Extremal *t*-intersecting families for direct products

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Abstract

In this paper, by shifting technique we study t-intersecting families for direct products where the ground set is divided into several parts. Assuming the size of each part is sufficiently large, we determine all extremal t-intersecting families for direct products. We also prove that every largest t-intersecting subfamily of a more general family introduced by Katona is trivial under certain conditions.

AMS classification: 05D05.

Key words: Erdős-Ko-Rado Theorem; direct products; *t*-intersecting families; cross *t*-intersecting families; shifting technique.

1 Introduction

Let *n* and *k* be two integers with $0 \leq k \leq n$. For an *n*-element set *X*, denote the set of all subsets and the collection of all *k*-subsets of *X* by 2^X and $\binom{X}{k}$, respectively. Given a positive integer *t*, we say a family $\mathscr{F} \subset 2^X$ is *t*-intersecting if $|A \cap B| \geq t$ for any $A, B \in \mathscr{F}$. A *t*-intersecting family is called *trivial* if every element of this family contains a fixed *t*-subset of *X*. When t = 1, we usually omit *t*. The famous Erdős-Ko-Rado theorem [7] states that if $\mathscr{F} \subset \binom{X}{k}$ is *t*-intersecting and $n > n_0(k, t)$, then

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t},$$

and the equality holds if and only if $\mathscr{F} = \{F \in {X \choose k} : T \subset F\}$ for some $T \in {X \choose t}$.

It is well-known that the smallest value of $n_0(k, t)$ is (t+1)(k-t+1), which was proved by Frankl [8] for $t \ge 15$, and confirmed by Wilson [20] for all t via the eigenvalue method. In [8], Frankl also put forward a conjecture about the

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maximum size of a *t*-intersecting subfamily of $\binom{X}{k}$ for n > 2k - t. This conjecture was proved by Ahlswede and Khachatrian [2].

The Erdős-Ko-Rado theorem has been extended to different mathematical objects, such as vector spaces [11, 18], attenuated spaces [12], permutation groups [6], 2-transitive groups [13], labeled sets [3] and partition sets [16].

In [9], Frankl studied intersecting families for direct products. For convenience, set $X = [n] := \{1, \ldots, n\}$ in the following. Let p, n_1, \ldots, n_p be positive integers such that $n = n_1 + \cdots + n_p$. Then X can be partitioned into p parts X_1, X_2, \ldots, X_p where

$$X_1 = [n_1], \ X_i = \left[\sum_{j \leq i} n_j\right] \setminus \left[\sum_{j \leq i-1} n_j\right], \quad i = 2, \dots, p.$$

For positive integers $k_i \in [n_i]$ with $k = k_1 + \cdots + k_p$, write

$$\mathscr{H}_1 := \begin{pmatrix} X_1, \dots, X_p \\ k_1, \dots, k_p \end{pmatrix} = \left\{ F \in \begin{pmatrix} X \\ k \end{pmatrix} : |F \cap X_i| = k_i, \ i = 1, \dots, p \right\}.$$

Observe that $|\mathscr{H}_1| = \prod_{j \in [p]} {n_j \choose k_j}$. For each $x \in X_l$, the size of $\{A \in \mathscr{H}_1 : x \in A\}$ is $k_l |\mathscr{H}_1|/n_l$. Frankl gave the maximum size of an intersecting subfamily of \mathscr{H}_1 by the eigenvalue method.

Theorem 1.1. ([9]) Suppose $\mathscr{F} \subset \mathscr{H}_1$ is an intersecting family and $n_i \ge 2k_i$ for $i = 1, \ldots, p$. Then

$$\frac{|\mathscr{F}|}{|\mathscr{H}_1|} \leqslant \max_{i \in [p]} \frac{k_i}{n_i}.$$

Recently, Kwan et al. [17] determined the maximum size of a non-trivially intersecting subfamily of \mathscr{H}_1 when n_1, \ldots, n_p are sufficiently large and so disproved a conjecture of Alon and Katona, which was also mentioned in [14]. The maximum sum of sizes of cross intersecting subfamilies of \mathscr{H}_1 was determined by Kong et al. [15]. Ahlswede et al. [1] completely determined the maximum size of a (t_1, \ldots, t_p) intersecting subfamily of \mathscr{H}_1 , in which any two sets intersect in at least t_i elements of X_i for some $i \in [p]$.

In this paper, we study *t*-intersecting subfamilies of \mathscr{H}_1 . One of our main results is the following.

Theorem 1.2. Suppose $\mathscr{F} \subset \mathscr{H}_1$ is a t-intersecting family. If $n_i > 2(t+1)pk_i^2$ for any $i \in [p]$, then

$$|\mathscr{F}| \leqslant \max_{\substack{t_1+\dots+t_p=t\\t_1,\dots,t_p\in\mathbb{N}}} \prod_{i\in[p]} \binom{n_i-t_i}{k_i-t_i}.$$

Moreover, the equality holds if and only if

$$\mathscr{F} = \{F \in \mathscr{H}_1 : T \subset F\},\$$

where $T \in {X \choose t}$ such that

$$\frac{k_i - |T \cap X_i|}{n_i - |T \cap X_i|} \leqslant \frac{k_j - |T \cap X_j| + 1}{n_j - |T \cap X_j| + 1}$$
(1)

for any $i \in [p]$ whenever $|T \cap X_j| \ge 1$.

We remark here that t-intersecting subfamilies of \mathscr{H}_1 with maximum size may not be trivial when n_1, \ldots, n_p are small. Under the condition that $p = t = 2, n_1 =$ 8, $n_2 = 10$ and $k_1 = k_2 = 4$, it is routine to check that the 2-intersecting family $\{A \in \mathscr{H}_1 : |A \cap [4]| \ge 3\}$ has a larger size than the largest trivially 2-intersecting subfamily of \mathscr{H}_1 .

In [14], Katona extended \mathscr{H}_1 to a more general case. For a non-empty finite set $\mathscr{R} \subset \underbrace{\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+}_n$, write

$$\mathscr{H}_2 := \bigcup_{(r_1,\dots,r_p)\in\mathscr{R}} \binom{X_1,\dots,X_p}{r_1,\dots,r_p}.$$

For convenience, let b and c denote the maximum and minimum of numbers appearing in some elements of \mathscr{R} , respectively. By the cyclic method, Katona proved the following result.

Theorem 1.3. ([14]) Suppose p = 2 and $n_1, n_2 \ge 9b^2$. If $\mathscr{F} \subset \mathscr{H}_2$ is intersecting, then $|\mathscr{F}|$ cannot exceed the size of the largest trivially intersecting subfamily of \mathscr{H}_2 .

Our another main result extends Katona's result.

Theorem 1.4. Suppose $t \leq c$. If $n_i > 2(t+1)pb^{t+2}$ for any $i \in [p]$, then every largest t-intersecting subfamily of \mathscr{H}_2 is trivial.

Write

$$\mathscr{H}_3 := \left\{ F \in \begin{pmatrix} X \\ k \end{pmatrix} : |F \cap X_i| \ge a_i, \ i = 1, \dots, p \right\},$$

where a_1, \ldots, a_p are integers with $a_1 + \cdots + a_p \leq k$ and $0 \leq a_i < n_i$. In [10], Frankl et al. put forward the following conjecture.

Conjecture 1.5. ([10]) If $n_i \ge 2a_i$ for all i and $n_i > k - \sum_{j=1}^p a_j + a_i$ for all but at most one $i \in [p]$ such that $a_i > 0$, then the largest intersecting subfamily of \mathscr{H}_3 is trivial.

As a corollary of Theorem 1.4, Conjecture 1.5 is true when a_1, \ldots, a_p are positive and each X_i has a size larger than $4p(k - \sum_{i=1}^p a_i + \max_{i \in [p]} a_i)^3$.

In Section 2, we will focus on the shifting technique and prove some useful results for direct products. In Section 3, we will give the proof of our main results.

2 Shifting technique for direct products

In this section, we investigate the shifting technique and prove some useful results for direct products.

For any $i, j \in X$ and $F \subset X$, define

$$\delta_{i,j}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & j \in F, i \notin F; \\ F, & otherwise. \end{cases}$$

Let $\Delta_{i,j}$ be the operation on a family $\mathscr{F} \subset 2^X$ defined by

$$\Delta_{i,j}(\mathscr{F}) = \{\delta_{i,j}(F) : F \in \mathscr{F}\} \cup \{F \in \mathscr{F} : \delta_{i,j}(F) \in \mathscr{F}\}.$$

We have $|\Delta_{i,j}(\mathscr{F})| = |\mathscr{F}|$.

A family $\mathscr{F} \subset 2^X$ is called *shifted* if $\Delta_{i,j}(\mathscr{F}) = \mathscr{F}$ holds for any $i, j \in X$ with i < j. By applying such operations repeatedly to a subfamily of 2^X we can get a shifted family.

We say two non-empty subfamilies \mathscr{A} and \mathscr{B} of 2^X are cross t-intersecting if $|A \cap B| \ge t$ for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$. The following lemma states that the shifting operation keeps such intersection property.

Lemma 2.1. ([4, Lemma 2.1]) Let \mathscr{A} and $\mathscr{B} \subset 2^X$ be cross *t*-intersecting families.

- (i) For any $i, j \in X$, $\Delta_{i,j}(\mathscr{A})$ and $\Delta_{i,j}(\mathscr{B})$ are still cross *t*-intersecting.
- (ii) If $t \leq r \leq s \leq n$, $\mathscr{A} \subset {\binom{X}{r}}$, $\mathscr{B} \subset {\binom{X}{s}}$, and \mathscr{A} and \mathscr{B} are shifted, then $|A \cap B \cap [r+s-t]| \ge t$ for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

For $\mathscr{F} \subset \mathscr{H}_2$, if $\Delta_{i,j}(\mathscr{F}) = \mathscr{F}$ holds for any $i, j \in X_l$ with i < j, we say \mathscr{F} is *l-shifted*. Similar to the single-part case, one gains an *l*-shifted family by doing the shifting operation repeatedly on \mathscr{F} . Notice that Lemma 2.1(i) still holds for $\mathscr{A} \subset \binom{X_1, \dots, X_p}{r_1, \dots, r_p}$ and $\mathscr{B} \subset \binom{X_1, \dots, X_p}{s_1, \dots, s_p}$.

For $l \in [p]$ and a positive integer $s \leq n_l$, denote the collection of the first s elements of X_l by $Q_l(s)$. The next lemma is an extension of Lemma 2.1(ii).

Lemma 2.2. Suppose $n_i > r_i + s_i - 1$ for any $i \in [p]$. Let $\mathscr{A} \subset {\binom{X_1,...,X_p}{r_1,...,r_p}}$ and $\mathscr{B} \subset {\binom{X_1,...,X_p}{s_1,...,s_p}}$ be cross t-intersecting families. If \mathscr{A} and \mathscr{B} are l-shifted for any $l \in [p]$, then

$$\sum_{i=1}^{p} |A \cap B \cap Q_i(r_i + s_i - 1)| \ge t$$

for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

Proof. For each $i \in [p]$, write

$$D_i := Q_i(r_i + s_i - 1) \setminus (A \cup B), \ E_i := (A \cap B \cap X_i) \setminus Q_i(r_i + s_i - 1).$$

Note that

$$r_i + s_i = |A \cap X_i| + |B \cap X_i| \ge 2|E_i| + |(A \cup B) \cap Q_i(r_i + s_i - 1)|,$$
(2)

$$|D_i| = r_i + s_i - 1 - |(A \cup B) \cap Q_i(r_i + s_i - 1)|.$$
(3)

If $|E_i| \neq \emptyset$, then $|D_i| \ge |E_i|$ from (2) and (3). Let G_i be an $|E_i|$ -subset of D_i . Write

$$C := \left(B \setminus \bigcup_{i \in [p]} E_i \right) \cup \left(\bigcup_{i \in [p]} G_i \right).$$

Observe that, for each $i \in [p]$,

$$C \cap A \cap X_i = ((B \setminus E_i) \cup G_i) \cap A \cap X_i = A \cap B \cap Q_i(r_i + s_i - 1).$$

When $E_i \neq \emptyset$, notice that $\max G_i < \min E_i$ and $|E_i| = |G_i|$. Thus C can be obtained by doing a series of shifting operations on B. Since \mathscr{B} is *l*-shifted for any $l \in [p]$, we have $C \in \mathscr{B}$. So $|A \cap C| \ge t$. Hence

$$\sum_{i=1}^{p} |A \cap B \cap Q_i(r_i + s_i - 1)| = \sum_{i=1}^{p} |A \cap C \cap X_i| = |A \cap C| \ge t,$$

as desired.

Given positive integers g, h with $g \ge 2h$, it is well-known that the Kneser graph KG(g, h) is the graph on the vertex set $\binom{[g]}{h}$, with an edge between two vertices if and only if they are disjoint. To characterize extremal structures in Theorems 1.2 and 1.4, we need a property of Kneser graphs which is derived from Theorem 1 in [5].

Lemma 2.3. For Kneser graphs $KG(g_1, h_1), \ldots, KG(g_w, h_w)$ with $g_i > 2h_i$ for any $i \in [w]$, their direct product $\prod_{i \in [w]} KG(g_i, h_i)$ is connected.

For $\mathscr{H} \subset 2^X$, we say $\mathscr{F} \subset \mathscr{H}$ is a *full t-star* in \mathscr{H} if \mathscr{F} is the collection of all sets in \mathscr{H} containing a fixed *t*-subset of X. For each $i \in [p]$, let b_i be the maximum number appearing in the *i*-th coordinate of some elements of \mathscr{R} .

Lemma 2.4. Let $\mathscr{F} \subset \mathscr{H}_2$ be a t-intersecting family. Suppose $n_m > 2(t+1)b_m$ for any $m \in [p]$. For $l \in [p]$ and $i, j \in X_l$, if $\Delta_{i,j}(\mathscr{F})$ is a full t-star in \mathscr{H}_2 , then \mathscr{F} is also a full t-star in \mathscr{H}_2 . *Proof.* For $\mathbf{r} = (r_1, \ldots, r_p) \in \mathscr{R}$, let \mathscr{F}_r denote $\mathscr{F} \cap \binom{X_1, \ldots, X_p}{r_1, \ldots, r_p}$ in the rest of the paper. Write

$$\mathscr{F}_{\boldsymbol{r}}(l) := \{F \setminus X_l : F \in \mathscr{F}_{\boldsymbol{r}}\}.$$

For each $R \in \mathscr{F}_{\boldsymbol{r}}(l)$, let

$$\mathscr{G}_R := \left\{ R' \in \begin{pmatrix} X_l \\ r_l \end{pmatrix} : R \cup R' \in \mathscr{F}_r \right\}.$$

Observe that

$$\mathscr{F}_{\boldsymbol{r}} = \bigcup_{R \in \mathscr{F}_{\boldsymbol{r}}(l)} \{ R \cup R' : R' \in \mathscr{G}_R \}, \ \Delta_{i,j}(\mathscr{F}_{\boldsymbol{r}}) = \bigcup_{R \in \mathscr{F}_{\boldsymbol{r}}(l)} \{ R \cup R'' : R'' \in \Delta_{i,j}(\mathscr{G}_R) \}.$$
(4)

By assumption, there exists $T_0 \in {\binom{X}{t}}$ such that $\Delta_{i,j}(\mathscr{F}) = \{F \in \mathscr{H}_2 : T_0 \subset F\}$, which implies that

$$\Delta_{i,j}(\mathscr{F}_{\mathbf{r}}) = \left\{ F \in \begin{pmatrix} X_1, \dots, X_p \\ r_1, \dots, r_p \end{pmatrix} : T_0 \subset F \right\}.$$
 (5)

We have $|\mathscr{G}_R| = |\Delta_{i,j}(\mathscr{G}_R)| = \binom{n_l - t_l}{r_l - t_l}$, where $t_l := |T_0 \cap X_l|$.

If $T_0 \cap X_l = \emptyset$, we get $\mathscr{G}_R = \Delta_{i,j}(\mathscr{G}_R)$ from $\Delta_{i,j}(\mathscr{G}_R) = \binom{X_l}{r_l}$. By (4), $\mathscr{F}_r = \Delta_{i,j}(\mathscr{F}_r)$. Hence $\mathscr{F} = \Delta_{i,j}(\mathscr{F})$, as desired.

Now suppose $T_0 \cap X_l \neq \emptyset$. By (5), we have

$$\mathscr{F}_{\boldsymbol{r}}(l) = \{ G \subset X \setminus X_l : T_0 \setminus X_l \subset G, \ |G \cap X_m| = r_m, \ m \in [p] \setminus \{l\} \}.$$

Note that $n_m > 2(t+1)r_m$ for any $m \in [p]$. Then given $R_0 \in \mathscr{F}_r(l)$, there exists $S_0 \in \mathscr{F}_r(l)$ such that $R_0 \cap S_0 = T_0 \setminus X_l$. Since \mathscr{F}_r is *t*-intersecting, \mathscr{G}_{R_0} and \mathscr{G}_{S_0} are cross t_l -intersecting families with $|\mathscr{G}_{R_0}||\mathscr{G}_{S_0}| = {n_l - t_l \choose r_l - t_l}^2$. By Theorem 1 in [19], we get

$$\mathscr{G}_{R_0} = \mathscr{G}_{S_0} = \left\{ G \in \binom{X_l}{r_l} : T'_l \subset G \right\}$$

for some $T'_l \in \binom{X_l}{t_l}$. Next we prove $\mathscr{G}_S = \mathscr{G}_{R_0}$ for any $S \in \mathscr{F}_r(l) \setminus \{R_0\}$.

For each $S \in \mathscr{F}_{\mathbf{r}}(l)$, we have $|(S \setminus T_0) \cap X_m| = r_m - t_m, m \in [p] \setminus \{l\}$. Thus the set $\{R \setminus T_0 : R \in \mathscr{F}_{\mathbf{r}}(l)\}$ can be seen as the vertex set of the graph $\prod_{m \in [p] \setminus \{l\}} KG(n_m - t_m, r_m - t_m)$. Notice that $n_m - t_m > 2(r_m - t_m)$. Suppose $S \neq R_0$. By Lemma 2.3, this graph contains a walk

$$R_0 \setminus T_0, A_1, \ldots, A_z = S \setminus T_0.$$

Let $B_0 = R_0$, $B_1 = A_1 \cup (T_0 \setminus X_l)$, ..., $B_z = S \in \mathscr{F}_r(l)$. Then $B_q \cap B_{q+1} = T_0 \setminus X_l$ for $q = 0, 1, \ldots, z - 1$. Consequently $\mathscr{G}_{R_0} = \mathscr{G}_{B_1} = \cdots = \mathscr{G}_S$.

For any $R \in \mathscr{F}_r(l)$, \mathscr{G}_R is the collection of all r_l -subsets of X_l containing T'_l . Hence $\langle w \rangle$

$$\mathscr{F}_{\boldsymbol{r}} = \left\{ R \cup R' : R \in \mathscr{F}_{\boldsymbol{r}}(l), \ T'_{l} \subset R' \in \binom{X_{l}}{r_{l}} \right\}$$

$$= \left\{ F \in \binom{X_{1}, \dots, X_{p}}{r_{1}, \dots, r_{p}} : T_{1} \subset F \right\},$$

$$(6)$$

where $T_1 := (T_0 \setminus X_l) \cup T'_l$.

For $\mathbf{s} = (s_1, \ldots, s_p) \in \mathscr{R}$, by (6), there exists $T_2 \in \binom{X}{t}$ such that $\mathscr{F}_{\mathbf{s}}$ is the collection of all sets in $\binom{X_1, \ldots, X_p}{s_1, \ldots, s_p}$ containing T_2 . Since $n_m > 2(t+1)b_m$ for any $m \in [p]$, there are $F_1 \in \mathscr{F}_r$ and $F_2 \in \mathscr{F}_s$ such that $(F_1 \setminus T_1) \cap (F_2 \setminus T_2) = \emptyset$. Then $t \leq |F_1 \cap F_2| = |T_1 \cap T_2| \leq t$, which implies that $T_1 = T_2$. Thus for any $s \in \mathscr{R}$, \mathscr{F}_s is the collection of all sets in $\binom{X_1,\ldots,X_p}{s_1,\ldots,s_p}$ containing T_1 , which implies that the desired result follows.

Proof of main results 3

In this section, we shall prove our main results.

Let $\mathscr{F} \subset \mathscr{H}_2$ be a *t*-intersecting family. If $\mathscr{F} = \emptyset$, there is nothing to prove. So suppose that $\mathscr{F} \neq \emptyset$. Besides, according to Lemma 2.4, we may assume that \mathscr{F} is *l*-shifted for any $l \in [p]$.

Recall that $b_i = \max_{(r_1, \dots, r_p) \in \mathscr{R}} r_i$ for $i = 1, \dots, p$. Write

$$K := \bigcup_{i=1}^{p} Q_i(2b_i - 1), \ \alpha(\mathscr{F}) := \min_{F \in \mathscr{F}} |F \cap K|.$$

We have $\alpha(\mathscr{F}) \ge t$. Indeed, since two non-empty subfamilies \mathscr{F}_r and \mathscr{F}_s are cross *t*-intersecting and *l*-shifted for any $l \in [p]$, by Lemma 2.2 we get

$$|F \cap K| \ge \sum_{i=1}^{p} |F \cap G \cap Q_i(2b_i - 1)| \ge t,$$
(7)

where $F \in \mathscr{F}_r$ and $G \in \mathscr{F}_s$.

Lemma 3.1. Suppose $\mathscr{F} \subset \mathscr{H}_2$ is a t-intersecting family. If $\alpha(\mathscr{F}) = t$ and \mathscr{F} is *l-shifted for any* $l \in [p]$ *, then*

$$|\mathscr{F}| \leqslant \max_{\substack{t_1 + \dots + t_p = t \\ t_1, \dots, t_p \in \mathbb{N}}} \sum_{(r_1, \dots, r_p) \in \mathscr{R}} \prod_{i \in [p]} \binom{n_i - t_i}{r_i - t_i}.$$
(8)

Moreover, when the equality holds, \mathscr{F} is a full t-star in \mathscr{H}_2 .

Proof. By assumption, there exists $F_0 \in \mathscr{F}$ such that $|F_0 \cap K| = t$. By (7), for any $G \in \mathscr{F}$, we have

$$F_0 \cap K = \bigcup_{i \in [p]} \left(F_0 \cap G \cap Q_i(2b_i - 1) \right) \subset G.$$
(9)

Therefore, for any $\boldsymbol{r} = (r_1, \ldots, r_p) \in \mathscr{R}$,

$$|\mathscr{F}_{\mathbf{r}}| \leqslant \prod_{i \in [p]} \binom{n_i - |F_0 \cap Q_i(2b_i - 1)|}{r_i - |F_0 \cap Q_i(2b_i - 1)|}.$$

Then (8) follows from $|\mathscr{F}| = \sum_{r \in \mathscr{R}} |\mathscr{F}_r|$. By (9), \mathscr{F} is a collection of some sets in \mathscr{H}_2 containing $F_0 \cap K$. So when the equality in (8) holds, \mathscr{F} is a full *t*-star in \mathscr{H}_2 .

For positive integers $t, p, n_1, \ldots, n_p, k_1, \ldots, k_p$ with $n_i > k_i$ and $k_1 + \cdots + k_p \ge t$, write

$$g_{t,p}(n_1, \dots, n_p; k_1, \dots, k_p) = \max_{\substack{t_1 + \dots + t_p = t \\ t_1, \dots, t_p \in \mathbb{N}}} \prod_{i \in [p]} \binom{n_i - t_i}{k_i - t_i}$$

Proof of Theorem 1.2. Notice that \mathscr{H}_1 is a special case of \mathscr{H}_2 . In view of Lemma 3.1, we show that

$$|\mathscr{F}| < g_{t,p}(n_1, \ldots, n_p; k_1, \ldots, k_p)$$

when $\alpha(\mathscr{F}) \ge t+1$. For convenience, if there is no confusion, we replace $\alpha(\mathscr{F})$ with α in the following.

By assumption, there exists $A_0 \in \mathscr{F}$ such that $|A_0 \cap K| = \alpha$. Then for $F \in \mathscr{F}$, we have $|F \cap K| \ge \alpha$ and $|F \cap K \cap A_0| \ge t$ by (7). Thus

$$\mathscr{F} \subset \bigcup_{J \in \binom{K}{\alpha}, \ |J \cap A_0| \ge t} \left\{ F \in \mathscr{H}_1 : J \subset F \right\}.$$
(10)

Let N be the collection of all non-negative integer solutions of the equation $x_1 + \dots + x_p = \alpha - t$. For each $H \in \binom{K \cap A_0}{t}$ and $\beta = (c_1, \dots, c_p) \in N$, let $\mathscr{J}(H, \beta)$ be the set of all $J \in \binom{K}{\alpha}$ with $H \subset J$ and $|(J \setminus H) \cap X_i| = c_i$. Denote the number of $F \in \mathscr{H}_1$ containing at least one element of $\mathscr{J}(H,\beta)$ by $f(H,\beta)$. For each $J \in \binom{K}{\alpha}$ satisfying $|J \cap A_0| \ge t$, observe that J is an element of some $\mathscr{J}(H,\beta)$. Then by (10), we have

$$|\mathscr{F}| \leqslant \sum_{H \in \binom{K \cap A_0}{t}} \sum_{\beta \in N} f(H, \beta).$$

Observe that

$$|\mathscr{J}(H,\beta)| \leqslant \prod_{i \in [p]} \binom{2k_i - 1}{c_i} \leqslant \prod_{i \in [p]} (2k_i)^{c_i}.$$

Thus

$$\frac{f(H,\beta)}{g_{t,p}(n_1,\ldots,n_p;k_1,\ldots,k_p)} \leqslant \frac{\left(\prod_{i\in[p]} (2k_i)^{c_i}\right) \cdot \left(\prod_{i\in[p]} \binom{n_i-|H\cap X_i|-c_i}{k_i-|H\cap X_i|}\right)}{\prod_{i\in[p]} \binom{n_i-|H\cap X_i|}{k_i-|H\cap X_i|}} \leqslant \prod_{i\in[p]} \left(\frac{2k_i^2}{n_i}\right)^{z_i},$$

where $(z_1, \ldots, z_p) \in N$ such that

$$\prod_{i \in [p]} \left(\frac{2k_i^2}{n_i}\right)^{z_i} = \max_{(c_1, \dots, c_p) \in N} \prod_{i \in [p]} \left(\frac{2k_i^2}{n_i}\right)^{c_i}$$

Note that $|N| = {\binom{\alpha - t + p - 1}{p - 1}}$ and

$$\binom{x}{y} = \prod_{i=y+1}^{x} (1 + \frac{y}{i-y}) \le (y+1)^{x-y}$$

for any positive integers x, y with $x \ge y + 1$. By above discussion, we obtain

$$\frac{|\mathscr{F}|}{g_{t,p}(n_1,\ldots,n_p;k_1,\ldots,k_p)} \leq {\binom{\alpha}{t}} {\binom{\alpha-t+p-1}{p-1}} \cdot \prod_{i\in[p]} \left(\frac{2k_i^2}{n_i}\right)^{z_i}$$
$$\leq \left((t+1)p\right)^{\alpha-t} \cdot \prod_{i\in[p]} \left(\frac{2k_i^2}{n_i}\right)^{z_i}$$
$$= \prod_{i\in[p]} \left(\frac{2(t+1)pk_i^2}{n_i}\right)^{z_i}.$$

Since $n_i > 2(t+1)pk_i^2$ for any $i \in [p]$, we have $|\mathscr{F}| < g_{t,p}(n_1, \ldots, n_p; k_1, \ldots, k_p)$, as desired. For each $S \in {X \choose t}$, write

$$\mathscr{P}(S) := \{ (i, j(i)) \in \mathbb{Z}^2 : i \in [p], \ 0 \leq j(i) < |S \cap X_i| \}.$$

Observe that

$$e(S) := \frac{\prod_{i \in [p]} \binom{n_i - |S \cap X_i|}{k_i - |S \cap X_i|}}{\prod_{i \in [p]} \binom{n_i}{k_i}} = \prod_{(i,j) \in \mathscr{P}(S)} \frac{k_i - j}{n_i - j}.$$
 (11)

Let T be a t-subset of X. To finish the proof, it is sufficient to show that $e(T) = \max_{S \in \binom{X}{i}} e(S)$ if and only if (1) holds for any $i \in [p]$ whenever $|T \cap X_j| \ge 1$.

Suppose that (1) holds for any $i \in [p]$ whenever $|X \cap T_j| \ge 1$. For each $S \in \binom{X}{t} \setminus \{T\}$, from

$$\frac{k_i}{n_i} > \frac{k_i - 1}{n_i - 1} > \dots > \frac{1}{n_i - k_i + 1},$$

we get

$$\min_{(i,j)\in\mathscr{P}(T)\setminus\mathscr{P}(S)}\frac{k_i-j}{n_i-j} \ge \max_{(i,j)\in\mathscr{P}(S)\setminus\mathscr{P}(T)}\frac{k_i-j}{n_i-j}.$$
(12)

By (11) and (12), we have $e(T)/e(S) \ge 1$. On the other hand, suppose $e(T) = \max_{S \in \binom{X}{t}} e(S)$. For each i, j with $|T \cap X_j| \ge 1$, let $T' := (T \setminus \{u\}) \cup \{v\} \in \binom{X}{t}$, where $u \in T \cap X_j$ and $v \in X_i \setminus T$. By (11), we have

$$\frac{k_i - |T \cap X_i|}{n_i - |T \cap X_i|} = \frac{e(T')}{e(T)} \cdot \frac{k_j - |T \cap X_j| + 1}{n_j - |T \cap X_j| + 1} \leqslant \frac{k_j - |T \cap X_j| + 1}{n_j - |T \cap X_j| + 1}.$$

Hence the desired result holds.

It is not intuitive to find $T \in {X \choose t}$ such that the size of $\{F \in \mathscr{H}_1 : T \subset F\}$ is $g_{t,p}(n_1, \ldots, n_p; k_1, \ldots, k_p)$. Thus we extract an algorithm about how to find all $|T \cap X_i|$ from the proof of Theorem 1.2.

Algorithm 1

1: **Input** $t, p, k_1, \dots, k_p, n_1, \dots, n_p$ 2: Let A be the collection of $\frac{k_i - j}{n_i - j}$ for all i, j with $i \in [p], j = 0, \dots, k_i - 1$ 3: Sort A in decreasing order a_1, a_2, \ldots 4: Let A(f) be the collection of (i, j) satisfying $\frac{k_i - j}{n_i - i} = f$ for $f \in A$ 5: Put $i \leftarrow 1, c \leftarrow 0, k \leftarrow 0, G \leftarrow \emptyset$ 6: while k < t do $k \leftarrow k + |A(a_i)|$ 7: if $k \leq t$ then 8: $G \leftarrow G \cup A(a_i)$ 9: else 10: $c \leftarrow |A(a_i)| - k + t$ $H \leftarrow \binom{A(a_i)}{c}$ 11: 12:end if 13: $i \leftarrow i + 1$ 14:15: end while

```
16: if c = 0 then
       for t_m do
17:
          t_m \leftarrow |\{(m,j): (m,j) \in G\}|
18:
       end for
19:
       Output
                       t_1,\ldots,t_p
20:
21: else
       for L \in H do
22:
          J \leftarrow G \cup L
23:
          for t_m do
24:
             t_m \leftarrow |\{(m,j) : (m,j) \in J\}|
25:
          end for
26:
                          t_1,\ldots,t_p
          Output
27:
       end for
28:
29: end if
```

Proof of Theorem 1.4. In consideration of Lemma 3.1, it is sufficient to show that (t_1, t_2)

$$|\mathscr{F}| < \max_{\substack{t_1 + \dots + t_p = t \\ t_1, \dots, t_p \in \mathbb{N}}} \sum_{(r_1, \dots, r_p) \in \mathscr{R}} \prod_{i \in [p]} \binom{n_i - t_i}{r_i - t_i}$$

when $\alpha(\mathscr{F}) \ge t+1$. W.o.l.g., suppose that $n_1 = \min_{i \in [p]} n_i$.

We may assume that $\mathscr{F}_r \neq \emptyset$ for some $r = (r_1, \ldots, r_p) \in \mathscr{R}$, otherwise there is nothing to prove. Observe that \mathscr{F}_r is *t*-intersecting and $\alpha(\mathscr{F}_r) \ge \alpha(\mathscr{F}) \ge t+1$. From the proof of Theorem 1.2, we get

$$\frac{|\mathscr{F}_{\boldsymbol{r}}|}{g_{t,p}(n_1,\ldots,n_p;r_1,\ldots,r_p)} \leqslant \prod_{i\in[p]} \left(\frac{2(t+1)pr_i^2}{n_i}\right)^{w_i} \leqslant \prod_{i\in[p]} \left(\frac{2(t+1)pb_i^2}{n_i}\right)^{q_i}, \quad (13)$$

where $w_1 + \cdots + w_p = \alpha(\mathscr{F}_r) - t$ and $q_1 + \cdots + q_p = \alpha(\mathscr{F}) - t$. Notice that there exist non-negative integers d_1, \ldots, d_p with $d_1 + \cdots + d_p = t$ such that

$$\frac{g_{t,p}(n_1,\ldots,n_p;r_1,\ldots,r_p)}{\binom{n_1-t}{r_1-t}\cdot\prod_{i=2}^p\binom{n_i}{r_i}} = \left(\prod_{i\in[p]}\left(\prod_{j=0}^{d_i-1}\frac{r_i-j}{n_i-j}\right)\right)\cdot\left(\prod_{j=0}^{t-1}\frac{n_1-j}{r_1-j}\right)\leqslant b^t.$$
 (14)

Combining (13) and (14), we derive

$$\frac{|\mathscr{F}_{\boldsymbol{r}}|}{\binom{n_1-t}{r_1-t}\cdot\prod_{i=2}^p\binom{n_i}{r_i}} \leqslant b^t\cdot\prod_{i\in[p]} \left(\frac{2(t+1)pb_i^2}{n_i}\right)^{q_i} \leqslant \left(\frac{2(t+1)pb^{t+2}}{n_1}\right)^{\alpha-t} < 1$$

from $n_1 > 2(t+1)pb^{t+2}$. Therefore, $|\mathscr{F}|$ is smaller than the number of sets in \mathscr{H}_2 containing [t], which implies that the desired result follows.

Acknowledgement

This research is supported by NSFC (11671043) and NSF of Hebei Province (A2019205092).

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