# Extremal $t$-intersecting families for direct products 

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#### Abstract

In this paper, by shifting technique we study $t$-intersecting families for direct products where the ground set is divided into several parts. Assuming the size of each part is sufficiently large, we determine all extremal $t$-intersecting families for direct products. We also prove that every largest $t$-intersecting subfamily of a more general family introduced by Katona is trivial under certain conditions.


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Key words: Erdős-Ko-Rado Theorem; direct products; $t$-intersecting families; cross $t$-intersecting families; shifting technique.

## 1 Introduction

Let $n$ and $k$ be two integers with $0 \leqslant k \leqslant n$. For an $n$-element set $X$, denote the set of all subsets and the collection of all $k$-subsets of $X$ by $2^{X}$ and $\binom{X}{k}$, respectively. Given a positive integer $t$, we say a family $\mathscr{F} \subset 2^{X}$ is $t$-intersecting if $|A \cap B| \geqslant t$ for any $A, B \in \mathscr{F}$. A $t$-intersecting family is called trivial if every element of this family contains a fixed $t$-subset of $X$. When $t=1$, we usually omit $t$. The famous Erdős-Ko-Rado theorem [7] states that if $\mathscr{F} \subset\binom{X}{k}$ is $t$-intersecting and $n>n_{0}(k, t)$, then

$$
|\mathscr{F}| \leqslant\binom{ n-t}{k-t}
$$

and the equality holds if and only if $\mathscr{F}=\left\{F \in\binom{X}{k}: T \subset F\right\}$ for some $T \in\binom{X}{t}$.
It is well-known that the smallest value of $n_{0}(k, t)$ is $(t+1)(k-t+1)$, which was proved by Frankl [8] for $t \geqslant 15$, and confirmed by Wilson [20] for all $t$ via the eigenvalue method. In [8], Frankl also put forward a conjecture about the

[^0]maximum size of a $t$-intersecting subfamily of $\binom{X}{k}$ for $n>2 k-t$. This conjecture was proved by Ahlswede and Khachatrian [2].

The Erdős-Ko-Rado theorem has been extended to different mathematical objects, such as vector spaces [11, 18], attenuated spaces [12], permutation groups [6], 2-transitive groups [13], labeled sets [3] and partition sets [16].

In [9], Frankl studied intersecting families for direct products. For convenience, set $X=[n]:=\{1, \ldots, n\}$ in the following. Let $p, n_{1}, \ldots, n_{p}$ be positive integers such that $n=n_{1}+\cdots+n_{p}$. Then $X$ can be partitioned into $p$ parts $X_{1}, X_{2}, \ldots, X_{p}$ where

$$
X_{1}=\left[n_{1}\right], \quad X_{i}=\left[\sum_{j \leqslant i} n_{j}\right] \backslash\left[\sum_{j \leqslant i-1} n_{j}\right], \quad i=2, \ldots, p .
$$

For positive integers $k_{i} \in\left[n_{i}\right]$ with $k=k_{1}+\cdots+k_{p}$, write

$$
\mathscr{H}_{1}:=\binom{X_{1}, \ldots, X_{p}}{k_{1}, \ldots, k_{p}}=\left\{F \in\binom{X}{k}:\left|F \cap X_{i}\right|=k_{i}, i=1, \ldots, p\right\} .
$$

Observe that $\left|\mathscr{H}_{1}\right|=\prod_{j \in[p]}\binom{n_{j}}{k_{j}}$. For each $x \in X_{l}$, the size of $\left\{A \in \mathscr{H}_{1}: x \in A\right\}$ is $k_{l}\left|\mathscr{H}_{1}\right| / n_{l}$. Frankl gave the maximum size of an intersecting subfamily of $\mathscr{H}_{1}$ by the eigenvalue method.

Theorem 1.1. ([9]) Suppose $\mathscr{F} \subset \mathscr{H}_{1}$ is an intersecting family and $n_{i} \geqslant 2 k_{i}$ for $i=1, \ldots, p$. Then

$$
\frac{|\mathscr{F}|}{\left|\mathscr{H}_{1}\right|} \leqslant \max _{i \in[p]} \frac{k_{i}}{n_{i}} .
$$

Recently, Kwan et al. [17] determined the maximum size of a non-trivially intersecting subfamily of $\mathscr{H}_{1}$ when $n_{1}, \ldots, n_{p}$ are sufficiently large and so disproved a conjecture of Alon and Katona, which was also mentioned in [14]. The maximum sum of sizes of cross intersecting subfamilies of $\mathscr{H}_{1}$ was determined by Kong et al. [15]. Ahlswede et al. [1] completely determined the maximum size of a $\left(t_{1}, \ldots, t_{p}\right)$ intersecting subfamily of $\mathscr{H}_{1}$, in which any two sets intersect in at least $t_{i}$ elements of $X_{i}$ for some $i \in[p]$.

In this paper, we study $t$-intersecting subfamilies of $\mathscr{H}_{1}$. One of our main results is the following.

Theorem 1.2. Suppose $\mathscr{F} \subset \mathscr{H}_{1}$ is a $t$-intersecting family. If $n_{i}>2(t+1) p k_{i}^{2}$ for any $i \in[p]$, then

$$
|\mathscr{F}| \leqslant \max _{\substack{t_{1}+\cdots+t_{p}=t \\ t_{1}, \ldots, t_{p} \in \mathbb{N}}} \prod_{i \in[p]}\binom{n_{i}-t_{i}}{k_{i}-t_{i}} .
$$

Moreover, the equality holds if and only if

$$
\mathscr{F}=\left\{F \in \mathscr{H}_{1}: T \subset F\right\},
$$

where $T \in\binom{X}{t}$ such that

$$
\begin{equation*}
\frac{k_{i}-\left|T \cap X_{i}\right|}{n_{i}-\left|T \cap X_{i}\right|} \leqslant \frac{k_{j}-\left|T \cap X_{j}\right|+1}{n_{j}-\left|T \cap X_{j}\right|+1} \tag{1}
\end{equation*}
$$

for any $i \in[p]$ whenever $\left|T \cap X_{j}\right| \geqslant 1$.
We remark here that $t$-intersecting subfamilies of $\mathscr{H}_{1}$ with maximum size may not be trivial when $n_{1}, \ldots, n_{p}$ are small. Under the condition that $p=t=2, n_{1}=$ 8, $n_{2}=10$ and $k_{1}=k_{2}=4$, it is routine to check that the 2-intersecting family $\left\{A \in \mathscr{H}_{1}:|A \cap[4]| \geqslant 3\right\}$ has a larger size than the largest trivially 2-intersecting subfamily of $\mathscr{H}_{1}$.

In [14], Katona extended $\mathscr{H}_{1}$ to a more general case. For a non-empty finite set $\mathscr{R} \subset \underbrace{\mathbb{Z}^{+} \times \cdots \times \mathbb{Z}^{+}}_{p}$, write

$$
\mathscr{H}_{2}:=\bigcup_{\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}}\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}} .
$$

For convenience, let $b$ and $c$ denote the maximum and minimum of numbers appearing in some elements of $\mathscr{R}$, respectively. By the cyclic method, Katona proved the following result.
Theorem 1.3. ([14]) Suppose $p=2$ and $n_{1}, n_{2} \geqslant 9 b^{2}$. If $\mathscr{F} \subset \mathscr{H}_{2}$ is intersecting, then $|\mathscr{F}|$ cannot exceed the size of the largest trivially intersecting subfamily of $\mathscr{H}_{2}$.

Our another main result extends Katona's result.
Theorem 1.4. Suppose $t \leqslant c$. If $n_{i}>2(t+1) p b^{t+2}$ for any $i \in[p]$, then every largest $t$-intersecting subfamily of $\mathscr{H}_{2}$ is trivial.

Write

$$
\mathscr{H}_{3}:=\left\{F \in\binom{X}{k}:\left|F \cap X_{i}\right| \geqslant a_{i}, i=1, \ldots, p\right\}
$$

where $a_{1}, \ldots, a_{p}$ are integers with $a_{1}+\cdots+a_{p} \leqslant k$ and $0 \leqslant a_{i}<n_{i}$. In [10], Frankl et al. put forward the following conjecture.

Conjecture 1.5. ([10]) If $n_{i} \geqslant 2 a_{i}$ for all $i$ and $n_{i}>k-\sum_{j=1}^{p} a_{j}+a_{i}$ for all but at most one $i \in[p]$ such that $a_{i}>0$, then the largest intersecting subfamily of $\mathscr{H}_{3}$ is trivial.

As a corollary of Theorem 1.4, Conjecture 1.5 is true when $a_{1}, \ldots, a_{p}$ are positive and each $X_{i}$ has a size larger than $4 p\left(k-\sum_{i=1}^{p} a_{i}+\max _{i \in[p]} a_{i}\right)^{3}$.

In Section 2, we will focus on the shifting technique and prove some useful results for direct products. In Section 3, we will give the proof of our main results.

## 2 Shifting technique for direct products

In this section, we investigate the shifting technique and prove some useful results for direct products.

For any $i, j \in X$ and $F \subset X$, define

$$
\delta_{i, j}(F)= \begin{cases}(F \backslash\{j\}) \cup\{i\}, & j \in F, i \notin F ; \\ F, & \text { otherwise } .\end{cases}
$$

Let $\Delta_{i, j}$ be the operation on a family $\mathscr{F} \subset 2^{X}$ defined by

$$
\Delta_{i, j}(\mathscr{F})=\left\{\delta_{i, j}(F): F \in \mathscr{F}\right\} \cup\left\{F \in \mathscr{F}: \delta_{i, j}(F) \in \mathscr{F}\right\} .
$$

We have $\left|\Delta_{i, j}(\mathscr{F})\right|=|\mathscr{F}|$.
A family $\mathscr{F} \subset 2^{X}$ is called shifted if $\Delta_{i, j}(\mathscr{F})=\mathscr{F}$ holds for any $i, j \in X$ with $i<j$. By applying such operations repeatedly to a subfamily of $2^{X}$ we can get a shifted family.

We say two non-empty subfamilies $\mathscr{A}$ and $\mathscr{B}$ of $2^{X}$ are crosst-intersecting if $|A \cap B| \geqslant t$ for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The following lemma states that the shifting operation keeps such intersection property.

Lemma 2.1. ([4, Lemma 2.1]) Let $\mathscr{A}$ and $\mathscr{B} \subset 2^{X}$ be cross $t$-intersecting families.
(i) For any $i, j \in X, \Delta_{i, j}(\mathscr{A})$ and $\Delta_{i, j}(\mathscr{B})$ are still cross $t$-intersecting.
(ii) If $t \leqslant r \leqslant s \leqslant n, \mathscr{A} \subset\binom{X}{r}, \mathscr{B} \subset\binom{X}{s}$, and $\mathscr{A}$ and $\mathscr{B}$ are shifted, then $|A \cap B \cap[r+s-t]| \geqslant t$ for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

For $\mathscr{F} \subset \mathscr{H}_{2}$, if $\Delta_{i, j}(\mathscr{F})=\mathscr{F}$ holds for any $i, j \in X_{l}$ with $i<j$, we say $\mathscr{F}$ is $l$-shifted. Similar to the single-part case, one gains an $l$-shifted family by doing the shifting operation repeatedly on $\mathscr{F}$. Notice that Lemma 2.1(i) still holds for $\mathscr{A} \subset\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}}$ and $\mathscr{B} \subset\binom{X_{1}, \ldots, X_{p}}{s_{1}, \ldots, s_{p}}$.

For $l \in[p]$ and a positive integer $s \leqslant n_{l}$, denote the collection of the first $s$ elements of $X_{l}$ by $Q_{l}(s)$. The next lemma is an extension of Lemma 2.1(ii).

Lemma 2.2. Suppose $n_{i}>r_{i}+s_{i}-1$ for any $i \in[p]$. Let $\mathscr{A} \subset\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}}$ and $\mathscr{B} \subset\binom{X_{1}, \ldots, X_{p}}{s_{1}, \ldots, s_{p}}$ be cross $t$-intersecting families. If $\mathscr{A}$ and $\mathscr{B}$ are l-shifted for any $l \in[p]$, then

$$
\sum_{i=1}^{p}\left|A \cap B \cap Q_{i}\left(r_{i}+s_{i}-1\right)\right| \geqslant t
$$

for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

Proof. For each $i \in[p]$, write

$$
D_{i}:=Q_{i}\left(r_{i}+s_{i}-1\right) \backslash(A \cup B), E_{i}:=\left(A \cap B \cap X_{i}\right) \backslash Q_{i}\left(r_{i}+s_{i}-1\right) .
$$

Note that

$$
\begin{gather*}
r_{i}+s_{i}=\left|A \cap X_{i}\right|+\left|B \cap X_{i}\right| \geqslant 2\left|E_{i}\right|+\left|(A \cup B) \cap Q_{i}\left(r_{i}+s_{i}-1\right)\right|,  \tag{2}\\
\left|D_{i}\right|=r_{i}+s_{i}-1-\left|(A \cup B) \cap Q_{i}\left(r_{i}+s_{i}-1\right)\right| . \tag{3}
\end{gather*}
$$

If $\left|E_{i}\right| \neq \emptyset$, then $\left|D_{i}\right| \geqslant\left|E_{i}\right|$ from (2) and (3).
Let $G_{i}$ be an $\left|E_{i}\right|$-subset of $D_{i}$. Write

$$
C:=\left(B \backslash \bigcup_{i \in[p]} E_{i}\right) \cup\left(\bigcup_{i \in[p]} G_{i}\right) .
$$

Observe that, for each $i \in[p]$,

$$
C \cap A \cap X_{i}=\left(\left(B \backslash E_{i}\right) \cup G_{i}\right) \cap A \cap X_{i}=A \cap B \cap Q_{i}\left(r_{i}+s_{i}-1\right) .
$$

When $E_{i} \neq \emptyset$, notice that $\max G_{i}<\min E_{i}$ and $\left|E_{i}\right|=\left|G_{i}\right|$. Thus $C$ can be obtained by doing a series of shifting operations on $B$. Since $\mathscr{B}$ is $l$-shifted for any $l \in[p]$, we have $C \in \mathscr{B}$. So $|A \cap C| \geqslant t$. Hence

$$
\sum_{i=1}^{p}\left|A \cap B \cap Q_{i}\left(r_{i}+s_{i}-1\right)\right|=\sum_{i=1}^{p}\left|A \cap C \cap X_{i}\right|=|A \cap C| \geqslant t
$$

as desired.
Given positive integers $g, h$ with $g \geqslant 2 h$, it is well-known that the Kneser graph $K G(g, h)$ is the graph on the vertex set $\binom{[g]}{h}$, with an edge between two vertices if and only if they are disjoint. To characterize extremal structures in Theorems 1.2 and 1.4 , we need a property of Kneser graphs which is derived from Theorem 1 in [5].

Lemma 2.3. For Kneser graphs $K G\left(g_{1}, h_{1}\right), \ldots, K G\left(g_{w}, h_{w}\right)$ with $g_{i}>2 h_{i}$ for any $i \in[w]$, their direct product $\prod_{i \in[w]} K G\left(g_{i}, h_{i}\right)$ is connected.

For $\mathscr{H} \subset 2^{X}$, we say $\mathscr{F} \subset \mathscr{H}$ is a full t-star in $\mathscr{H}$ if $\mathscr{F}$ is the collection of all sets in $\mathscr{H}$ containing a fixed $t$-subset of $X$. For each $i \in[p]$, let $b_{i}$ be the maximum number appearing in the $i$-th coordinate of some elements of $\mathscr{R}$.

Lemma 2.4. Let $\mathscr{F} \subset \mathscr{H}_{2}$ be a t-intersecting family. Suppose $n_{m}>2(t+1) b_{m}$ for any $m \in[p]$. For $l \in[p]$ and $i, j \in X_{l}$, if $\Delta_{i, j}(\mathscr{F})$ is a full $t$-star in $\mathscr{H}_{2}$, then $\mathscr{F}$ is also a full t-star in $\mathscr{H}_{2}$.

Proof. For $\boldsymbol{r}=\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}$, let $\mathscr{F}_{r}$ denote $\mathscr{F} \cap\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}}$ in the rest of the paper. Write

$$
\mathscr{F}_{r}(l):=\left\{F \backslash X_{l}: F \in \mathscr{F}_{r}\right\} .
$$

For each $R \in \mathscr{F}_{r}(l)$, let

$$
\mathscr{G}_{R}:=\left\{R^{\prime} \in\binom{X_{l}}{r_{l}}: R \cup R^{\prime} \in \mathscr{F}_{r}\right\} .
$$

Observe that

$$
\begin{equation*}
\mathscr{F}_{r}=\bigcup_{R \in \mathscr{F}_{r}(l)}\left\{R \cup R^{\prime}: R^{\prime} \in \mathscr{G}_{R}\right\}, \Delta_{i, j}\left(\mathscr{F}_{r}\right)=\bigcup_{R \in \mathscr{F}_{r}(l)}\left\{R \cup R^{\prime \prime}: R^{\prime \prime} \in \Delta_{i, j}\left(\mathscr{G}_{R}\right)\right\} . \tag{4}
\end{equation*}
$$

By assumption, there exists $T_{0} \in\binom{X}{t}$ such that $\Delta_{i, j}(\mathscr{F})=\left\{F \in \mathscr{H}_{2}: T_{0} \subset F\right\}$, which implies that

$$
\begin{equation*}
\Delta_{i, j}\left(\mathscr{F}_{r}\right)=\left\{F \in\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}}: T_{0} \subset F\right\} . \tag{5}
\end{equation*}
$$

We have $\left|\mathscr{G}_{R}\right|=\left|\Delta_{i, j}\left(\mathscr{G}_{R}\right)\right|=\binom{n_{l}-t_{l}}{r_{l}-t_{l}}$, where $t_{l}:=\left|T_{0} \cap X_{l}\right|$.
If $T_{0} \cap X_{l}=\emptyset$, we get $\mathscr{G}_{R}=\Delta_{i, j}\left(\mathscr{G}_{R}\right)$ from $\Delta_{i, j}\left(\mathscr{G}_{R}\right)=\binom{X_{l}}{r_{l}}$. By (4), $\mathscr{F}_{r}=$ $\Delta_{i, j}\left(\mathscr{F}_{r}\right)$. Hence $\mathscr{F}=\Delta_{i, j}(\mathscr{F})$, as desired.

Now suppose $T_{0} \cap X_{l} \neq \emptyset$. By (5), we have

$$
\mathscr{F}_{r}(l)=\left\{G \subset X \backslash X_{l}: T_{0} \backslash X_{l} \subset G,\left|G \cap X_{m}\right|=r_{m}, m \in[p] \backslash\{l\}\right\} .
$$

Note that $n_{m}>2(t+1) r_{m}$ for any $m \in[p]$. Then given $R_{0} \in \mathscr{F}_{r}(l)$, there exists $S_{0} \in \mathscr{F}_{r}(l)$ such that $R_{0} \cap S_{0}=T_{0} \backslash X_{l}$. Since $\mathscr{F}_{r}$ is $t$-intersecting, $\mathscr{G}_{R_{0}}$ and $\mathscr{G}_{S_{0}}$ are cross $t_{l}$-intersecting families with $\left|\mathscr{G}_{R_{0}}\right|\left|\mathscr{G}_{S_{0}}\right|=\binom{n_{l}-t_{l}}{r_{l}-t_{l}}^{2}$. By Theorem 1 in [19], we get

$$
\mathscr{G}_{R_{0}}=\mathscr{G}_{S_{0}}=\left\{G \in\binom{X_{l}}{r_{l}}: T_{l}^{\prime} \subset G\right\}
$$

for some $T_{l}^{\prime} \in\binom{X_{l}}{t_{l}}$. Next we prove $\mathscr{G}_{S}=\mathscr{G}_{R_{0}}$ for any $S \in \mathscr{F}_{r}(l) \backslash\left\{R_{0}\right\}$.
For each $S \in \mathscr{F}_{r}(l)$, we have $\left|\left(S \backslash T_{0}\right) \cap X_{m}\right|=r_{m}-t_{m}, m \in[p] \backslash\{l\}$. Thus the set $\left\{R \backslash T_{0}: R \in \mathscr{F}_{r}(l)\right\}$ can be seen as the vertex set of the graph $\prod_{m \in[p] \backslash\{ \}\}} K G\left(n_{m}-t_{m}, r_{m}-t_{m}\right)$. Notice that $n_{m}-t_{m}>2\left(r_{m}-t_{m}\right)$. Suppose $S \neq R_{0}$. By Lemma 2.3, this graph contains a walk

$$
R_{0} \backslash T_{0}, A_{1}, \ldots, A_{z}=S \backslash T_{0}
$$

Let $B_{0}=R_{0}, B_{1}=A_{1} \cup\left(T_{0} \backslash X_{l}\right), \ldots, B_{z}=S \in \mathscr{F}_{r}(l)$. Then $B_{q} \cap B_{q+1}=T_{0} \backslash X_{l}$ for $q=0,1, \ldots, z-1$. Consequently $\mathscr{G}_{R_{0}}=\mathscr{G}_{B_{1}}=\cdots=\mathscr{G}_{S}$.

For any $R \in \mathscr{F}_{r}(l), \mathscr{G}_{R}$ is the collection of all $r_{l}$-subsets of $X_{l}$ containing $T_{l}^{\prime}$. Hence

$$
\begin{align*}
\mathscr{F}_{r} & =\left\{R \cup R^{\prime}: R \in \mathscr{F}_{r}(l), T_{l}^{\prime} \subset R^{\prime} \in\binom{X_{l}}{r_{l}}\right\} \\
& =\left\{F \in\binom{X_{1}, \ldots, X_{p}}{r_{1}, \ldots, r_{p}}: T_{1} \subset F\right\} \tag{6}
\end{align*}
$$

where $T_{1}:=\left(T_{0} \backslash X_{l}\right) \cup T_{l}^{\prime}$.
For $\boldsymbol{s}=\left(s_{1}, \ldots, s_{p}\right) \in \mathscr{R}$, by (6), there exists $T_{2} \in\binom{X}{t}$ such that $\mathscr{F}_{s}$ is the collection of all sets in $\binom{X_{1}, \ldots, X_{p}}{s_{1}, \ldots, s_{p}}$ containing $T_{2}$. Since $n_{m}>2(t+1) b_{m}$ for any $m \in[p]$, there are $F_{1} \in \mathscr{F}_{r}$ and $F_{2} \in \mathscr{F}_{s}$ such that $\left(F_{1} \backslash T_{1}\right) \cap\left(F_{2} \backslash T_{2}\right)=\emptyset$. Then $t \leqslant\left|F_{1} \cap F_{2}\right|=\left|T_{1} \cap T_{2}\right| \leqslant t$, which implies that $T_{1}=T_{2}$. Thus for any $s \in \mathscr{R}$, $\mathscr{F}_{s}$ is the collection of all sets in $\binom{X_{1}, \ldots, X_{p}}{s_{1}, \ldots, s_{p}}$ containing $T_{1}$, which implies that the desired result follows.

## 3 Proof of main results

In this section, we shall prove our main results.
Let $\mathscr{F} \subset \mathscr{H}_{2}$ be a $t$-intersecting family. If $\mathscr{F}=\emptyset$, there is nothing to prove. So suppose that $\mathscr{F} \neq \emptyset$. Besides, according to Lemma 2.4, we may assume that $\mathscr{F}$ is $l$-shifted for any $l \in[p]$.

Recall that $b_{i}=\max _{\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}} r_{i}$ for $i=1, \ldots, p$. Write

$$
K:=\bigcup_{i=1}^{p} Q_{i}\left(2 b_{i}-1\right), \alpha(\mathscr{F}):=\min _{F \in \mathscr{F}}|F \cap K| .
$$

We have $\alpha(\mathscr{F}) \geqslant t$. Indeed, since two non-empty subfamilies $\mathscr{F}_{r}$ and $\mathscr{F}_{s}$ are cross $t$-intersecting and $l$-shifted for any $l \in[p]$, by Lemma 2.2 we get

$$
\begin{equation*}
|F \cap K| \geqslant \sum_{i=1}^{p}\left|F \cap G \cap Q_{i}\left(2 b_{i}-1\right)\right| \geqslant t \tag{7}
\end{equation*}
$$

where $F \in \mathscr{F}_{r}$ and $G \in \mathscr{F}_{s}$.
Lemma 3.1. Suppose $\mathscr{F} \subset \mathscr{H}_{2}$ is a $t$-intersecting family. If $\alpha(\mathscr{F})=t$ and $\mathscr{F}$ is $l$-shifted for any $l \in[p]$, then

$$
\begin{equation*}
|\mathscr{F}| \leqslant \max _{\substack{t_{1}+\ldots+t_{p}=t \\ t_{1}, \ldots, t_{p} \in \mathbb{N}}} \sum_{\substack{\left.r_{1}, \ldots, r_{p}\right) \in \mathscr{R}}} \prod_{i \in[p]}\binom{n_{i}-t_{i}}{r_{i}-t_{i}} . \tag{8}
\end{equation*}
$$

Moreover, when the equality holds, $\mathscr{F}$ is a full t-star in $\mathscr{H}_{2}$.

Proof. By assumption, there exists $F_{0} \in \mathscr{F}$ such that $\left|F_{0} \cap K\right|=t$. By (7), for any $G \in \mathscr{F}$, we have

$$
\begin{equation*}
F_{0} \cap K=\bigcup_{i \in[p]}\left(F_{0} \cap G \cap Q_{i}\left(2 b_{i}-1\right)\right) \subset G \tag{9}
\end{equation*}
$$

Therefore, for any $\boldsymbol{r}=\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}$,

$$
\left|\mathscr{F}_{r}\right| \leqslant \prod_{i \in[p]}\binom{n_{i}-\left|F_{0} \cap Q_{i}\left(2 b_{i}-1\right)\right|}{r_{i}-\left|F_{0} \cap Q_{i}\left(2 b_{i}-1\right)\right|} .
$$

Then (8) follows from $|\mathscr{F}|=\sum_{r \in \mathscr{R}}\left|\mathscr{F}_{r}\right|$.
By (9), $\mathscr{F}$ is a collection of some sets in $\mathscr{H}_{2}$ containing $F_{0} \cap K$. So when the equality in (8) holds, $\mathscr{F}$ is a full $t$-star in $\mathscr{H}_{2}$.

For positive integers $t, p, n_{1}, \ldots, n_{p}, k_{1}, \ldots, k_{p}$ with $n_{i}>k_{i}$ and $k_{1}+\cdots+k_{p} \geqslant t$, write

$$
g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)=\max _{\substack{t_{1}+\cdots+t_{p}=t \\ t_{1}, \ldots, t_{p} \in \mathbb{N}}} \prod_{i \in[p]}\binom{n_{i}-t_{i}}{k_{i}-t_{i}} .
$$

Proof of Theorem 1.2. Notice that $\mathscr{H}_{1}$ is a special case of $\mathscr{H}_{2}$. In view of Lemma 3.1, we show that

$$
|\mathscr{F}|<g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)
$$

when $\alpha(\mathscr{F}) \geqslant t+1$. For convenience, if there is no confusion, we replace $\alpha(\mathscr{F})$ with $\alpha$ in the following.

By assumption, there exists $A_{0} \in \mathscr{F}$ such that $\left|A_{0} \cap K\right|=\alpha$. Then for $F \in \mathscr{F}$, we have $|F \cap K| \geqslant \alpha$ and $\left|F \cap K \cap A_{0}\right| \geqslant t$ by (7). Thus

$$
\begin{equation*}
\mathscr{F} \subset \bigcup_{J \in\binom{K}{\alpha},\left|J \cap A_{0}\right| \geqslant t}\left\{F \in \mathscr{H}_{1}: J \subset F\right\} . \tag{10}
\end{equation*}
$$

Let $N$ be the collection of all non-negative integer solutions of the equation $x_{1}+\cdots+x_{p}=\alpha-t$. For each $H \in\binom{K \cap A_{0}}{t}$ and $\beta=\left(c_{1}, \ldots, c_{p}\right) \in N$, let $\mathscr{J}(H, \beta)$ be the set of all $J \in\binom{K}{\alpha}$ with $H \subset J$ and $\left|(J \backslash H) \cap X_{i}\right|=c_{i}$. Denote the number of $F \in \mathscr{H}_{1}$ containing at least one element of $\mathscr{J}(H, \beta)$ by $f(H, \beta)$. For each $J \in\binom{K}{\alpha}$ satisfying $\left|J \cap A_{0}\right| \geqslant t$, observe that $J$ is an element of some $\mathscr{J}(H, \beta)$. Then by (10), we have

$$
|\mathscr{F}| \leqslant \sum_{H \in\binom{K \cap A_{0}}{t}} \sum_{\beta \in N} f(H, \beta) .
$$

Observe that

$$
|\mathscr{J}(H, \beta)| \leqslant \prod_{i \in[p]}\binom{2 k_{i}-1}{c_{i}} \leqslant \prod_{i \in[p]}\left(2 k_{i}\right)^{c_{i}} .
$$

Thus
$\frac{f(H, \beta)}{g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)} \leqslant \frac{\left(\prod_{i \in[p]}\left(2 k_{i}\right)^{c_{i}}\right) \cdot\left(\prod_{i \in[p]}\binom{n_{i}-\left|H \cap X_{i}\right|-c_{i}}{k_{i}-\left|H \cap X_{i}\right|-c_{i}}\right.}{\prod_{i \in[p]}\binom{n_{i}-\left|H \cap X_{i}\right|}{k_{i}-\left|H \cap X_{i}\right|}} \leqslant \prod_{i \in[p]}\left(\frac{2 k_{i}^{2}}{n_{i}}\right)^{z_{i}}$,
where $\left(z_{1}, \ldots, z_{p}\right) \in N$ such that

$$
\prod_{i \in[p]}\left(\frac{2 k_{i}^{2}}{n_{i}}\right)^{z_{i}}=\max _{\left(c_{1}, \ldots, c_{p}\right) \in N} \prod_{i \in[p]}\left(\frac{2 k_{i}^{2}}{n_{i}}\right)^{c_{i}}
$$

Note that $|N|=\binom{\alpha-t+p-1}{p-1}$ and

$$
\binom{x}{y}=\prod_{i=y+1}^{x}\left(1+\frac{y}{i-y}\right) \leqslant(y+1)^{x-y}
$$

for any positive integers $x, y$ with $x \geqslant y+1$. By above discussion, we obtain

$$
\begin{aligned}
\frac{|\mathscr{F}|}{g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)} & \leqslant\binom{\alpha}{t}\binom{\alpha-t+p-1}{p-1} \cdot \prod_{i \in[p]}\left(\frac{2 k_{i}^{2}}{n_{i}}\right)^{z_{i}} \\
& \leqslant((t+1) p)^{\alpha-t} \cdot \prod_{i \in[p]}\left(\frac{2 k_{i}^{2}}{n_{i}}\right)^{z_{i}} \\
& =\prod_{i \in[p]}\left(\frac{2(t+1) p k_{i}^{2}}{n_{i}}\right)^{z_{i}} .
\end{aligned}
$$

Since $n_{i}>2(t+1) p k_{i}^{2}$ for any $i \in[p]$, we have $|\mathscr{F}|<g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)$, as desired.

For each $S \in\binom{X}{t}$, write

$$
\mathscr{P}(S):=\left\{(i, j(i)) \in \mathbb{Z}^{2}: i \in[p], 0 \leqslant j(i)<\left|S \cap X_{i}\right|\right\} .
$$

Observe that

$$
\begin{equation*}
e(S):=\frac{\prod_{i \in[p]}\binom{n_{i}-\left|S \cap X_{i}\right|}{k_{i}-\mid S \cap X_{i}}}{\prod_{i \in[p]}\binom{n_{i}}{k_{i}}}=\prod_{(i, j) \in \mathscr{P}(S)} \frac{k_{i}-j}{n_{i}-j} . \tag{11}
\end{equation*}
$$

Let $T$ be a $t$-subset of $X$. To finish the proof, it is sufficient to show that $e(T)=$ $\max _{S \in\binom{X}{t}} e(S)$ if and only if (1) holds for any $i \in[p]$ whenever $\left|T \cap X_{j}\right| \geqslant 1$.

Suppose that (1) holds for any $i \in[p]$ whenever $\left|X \cap T_{j}\right| \geqslant 1$. For each $S \in\binom{X}{t} \backslash\{T\}$, from

$$
\frac{k_{i}}{n_{i}}>\frac{k_{i}-1}{n_{i}-1}>\cdots>\frac{1}{n_{i}-k_{i}+1},
$$

we get

$$
\begin{equation*}
\min _{(i, j) \in \mathscr{P}(T) \backslash \mathscr{P}(S)} \frac{k_{i}-j}{n_{i}-j} \geqslant \max _{(i, j) \in \mathscr{P}(S) \backslash \mathscr{P}(T)} \frac{k_{i}-j}{n_{i}-j} . \tag{12}
\end{equation*}
$$

By (11) and (12), we have $e(T) / e(S) \geqslant 1$. On the other hand, suppose $e(T)=$ $\max _{S \in\binom{X}{t}} e(S)$. For each $i, j$ with $\left|T \cap X_{j}\right| \geqslant 1$, let $T^{\prime}:=(T \backslash\{u\}) \cup\{v\} \in\binom{X}{t}$, where $u \in T \cap X_{j}$ and $v \in X_{i} \backslash T$. By (11), we have

$$
\frac{k_{i}-\left|T \cap X_{i}\right|}{n_{i}-\left|T \cap X_{i}\right|}=\frac{e\left(T^{\prime}\right)}{e(T)} \cdot \frac{k_{j}-\left|T \cap X_{j}\right|+1}{n_{j}-\left|T \cap X_{j}\right|+1} \leqslant \frac{k_{j}-\left|T \cap X_{j}\right|+1}{n_{j}-\left|T \cap X_{j}\right|+1} .
$$

Hence the desired result holds.
It is not intuitive to find $T \in\binom{X}{t}$ such that the size of $\left\{F \in \mathscr{H}_{1}: T \subset F\right\}$ is $g_{t, p}\left(n_{1}, \ldots, n_{p} ; k_{1}, \ldots, k_{p}\right)$. Thus we extract an algorithm about how to find all $\left|T \cap X_{i}\right|$ from the proof of Theorem 1.2.

```
Algorithm 1
    Input \(t, p, k_{1}, \ldots, k_{p}, n_{1}, \ldots, n_{p}\)
    Let \(A\) be the collection of \(\frac{k_{i}-j}{n_{i}-j}\) for all \(i, j\) with \(i \in[p], j=0, \ldots, k_{i}-1\)
    Sort \(A\) in decreasing order \(a_{1}, a_{2}, \ldots\)
    Let \(A(f)\) be the collection of \((i, j)\) satisfying \(\frac{k_{i}-j}{n_{i}-j}=f\) for \(f \in A\)
    Put \(i \leftarrow 1, c \leftarrow 0, k \leftarrow 0, G \leftarrow \emptyset\)
    while \(k<t\) do
        \(k \leftarrow k+\left|A\left(a_{i}\right)\right|\)
        if \(k \leqslant t\) then
                \(G \leftarrow G \cup A\left(a_{i}\right)\)
        else
            \(c \leftarrow\left|A\left(a_{i}\right)\right|-k+t\)
            \(H \leftarrow\binom{A\left(a_{i}\right)}{c}\)
        end if
        \(i \leftarrow i+1\)
    end while
```

```
if \(c=0\) then
    for \(t_{m}\) do
            \(t_{m} \leftarrow|\{(m, j):(m, j) \in G\}|\)
        end for
    Output \(\quad t_{1}, \ldots, t_{p}\)
    else
        for \(L \in H\) do
            \(J \leftarrow G \cup L\)
            for \(t_{m}\) do
                \(t_{m} \leftarrow|\{(m, j):(m, j) \in J\}|\)
            end for
            Output \(\quad t_{1}, \ldots, t_{p}\)
        end for
    end if
```

Proof of Theorem 1.4. In consideration of Lemma 3.1, it is sufficient to show that

$$
|\mathscr{F}|<\max _{\substack{t_{1}+\cdots+t_{p}=t \\ t_{1}, \ldots, t_{p} \in \mathbb{N}}} \sum_{\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}} \prod_{i \in[p]}\binom{n_{i}-t_{i}}{r_{i}-t_{i}}
$$

when $\alpha(\mathscr{F}) \geqslant t+1$. W.o.l.g., suppose that $n_{1}=\min _{i \in[p]} n_{i}$.
We may assume that $\mathscr{F}_{r} \neq \emptyset$ for some $\boldsymbol{r}=\left(r_{1}, \ldots, r_{p}\right) \in \mathscr{R}$, otherwise there is nothing to prove. Observe that $\mathscr{F}_{r}$ is $t$-intersecting and $\alpha\left(\mathscr{F}_{r}\right) \geqslant \alpha(\mathscr{F}) \geqslant t+1$. From the proof of Theorem 1.2, we get

$$
\begin{equation*}
\frac{\left|\mathscr{F}_{r}\right|}{g_{t, p}\left(n_{1}, \ldots, n_{p} ; r_{1}, \ldots, r_{p}\right)} \leqslant \prod_{i \in[p]}\left(\frac{2(t+1) p r_{i}^{2}}{n_{i}}\right)^{w_{i}} \leqslant \prod_{i \in[p]}\left(\frac{2(t+1) p b_{i}^{2}}{n_{i}}\right)^{q_{i}}, \tag{13}
\end{equation*}
$$

where $w_{1}+\cdots+w_{p}=\alpha\left(\mathscr{F}_{r}\right)-t$ and $q_{1}+\cdots+q_{p}=\alpha(\mathscr{F})-t$. Notice that there exist non-negative integers $d_{1}, \ldots, d_{p}$ with $d_{1}+\cdots+d_{p}=t$ such that

$$
\begin{equation*}
\frac{g_{t, p}\left(n_{1}, \ldots, n_{p} ; r_{1}, \ldots, r_{p}\right)}{\binom{n_{1}-t}{r_{1}-t} \cdot \prod_{i=2}^{p}\binom{n_{i}}{r_{i}}}=\left(\prod_{i \in[p]}\left(\prod_{j=0}^{d_{i}-1} \frac{r_{i}-j}{n_{i}-j}\right)\right) \cdot\left(\prod_{j=0}^{t-1} \frac{n_{1}-j}{r_{1}-j}\right) \leqslant b^{t} . \tag{14}
\end{equation*}
$$

Combining (13) and (14), we derive

$$
\frac{\left|\mathscr{F}_{r}\right|}{\binom{n_{1}-t}{r_{1}-t} \cdot \prod_{i=2}^{p}\binom{n_{i}}{r_{i}}} \leqslant b^{t} \cdot \prod_{i \in[p]}\left(\frac{2(t+1) p b_{i}^{2}}{n_{i}}\right)^{q_{i}} \leqslant\left(\frac{2(t+1) p b^{t+2}}{n_{1}}\right)^{\alpha-t}<1
$$

from $n_{1}>2(t+1) p b^{t+2}$. Therefore, $|\mathscr{F}|$ is smaller than the number of sets in $\mathscr{H}_{2}$ containing $[t]$, which implies that the desired result follows.

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