Now, we can obtain a new 3-matching assignment $\mathscr{M}^*$ for G by adding an edge $(x_3, 1)(x_2, 2)$ to $\mathscr{M}$. By the adjacency of cycles, $x_2 x_3$ is only contained in an unique triangle $x_1 x_2 x_3$, so the addition of $(x_3, 1)(x_2, 2)$ does not make $\mathscr{M}^*$ inconsistent on closed 3-walk, but this contradicts the condition (2). $\square$

Lemma 2.5. Let w_0, w_1, w_2, w_3 and w_4 be five consecutive vertices on a 5^+-face. If w_1, w_2, w_3 and w_4 are all 3-vertices, and $w_1 w_2$ is incident with a 3-face $w w_1 w_2$, then at least one vertex in $\{w_1, w_2, w_3, w_4\}$ is on the outer cycle $\mathcal{C}$.

Proof. Suppose to the contrary that none of $\{w_1, w_2, w_3, w_4\}$ is on the outer cycle $\mathcal{C}$. Let w' be the neighbor of w_3 other than w_2, w_4, and let $G^* = G - \{w_1, w_2, w_3, w_4\}$. It is observed that $w_0, w_1, w_2, w_3, w_4, w$ and w' are seven distinct vertices. We claim that the distance between w_0 and w' is at least nine in G^*. Let P be a shortest path between w_0 and w' in G^*. It is observed that $\mathcal{Q} = P \cup w_0 w_1 w_2 w_3 w'$ is a cycle. If w is on the path P, then $P[w_0, w] \cup w_0 w_1 w$ and $P[w, w'] \cup w w_2 w_3 w'$ are all cycles (see Fig. 3), which implies that these two cycles have length at least nine and $|E(P)| \geq (9-2) + (9-3) = 13$. If w is not on the path P, then $\mathcal{Q}$ is a separating normal cycle (note that w and w_4 are in different sides of the cycle $\mathcal{Q}$) and it has length at least 13, which implies that $|E(P)| = |\mathcal{Q}| - 4 \geq 13 - 4 = 9$. Therefore, the distance between w_0 and w' is at least nine in G^*.

By Lemma 2.4 and Lemma 1.1, we may assume that all the edges incident with the vertices in $\{w_1, w_2, w_3\}$ are straight. Let G' be the graph obtained from G^* by identifying w_0 and w', and let $\mathscr{M}'$ be the restriction of $\mathscr{M}$ on $E(G')$. Since the distance between w_0 and w' is at least nine in G^*, the graph G' has no loops, no multiple edges and no new 8^--cycles, thus G' is a simple plane graph satisfying the assumption of Theorem 1.6, and $\mathscr{M}'$ is consistent on every closed walk of length three in G'. Moreover, $\mathcal{C}$ is also a normal cycle of G' and it has no chords in G'. This implies that ϕ is an $\mathscr{M}'$-coloring of $G'[S]$. Since $|V(G')| < |V(G)|$, the $\mathscr{M}'$-coloring ϕ of $G'[S]$ can be extended to an $\mathscr{M}'$-coloring φ of G'. Since w_3 and w_4 are all 3-vertices, we can extend φ to w_4 and w_3 in order. Recall that all the edges incident

with vertices in $\{w_1, w_2, w_3\}$ are straight, thus we may assume that w_0 and w_3 have distinct colors, and then we can further extend the coloring to w_2 and w_1, a contradiction. $\square$

Let w be a vertex on the outer cycle $\mathcal{C}$, and let $w_1, w_2, \ldots, w_k$ be consecutive neighbors in a cyclic order. If f is a face in $\mathcal{N}$ incident with $w w_i$ and $w w_{i+1}$, but neither $w w_i$ nor $w w_{i+1}$ is an edge of $\mathcal{C}$, then we call f a **special** face (at w). A 4-face is a 4^***-face** if it has three common vertices with $\mathcal{C}$. An internal 3-vertex is **bad** if it is incident with a 3-face which is not special, **light** if it is incident with an internal 4-face or a 4^*-face or a special 3-face, **good** if it is neither bad nor light. According to Lemma 2.5, we have the following result on bad vertices.

Lemma 2.6. There are no five consecutive bad vertices on the boundary of a 5^+-face.

Lemma 2.7. If a 4-face in $\mathcal{N}$ has exactly two common vertices with $\mathcal{C}$, then these two vertices are consecutive on the 4-face.

Proof. Suppose that f is a 4-face in $\mathcal{N}$ that has exactly two common vertices with $\mathcal{C}$. If these two vertices are not consecutive on the 4-face, then there exists a separating normal 8^--cycle, a contradiction. $\square$

Assume that $f = [v_1 v_2 \ldots v_l]$ is an internal $(3, 3, 3, 3)$-face or an internal $(3, 3, 3, 3, 3)$-face. Let u_i be the third neighbor of v_i for $1 \le i \le l$. Note that every 8^--cycle has no chords, and there are no separating 4- or 5-cycles. It is observed that $\{u_i, v_i \mid 1 \le i \le l\}$ contains $2l$ distinct vertices. Let Γ be the graph $G \setminus (V(f) \setminus \{v_3\})$, and let G^* be the graph obtained from Γ by identifying u_1 and v_3 into a new vertex z.

Lemma 2.8. The graph G^* is in the class $\mathscr{G}$, and $\mathcal{C}$ is an induced cycle of G^*.

Proof. Let P be an arbitrary path between u_1 and v_3 in Γ. It is observed that $\mathcal{Q} = P \cup u_1 v_1 v_2 v_3$ is a cycle of length $|E(P)| + 3$.

(i) Assume that u_2 is not on the path P. It is easy to check that $|E(P)| \le 8$ only if $|E(P)| = 8$ and $\mathcal{Q}$ is a separating abnormal 11-cycle (note that u_2 and v_l are in different sides of the cycle $\mathcal{Q}$). Note that there are no 3-cycles normally adjacent to abnormal cycles. Even though the identification may make the path P becomes an 8-cycle, but the new cycle is not normally adjacent to any 3-cycle in G^*.

(ii) Assume that u_2 is on the path P. Then $Q_1 = P[u_1, u_2] \cup u_1 v_1 v_2 u_2$ and $Q_2 = P[u_2, v_3] \cup u_2 v_2 v_3$ are all cycles, which implies that each of these two cycles has length at least six and $|E(P)| \ge (6-2) + (6-3) = 7$. It is easy to check that $|E(P)| = 7$ only if $|Q_1| = |Q_2| = 6$, and $|E(P)| = 8$ only if $|Q_i| = 6$ and $|Q_{3-i}| = 7$. Recall that there are no 3-cycles normally adjacent to 8^--cycles, and there are no 4-cycles normally adjacent to 6^--cycles. Therefore, even though the identification may make the path P become a 7- or 8-cycle, but the new cycle is not normally adjacent to any 3-cycle in G^*.

According to the above arguments on the two cases, G^* satisfies the requirement for adjacency in Theorem 1.6, and the distance of u_1 and v_3 in Γ is at least seven. Note that v_3 is an internal vertex, $\mathcal{C}$ is still a 12^--cycle in G^*. It is easy to check that $\mathcal{C}$ is also a normal 12^--cycle in G^*, for otherwise, two new 7-cycles are created to make $\mathcal{C}$ abnormal in G^*, but there is a separating normal 10-cycle in G, a contradiction.

Suppose that the identification creates a chord for $\mathcal{C}$. Since v_3 is an internal vertex, the vertex u_1 is on the outer cycle $\mathcal{C}$. Note that u_3 is also on the outer cycle, the two vertices u_1 and u_3 divide the cycle $\mathcal{C}$ into two paths P' and P''. Since the distance of u_1 and v_3 in Γ is at least seven, we have that $|E(P')| = |E(P'')| = 6$. Thus, $P' \cup u_1 v_1 v_2 v_3 u_3$ or $P'' \cup u_1 v_1 v_2 v_3 u_3$ is a separating normal 10-cycle in G, a contradiction. Hence, the identification does not create chords of $\mathcal{C}$. $\square$

Lemma 2.9. There are no internal $(3, 3, 3, 3)$-faces or $(3, 3, 3, 3, 3)$-faces.

Proof. Suppose to the contrary that $f = [v_1 v_2 \ldots v_l]$ is an internal $(3, 3, 3, 3)$-face or an internal $(3, 3, 3, 3, 3)$-face. All the related vertices are labeled as the above. Noting that none of $u_1 v_1, v_1 v_2, v_2 v_3, v_3 u_3$ is in a triangle, it follows that these four edges are full. So we may assume that each of these four edges is straight. Let H^* be the cover obtained from $H - \cup L_v$ by identifying (u_1, j) and (v_3, j) into a new vertex (z, j), where $1 \le j \le 3$, and the union is taken over $v \in V(f) \setminus \{v_3\}$. By the minimality of G, the precoloring ϕ can be extended to a coloring T^* of the cover H^*. Without loss of generality, we may assume that $(z, 1) \in T$. This coloring naturally gives a coloring T of $H - \cup_{x \in V(f)} L_x$. For each $x \in V(f)$,

let $L'_x = L_x \setminus \{(x, j_x) \mid (x, j_x)$ is the neighbor of a vertex in $T\}$. Let H' be the subgraph of H induced by $\cup_{x \in V(f)} L'_x$. Note that $(v_1, 1) \notin L'_{v_1}$ and $(v_3, 1) \in L'_{v_3}$. Recall that $v_1 v_2$ and $v_2 v_3$ are full and straight, thus H' is neither a circular ladder nor a Möbius ladder. By Theorem 1.13, the cover H' has a coloring T'. Therefore, $T \cup T'$ is a coloring of the cover H, a contradiction. $\square$

We give the initial charge $\mu(v) = \deg(v) - 4$ for any $v \in V(G)$, $\mu(f) = \deg(f) - 4$ for any face $f \in F(G)$ other than outer face D, and $\mu(D) = \deg(D) + 4$ for the outer face D. By the Euler formula, the sum of the initial charges is zero. That is,

$$\sum_{v \in V(G)} \big(\deg(v) - 4 \big) + \sum_{f \in F(G) \setminus D} \big(\deg(f) - 4 \big) + (\deg(D) + 4) = 0. \tag{3}$$

Next, we give the discharging rules to redistribute the charges, preserving the sum, such that the final charge of every element in $V(G) \cup F(G)$ is nonnegative, and at least one element in $V(G) \cup F(G)$ has positive final charge. This leads a contradiction to complete the proof.

A **small face** is a 5^--face. We say that a face is a **negative face** if it is a non-special 3-face, or an internal $(3, 3, 3, 3, 4)$-face, or an internal 4-face.

The followings are the discharging rules.

R1. Each non-special 3-face receives $\frac{1}{3}$ from each incident internal vertex.

R2. Let w be an internal 3-vertex.

    (a) If w is a bad vertex, then it receives $\frac{2}{3}$ from each incident 9^+-face.

    (b) If w is incident with a special 3-face or a 4^*-face, then it receives $\frac{1}{2}$ from each incident 7^+-face.

    (c) If w is incident with an internal 4-face, then it receives $\frac{1}{9}$ from the incident 4-face, and $\frac{4}{9}$ from each incident 7^+-face.

    (d) If w is a good vertex, then it receives $\frac{1}{3}$ from each incident face.

R3. Let w be an internal 4-vertex. Then w sends $\frac{1}{3}$ to each incident internal $(3, 3, 3, 3, 4)$-face and internal 4-face.

    (a) If w is incident with two negative faces, then it receives $\frac{1}{3}$ from each of the other two incident faces.

    (b) If w is incident with exactly one negative face f, and the opposite face at w is a 9^+-face, then w receives $\frac{1}{3}$ from the opposite face.

    (c) If w is incident with exactly one negative face f, and the opposite face at w is an 8^--face, then w receives $\frac{1}{6}$ from each face at w that is not the opposite face.

R4. Each internal 5^+-vertex sends $\frac{1}{3}$ to each incident 5-face.

R5. Each 2-vertex on the outer cycle $\mathcal{C}$ receives $\frac{2}{3}$ from the incident face in $\mathcal{N}$ and $\frac{4}{3}$ from the outer face.

R6. Each 3-vertex on the outer cycle $\mathcal{C}$ receives $\frac{4}{3}$ from the outer face, and sends $\frac{1}{3}$ to each incident 4^--face in $\mathcal{N}$ and $\frac{1}{6}$ to each incident 5- or 6-face in $\mathcal{N}$.

R7. Each 4-vertex on the outer cycle $\mathcal{C}$ receives 1 from the outer face, and sends 1 to each incident special 4^--face, and $\frac{1}{3}$ to each of the other incident 6^--face in $\mathcal{N}$.

R8. Each 5^+-vertex on the outer cycle $\mathcal{C}$ receives 1 from the outer face, and sends 1 to each incident special 4^--face, and $\frac{1}{2}$ to each of the other incident 6^--face in $\mathcal{N}$.

Lemma 2.10. Every face other than D has nonnegative final charge.

Proof. According to the discharging rules, inner 3-faces never give charges. If f is a special 3-face, then $\mu'(f) = 3 - 4 + 1 = 0$ by R7 and R8. If f is a non-special 3-face having no vertex on the outer cycle $\mathcal{C}$, then $\mu'(f) = 3 - 4 + 3 \times \frac{1}{3} = 0$ by R1. If f is a non-special 3-face having two vertices on the outer cycle $\mathcal{C}$, then f has a common edge with the outer face by Lemma 2.2(f), and then $\mu'(f) \geq 3 - 4 + 3 \times \frac{1}{3} = 0$ by R1, R6, R7 and R8. Note that no inner 3-faces have three common vertices with $\mathcal{C}$.

Let f be a 4-face. If f is an internal face, then it is incident with at least one 4^+-vertex by Lemma 2.9, and then $\mu'(f) \geq 4 - 4 + \frac{1}{3} - 3 \times \frac{1}{9} = 0$ by R2c and R3. So we may assume that f is a face in $\mathcal{N}$. If f has exactly one common vertex with $\mathcal{C}$, then it receives 1 from the vertex on the outer cycle $\mathcal{C}$, and then $\mu'(f) \geq 4 - 4 + 1 - 3 \times \frac{1}{3} = 0$ by R2d, R7 and R8. If f has exactly two common vertices with the outer cycle $\mathcal{C}$, then these two vertices are consecutive on f by Lemma 2.7, and then $\mu'(f) \geq 4 - 4 + 2 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$ by R2d, R6, R7 and R8. If f has exactly three common vertices with the outer cycle $\mathcal{C}$, i.e., f is a 4^*-face, then $\mu'(f) \geq 4 - 4 + 2 \times \frac{1}{3} - \frac{2}{3} = 0$ by R5, R7 and R8.

Let f be a 5-face. By Lemma 2.9, f cannot be incident with five internal 3-vertices. If f is an internal $(3, 3, 3, 3, 4^+)$-face, then f receives $\frac{1}{3}$ from the incident 4^+-vertex and sends $\frac{1}{3}$ to each incident 3-vertex, thus $\mu'(f) = 5 - 4 + \frac{1}{3} - 4 \times \frac{1}{3} = 0$ by R2d, R3 and R4. If f is an internal face incident with at least two 4^+-vertices, then $\mu'(f) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$ by R2d. So we may assume that f is a face in $\mathcal{N}$. If f has exactly one common vertex with the outer cycle $\mathcal{C}$, then it receives at least $\frac{1}{3}$ from the vertex on the outer cycle $\mathcal{C}$, and then $\mu'(f) \geq 5 - 4 + \frac{1}{3} - 4 \times \frac{1}{3} = 0$ by R2d, R7 and R8. If f has at least two common vertices with the outer cycle $\mathcal{C}$, and it is not incident with any 2-vertex, then it receives at least $\frac{1}{6}$ from each incident vertex on the outer cycle $\mathcal{C}$, and then $\mu'(f) \geq 5 - 4 + 2 \times \frac{1}{6} - 3 \times \frac{1}{3} = \frac{1}{3}$ by R2d, R6, R7 and R8. If f is incident with exactly one 2-vertex, then $\mu'(f) \geq 5 - 4 + 2 \times \frac{1}{6} - \frac{2}{3} - 2 \times \frac{1}{3} = 0$ by R2d, R6, R7 and R8. By Lemma 2.2(g), f cannot be incident with more than one 2-vertex.

Let f be an internal 6-face. Note that there are no 4^--faces adjacent to 6-faces, it follows that f sends at most $\frac{1}{3}$ to each incident vertex, thus $\mu'(f) \geq 6 - 4 - 6 \times \frac{1}{3} = 0$.

Let $f = [x_1 x_2 \ldots x_6]$ be a 6-face in $\mathcal{N}$. (i) Suppose that f is incident with three 2-vertices and an internal vertex x_6. If x_6 has at least three neighbors on the outer cycle $\mathcal{C}$, then $\mathcal{C}$ is an abnormal cycle (see Fig. 1a, Fig. 1b and Fig. 1c), a contradiction. So x_6 has a neighbor not on $\mathcal{C}$, and $\mathcal{C}' = (\mathcal{C} - \{x_2, x_3, x_4\}) \cup \{x_1 x_6, x_5 x_6\}$ is a separating 10^--cycle, a contradiction. (ii) Suppose that f is incident with one or two 2-vertices. Thus, f receives at least $\frac{1}{6}$ from each incident 3^+-vertex on the outer cycle $\mathcal{C}$, and sends at most $\frac{1}{3}$ to each incident internal vertex, which implies that $\mu'(f) \geq 6 - 4 + 2 \times \frac{1}{6} - 2 \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{1}{3}$. (iii) If f is not incident with any 2-vertex, then it sends at most $\frac{1}{3}$ to each incident internal vertex, but does not send any charge to the incident vertices on the outer cycle $\mathcal{C}$, thus $\mu'(f) \geq 6 - 4 - 5 \times \frac{1}{3} > 0$.

Let $f = [x_1 x_2 \ldots x_7]$ be a 7-face. Suppose that f is an internal face. Since there are no 3-faces adjacent to 7-faces, we have that f sends at most $\frac{4}{9}$ to each incident internal vertex. Note that seven is an odd number, f is incident with at most six 3-vertices which are incident with internal 4-faces, thus $\mu'(f) \geq 7 - 4 - 6 \times \frac{4}{9} - \frac{1}{3} = 0$. Now, assume that f is a face in $\mathcal{N}$. If f is not incident with any 2-vertex, then $\mu'(f) \geq 7 - 4 - 6 \times \frac{1}{2} = 0$. If f is incident with one or two 2-vertices, then $\mu'(f) \geq 7 - 4 - 2 \times \frac{2}{3} - 3 \times \frac{1}{2} > 0$. Consider the case f is incident with exactly three 2-vertices. It is observed that the three 2-vertices are consecutive on the outer cycle, say x_2, x_3 and x_4. By the discharging rules, f sends at most $\frac{1}{2}$ to each of x_6 and x_7. It follows that $\mu'(f) \geq 7 - 4 - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$. Furthermore, $\mu'(f) = 0$ only if both x_6 and x_7 are 3-vertices, and each of which is incident with a 4^*-face, which implies that $\mathcal{C}$ is abnormal (see Fig. 1f), a contradiction. The final case: f is incident with four 2-vertices, say x_2, x_3, x_4 and x_5. Then it is incident with exactly one internal vertex x_7. Note that x_7 has degree at least three. If x_7 has another neighbor on the outer cycle other than x_1 and x_6, then $\mathcal{C}$ is abnormal (see Fig. 1b and Fig. 1e), a contradiction. If x_7 has a neighbor not on the outer cycle, then $(\mathcal{C} \setminus \{x_2, x_3, x_4, x_5\}) \cup \{x_1 x_7, x_6 x_7\}$ is a separating normal 9^--cycle, a contradiction. In summary, every 7-face has nonnegative final charge, and each 7-face adjacent to D has positive final charge.

Let f be an 8-face. By Lemma 2.3, there are no 3-faces adjacent to 8-faces, so f sends at most $\frac{1}{2}$ to each incident internal vertex. If f is an internal face, then $\mu'(f) \geq 8 - 4 - 8 \times \frac{1}{2} = 0$. So we may assume that f is an 8-face in $\mathcal{N}$. If f is not incident with any 2-vertex, then $\mu'(f) \geq 8 - 4 - 7 \times \frac{1}{2} = \frac{1}{2}$. Suppose that f is incident with at least one 2-vertex. It follows that f has at least two common 3^+-vertices with the outer cycle $\mathcal{C}$, and it sends nothing to these vertices. Thus, f sends $\frac{2}{3}$ to each incident 2-vertex, and sends at most $\frac{1}{2}$ to each incident internal vertex, which implies that $\mu'(f) \geq 8 - 4 - 5 \times \frac{2}{3} - \frac{1}{2} > 0$.

Let f be a k-face with $k \geq 9$. By the discharging rules, it is easy to show the following fact.

Fact-1. The face f sends nothing to the 3^+-vertices on the outer cycle $\mathcal{C}$, and sends at most $\frac{1}{3}$ to each incident internal 4-vertex, and sends at most $\frac{2}{3}$ to any vertex.

If f is incident with some 2-vertices, then f is incident with at least two 3^+-vertices on the outer cycle $\mathcal{C}$ and it sends nothing to these vertices, which implies that

$$\mu'(f) \geq k - 4 - (k-2) \times \frac{2}{3} = \frac{1}{3}(k-8) > 0. \tag{4}$$

So we may assume that f is not incident with any 2-vertex.

Let α be the number of incident bad vertices, β be the number of incident light vertices, and let ρ be the number of incident internal 3-vertices. It is observed that $\alpha + \beta \leq \rho$. By Lemma 2.5, we can easily show the following fact on the parameters α and β.

Fact-2. If $\alpha \geq 3$, then $\alpha + \beta \leq \rho \leq k - 2$.

If $\alpha \leq 3$, then $\mu'(f) \geq k - 4 - 3 \times \frac{2}{3} - (k-3) \times \frac{1}{2} = \frac{k-9}{2} \geq 0$. If $\alpha \geq 4$ and $k \geq 10$, then $\mu'(f) \geq k - 4 - (k-2) \times \frac{2}{3} - 2 \times \frac{1}{3} = \frac{1}{3}(k-10) \geq 0$. It remains to assume that $\alpha \geq 4$, $k = 9$ and $f = [w_1 w_2 \ldots w_9]$.

Suppose that $\alpha = 7$. By Lemma 2.6, the two non-bad vertices divide the bad vertices on f into two parts, one consisting of three consecutive bad vertices and the other consisting of four consecutive bad vertices. Without loss of generality, we may assume that none of w_1 and w_6 is a bad vertex. By Lemma 2.5, $w_1 w_2$, $w_3 w_4$ and $w_5 w_6$ are incident with 3-faces. By symmetry, we may further assume that $w_6 w_7$ and $w_8 w_9$ are incident with 3-faces. By Lemma 2.5, w_1 cannot be an internal 3-vertex. If w_1 is an internal 5^+-vertex or on the outer cycle $\mathcal{C}$, then f sends nothing to w_1. If w_1 is an internal 4-vertex, then w_1 is incident with three 9^+-faces, and it receives nothing from f by R3b. Thus, f sends nothing to w_1 in all cases, which implies that $\mu'(f) \geq 9 - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$.

If $4 \leq \alpha \leq 5$, then $\mu'(f) \geq 9 - 4 - \alpha \times \frac{2}{3} - (\rho - \alpha) \times \frac{1}{2} - (9 - \rho) \times \frac{1}{3} = 2 - \frac{\alpha + \rho}{6} \geq 2 - \frac{5+7}{6} = 0$. It remains to assume that $\alpha = 6$. Recall that f is not incident with any 2-vertex. If f has a common vertex with the outer cycle $\mathcal{C}$, then f sends nothing to this vertex and $\mu'(f) \geq 9 - 4 - 6 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$. So we may assume that f is an internal 9-face. If f is not incident with any light vertex, then $\mu'(f) \geq 9 - 4 - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$. So we may assume that there is a light vertex on f. By the definitions of bad vertices and light vertices, a light vertex cannot be adjacent to two bad vertices on f, thus a light vertex must be adjacent to a non-bad vertex on f, which implies that the bad vertices on f are divided into two parts by Lemma 2.6. Moreover, each part has at most four consecutive bad vertices, thus each part has at least two consecutive bad vertices. Without loss of generality, we may assume that w_1 is a light vertex and w_9 is a non-bad vertex. Then w_2 and w_8 must be bad vertices. Since bad vertex and light vertex cannot be incident with the same 3-face, $w_2 w_3$ is incident with a 3-face and $w_1 w_9$ is incident with a 4^--face. By Lemma 2.5 and Lemma 2.6, w_3 is also a bad vertex and w_9 is not an internal 3-vertex. Hence, w_9 must be an internal 4^+-vertex. Since w_8 is a bad vertex, either $w_7 w_8$ or $w_8 w_9$ is incident with a 3-face. If $w_7 w_8$ is incident with a 3-face, then f sends nothing to w_9, which implies that $\mu'(f) \geq 9 - 4 - 6 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$. In the other case, $w_8 w_9$ is incident with a 3-face. Then $w_6 w_7$ is incident with a 3-face. By Lemma 2.5, one vertex in $\{w_4, w_5\}$ is an internal 4^+-vertex, and the other is a bad 3-vertex, thus $w_4 w_5$ is incident with a 3-face. Whenever w_4 or w_5 is an internal 4^+-vertex, it receives nothing from f by R3b, which implies that $\mu'(f) \geq 9 - 4 - 6 \times \frac{2}{3} - \frac{1}{2} - \frac{1}{3} > 0$. $\square$

Lemma 2.11. Every vertex has nonnegative final charge.

Proof. If v is a 2-vertex, then it is on the outer cycle, and it receives $\frac{2}{3}$ from the incident face in $\mathcal{N}$ and $\frac{4}{3}$ from the outer face D by R5, which implies that $\mu'(v) = 2 - 4 + \frac{2}{3} + \frac{4}{3} = 0$. Let v be a 3-vertex on the outer cycle. By the adjacency of the faces, v is incident with a 4^--face and a 7^+-face, or it is incident with two 5^+-faces. Thus, $\mu'(v) \geq 3 - 4 + \frac{4}{3} - \max\left\{\frac{1}{3}, 2 \times \frac{1}{6}\right\} = 0$ by R6. If v is a 4-vertex on the outer cycle, then it receives 1 from the outer face, sends 1 to an incident special 4^--face and at most $\frac{1}{3}$ to each of the other incident 6^--face in $\mathcal{N}$ by R7, which implies that $\mu'(v) \geq 4 - 4 + 1 - \max\{1, 3 \times \frac{1}{3}\} = 0$. If v

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is a 5^+-vertex on the outer cycle, then it receives 1 from the outer face, and averagely sends at most $\frac{1}{2}$ to each incident face in $\mathcal{N}$, and then $\mu'(v) \geq \deg(v) - 4 + 1 - (\deg(v) - 1) \times \frac{1}{2} = \frac{\deg(v)-5}{2} \geq 0$.

If v is a bad vertex, then $\mu'(v) = 3 - 4 + 2 \times \frac{2}{3} - \frac{1}{3} = 0$ by R1 and R2a. If v is incident with a 4^*-face or a special 3-face, then $\mu'(v) = 3 - 4 + 2 \times \frac{1}{2} = 0$ by R2b. If v is incident with an internal 4-face, then $\mu'(v) = 3 - 4 + \frac{1}{9} + 2 \times \frac{4}{9} = 0$ by R2c. If v is a good vertex, then $\mu'(v) = 3 - 4 + 3 \times \frac{1}{3} = 0$ by R2d. If v is an internal 4-vertex which is incident with two negative faces, then $\mu'(v) = 4 - 4 + 2 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$ by R1 and R3. If v is an internal 4-vertex which is incident with exactly one negative face and a 9^+-face at the opposite side, then $\mu'(v) = 4 - 4 + \frac{1}{3} - \frac{1}{3} = 0$ by R1 and R3. If v is incident with exactly one negative face and an 8^--face at the opposite side, then $\mu'(v) = 4 - 4 + 2 \times \frac{1}{6} - \frac{1}{3} = 0$ by R1 and R3. If v is an internal 4-vertex which is not incident with any negative face, then $\mu'(v) = 4 - 4 = 0$. Note that there are no adjacent 5^--faces, thus every vertex v is incident with at most $\lfloor \frac{\deg(v)}{2} \rfloor$ small faces. If v is an internal 5^+-vertex, then it sends at most $\frac{1}{3}$ to each incident 5^--face, which implies that $\mu'(v) \geq \deg(v) - 4 - \frac{1}{3} \times \lfloor \frac{\deg(v)}{2} \rfloor > 0$. $\qquad\square$

Lemma 2.12. The outer face D has nonnegative charge, and there exists an element in $V(G) \cup F(G)$ having positive final charge.

By R5–R8, the outer face D sends $\frac{4}{3}$ to each incident 3^--vertex, and sends 1 to each incident 4^+-vertex, thus $\mu'(D) \geq |D| + 4 - \frac{4}{3}|D| = \frac{1}{3}(12 - |D|) \geq 0$. Therefore, every element in $V(G) \cup F(G)$ has a nonnegative final charge. In particular, $\mu'(D) = 0$ holds if and only if $|D| = 12$ and each vertex incident with D receives $\frac{4}{3}$ from D. So we may assume that $|D| = 12$ and each vertex on the outer cycle $\mathcal{C}$ is a 3^--vertex.

Let f be an arbitrary k-face adjacent to D. By the discharging rules, f sends nothing to 3^+-vertices on the outer cycle, and sends at most $\frac{2}{3}$ to each of the other incident vertex. If $k \geq 9$, then $\mu'(f) \geq k - 4 - (k - 2) \times \frac{2}{3} > 0$. Recall that every 6-, 7- and 8-face adjacent to D has a positive final charge, thus D is not adjacent to any 6^+-face. Therefore, there exists a 3-vertex on the outer cycle such that it is incident with two adjacent 5^--faces, a contradiction. $\qquad\square$

3 IF-coloring

For IF-coloring, a vertex is **colored with** I if the image is I in a mapping, or it is in the part I of an (I, F)-partition. An F**-path** is a path whose vertices are all colored with F, and an F**-cycle** is a cycle whose vertices are all colored with F.

Given a graph G and a cycle C in G, an IF-coloring ϕ_C of $G[V(C)]$ can **be superextended** to G if there exists an IF-coloring ϕ_G of G that extends ϕ_C with the property that there are no F-paths (having at least one vertex not on C) linking two vertices of C. We say that C is **superextendable** to G if every IF-coloring of $G[V(C)]$ can be superextended to G. For convenience, we also say that a vertex w is **superextendable** to G if every IF-coloring of w can be extended to an IF-coloring of G.

For an IF-coloring, a vertex v is F**-reachable** to a cycle if there is a path from v to a vertex on the cycle such that all the vertices on the path are colored with F.

Instead of proving Theorem 1.12, we prove the following stronger result.

Theorem 3.1. Let G be a graph in $\mathscr{G}$. Let S be a set of vertices of G such that $|S| \leq 1$ or S consists of all vertices on a normal cycle of G. If $|S| \leq 12$, then every IF-coloring ϕ of $G[S]$ can be superextended to an IF-coloring of G.

Remark 3. Analogously, not every IF-coloring of the 11- or 12-cycle can be superextended to the whole graph. Thus, we require the condition that S consists of all vertices on a "normal" cycle.

The following result is a direct consequence of Theorem 3.1.

Theorem 3.2. Every graph in $\mathscr{G}$ is IF-colorable.

Sketch of a proof for Theorem 3.1. Suppose that G is a minimal counterexample to Theorem 3.1. That is, there exists an IF-coloring of $G[S]$ that cannot be superextended to an IF-coloring of G such that $|V(G)| + |E(G)| - |S|$ is minimized.

Lemma 3.1. Every 8^--cycle has no chords.

Lemma 3.2.

(a) $S \neq V(G)$.

(b) G is 2-connected, and thus the boundary of every face is a cycle.

(c) Each vertex not in S has degree at least three.

(d) Either $|S| = 1$ or $G[S]$ is an induced cycle of G.

(e) There are no separating normal k-cycles for $3 \leq k \leq 12$. Thus, every edge on an abnormal cycle is incident with a 4-, 5-, 6- or 7-face.

(f) $G[S]$ is an induced cycle of G. For convenience, we can redraw the graph G such that $G[S]$ is the outer cycle $\mathcal{C}$ of G. Let D be the outer face which is bounded by $G[S]$.

(g) Every 5-face ($\neq$ outer face) is incident with at most one 2-vertex.

Proof. The proof is similar to the one in Lemma 2.2, so the reader can do it as an exercise, or find it in the arXiv version. $\square$

Lemma 3.3. There are no 3-faces adjacent to 8^--faces, no 4-faces adjacent to 6^--faces, and no adjacent 5-faces.

Proof. The proof is the same with that in Lemma 2.1. $\square$

Lemma 3.4. Let w_0, w_1, w_2, w_3 and w_4 be five consecutive vertices on a 5^+-face. If w_1, w_2, w_3 and w_4 are all 3-vertices, and $w_1 w_2$ is incident with a 3-face $w w_1 w_2$, then at least one vertex in $\{w_1, w_2, w_3, w_4\}$ is on the outer cycle $\mathcal{C}$.

Proof. Suppose to the contrary that none of $\{w_1, w_2, w_3, w_4\}$ is on the outer cycle $\mathcal{C}$. Let w' be the neighbor of w_3 other than w_2, w_4, and let $G^* = G - \{w_1, w_2, w_3, w_4\}$. It is observed that $w_0, w_1, w_2, w_3, w_4, w$ and w' are seven distinct vertices. Similar to Lemma 2.5, we can prove that the distance between w_0 and w' is at least nine in G^*.

Let G' be the graph obtained from G^* by identifying w_0 and w'. Since the distance between w_0 and w' is at least nine in G^*, the graph G' has no loop, no multiple edge and no new 8^--cycle, thus G' is a simple plane graph satisfying the assumption of Theorem 3.1. Moreover, $\mathcal{C}$ is also a normal cycle of G' and it has no chords in G'. This implies that ϕ is an IF-coloring of $G'[S]$. Since $|V(G')| < |V(G)|$, the IF-coloring ϕ of $G'[S]$ can be superextended to an IF-coloring φ of G'. We color w_0 and w' with the same color as the new vertex in G'. If one neighbor of w_4 is colored with I, then we color w_4 with F; otherwise, we color w_4 with I.

If $\varphi(w_0) = \varphi(w') = I$, then let $\varphi(w_1) = \varphi(w_3) = F$ and $\varphi(w_2) \neq \varphi(w)$. So we may assume that $\varphi(w_0) = \varphi(w') = F$, and let $\varphi(w_3) \neq \varphi(w_4)$. If $\varphi(w) = I$, then let $\varphi(w_1) = \varphi(w_2) = F$. So we may assume that $\varphi(w) = F$. If $\varphi(w_3) = I$, then let $\varphi(w_1) = I$ and $\varphi(w_2) = F$. In the final, we may assume that $\varphi(w_0) = \varphi(w) = \varphi(w') = \varphi(w_3) = F$. It is observed that there are no F-paths in H between w' and w_0. Meanwhile, at least one of w' and w_0 is not reachable to $\mathcal{C}$. When we color w_1 with F, it is a superextension to $G - w_2$, we can color w_2 with I. Otherwise, there is an F-path in H between w and w_0, or both w and w_0 are reachable to $\mathcal{C}$. It follows that there are no F-paths between w and w', and at least one of w and w' is not reachable to $\mathcal{C}$. Then let $\varphi(w_1) = I$ and $\varphi(w_2) = F$.

It is easy to check that the resulting coloring is always a superextension to G, a contradiction. $\square$

Let w be a vertex on the outer cycle $\mathcal{C}$, and let $w_1, w_2, \ldots, w_k$ be consecutive neighbors in a cyclic order. If f is a face in $\mathcal{N}$ incident with $w w_i$ and $w w_{i+1}$, but neither $w w_i$ nor $w w_{i+1}$ is an edge of $\mathcal{C}$, then we call f a **special** face (at w). A 4-face is a 4^*-**face** if it has three common vertices with $\mathcal{C}$. An internal 3-vertex is **bad** if it is incident with a 3-face which is not special, **light** if it is incident with an internal 4-face or a 4^*-face or a special 3-face, **good** if it is neither bad nor light. According to Lemma 3.4, we have the following result on bad vertices.

Lemma 3.5. There are no five consecutive bad vertices on the boundary of a 5^+-face.

Lemma 3.6. If a 4-face in $\mathcal{N}$ has exactly two common vertices with $\mathcal{C}$, then these two vertices are consecutive on the 4-face.

Proof. Suppose that f is a 4-face in $\mathcal{N}$ that has exactly two common vertices with $\mathcal{C}$. If these two vertices are not consecutive on the 4-face, then there exists a separating normal 8^--cycle, a contradiction. $\square$

Assume that $f = [v_1 v_2 \ldots v_l]$ is an internal $(3,3,3,3)$-face or an internal $(3,3,3,3)$-face. Let u_i be the third neighbor of v_i for $1 \leq i \leq l$. Note that every 8^--cycle has no chords, and there are no separating 4- or 5-cycles. It is observed that $\{u_i, v_i \mid 1 \leq i \leq l\}$ contains $2l$ distinct vertices. Let Γ be the graph $G \setminus (V(f) \setminus \{v_3\})$, and let G^* be the graph obtained from Γ by identifying u_1 and v_3 into a new vertex z.

Lemma 3.7. The graph G^* is in the class $\mathscr{G}$, and $\mathcal{C}$ is an induced cycle of G^*.

Proof. The proof is the same with that in Lemma 2.8. $\square$

Lemma 3.8. There are no internal $(3,3,3,3)$-faces or $(3,3,3,3,3)$-face.

Proof. We know that $(G^*, \mathcal{C}, \phi)$ satisfies all the requirements in Theorem 3.1. By the minimality of G, the IF-coloring ϕ can be superextended to an IF-coloring φ of G^*. We color u_1 and v_3 with the same color as the new vertex by the identification.

- $l = 5$.

Assume that $\varphi(u_1) = \varphi(v_3) = I$. Then let $\varphi(v_1) = \varphi(v_2) = \varphi(v_4) = F$. Furthermore, we color v_5 such that $\varphi(v_5) \neq \varphi(u_5)$, this is invalid only if $u_2 v_2 v_1 v_5 v_4 u_4$ is on an F-cycle or is on an F-path linking two vertices of $\mathcal{C}$. In both cases, we recolor v_2 and v_4 with I, and v_3 with F.

Assume that $\varphi(u_1) = \varphi(v_3) = F$. If $\varphi(u_4) = I$, then let $\varphi(v_1) = \varphi(v_4) = F$, $\varphi(v_2) \neq \varphi(u_2)$ and $\varphi(v_5) \neq \varphi(u_5)$, this is invalid only if v_1, v_2, v_3, v_4, v_5 are all colored with F. In this case, we recolor v_1 with I. So we may assume that $\varphi(u_4) = F$. If $\varphi(u_2) = I$ or $\varphi(u_3) = I$, then let $\varphi(v_1) = \varphi(v_4) = I$ and $\varphi(v_2) = \varphi(v_5) = F$. So we may assume that $\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = \varphi(u_4) = F$. If $\varphi(u_5) = I$, then let $\varphi(v_2) = \varphi(v_4) = I$, and $\varphi(v_1) = \varphi(v_5) = F$. Now, we may assume that all the vertices u_i are colored with F. Then let $\varphi(v_3) = \varphi(v_5) = I$ and $\varphi(v_1) = \varphi(v_2) = \varphi(v_4) = F$, this is invalid only if $u_1 v_1 v_2 u_2$ is on an F-cycle or an F-path linking two vertices of $\mathcal{C}$. In this case, let $\varphi(v_1) = \varphi(v_4) = I$ and $\varphi(v_2) = \varphi(v_3) = \varphi(v_5) = F$.

- $l = 4$.

Assume that $\varphi(u_1) = \varphi(v_3) = I$. Then let $\varphi(v_1) = \varphi(v_2) = \varphi(v_4) = F$, this is invalid only if $u_2 v_2 v_1 v_4 u_4$ is on an F-cycle or is on an F-path linking two vertices of $\mathcal{C}$. In both cases, let $\varphi(v_1) = \varphi(v_3) = F$, $\varphi(v_2) = \varphi(v_4) = I$.

Assume that $\varphi(u_1) = \varphi(v_3) = F$. Then let $\varphi(v_1) = F$, $\varphi(v_2) \neq \varphi(u_2)$ and $\varphi(v_4) \neq \varphi(u_4)$, this is invalid only if $\varphi(u_2) = \varphi(u_4) = I$. In this case, we recolor v_1 with I. $\square$

Note that Lemma 2.4 is only prepared for Lemma 2.5, so we do not need it here. All the structural lemmas are the same in the proof processes, so the discharging part is the same with that in Theorem 3.1.
$\square$

4 Final discussion

If we can relax the normally adjacent 5-cycles in Theorem 1.6, then Theorem 1.6 implies that every planar graph without 3-, 6-, 7-cycles is DP-3-colorable. On the other hand, if we can allow the normally adjacent 4-cycles in Theorem 1.6, then Theorem 1.6 implies that every planar graph without 3-, 7-, 8-cycles is DP-3-colorable. But until now, we don't know whether every planar graph without 3-, 6-, 7-cycles is DP-3-colorable or not, and we don't know whether every planar graph without 3-, 7-, 8-cycles is DP-3-colorable or not. It seems that it is not easy to solve these two problems.

Dvořák, Lidický, and Škrekovski [4] proved that every planar graph without 3-, 6-, 7-cycles is 3-choosable. The same authors [3] also proved that every planar graph without 3-, 7-, 8-cycles is 3-choosable. So it is interesting to consider the following problems.

Problem 1. Is every planar graph without 3-, 6-, 7-cycles DP-3-colorable?

Problem 2. Is every planar graph without 3-, 7-, 8-cycles DP-3-colorable?

Problem 3. Does every planar graph without 3-, 6-, 7-cycles have an *IF*-coloring?

Problem 4. Does every planar graph without 3-, 7-, 8-cycles have an *IF*-coloring?

Added Note. Recently, Han et al. [7] proved that every triangle-free planar graph without 4-cycles normally adjacent to 4- and 5-cycles is DP-3-colorable. This solves Problem 1.

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Appendix

Proof of Lemma 3.2. (a) Suppose to the contrary that $S = V(G)$. Every *IF*-coloring of $G[S]$ is an *IF*-coloring of G, a contradiction.

(b) It is observed that G is connected. Suppose to the contrary that G has a cut-vertex w. We may assume that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{w\}$. By the assumption of the set S, either $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$. We may assume that $S \subseteq V(G_1)$. By the minimality of G, the *IF*-coloring ϕ of $G[S]$ can be superextended to an *IF*-coloring ϕ_1 of G_1, and $\phi_1(w)$ can be superextended to an *IF*-coloring ϕ_2 of G_2. These two colorings ϕ_1 and ϕ_2 together give an *IF*-coloring of G whose restriction on $G[S]$ is ϕ, a contradiction.

(c) Suppose that there exists a vertex w not in S having degree two. By the minimality of G, the *IF*-coloring of $G[S]$ can be superextended to an *IF*-coloring of $G - w$. If the two neighbors of w are colored with F, then we color w with I; otherwise, we color w with F.

(d) If $S = \emptyset$, then we put any vertex into S to make $|S| = 1$. Suppose that $S = V(\mathcal{Q})$ and $\mathcal{Q}$ is a cycle with a chord uv. It is observed that the *IF*-coloring of $G[S]$ is also an *IF*-coloring of the induced subgraph in $G - uv$. By the minimality of G, the *IF*-coloring ϕ of $G[S]$ can be superextended to an *IF*-coloring of $G - uv$, and hence it is also a superextension of G, a contradiction.

(e) We first show that $G[S]$ cannot be a separating cycle. Suppose to the contrary that $G[S]$ is a separating (normal) cycle $\mathcal{O}$. By the minimality of G, the *IF*-coloring ϕ of $\mathcal{O}$ can be superextended to an *IF*-coloring ϕ_1 of $\text{Int}(\mathcal{O})$, and another *IF*-coloring ϕ_2 of $\text{Ext}(\mathcal{O})$. These two colorings ϕ_1 and ϕ_2 together give a superextension, a contradiction.

Thus, either $|S| = 1$ or S consists of all vertices on a face of G. Let $\mathcal{Q}$ be a separating normal k-cycle with $3 \leq k \leq 12$. Thus, we may assume that $S \subseteq \text{Ext}(\mathcal{Q})$. By the minimality of G, the *IF*-coloring ϕ of $G[S]$ can be superextended to an *IF*-coloring φ_1 of $\text{Ext}(\mathcal{Q})$. Similarly, the restriction of φ_1 on $\mathcal{Q}$ can be superextended to an *IF*-coloring φ_2 of $\text{Int}(\mathcal{Q})$. These two colorings φ_1 and φ_2 together give a superextension of G, a contradiction.

(f) According to (d), suppose to the contrary that $S = \{w\}$. We first assume that w is on a 10^--cycle $\mathcal{Q}$. Without loss of generality, we may assume that $\mathcal{Q}$ is a shortest cycle containing w. Then $\mathcal{Q}$ is an induced cycle. By (e), we may assume that $\text{ext}(\mathcal{Q}) = \emptyset$ and $\mathcal{Q}$ is the outer cycle. By (c) and $\mathcal{Q}$ is an induced cycle, every vertex on $\mathcal{Q}$ other than w has a neighbor in $\text{int}(\mathcal{Q})$, which implies that $\text{int}(\mathcal{Q}) \neq \emptyset$. By the minimality of G, the *IF*-coloring ϕ of $\{w\}$ can be superextended to an *IF*-coloring ϕ_1 of $\mathcal{Q}$. By the minimality of G, the *IF*-coloring ϕ_1 of $\mathcal{Q}$ can be further superextended to an *IF*-coloring of G, a contradiction.

So we may assume that every cycle containing w has length at least 11. Let w be incident with a face $[w_1ww_2\ldots w_1]$. Let G' be obtained from G by adding a chord w_1w_2 in the face, let $S' = \{w, w_1, w_2\}$. We can easily check that G' is a plane graph satisfying the assumption of Theorem 3.1. By the minimality of G, the *IF*-coloring ϕ of $\{w\}$ can be superextended to an *IF*-coloring ϕ_1 of $G'[S']$. By the minimality of G, the *IF*-coloring ϕ_1 of $G'[S']$ can be further superextended to an *IF*-coloring φ of G'. It is observed that φ is an *IF*-coloring of G, a contradiction.

(g) Note that every 2-vertex and its two neighbors are all on the outer cycle. Suppose to the contrary that $f = [x_1x_2x_3x_4x_5]$ is a 5-face which is incident with two 2-vertices. Note that the two 2-vertices must be adjacent on the outer cycle, say x_2 and x_3. It follows that x_1 and x_4 are on the outer cycle $\mathcal{C}$. If x_5 has three neighbors on $\mathcal{C}$, then $\mathcal{C}$ is abnormal (see Fig. 1a and Fig. 1b), a contradiction. Thus, x_5 has a neighbor not on the outer cycle $\mathcal{C}$, and $\mathcal{C}' = (\mathcal{C} - \{x_2, x_3\}) \cup \{x_1x_5, x_4x_5\}$ is a separating 11^--cycle. By Lemma 3.2(e), $\mathcal{C}'$ is an abnormal 11-cycle (see Fig. 1a). It follows that $\mathcal{C}$ is an abnormal 12-cycle (see Fig. 1d), a contradiction. $\square$