# List homomorphism problems for signed trees ${ }^{\star}$ 

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#### Abstract

We consider homomorphisms of signed graphs from a computational perspective. In particular, we study the list homomorphism problem seeking a homomorphism of an input signed graph $(G, \sigma)$, equipped with lists $L(v) \subseteq V(H), v \in$ $V(G)$, of allowed images, to a fixed target signed graph $(H, \pi)$. The complexity of the similar homomorphism problem without lists (corresponding to all lists being $L(v)=V(H)$ ) has been previously classified by Brewster and Siggers, but the list version remains open and appears difficult. We illustrate this difficulty by classifying the complexity of the problem when $H$ is a tree (with possible loops). The tools we develop will be useful for classifications of other classes of signed graphs, and in a future companion paper we will illustrate this by using them to classify the complexity for certain irreflexive signed graphs. The structure of the signed trees in the polynomial cases is interesting, suggesting that the class of general signed graphs for which the problems are polynomial may have nice structure, analogous to the so-called bi-arc graphs (which characterized the polynomial cases of list homomorphisms to unsigned graphs).


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## 1. Motivation

We investigate a problem at the confluence of two popular topics - graph homomorphisms and signed graphs. Their interplay was first considered in an unpublished manuscript of Guenin 17, and has since become an established field of study [25].

We now introduce the two topics separately. In the study of computational aspects of graph homomorphisms, the central problem is one of existence does an input graph $G$ admit a homomorphism to a fixed target graph $H$ ? (The graphs considered here are undirected graphs with possible loops but no parallel edges.) This is known as the graph homomorphism problem. It was shown in [20] that this problem is polynomial-time solvable when $H$ has a loop or is bipartite, and is NP-complete otherwise. This is known as the dichotomy of graph homomorphisms (see [21]). The core of a graph $H$ is a subgraph of $H$ with the smallest number of vertices to which $H$ admits a homomorphism; note that such a subgraph is unique up to isomorphism. A graph with a loop has a vertex with a loop as its core, and a (non-empty) bipartite graph has an edge as its core. Thus an equivalent way of stating the graph dichotomy result is that the problem is polynomial-time solvable when the core of $H$ has at most one edge, and is NP-complete otherwise.

Now suppose the input graph $G$ is equipped with lists, $L(v) \subseteq V(H), v \in$ $V(G)$, and we ask if there is a homomorphism $f$ of $G$ to $H$ such that each $f(v) \in L(v)$. This is known as the graph list homomorphism problem. This problem also has a dichotomy of possible complexities [12] - it is polynomialtime solvable when $H$ is a so-called bi-arc graph and is NP-complete otherwise. Bi -arc graphs have turned out to be an interesting class of graphs; for instance, when $H$ is a reflexive graph (each vertex has a loop), $H$ is a bi-arc graph if and only if it is an interval graph 10 .

These kinds of complexity questions found their most general formulation in the context of constraint satisfaction problems. The Feder-Vardi dichotomy conjecture [15] claimed that every constraint satisfaction problem with a fixed template $H$ is polynomial-time solvable or NP-complete. After a quarter century of concerted effort by researchers in theoretical computer science, universal algebra, logic, and graph theory, the conjecture was proved in 2017, independently by Bulatov [9] and Zhuk [31. This exciting development focused research attention on additional homomorphism type dichotomies, including ones for signed graphs [6, 8, 16].

The study of signed graphs goes back to [18, 19], and has been most notably investigated in [26, [27, 28, 29, 30, from the point of view of colourings, matroids, or embeddings. Following Guenin, homomorphisms of signed graphs have been pioneered in [7] and [24]. The computational aspects of existence of homomorphisms in signed graphs - given a fixed signed graph $(H, \pi)$, does an input signed graph $(G, \sigma)$ admit a homomorphism to $(H, \pi)$ - were studied in [6, 16], and eventually a complete dichotomy classification was obtained in [8]. It is surprisingly similar to the second way we stated the graph dichotomy result above, see Theorem 5 and the discussion following it.

Although typically homomorphism problems tend to be easier to classify with lists than without lists (lists allow for recursion to subgraphs), the complexity of the list homomorphism problem for signed graphs appears difficult to classify [4, 8. If the analogy to (unsigned) graphs holds again, then the tractable cases of the problem should identify an interesting class of signed graphs, generalizing bi-arc graphs. In this paper, we begin the exploration of this concept, focusing on the case of signed trees. We find that there is interesting structure to the tractable cases.

## 2. Terminology and notation

A signed graph is a graph $G$, with possible loops and multiple edges (at most two loops per vertex and at most two edges between a pair of vertices), together with a mapping $\sigma: E(G) \rightarrow\{+,-\}$, assigning a sign (+ or - ) to each edge and each loop of $G$, so that different loops at a vertex have different signs, and similarly for different edges between the same two vertices. For convenience, we shall usually consider an edge to mean an edge or a loop, and to emphasize otherwise we shall call it a non-loop edge. Thus we can say, for example, that each edge of a signed graph has a sign, meaning both loops and non-loop edges. We denote a signed graph by $(G, \sigma)$, and call $G$ its underlying graph and $\sigma$ its signature. When the signature name is not needed, we denote the signed graph $(G, \sigma)$ by $\widehat{G}$ to emphasize that it has a signature even though we do not give it a name. We will usually view signs of edges as colours, and call positive edges blue, and negative edges red. It will be convenient to call a red-blue pair of edges with the same endpoint(s) a bicoloured edge (this includes loops as well as non-loop edges); however, formally they are two distinct edges. By contrast, we call edges that are not part of such a pair unicoloured; moreover, when we refer to an edge as blue or red we shall always mean the edge is unicoloured blue or red. We also call an edge at least blue if it is either blue or bicoloured, and similarly for at least red edges. The terms at least positive and at least negative are used in the same sense. Treating a pair of red-blue edges as one bicoloured edge is advantageous in many descriptions, but introduces an ambiguity when discussing walks, since a walk in a signed graph could be seen as a sequence of incident vertices and edges, and so selecting just one edge from a red-blue pair, or it could be interpreted as a sequence of consecutively adjacent vertices, and hence contain some bicoloured edges. This creates particular problem for cycles, since in the former view, a bicoloured edge would be seen as a cycle of length two, with one red edge and one blue edge. In the literature, the former approach is more common, but here we take the latter approach. Of course, the two views coincide if only walks of unicoloured edges are considered. The sign of a walk consisting of unicoloured edges $\widehat{G}$ is the product of the signs of its edges. Thus a walk of unicoloured edges is negative if it has an odd number of negative (red) edges, and positive if it has an even number of negative (red) edges. In the case of unicoloured cycles, we also call a negative cycle unbalanced and a positive cycle balanced. Note that a vertex with a red loop is a cycle with one negative edge, and hence is unbalanced. A uni-balanced signed graph is a signed graph
without unbalanced cycles, i.e., a signed graph in which all unicoloured cycles (if any) have an even number of red edges. An anti-uni-balanced signed graph is a signed graph in which each unicoloured cycle has an even number of blue edges. Thus we have a symmetry to viewing the signs as colours, in particular $\widehat{G}$ is uni-balanced if and only if $\widehat{G}^{\prime}$, obtained from $\widehat{G}$ by exchanging the colour of each edge, is anti-uni-balanced. We introduce the qualifier "uni-" because the notion of a balanced signed graph is well established in the literature: it means a signed graph without any unbalanced cycles in the classical view, including the two-cycles formed by red-blue pairs of edges. Thus a balanced signed graph is a uni-balanced signed graph without bicoloured edges and loops.

We now define the switching operation. This operation can be applied to any vertex of a signed graph and it negates the signs of all its incident nonloop edges. (The signs of loops are unchanged by switching.) We say that two signatures $\sigma_{1}, \sigma_{2}$ of a graph $G$ are switching equivalent if we can obtain ( $G, \sigma_{2}$ ) from $\left(G, \sigma_{1}\right)$ by a sequence of switchings. In that case we also say that the two signed graphs $\left(G, \sigma_{1}\right)$ and $\left(G, \sigma_{2}\right)$ are switching equivalent. (We note a sequence of switchings may also be realized by negating all the edges of a single edge cut.) In a very formal way, a signed graph is an equivalence class under the switching equivalence, and we sometimes use the notation $\widehat{G}$ to mean the entire class.

It was proved by Zaslavsky [27] that two signatures of $G$ are switching equivalent if and only if they define exactly the same set of negative (or positive) cycles. It is easy to conclude that a uni-balanced signed graph is switching equivalent to a signed graph with all edges and loops at least blue, and an anti-uni-balanced signed graph is switching equivalent to a signed graph with all edges and loops at least red.

We now consider homomorphisms of signed graphs. Since signed graphs $\widehat{G}, \widehat{H}$ can be viewed as equivalence classes, a homomorphism of signed graphs $\widehat{G}$ to $\widehat{H}$ should be a homomorphism of one representative $(G, \sigma)$ of $\widehat{G}$ to one representative $(H, \pi)$ of $\widehat{H}$. It is easy to see that this definition can be simplified by prescribing any fixed representative $(H, \pi)$ of $\widehat{H}$. In other words, we now consider mapping all possible representatives $\left(G, \sigma^{\prime}\right)$ of $\widehat{G}$ to one fixed representative $(H, \pi)$ of $\widehat{H}$. At this point, a homomorphism $f$ of one concrete $\left(G, \sigma^{\prime}\right)$ to $(H, \pi)$ is just a homomorphism of the underlying graph $G$ to the underlying graph $H$ preserving the edge colours. Since there are multiple edges, we can either consider $f$ to be a mapping of vertices to vertices and edges to edges, preserving vertex-edge incidences and edge-colours, as in [25], or simply state that blue edges map to edges that are at least blue, red edges map to edges that are at least red, and bicoloured edges map to bicoloured edges. Formally, we state it as follows.

Definition 1. We say that a mapping $f: V(G) \rightarrow V(H)$ is a homomorphism of the signed graph $(G, \sigma)$ to the signed graph $(H, \pi)$, written as $f:(G, \sigma) \rightarrow$ $(H, \pi)$, if there exists a signed graph $\left(G, \sigma^{\prime}\right)$, switching equivalent to $(G, \sigma)$, such that whenever the edge $u v$ is at least positive in $\left(G, \sigma^{\prime}\right)$, then $f(u) f(v)$ is an edge that is at least positive in $(H, \pi)$, and whenever the edge $u v$ is at least negative in $\left(G, \sigma^{\prime}\right)$, then $f(u) f(v)$ is an edge that is at least negative in $(H, \pi)$.

There is an equivalent alternative definition (see [25]). A homomorphism of the signed graph $(G, \sigma)$ to the signed graph $(H, \pi)$ is a homomorphism $f$ of the underlying graph $G$ to the underlying graph $H$, which maps bicoloured edges of $(G, \sigma)$ to bicoloured edges of $(H, \pi)$, and which for any closed walk $W$ in $(G, \sigma)$ with only unicoloured edges for which the image walk $f(W)$ has also only unicoloured edges, the sign of $f(W)$ in $(H, \pi)$ is the same as the sign of $W$ in $(G, \sigma)$. (In other words, negative closed walks map to negative closed walks and positive closed walks map to positive closed walks.) This definition does not require switching the input graph before mapping it. The equivalence of the two definitions follows from the theorem of Zaslavsky 27] cited above. That result is constructive, and the actual switching required to produce the switching equivalent signed graph $\left(G, \sigma^{\prime}\right)$ can be found in polynomial time [25].

We deduce the following fact.
Lemma 2. Suppose $(G, \sigma)$ and $(H, \pi)$ are signed graphs, and $f$ is a mapping of the vertices of $G$ to the vertices of $H$. Then $f$ is a homomorphism of the signed graph $(G, \sigma)$ to the signed graph $(H, \pi)$ if and only if $f$ is a homomorphism of the underlying graph $G$ to the underlying graph $H$, which moreover maps bicoloured edges of $(G, \sigma)$ to bicoloured edges of $(H, \pi)$, and for any closed walk $W$ in $(G, \sigma)$ with only unicoloured edges for which the image walk $f(W)$ has also only unicoloured edges, the signs of $W$ and $f(W)$ are the same.

Note that each negative closed walk contains a negative cycle, and in particular an irreflexive tree $(H, \pi)$ has no negative closed walks except for those using bicoloured edges. Thus if $(H, \pi)$ is an irreflexive tree, then the condition simplifies to having no negative cycle of $(G, \sigma)$ mapped to unicoloured edges in $(H, \pi)$ (because the image would be a positive closed walk). For reflexive trees, the condition requires that no negative cycle of $(G, \sigma)$ maps to a positive closed walk in $(H, \pi)$, and no positive cycle of $(G, \sigma)$ maps to a negative closed walk.

For our purposes, the simpler Definition 1 is sufficient. Note that whether an edge is unicoloured or bicoloured is independent of switching, and that a homomorphism can map a unicoloured edge or loop in $\widehat{G}$ to a bicoloured edge or loop in $\widehat{H}$ but not conversely.

Let $\widehat{H}$ be a fixed signed graph. The homomorphism problem S-Hom $(\widehat{H})$ takes as input a signed graph $\widehat{G}$ and asks whether there exists a homomorphism of $\widehat{G}$ to $\widehat{H}$. The formal definition of the list homomorphism problems for signed graphs is very similar.
Definition 3. Let $\widehat{H}$ be a fixed signed graph. The list homomorphism problem $\operatorname{List-S-Hom}(\widehat{H})$ takes as input a signed graph $\widehat{G}$ with lists $L(v) \subseteq V(H)$ for every $v \in V(G)$, and asks whether there exists a homomorphism $f$ of $\widehat{G}$ to $\widehat{H}$ such that $f(v) \in L(v)$ for every $v \in V(G)$.

We note that when $\widehat{H}$ and $\widehat{H}^{\prime}$ are switching equivalent signed graphs, then any homomorphism of an input signed graph $\widehat{G}$ to $\widehat{H}$ is also a homomorphism to $\widehat{H}^{\prime}$, and therefore the problems S- $\operatorname{Hom}(\widehat{H})$ and $\operatorname{S-Hom}\left(\widehat{H}^{\prime}\right)$, as well as the problems List-S-Hom $(\widehat{H})$ and List-S-Hom $\left(\widehat{H}^{\prime}\right)$, are equivalent.

We call a signed graph $\widehat{H}$ connected if the underlying graph $H$ is connected. We call $\widehat{H}$ reflexive if each vertex of $H$ has a loop, and irreflexive if no vertex has a loop. We call $\widehat{H}$ a signed tree if $H$, with any existing loops removed, is a tree.

We may assume that the target signed graph $\widehat{H}$ is connected. This implies no loss of generality for list homomorphism problems, as each component of an input signed graph $\widehat{G}$ can only be mapped to one component of a target signed graph $\widehat{H}$.

## 3. More background and connections to constraint satisfaction

We now briefly introduce the constraint satisfaction problems, in the format used in [15. A relational system $G$ consists of a set $V(G)$ of vertices and a family of relations $R_{1}, R_{2}, \ldots, R_{k}$ on $V(G)$. Assume $G$ is a relational system with relations $R_{1}, R_{2}, \ldots, R_{k}$ and $H$ a relational system with relations $S_{1}, S_{2}, \ldots, S_{k}$, where the arity of the corresponding relations $R_{i}$ and $S_{i}$ is the same for all $i=1,2, \ldots, k$. A homomorphism of $G$ to $H$ is a mapping $f: V(G) \rightarrow V(H)$ that preserves all relations, i.e., satisfies $\left(v_{1}, v_{2}, \ldots\right) \in R_{i} \Longrightarrow\left(f\left(v_{1}\right), f\left(v_{2}\right), \ldots\right) \in$ $S_{i}$, for all $i=1,2, \ldots, k$. The constraint satisfaction problem with fixed template $H$ asks whether or not an input relational system $G$, with the same arities of corresponding relations as $H$, admits a homomorphism to $H$.

Note that when $H$ has a single relation $S$, which is binary and symmetric, then we obtain the graph homomorphism problem referred to at the beginning of Section 1 When $H$ has a single relation $S$, which is an arbitrary binary relation, we obtain the digraph homomorphism problem [2] which is in a certain sense [15] as difficult to classify as the general constraint satisfaction problem. When $H$ has two relations,+- , then we obtain a problem that is superficially similar to the homomorphism problem for signed graphs, except that switching is not allowed. This problem is called the edge-coloured graph homomorphism problem [5], and it turns out to be similar to the digraph homomorphism problem in that it is difficult to classify [6]. On the other hand, the homomorphism problem for signed graphs [6, 8, 16, seems easier to classify, and exhibits a dichotomy similar to the graph dichotomy classification, see Theorem 5 .

List homomorphism problems are also special cases of constraint satisfaction problems, as lists can be replaced by unary relations. Consider first the case of graphs. Suppose $H$ is a fixed graph, and form the relational system $H^{\#}$ with vertices $V(H)$ and the following relations: one binary relation $E(H)$ (this is a symmetric relation corresponding to the undirected edges of the graph $H$ ), and $2^{|V(H)|}-1$ unary relations $R_{X}$ on $V(H)$, each consisting of a different nonempty subset $X$ of $V(H)$. The constraint satisfaction problem with template $H^{\#}$ has inputs $G$ with a symmetric binary relation $E(G)$ (a graph) and unary relations $S_{X}, X \subseteq V(H)$, and the question is whether or not a homomorphism exists. If a vertex $v \in V(G)$ is in the relation $S_{X}$ corresponding to $R_{X}$, then any mapping preserving the relations must map $v$ to a vertex in $X$; thus imposing the relation $S_{X}$ on $v \in V(G)$ amounts to setting $L(v)=X$. Therefore the
list homomorphism problem for the graph $H$ is formulated as the constraint satisfaction problem for the template $H^{\#}$.

Such a translation is also possible for homomorphism of signed graphs. Brewster and Graves introduced a useful construction. The switching graph $\left(H^{+}, \pi^{+}\right)$ has two vertices $v_{1}, v_{2}$ for each vertex $v$ of $(H, \pi)$, and each edge $v w$ of $(H, \pi)$ gives rise to edges $v_{1} w_{1}, v_{2} w_{2}$ of colour $\pi(v w)$ and edges $v_{1} w_{2}, v_{2} w_{1}$ of the opposite colour. (This definition applies also for loops, i.e., when $v=w$.) Then each homomorphism of the signed graph $(G, \sigma)$ to the signed graph $(H, \pi)$ corresponds to a homomorphism of the edge-coloured graph $(G, \sigma)$ to the edgecoloured graph $\left(H^{+}, \pi^{+}\right)$and conversely. For list homomorphisms of signed graphs, we can use the same transformation, modifying the lists of the input signed graph. If $(G, \sigma)$ has lists $L(v), v \in V(G)$, then the new lists $L^{+}(v), v \in$ $V(G)$, are defined as follows: for any $x \in L(v)$ with $v \in V(G)$, we place both $x_{1}$ and $x_{2}$ in $L^{+}(v)$. It is easy to see that the signed graph $(G, \sigma)$ has a list homomorphism to the signed graph $(H, \pi)$ with respect to the lists $L$ if and only if the edge-coloured graph $(G, \sigma)$ has a list homomorphism to the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$with respect to the lists $L^{+}$. The new lists $L^{+}$are symmetric sets in $H^{+}$, meaning that for any $x \in V(H), v \in V(G)$, we have $x_{1} \in L^{+}(v)$ if and only if we have $x_{2} \in L^{+}(v)$. Thus we obtain the list homomorphism problem for the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$, restricted to input instances $(G, \sigma)$ with lists $L$ that are symmetric in $H^{+}$. As above, we can transform this list homomorphism problem for the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$, to a constraint satisfaction problem. The details are similar to the construction of $H^{\#}$, except this time the new template $\left(H^{+}, \pi^{+}\right)^{*}$ is obtained by adding unary relations $R_{X}=X$ only for sets $X \subseteq V\left(H^{+}\right)$that are symmetric in $H^{+}$.

We conclude that our problems List-S- $\operatorname{Hom}(\widehat{H})$ fit into the general constraint satisfaction framework, and therefore it follows from 9, 31 that dichotomy holds for problems List-S- $\operatorname{Hom}(\widehat{H})$. We therefore ask which problems $\operatorname{List}-\mathrm{S}-\operatorname{Hom}(\widehat{H})$ are polynomial-time solvable and which are NP-complete.

The solution of the Feder-Vardi dichotomy conjecture involved an algebraic classification of the complexity pioneered by Jeavons [22. A key role in this is played by the notion of a polymorphism of a relational structure $H$. If $H$ is a digraph, then a polymorphism of $H$ is a homomorphism $f$ of some power $H^{t}$ to $H$, i.e., a function $f$ that assigns to each ordered $t$-tuple $\left(v_{1}, v_{2}, \ldots, v_{t}\right)$ of vertices of $H$ a vertex $f\left(v_{1}, v_{2}, \ldots, v_{t}\right)$ such that two coordinate-wise adjacent tuples obtain adjacent images. For general templates, all relations must be similarly preserved. A polymorphism of order $t=3$ is a majority if $f(v, v, w)=$ $f(v, w, v)=f(w, v, v)=v$ for all $v, w$. A Siggers polymorphism is a polymorphism of order $t=4$, if $f(a, r, e, a)=f(r, a, r, e)$ for all $a, r, e$. One formulation of the dichotomy theorem proved by Bulatov [9] and Zhuk [31] states that the constraint satisfaction problem for the template $H$ is polynomial-time solvable if $H$ admits a Siggers polymorphism, and is NP-complete otherwise. Majority polymorphisms are less powerful, but it is known [15] that if $H$ admits a majority then the constraint satisfaction problem for the template $H$ is polynomial-time solvable. Moreover, it was shown in [12] that a graph $H$ is a bi-arc graph if
and only if the associated relational system $H^{*}$ admits a majority polymorphism. Thus the list homomorphism problem for a graph $H$ with possible loops is polynomial-time solvable if $H^{*}$ admits a majority polymorphism, and is NPcomplete otherwise. It was observed in [23] that this is not true for signed graphs.

There is a convenient way to think of polymorphisms $f$ of the relational system $\left(H^{+}, \pi^{+}\right)^{*}$. A mapping $f$ is a polymorphism of $\left(H^{+}, \pi^{+}\right)^{*}$ if and only if it is a polymorphism of the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$and if, for any symmetric set $X \subseteq V\left(H^{+}\right)$, we have $x_{1}, x_{2}, \ldots, x_{t} \in X$ then also $f\left(x_{1}, x_{2}, \ldots, x_{t}\right) \in X$. We call such polymorphisms of $\left(H^{+}, \pi^{+}\right)$semi-conservative.

We can apply the dichotomy result of [9, 31] to obtain an algebraic classification.

Theorem 4. For any signed graph $(H, \pi)$, the problem List-S-Hom $(H, \pi)$ is polynomial-time solvable if $\left(H^{+}, \pi^{+}\right)$admits a semi-conservative Siggers polymorphism, and is NP-complete otherwise.

As mentioned above, one can not replace the semi-conservative Siggers polymorphism by a semi-conservative majority polymorphism [23]. We focus in this paper on seeking a graph theoretic classification, at least for some classes of signed graphs.

## 4. Basic facts

We first mention the dichotomy classification of the problems S-Hom $(\widehat{H})$ from [8]. A subgraph $\widehat{G}$ of the signed graph $\widehat{H}$ is the $s$-core of $\widehat{H}$ if there is homomorphism $f: \widehat{H} \rightarrow \widehat{G}$, and every homomorphism $\widehat{G} \rightarrow \widehat{G}$ is a bijection on $V(G)$. The letter $s$ stands for signed. It is again easy to see that the $s$-core is unique up to isomorphism and switching equivalence.

Theorem 5. 8 The problem $\operatorname{S-Hom}(\widehat{H})$ is polynomial-time solvable if the score of $\widehat{H}$ has at most two edges, and is NP-complete otherwise.

When the signature $\pi$ has all edges positive, the problem $\operatorname{S-Hom}(H, \pi)$ is equivalent to the unsigned graph homomorphism problem, and the s-core of $(H, \pi)$ is just the core of $H$. To compare Theorem 5 with the graph dichotomy theorem of [20] as discussed at the beginning Section 1] we observe that the core of a graph cannot have exactly two edges, as a core must be either a single vertex (possibly with a loop), or a single edge, or a graph with at least three edges. Thus Theorem 5 is stronger than the graph dichotomy theorem from [20], which states that the graph homomorphism problem to $H$ is polynomial-time solvable if the core of $H$ has at most one edge and is NP-complete otherwise. (However, we note that the proof of Theorem 5in [8] uses the graph dichotomy theorem [20].)

Observe that an instance of the problem $\mathrm{S}-\operatorname{Hom}(\widehat{H})$ can be also viewed as an instance of List-S-Hom $(\widehat{H})$ with all lists $L(v)=V(H)$, therefore if

S-Hom $(\widehat{H})$ is NP-complete, then so is List-S- $\operatorname{Hom}(\widehat{H})$. Moreover, if $\widehat{H}^{\prime}$ is an induced subgraph of $\widehat{H}$, then any instance of List-S-Hom $\left(\widehat{H}^{\prime}\right)$ can be viewed as an instance of List-S-Hom $(\widehat{H})$ (with the same lists), therefore if the problem List-S-Hom $\left(\widehat{H}^{\prime}\right)$ is NP-complete, then so is the problem List-S-Hom $(\widehat{H})$. This yields the NP-completeness of List-S-Hom $(\widehat{H})$ for all signed graphs $(\widehat{H})$ that contain an induced subgraph $\widehat{H}^{\prime}$ whose s-core has more than two edges. Furthermore, when the signed graph $\widehat{H}$ is uni-balanced, then we may assume that all edges are at least blue, and the list homomorphism problem for $H$ can be reduced to List-S-Hom $(\widehat{H})$. In particular, we emphasize that List-S-Hom $(\widehat{H})$ is NP-complete if $\widehat{H}$ is a uni-balanced signed graph (or, by a symmetric argument, an anti-uni-balanced signed graph), and the underlying graph $H$ is not a bi-arc graph [12].

Next we focus on the class of signed graphs that have no bicoloured loops and no bicoloured edges. In this case, the following simple dichotomy describes the classification. (This result was previously announced in [4.) It follows from our earlier remarks that these signed graphs are balanced if and only if they are uni-balanced, and similarly they are anti-balanced if and only if they are anti-uni-balanced.

Theorem 6. Suppose $\widehat{H}$ is a connected signed graph without bicoloured loops and edges. If the underlying graph $H$ is a bi-arc graph, and $\widehat{H}$ is balanced or anti-balanced, then the problem List-S-Hom $(\widehat{H})$ is polynomial-time solvable. Otherwise, the problem is NP-complete.

Proof. The polynomial cases follow from Feder et al. [12, by the following argument. Suppose $\widehat{H}$ is balanced; we may assume all edges are blue. In [27], there is a polynomial-time algorithm to decide if the input signed graph $\widehat{G}$ is balanced. If it is not balanced, there is no homomorphism of $\widehat{G}$ to $\widehat{H}$. Otherwise, we may assume that $\widehat{G}$ has also all edges blue and hence there is a homomorphism of $\widehat{G}$ to $\widehat{H}$ if and only if there is a homomorphism of $G$ to $H$. Since $H$ is a bi-arc graph, this can be decided in polynomial time by the algorithm in [12]. The argument is similar if $\widehat{H}$ is anti-balanced. Otherwise, $\widehat{H}$ contains a cycle which cannot be switched to a blue cycle and a cycle which cannot be switched to a red cycle, in which case the s-core of $\widehat{H}$ contains at least three edges. (This is true even if the cycles are just loops.)

We have observed that List-S-Hom $(\widehat{H})$ is NP-complete if the s-core of $\widehat{H}$ has more than two edges. Thus we will focus on signed graphs $\widehat{H}$ whose s-cores have at most two edges. This is not as simple as it sounds, as there are many complex signed graphs with this property, including, for example, all irreflexive bipartite signed graphs that contain a bicoloured edge, and all signed graphs that contain a bicoloured loop. That these cases are not easy underlines the fact that the assumptions in Theorem 6 cannot be weakened without significant new breakthroughs. Consider, for example, allowing bicoloured edges but not bicoloured loops. In this situation, we may focus on the case when there is
a bicoloured edge (else Theorem 6 applies), and so if there is any loop at all, the s-core would have more than two edges. Thus we consider irreflexive signed graphs. The s-core is still too big if the underlying graph has an odd cycle. So in this case it remains to classify the irreflexive bipartite signed graphs that contain a bicoloured edge. Even this case is complex. We explore homomorphisms to irreflexive bipartite signed graphs in a companion paper.

In this paper we focus on List-S-Hom $(\widehat{H})$ when the underlying graph of $\widehat{H}$ is a tree with possible loops. We have treated the special cases of reflexive and irreflexive trees in 4] and in the conference version of this paper [3]; in the general case presented here, the polynomial algorithms are much more involved and technical, and the structure of these trees turns out to be surprisingly complex. Nevertheless, they seem to suggest that nice characterizations may be possible, in analogy to bi-arc trees 13, see also 12 .

We now introduce our basic tool for proving NP-completeness.
Definition 7. Let $(U, D)$ be two walks in $\widehat{H}$ of equal length, say $U$, with vertices $u=u_{0}, u_{1}, \ldots, u_{k}=v$ and $D$, with vertices $u=d_{0}, d_{1}, \ldots, d_{k}=v$. We say that $(U, D)$ is a chain, provided $u u_{1}, d_{k-1} v$ are unicoloured edges and $u d_{1}, u_{k-1} v$ are bicoloured edges, and for each $i, 1 \leq i \leq k-2$, we have

1. both $u_{i} u_{i+1}$ and $d_{i} d_{i+1}$ are edges of $\widehat{H}$ while $d_{i} u_{i+1}$ is not an edge of $\widehat{H}$, or
2. both $u_{i} u_{i+1}$ and $d_{i} d_{i+1}$ are bicoloured edges of $\widehat{H}$ while $d_{i} u_{i+1}$ is not a bicoloured edge of $\widehat{H}$.

Theorem 8. If a signed graph $\widehat{H}$ contains a chain, then List-S-Hom $(\widehat{H})$ is NP-complete.

Proof. Suppose that $\widehat{H}$ has a chain $(U, D)$ as specified above. We shall reduce from Not-All-EQUAL SAT. (Each clause has three unnegated variables, and we seek a truth assignment in which at least one variable is true and at least one is false, in each clause.) For each clause $x \vee y \vee z$, we take three vertices $x, y, z$, each with list $\{u\}$, and three vertices $x^{\prime}, y^{\prime}, z^{\prime}$, each with list $\{v\}$. For the triple $x, y, z$, we add three new vertices $p(x, y), p(y, z)$, and $p(z, x)$, each with list $\left\{u_{1}, d_{1}\right\}$, and for the triple $x^{\prime}, y^{\prime}, z^{\prime}$, we add three new vertices $p\left(x^{\prime}, y^{\prime}\right), p\left(y^{\prime}, z^{\prime}\right), p\left(z^{\prime}, x^{\prime}\right)$, each with list $\left\{u_{k-1}, d_{k-1}\right\}$. We connect these vertices as follows:

- $p(x, y)$ adjacent to $x$ by a red edge and to $y$ by a blue edge,
- $p(y, z)$ adjacent to $y$ by a red edge and to $z$ by a blue edge,
- $p(z, x)$ adjacent to $z$ by a red edge and to $x$ by a blue edge.

Analogously, the hexagon $x^{\prime}, p\left(x^{\prime}, y^{\prime}\right), y^{\prime}, p\left(y^{\prime}, z^{\prime}\right), z^{\prime}, p\left(z^{\prime}, x^{\prime}\right)$ will also be alternating in blue and red colours, with (say) $p\left(x^{\prime}, y^{\prime}\right)$ adjacent to $x^{\prime}$ by a red edge.

Moreover, we join each pair of vertices $p(x, y)$ and $p\left(x^{\prime}, y^{\prime}\right)$ by a separate path $P(x, y)$ with $k-1$ vertices, say $p(x, y)=a_{1}, a_{2}, \ldots, a_{k-2}, a_{k-1}=p\left(x^{\prime}, y^{\prime}\right)$,
where $a_{i}$ has list $\left\{u_{i}, d_{i}\right\}$ and the edge $a_{i} a_{i+1}$ is blue unless both $u_{i} u_{i+1}$ and $d_{i} d_{i+1}$ are bicoloured, in which case $a_{i} a_{i+1}$ is also bicoloured. Paths $P(z, x)$ and $P(y, z)$ are defined analogously. See Figure 1 for an illustration.

We observe for future reference that the path $x, p(x, y)=a_{1}, a_{2}, \ldots, a_{k-2}$, $a_{k-1}=p\left(x^{\prime}, y^{\prime}\right), x^{\prime}$, when considered by itself, admits a list homomorphism both to $U$ and to $D$. (To see this invoke Zaslavsky's theorem characterizing switching equivalent signatures, and use the fact that both $U$ and $D$ contain a bicoloured edge.) Further we note there is no list homomorphism to any other subgraph of $U \cup D$ where $p(x, y)$ maps to $d_{1}$ and $p\left(x^{\prime}, y^{\prime}\right)$ maps to $u_{k-1}$. (This follows from the conditions in the definition of chain.)

If $x$ occurs in several clauses, we now have several vertices corresponding to $x$. We link all these occurrences by a new vertex $p(x)$ with the list $\left\{u_{1}\right\}$, and joined by blue edges to all the occurrences of $x$. Since the edge $u u_{1}$ is unicoloured, this will ensure that all occurrences of the vertex $x$ are switched or no occurrence of $x$ is switched.

We denote the resulting graph $(G, \sigma)$. We now claim that this instance of Not-All-Equal SAT is satisfiable if and only if $(G, \sigma)$ admits a list homomorphism to $(H, \pi)$.

Let $\widehat{G}(x, y, z)$ denote the subgraph of $(G, \sigma)$ induced by $P(x, y), P(z, x), P(y, z)$ and $x, y, z, x^{\prime}, y^{\prime}, z^{\prime}$. We claim that
(i) any list homomorphism of $\widehat{G}(x, y, z)$ to $U \cup D$ must switch at either one or two of the vertices $x, y, z$, and that
(ii) there are list homomorphisms of $\widehat{G}(x, y, z)$ to $U \cup D$ that switch at any one or any two of the vertices $x, y, z$.

Once this claim is proved, we can associate with every truth assignment a list homomorphism of $\widehat{G}(x, y, z)$ to $U \cup D$ where a vertex corresponding to a variable is switched if and only if that variable is true, and conversely, setting a variable true if its corresponding vertex was switched in the list homomorphism. Since we have ensured that all occurrences of a variable are switched or all are not switched, we conclude that all occurrences of a variable take on the same truth value.


Figure 1: The clause gadget for clause $(x \vee y \vee z)$ in Theorem 8

We now prove (i). Since the lists are so restrictive, any list homomorphism is fully described by what happens to the paths $P(x, y), P(z, x), P(y, z)$, and whether or not the vertices corresponding to $x, y, z$ (and $x^{\prime}, y^{\prime}, z^{\prime}$ ) are switched. Note that $U$ begins with a unicoloured edge and $D$ ends with a unicoloured edge. If neither $x$ nor $y$ or both $x$ and $y$ are switched, the edges $x p(x, y)$ and $p(x, y) y$ are different colours, and in any list homomorphism of $\widehat{G}(x, y, z)$ to $U \cup D$ we must map $P(x, y)$ to $D$. In particular, $x^{\prime} p\left(x^{\prime}, y^{\prime}\right)$ and $p\left(x^{\prime}, y^{\prime}\right) y^{\prime}$ map to a unicoloured edge and must have the same colour. Thus, if none or all of the vertices $x, y, z$ were switched, then the hexagon $x^{\prime}, p\left(x^{\prime}, y^{\prime}\right), y^{\prime}, p\left(y^{\prime}, z^{\prime}\right), z^{\prime}, p\left(z^{\prime}, x^{\prime}\right)$ has an even number of red and even number of blue edges, which is impossible. (It started with an odd number of each.)

For (ii), it remains to show that one or two of the vertices $x, y, z$ can be switched under a list homomorphism of $\widehat{G}(x, y, z)$ to $U \cup D$. Suppose first that only one was switched; by symmetry assume it was $x$ (so $y$ and $z$ were not switched). Now edges $x, p(x, y)$ and $p(x, y), y$ have the same colour, and $z, p(z, x)$ and $p(z, x), x$ have the same colour. By the above observation, we can map $P(x, y)$ and $P(z, x)$ to $U$, and map $P(y, z)$ to $D$. Note that the switchings necessary for these list homomorphisms affect disjoint sets of vertices (the paths $P(x, y), P(y, z), P(z, x))$, so the observation applies. If two vertices, say $y$ and $z$ were switched, the argument is almost the same and we omit it.

## 5. Irreflexive trees

In this section, $\widehat{H}$ will always be an irreflexive tree. As trees do not have any cycles, $\widehat{H}$ is trivially uni-balanced, and hence we may assume that all edges are at least blue.

Lemma 9. If the underlying graph $H$ contains the graph $F_{1}$ in Figure 2, then $\operatorname{List-S-Hom}(\widehat{H})$ is NP-complete.

Proof. If the underlying graph $H$ contains the graph $F_{1}$ in Figure 2, then $H$ is not a bi-arc graph by [12], whence List-S- $\operatorname{Hom}(\widehat{H})$ is NP-complete by the remarks following Theorem 5

Lemma 10. If $\widehat{H}$ contains one of the signed graphs in family $\mathcal{F}$ from Figure 3 as an induced subgraph, then List-S- $\operatorname{Hom}(\widehat{H})$ is $N P$-complete


Figure 2: The subgraph $F_{1}$.


Figure 3: The family $\mathcal{F}$ of signed irreflexive trees with NP-complete problems.

Proof. For each signed tree in the family $\mathcal{F}$ we specify a chain either directly in Figure 3, or (in case a)) indirectly by reference to a more general situation addressed in Figure 9, By Theorem 8, these signed trees yield NP-complete list homomorphism problems. Thus any signed graph $\widehat{H}$ that contains one of them as an induced subgraph has also the problem List-S-HOM $(\widehat{H})$ NP-complete.

An irreflexive tree $H$ is a 2 -caterpillar if it contains a path $P=v_{1} v_{2} \ldots v_{k}$, such that each vertex of $H$ is either on $P$, or is a child of a vertex on $P$, or is a grandchild of a vertex on $P$, i.e., is adjacent to a child of a vertex on $P$. We also say that $H$ is a 2 -caterpillar with respect to the spine $P$. (Note that the same tree $H$ can be a 2-caterpillar with respect to different spines $P$.) In such a situation, let $T_{1}, T_{2}, \ldots, T_{\ell}$ be the connected components of $H \backslash P$. Each $T_{i}$ is a star adjacent to a unique vertex $v_{j}$ on $P$. The tree $T_{i}$ together with the edge joining it to $v_{j}$ is called a rooted subtree of $H$ (with respect to the spine $P)$, and is considered to be rooted at $v_{j}$. Note that there can be several rooted subtrees with the same root vertex $v_{j}$ on the spine, but each rooted subtree at $v_{j}$ contains a unique child of $P$ (and possibly no grandchildren, or possibly several grandchildren).

We also use the term 2-caterpillar for any signed graph to mean that the underlying graph, with loops removed, is a 2 -caterpillar.

If $H$ is a 2-caterpillar with respect to the spine $P$, and additionally the bicoloured edges of $\widehat{H}$ form a connected subgraph, and there exists an integer $d$, with $1 \leq d \leq k$, such that:

- all edges on the path $v_{1} v_{2} \ldots v_{d}$ are bicoloured, and all edges on the path $v_{d} v_{d+1} \ldots v_{k}$ are blue,
- the edges of all subtrees rooted at $v_{1}, v_{2}, \ldots, v_{d-1}$ are bicoloured, except
possibly edges incident to leaves, and
- the edges of all subtrees rooted at $v_{d+1}, \ldots, v_{k}$ are all blue,
then we call $\widehat{H}$ a good 2-caterpillar with respect to $P=v_{1} v_{2} \ldots v_{k}$.
The vertex $v_{d}$ is called the dividing vertex of $\widehat{H}$. Note that the subtrees rooted at $v_{d}$ are not limited by any condition except the connectivity of the subgraph formed by the bicoloured edges. A typical example of a good 2-caterpillar is depicted in Figure 4

Lemma 11. Let $\widehat{H}$ be an irreflexive signed tree. Then $\widehat{H}$ is a good 2-caterpillar if and only if it does not contain any of the graphs from family $\mathcal{F}$ in Figure 3 as an induced subgraph, and the underlying graph $H$ does not contain the graph $F_{1}$ in Figure 2.

Proof. It is easy to check that none of the depicted signed graphs admits a suitable spine, and hence they are not good 2-caterpillars. So assume $\widehat{H}$ does not contain any of the graphs in Figure 3 as an induced subgraph, and the underlying graph $H$ does not contain the graph $F_{1}$ in Figure 2 as a subgraph. If $H$ is not a 2-caterpillar with respect to any spine, then the underlying graph $H$ contains the tree in Figure 2. The bicoloured edges of $\widehat{H}$ induce a connected subgraph, since there is no subgraph of type a) from Figure 3 . Similarly, the unicoloured edges between two non-leaf vertices of $\widehat{H}$ induce a connected subgraph, since there is no subgraph of type b) in Figure 3. Let $\widehat{H^{\prime}}$ denote the signed tree obtained from $\widehat{H}$ by removing all vertices that are leaves incident with a unicoloured edge. Then we can conclude that there is a vertex $v_{d}$ that separates unicoloured and bicoloured edges in $\widehat{H^{\prime}}$.

The absence of classes b) and c) from Figure 3 also ensures that, there is a suitable spine $P=v_{1} v_{2} \ldots v_{k}$ with bicoloured edges on $v_{1} v_{2} \ldots v_{d}$ and blue edges on $v_{d} v_{d+1} \ldots v_{k}$, and with all subtrees of height two rooted at $v_{1}, \ldots, v_{d-1}$ attached to $P$ with a bicoloured edge. (For this, we note that in the case c), as long as the edges 34 and 45 are bicoloured, the subtree still yields an NPcomplete problem even with any of the edges $12,23,56,67$ unicoloured.) The absence of graphs d) and e) in Figure 3 ensures that all subtrees rooted at $v_{d+1}, \ldots, v_{k}$ have all edges blue.

Theorem 12. Let $\widehat{H}$ be an irreflexive tree. If $\widehat{H}$ is a good 2-caterpillar, then List-S-Hom $(\widehat{H})$ is polynomial-time solvable. Otherwise, $H$ contains a copy of $F_{1}$, or $\widehat{H}$ contains one of the signed graphs in family $\mathcal{F}$ as an induced subgraph, and the problem is NP-complete.


Figure 4: An example of a good 2-caterpillar.

The second claim follows from Lemmas 9,10 and 11 . We prove the first claim in a sequence of lemmas. Suppose that $H$ is a good 2-caterpillar with respect to the spine $P=v_{1} v_{2} \ldots v_{k}$. Since $H$ is bipartite, we may distinguish its vertices as black and white. We may assume that the input signed graph $\widehat{G}$ is connected and bipartite, and the lists of the black vertices of $G$ contain only black vertices of $H$, and similarly for white vertices. (Since $G$ is connected, there are only two possible assignments of black and white colours to its vertices, and we consider each separately.)

We distinguish four types of rooted subtrees of $\widehat{H}$ with respect to the spine $P$.

- Type $T_{1}$ : a bicoloured edge $v_{i} x$;
- Type $T_{2}$ : a bicoloured edge $v_{i} x$, bicoloured edges $x z_{j}$ for a set of vertices $z_{j}$, and blue edges $x t_{j}$ for another set of vertices $t_{j}$;
- Type $T_{3}$ : a blue edge $v_{i} x$ and blue edges $x t_{j}$ for a set of vertices $t_{j}$; and
- Type $T_{4}$ : a blue edge $v_{i} x$.

In the types $T_{2}$ and $T_{3}$ we assume that they are not of type $T_{1}$ or $T_{4}$, i.e., that at least some $z_{j}$ or $t_{j}$ exist; but we allow in $T_{2}$ either the set of $z_{j}$ or the set of $t_{j}$ to be empty.

Recall that we assume that all edges of $\widehat{H}$ are at least blue. Since the underlying graph $H$ is bipartite, we have also distinguished its vertices as black and white; we assume that $v_{1}$ is white.

A bipartite min ordering of the bipartite graph $H$ is a pair $<_{b},<_{w}$, where $<_{b}$ is a linear ordering of the black vertices and $<_{w}$ is a linear ordering of the white vertices, such that for white vertices $x<_{w} x^{\prime}$ and black vertices $y<_{b} y^{\prime}$, if $x y^{\prime}, x^{\prime} y$ are both edges in $H$, then $x y$ is also an edge in $H$. It is known [15] that if a bipartite graph $H$ has a bipartite min ordering, then the list homomorphism problem for $H$ can be solved in polynomial time as follows. First apply the arc consistency test, which repeatedly visits edges $x y$ and removes from $L(x)$ any vertex of $H$ not adjacent to some vertex of $L(y)$, and similarly removes from $L(y)$ any vertex of $H$ not adjacent to some vertex of $L(x)$. After arc consistency, if there is an empty list, no list homomorphism exists, and if all lists are non-empty, choosing the minimum element of each list, according to $<_{b}$ or $<_{w}$, defines a list homomorphism as required. We call a bipartite min ordering of the signed irreflexive tree $\widehat{H}$ special if for any black vertices $x, x^{\prime}$ and white vertices $y, y^{\prime}$, if $x y$ is bicoloured and $x y^{\prime}$ is blue, then $y<_{w} y^{\prime}$, and if $x y$ is bicoloured and $x^{\prime} y$ is blue, then $x<_{b} x^{\prime}$. In other words, the bicoloured neighbours of any vertex appear before its unicoloured neighbours, both in $<_{b}$ and in $<_{w}$.

Lemma 13. Every good 2-caterpillar $\widehat{H}$ admits a special bipartite min ordering.
Proof. Let us first observe that any 2-caterpillar admits a bipartite min ordering $<_{b},<_{w}$ with $v_{1}<_{w} v_{3}<_{w} v_{5}, \ldots$ and $v_{2}<_{b} v_{4}<_{b} v_{6}, \ldots$, in which the
vertices of each subtree rooted at a vertex $v_{i}$ are placed as follows: all non-leaf children of $v_{i}$, as well as all leaf children of $v_{i}$ adjacent to $v_{i}$ by bicoloured edges, are ordered between $v_{i-1}$ and $v_{i+1}$, all leaf children of $v_{i}$ adjacent to $v_{i}$ by unicoloured edges are ordered between $v_{i+1}$ and $v_{i+3}$, and all grandchildren of $v_{i}$ are ordered between $v_{i}$ and $v_{i+2}$. Moreover, we ensure that the order of the grandchildren conforms to the order of the children, i.e., if a child $a$ of $v_{i}$ is ordered before a child $b$ of $v_{i}$ then the children of $a$ are all ordered before the children of $b$. Finally, all children of $v_{i}$ are ordered after all the grandchildren of $v_{i-1}$. See Figure 5 for an illustration.

It remains to ensure that the bipartite min ordering we choose is in fact a special bipartite min ordering, i.e., that each vertex has its neighbours joined by bicoloured edges ordered before its neighbours joined by unicoloured edges. Therefore the subtrees rooted at each $v_{i}$ are handled as follows. We will order first the vertices of subtrees of type $T_{1}$, one at a time, then order the vertices of subtrees of type $T_{2}$, one at a time, then the vertices of subtrees of type $T_{3}$, one at a time, and finally the vertices of subtrees of type $T_{4}$, one at a time. Each subtree of type $T_{1}$ consists of only one bicoloured edge, and we order these consecutively between $v_{i-1}$ and $v_{i+1}$. Next in order will come the children of $v_{i}$ in subtrees of type $T_{2}$, still before $v_{i+1}$, and in each of these subtrees we order first the grandchildren of $v_{i}$ incident to a bicoloured edge before those incident to a unicoloured edge. We order the subtrees of type $T_{3}$ similarly. Note that by the definition of a good 2-caterpillar, the subtrees of type $T_{3}$ can only be rooted at vertices $v_{i}$ with $d \leq i \leq k$. Thus, if we have a blue child of $v_{i}$ ordered before $v_{i+1}$, then $v_{i} v_{i+1}$ is unicoloured. Finally, for subtrees of type $T_{4}$, we order their vertices (each a child of $v_{i}$ ) right after $v_{i+1}$.

Lemma 14. If a signed irreflexive tree $\widehat{H}$ admits a special bipartite min ordering, then List-S-Hom $(\widehat{H})$ is polynomial-time solvable.

Proof. We describe a polynomial-time algorithm. Suppose $\widehat{G}$ is the input signed graph; we may assume $\widehat{G}$ is connected, bipartite, and such that the black vertices have lists with only the black vertices of $\widehat{H}$, and similarly for the white vertices. The first step is to perform the arc consistency test for the existence of a homomorphism of the underlying graphs $G$ to $H$, using the special bipartite min ordering $<_{b},<_{w}$. We also perform the bicoloured arc consistency test, which repeatedly visits bicoloured edges $x y$ of $G$ and removes from $L(x)$ any vertex of $H$ not adjacent to some vertex of $L(y)$ by a bicoloured edge, and similarly


Figure 5: An example of special bipartite min ordering.
removes from $L(y)$ any vertex of $H$ not adjacent to some vertex of $L(x)$ by a bicoloured edge. If this yields an empty list, there is no list homomorphism of the underlying graphs, and hence no list homomorphism of signed graphs. Otherwise, the minima of all lists define a list homomorphism $f: G \rightarrow H$ of the underlying graphs, by [15]. By the bicoloured arc consistency test, the minimum choices imply that the image of a bicoloured edge under $f$ is also a bicoloured edge. According to Lemma 2 and the remark following it, $f$ is also a list homomorphism of signed graphs unless a negative cycle $C$ of unicoloured edges of $\widehat{G}$ maps to a closed walk $f(C)$ of blue edges in $\widehat{H}$. Now we make use of the properties of special bipartite min ordering to repair the situation, if possible. Note that the fact that we choose minimum possible values for $f$ means that we cannot map $C$ lower in the orders $<_{b},<_{w}$. We consider three possible cases.

- At least one of the edges of $f(C)$ is in a subtree $T$ of type $T_{2}$ rooted at some $v_{i}$, with $i \leq d$ :
In this case, all edges of $f(C)$ must be in $T$, since the edge of $T$ incident to $v_{i}$ is bicoloured. Assume without loss of generality that $v_{i}$ is white, $x$ is the unique child of $v_{i}$ in $T$, and $x t_{1}, \ldots, x t_{m}$ are the blue edges of $T$, where $x$ is black and $t_{1}, \ldots, t_{m}$ are white. Since $f(C)$ is included in the edges $x t_{1}, \ldots, x t_{m}$ and $v_{i}$ precedes in $<_{w}$ all vertices $t_{1}, \ldots, t_{m}$, the final lists of the white vertices in $C$ do not include $v_{i}$ (since we assigned the minimum value in each list). Therefore under any homomorphism the image of the connected graph $C$ either is included in the set of edges $x t_{1}, \ldots, x t_{m}$, or is disjoint from this set of edges. Since we have already explored the first possibility, we can delete the vertices $t_{1}, \ldots, t_{m}$ from the lists of all white vertices of $C$ and repeat the arc consistency test. This will check whether there is possibly another list homomorphism of graphs $G \rightarrow H$, which is also a homomorphism of signed graphs $\widehat{G} \rightarrow \widehat{H}$.
- At least one of the edges of $f(C)$ is in a subtree of type $T_{4}$ rooted at some $v_{i}, i \leq d-1$ :
In this case, all edges of $f(C)$ must be in subtrees of type $T_{4}$ rooted at the same $v_{i}$. Assume again, without loss of generality, that $v_{i}$ is white and the subtrees consist of the blue edges $v_{i} x_{1}, v_{i} x_{2}, \ldots, v_{i} x_{m}$, with each $x_{j}$ black. Since $<_{b},<_{w}$ is a special bipartite min ordering, all vertices adjacent to $v_{i}$ by a bicoloured edge are smaller in $<_{b}$ than $x_{1}, \ldots, x_{m}$. Therefore no such vertex can be in a list of a black vertex in $C$. This again means that the image of $C$ is either included in the set of edges $v_{i} x_{1}, v_{i} x_{2}, \ldots, v_{i} x_{m}$, or is disjoint from this set of edges. We can delete all vertices $x_{1}, \ldots, x_{m}$ from the lists of all black vertices of $C$ and repeat as above.
- The edges of $f(C)$ are included in the set of edges on the path $v_{d} v_{d+1} \ldots v_{k}$ and in the subtrees of types $T_{3}$ or $T_{4}$ rooted at $v_{d}, \ldots, v_{k}$ :
In this case, the vertices in the lists of the cycle $C$ are joined only by blue edges, and there is no homomorphism of signed graphs $\widehat{G} \rightarrow \widehat{H}$.

After we modified the image of one negative cycle $C$ of $\widehat{H}$, we proceed to modify another, until we either obtain a homomorphism of signed graph, or find that no such homomorphism exists. The algorithm is polynomial, because arc consistency can be performed in linear time [15], and each modification removes at least one vertex of $H$ from the list of at least one vertex of $G$. Recall that the graph $H$ is fixed, and hence its number of vertices is a constant $k$. If $G$ has $n$ vertices, then this step will be performed at most $k n$ times.

## 6. Reflexive trees

We now turn to reflexive trees, and hence in this section, $\widehat{H}$ will always be a reflexive tree. We may have red, blue, or bicoloured loops, but we may again assume that all non-loop unicoloured edges are of the same colour (blue or red).

Lemma 15. If $\widehat{H}$ contains one of the reflexive trees from the family $\mathcal{G}$ in Figure 6 as an induced subgraph, then List-S- $\operatorname{HOM}(\widehat{H})$ is NP-complete.

Proof. The signed trees in a), b) and c) are themselves s-cores with more than two edges, so it follows from Theorem 5 that they yield NP-complete problems. The signed trees in d), f), g), and h) have chains indicated in Figure 6, and hence also yield NP-complete problems by Theorem 8. The remaining cases are again handled in more general context in the next section, as indicated in Figure 6

The next lemma is used to prove that in all polynomial cases $\widehat{H}$ is a caterpillar. Although this section is restricted to reflexive graphs, we will prove it in greater generality for future use in a later section. To that end let $F_{2}$ be the graph in Figure 7 where each loop on the three leaves may or may not be present. Thus, $F_{2}$ represents a family of graphs, but we will abuse notation and simply refer to $F_{2}$ as any member of that family.

Lemma 16. If the underlying graph $H$ contains the graph $F_{2}$ in Figure 7 , then the problem List-S-HOM $(\widehat{H})$ is NP-complete.

Proof. Deciding if there exists a list homomorphism (of an unsigned graph) to the graph $F_{2}$ is NP-complete, as stated in [12] (and proved using results in 11] and [13]). It would be natural to attempt a direct reduction of $\operatorname{LIST-HOM}\left(F_{2}\right)$ to $\operatorname{List}-\mathrm{S}-\operatorname{Hom}(\widehat{H})$, as we have done here for the proof of Lemma 9. However, this is complicated by the fact that the loops in $\widehat{H}$ can be red, blue, or bicoloured. Therefore, below we proceed on a different path, adapting to our setting the proof of the reflexive case from from [10] (see Theorem 2.3 in that paper).

Suppose that $\widehat{F}_{2}$ is a subgraph of $\widehat{H}$ with underlying graph $F_{2}$, and suppose that $\widehat{F}_{2}$ has been switched so that all non-loop edges are at least blue. Label
a) $\underset{1}{8} \quad \begin{gathered}i \\ 0\end{gathered}$
d)

e)

f)

g)

b)

c)

$U=2-2-3-2$ $D=2-1-2-2$

Special case of $g$ ) of family $\mathcal{J}$ in Figure 9. $k \geq 2$

Special case of f) of family $\mathcal{J}$ in Figure 9. $k \geq 3$
h)

i) $\begin{array}{llllll}8 & 8 & \ddots & 8 & 8 \\ 1 & 2 & 3 & 4 & 5\end{array}$

k)

1)

m)


$$
\begin{aligned}
& U=2-1-1-2-\ldots-(k-1) \\
& D=2-3-\ldots-k-k-(k-1) \\
& k \geq 4
\end{aligned}
$$

Special case of b) of family $\mathcal{J}$ in Figure 9.

Special case of c) of family $\mathcal{J}$ in Figure 9. Special case of k) of family $\mathcal{J}$ in Figure 9. Special case of h) of family $\mathcal{J}$ in Figure 9.
Special case of e) in family $\mathcal{J}$ in Figure 9. $k \geq 4$ Special case of d) of family $\mathcal{J}$ in Figure 9.

Figure 6: The family $\mathcal{G}$ of signed reflexive trees with NP-complete problems. (The dotted loops can be either blue, red or bicoloured.)


Figure 7: The subgraph $F_{2}$.
the leaves of $\widehat{F}_{2}$ by $0,1,2$, and their respective neighbours by $0^{+}, 1^{+}, 2^{+}$, and finally label the central vertex by $c$.

If all the unicoloured loops in $\widehat{F}_{2}$ are blue, then we may restrict the input to blue (there is no advantage to switching). The NP-complete problem $\operatorname{List}-\operatorname{Hom}\left(F_{2}\right)[12$ reduces to List-S- $\operatorname{Hom}(\widehat{H})$.

Similarly, if all unicoloured loops in $\widehat{F}_{2}$ are red, then we can switch all nonloop edges to red and apply the same logic. Thus we assume that there are both blue and red unicoloured loops in $\widehat{F}_{2}$.

We first prove that if some edge $c i^{+}, i \in\{0,1,2\}$ is not bicoloured, then $\operatorname{List-S-HOM}\left(\widehat{F}_{2}\right)$ is NP-complete by showing that the copy of $\widehat{F}_{2}$ contains a member of the family $\mathcal{G}$ or $\mathcal{J}$.

Note, any path between a blue loop and a red loop must have a vertex with a bicoloured loop; otherwise, $\widehat{F}_{2}$ contains a) or b) from family $\mathcal{G}$. Thus at least one of $c, 0^{+}, 1^{+}, 2^{+}$has a bicoloured loop.

Next, if none of the edges $\mathrm{Ci}^{+}$are bicoloured, then we either have a copy of c) from family $\mathcal{J}$ when there is a bicoloured loop at some $i^{+}$or a copy of b) when there is bicoloured loop at $c$. If one of the $c i^{+}$edges is bicoloured, then we have a copy of k) from family $\mathcal{J}$. Finally if two of the edges are bicoloured, then we have a copy of $h$ ) from family $\mathcal{J}$. (We note that the chain in h ) is applicable even if the edges 12 or 45 are unicoloured.) Thus, all edges $c i^{+}$are bicoloured.

We now finish the proof using a modification of the proof in [10]. Given distinct $i$ and $j$ in $\{0,1,2\}$ and distinct subsets $I$ and $J$ of $\{0,1,2\}$, an $(i, I, j, J)$ chooser is a path $\widehat{P}$ with endpoints $a$ and $b$, together with a list assignment $L$, such that the following statement holds. For each list homomorphism $f$ from $\widehat{P}$ to $\widehat{F}_{2}$, either $f(a)=i$ and $f(b) \in I$ or $f(a)=j$ and $f(b) \in J$. Moreover, for each $i^{\prime} \in I$ and $j^{\prime} \in J$, there are list homomorphisms $g_{1}, g_{2}$ from $\widehat{P}$ to $\widehat{F}_{2}$ such that $g_{1}(a)=i, g_{1}(b)=i^{\prime}$ and $g_{2}(a)=j, g_{2}(b)=j^{\prime}$.

Suppose $\widehat{P}$ is a $(0,\{0,1\}, 1,\{1,2\})$-chooser, $\widehat{P}^{\prime}$ is $(0,\{1,2\}, 1,\{2,0\})$-chooser, and $\widehat{P}^{\prime \prime}$ is a $(0,\{2,0\}, 1,\{0,1\})$-chooser. Let $\widehat{T}$ be the tree obtained by identifying the $b$ vertices in the three choosers and labelling the leaves respectively as $a, a^{\prime}, a^{\prime \prime}$. It is easy to verify that $\widehat{T}$ admits a list-homomorphism to $\widehat{F}_{2}$ if, and only if, the triple $\left(a, a^{\prime}, a^{\prime \prime}\right)$ does not map to either $(0,0,0)$ or $(1,1,1)$. Consequently, we can reduce an instance of Not-All-Equal SAT to List-S-Hom $\left(\widehat{F}_{2}\right)$. For each clause in the instance, create a copy of $\widehat{T}$ and identify the vertices ( $a, a^{\prime}, a^{\prime \prime}$ ) with the three literals in the clause.

It remains to construct the choosers. First, we build a ( $0,\{0,2\}, 1,\{1,2\}$ )chooser. By symmetry we then have $(i,\{i, k\}, j,\{j, k\})$-choosers for any distinct $i, j, k \in\{0,1,2\}$. Let $Q$ be a path on $q_{0}, q_{1}, \ldots, q_{10}$ with lists

$$
\begin{array}{ll}
L\left(q_{0}\right)=\{0,1\} & L\left(q_{6}\right)=\left\{0^{+}, 2^{+}, 1\right\} \\
L\left(q_{1}\right)=\left\{0^{+}, 1^{+}\right\} & L\left(q_{7}\right)=\left\{0^{+}, c, 1^{+}\right\} \\
L\left(q_{2}\right)=\left\{0,1^{+}\right\} & L\left(q_{8}\right)=\left\{0,2^{+}, 1\right\} \\
L\left(q_{3}\right)=\left\{0^{+}, c, 1^{+}\right\} & L\left(q_{9}\right)=\left\{0^{+}, 2^{+}, 1^{+}\right\} \\
L\left(q_{4}\right)=\left\{0,2^{+}, 1\right\} & L\left(q_{10}\right)=\{0,2,1\} \\
L\left(q_{5}\right)=\left\{0^{+}, 2,1^{+}\right\} &
\end{array}
$$

The path $\widehat{Q}$ has all edges blue. In mapping $\widehat{Q}$ to $\widehat{F_{2}}$ first suppose $q_{0}$ maps to 0 . Then $q_{10}$ either maps to 0 , in which case the loop $0^{+}$is traversed twice, or $q_{10}$ maps to 2 , in which case the loop at $0^{+}$and the loop at $2^{+}$are each traversed once. In the both cases if the loop at $0^{+}$is unicoloured red, then switch at $q_{6}$. In the latter case, if there is a red loop at $2^{+}$, then we switch at $q_{8}$. Note in the latter case the bicoloured edges $0^{+} c$ and $c 2^{+}$allow the edges $q_{6} q_{7}$ and $q_{7} q_{8}$ to be of either colour. A similar reasoning shows $\widehat{Q}$ can map to $\widehat{F_{2}}$ with $q_{0}$ mapping to 1 and $q_{10}$ mapping to either 1 or 2 but not to 0 . Thus $\widehat{Q}$ is a ( $0,\{0,2\}, 1,\{1,2\}$ )-chooser.

The $(0,\{0\}, 1,\{2\})$-chooser $\widehat{R}$ is a path with vertices $r_{0}, \ldots, r_{6}$ and lists

$$
\{0,1\},\left\{0^{+}, 1^{+}\right\},\left\{0,1^{+}\right\},\left\{0^{+}, c\right\},\left\{0,2^{+}\right\},\left\{0^{+}, 2^{+}\right\},\{0,2\}
$$

All edges are blue. When $\widehat{R}$ maps to the edge $00^{+}$, no switching is required as $00^{+}$is at least blue. When $\widehat{R}$ maps to the path $1,1^{+}, 1^{+}, c, 2^{+}, 2^{+}, 2$, switching at $r_{2}$ (respectively $r_{4}$ ) is required when there is a unicoloured red loop at $1^{+}$ (respectively $2^{+}$).

The required choosers are defined as follows. First, $\widehat{P}$ is the $(0,\{0\}, 1,\{2\})$ chooser followed by the $(0,\{0,1\}, 2,\{1,2\})$-chooser. Next $\widehat{P}^{\prime}$ is the concatenation of the $(0,\{0\}, 1,\{2\})$-chooser, the $(0,\{1\}, 2,\{2\})$-chooser, the $(1,\{1\}, 2,\{0\})$ chooser, and the $(1,\{1,2\}, 0,\{0,2\})$-chooser. Finally $\widehat{P}^{\prime \prime}$ is the concatenation of the $(0,\{2\}, 1,\{1\})$-chooser and the $(2,\{0,2\}, 1,\{0,1\})$-chooser.

A tree $H$ is a caterpillar if it contains a path $P=v_{1} \ldots v_{k}$ such that each vertex of $H$ is on $P$ or is adjacent to a vertex of $P$. Note that the path $P$, which we again call the spine of $H$, is not unique, and we sometimes make it explicit by saying that $H$ is a caterpillar with spine $P$. A vertex $x$ not on $P$ is adjacent to a unique neighbour $v_{i}$ on $P$, and we call the edge $v_{i} x$ (with the loop at $x$ ) the subtree rooted at $v_{i}$. A vertex on the spine can have more than one subtree rooted at it. We say that a signed graph $\widehat{H}$ whose underlying graph $H$ is a reflexive caterpillar is a good caterpillar with respect to the spine $v_{1} \ldots v_{k}$ if the bicoloured edges of $\widehat{H}$ form a connected subgraph, the unicoloured non-loop edges all have the same colour $c$, and there exists an integer $d$, with $1 \leq d \leq k$, such that

- all edges on the path $v_{1} v_{2} \ldots v_{d}$ are bicoloured, and all edges on the path $v_{d} v_{d+1} \ldots v_{k}$ are unicoloured with colour $c$,
- all loops at the vertices $v_{1}, \ldots, v_{d-1}$ and all non-loop edges of the subtrees rooted at these vertices are bicoloured,
- all loops at the vertices $v_{d+1}, \ldots, v_{k}$ and all edges and loops of the subtrees rooted at these vertices are unicoloured with colour $c$,
- if $v_{d}$ has a bicoloured loop, then all children of $v_{d}$ with bicoloured loops are adjacent to $v_{d}$ by bicoloured edges,


Figure 8: Two good caterpillars with preferred colour blue: with $d<k$ (left), with $d=k$ (right).

- if $v_{d}$ has a unicoloured loop of colour $c$, then all children of $v_{d}$ have unicoloured loops of colour $c$, and are adjacent to $v_{d}$ by unicoloured edges, and
- if $d<k$, then the loops of all children of $v_{d}$ adjacent to $v_{d}$ by unicoloured edges also have colour $c$.
The vertex $v_{d}$ will again be called the dividing vertex. We also say that $\widehat{H}$ is a good caterpillar with preferred colour c. Figure 8 (on the left) shows an example of good caterpillar with preferred colour blue. We emphasize that in the case $d=k$ (depicted in Figure 8 on the right), it is possible (if $v_{d}$ has a bicoloured loop) that $v_{d}$ has some children with red loops and some with blue loops, adjacent to $v_{d}$ by unicoloured edges.

Let $\mathcal{G}$ be the family of signed graphs depicted in Figure 6, together with the family of complementary signed graphs where all unicoloured edges and loops are red, rather than blue, and vice versa. Note that the complementary signed graphs are not switching equivalent to the original signed graphs because switching does not change the colour of loops.

Lemma 17. Let $\widehat{H}$ be a reflexive signed tree. Then $\widehat{H}$ is a good caterpillar if and only if it does not contain any of the graphs in the family $\mathcal{G}$ as an induced subgraph, and the underlying graph $H$ does not contain the graph $F_{2}$.

Proof. It is easy to see that none of the signed reflexive trees in Figure 6 is a good caterpillar. By symmetry, the same is true for their complementary signed graphs. It is also clear that the graph $F_{2}$ (from Figure 7) is not a caterpillar. We proceed to show that if the signed reflexive trees from family $\mathcal{G}$ in Figure 6 are excluded as induced subgraphs, then $\widehat{H}$ is a good caterpillar with preferred colour blue. (The complementary exclusions produce a good caterpillar with preferred colour red.) Since the graphs g) are absent, the bicoloured non-loop edges induce a connected subgraph. The exclusion of family h) similarly ensures that all unicoloured non-loop edges induce a connected subgraph. By grouping all bicoloured non-loop edges before all unicoloured non-loop edges, we conclude that there exists a spine $P=v_{1} \ldots v_{k}$, and a dividing vertex $v_{d}$. Thus, all edges between $v_{1}, \ldots, v_{d-1}$, and (since l) is excluded) all edges to their children, are bicoloured. The exclusion of b) and c) ensures each bicoloured non-loop edge has a bicoloured loop on (at least) one of its endpoints. Forbidding the family d) ensures the vertices $v_{1}, \ldots, v_{d-1}$ all have bicoloured loops.

The subgraph induced by $v_{d+1}, \ldots, v_{k}$ and their children must contain only blue edges since the edge $v_{d} v_{d+1}$ is blue and the bicoloured edges induce a connected subgraph. Forbidding a), e), and f) implies all the loops in this subgraph are also blue. (Recall that when we say blue we always mean unicoloured blue.)

Now we distinguish two cases. If $v_{d}$ has a blue loop then by excluding families a), b), c) and d) we conclude that all edges to its children are blue and all loops of its children are also blue. In the case $v_{1}=v_{d}$, if there is a bicoloured loop on exactly one leaf of $v_{1}$ (respectively $v_{k}$ ), we renumber the vertices so that this leaf becomes the first vertex of the spine, $v_{1}$. (If it was a leaf of $v_{1}$, this involves a small shift of subscripts, if it was a leaf of $v_{k}$, it also involves a reversal of the ordering of subscripts.)

Now suppose that $v_{d}$ has a bicoloured loop. Excluding family e) ensures that any child of $v_{d}$ with a bicoloured loop must be adjacent to $v_{d}$ by a bicoloured edge.

Finally if $d<k$, case m) implies that we can choose the spine so that no child of $v_{d}$ has a red loop.

Families i), j), and k) ensure when there is a single bicoloured loop or a single bicoloured non-loop edge, the spine can be chosen to begin with this loop or edge.

Theorem 18. Let $\widehat{H}$ be a reflexive tree. If $\widehat{H}$ is a good caterpillar, then the problem List-S-Hom $(\widehat{H})$ is polynomial-time solvable. Otherwise, $H$ contains $F_{2}$ from Figure 7 , or $\widehat{H}$ contains one of the signed graphs in family $\mathcal{G}$ as an induced subgraph, and the problem is NP-complete.

Suppose that $\widehat{H}$ is not a good caterpillar. If $H$ is not a caterpillar, then it contains $F_{2}$ from Figure 7, and the problem is NP-complete by Lemma 16 Otherwise, $\widehat{H}$ contains an induced subgraph from $\mathcal{G}$, and the problem is NPcomplete by Lemma 15 .

We prove the first statement. Thus assume that $\widehat{H}$ is a good caterpillar, with spine $v_{1} \ldots v_{k}$ and dividing vertex $v_{d}$. By symmetry, we may assume it is a good caterpillar with preferred colour blue. We distinguish three types of rooted subtrees.

- Type $T_{1}$ : a bicoloured edge $v_{i} x$ with a bicoloured loop on $x$;
- Type $T_{2}:$ a bicoloured edge $v_{i} x$ with a unicoloured loop on $x$;
- Type $T_{3}$ : a blue edge $v_{i} x$ with a unicoloured loop on $x$.

There is a general version of min ordering we can use in this context. A min ordering of a graph $H$ is a linear ordering $<$ of the vertices of $H$, such that for vertices $x<x^{\prime}, y<y^{\prime}$, if $x y^{\prime}, x^{\prime} y$ are both edges in $H$, then $x y$ is also an edge in $H$. It is again the case that if a graph $H$ admits a min ordering, then the list homomorphism problem for $H$ can be solved in polynomial time by arc consistency followed by making the minimum choice in each list 15. Suppose again that $\widehat{H}$ is a good caterpillar with spine $v_{1} \ldots v_{k}$ and preferred colour blue.

A special min ordering of $\hat{H}$ is a min ordering of the underlying graph $H$ such that for any vertices $v_{i}, x, x^{\prime}$ with edges $v_{i} x, v_{i} x^{\prime}$ we have $x<x^{\prime}$ if

- the edge $v_{i} x$ is bicoloured and the edge $v_{i} x^{\prime}$ is blue, or
- $x$ has a bicoloured loop and $x^{\prime}$ a unicoloured loop, or
- $x$ has a blue loop and $x^{\prime}$ has a red loop.

Lemma 19. Every good caterpillar $\widehat{H}$ admits a special min ordering.
Proof. It is again easy to see that the ordering $v_{1}<v_{2}<\ldots<v_{k}$ of $V(\widehat{H})$ in which the children of each $v_{i}$ are ordered between $v_{i}$ and $v_{i+1}$ is a min ordering of the underlying graph $H$. We may again assume that $\widehat{H}$ has preferred colour blue. To ensure that $<$ is a special min ordering of $\widehat{H}$, we make sure that after each vertex $v_{i}$ with $i=1,2, \ldots, d-1$, we first list the leaves of subtrees of type $T_{1}$, then the leaves of subtrees of type $T_{2}$ with blue loop, and last the leaves of subtrees of type $T_{2}$ with red loop. If $d=k$, then we proceed the same way also after $v_{d}$, and then we list the leaves of subtrees of type $T_{3}$ with blue loop, and last the leaves of subtrees of type $T_{3}$ with red loop. If $d<k$, we list after $v_{d}$ first the leaves of subtrees of type $T_{1}$, then the leaves of subtrees of type $T_{2}$ with blue loop, then the leaves of subtrees of type $T_{2}$ with red loop, and last the leaves of subtrees of type $T_{3}$. For vertices $v_{i}, i>d$, there are only subtrees of type $T_{3}$, and their leaves can be listed in any order.

We now describe our polynomial-time algorithm. As in the irreflexive case, we first perform the arc consistency test to check for the existence of a homomorphism of the underlying graphs $(G$ to $H)$. Then we also perform the bicoloured arc consistency test. If we obtain an empty list, there is no list homomorphism. Otherwise, taking again the minima of all lists (in the special min ordering $<$ ) defines a list homomorphism $f: G \rightarrow H$ of the underlying graphs by [15], and again by bicoloured arc consistency test we have that $f$ maps bicoloured edges of $\widehat{G}$ to bicoloured edges of $\widehat{H}$. Therefore, by Lemma 2 and the remarks following it, $f$ is also a list homomorphism of the signed graphs $\widehat{G} \rightarrow \widehat{H}$, unless a negative cycle $C$ of unicoloured edges of $\widehat{G}$ maps to a positive closed walk $f(C)$ of unicoloured edges in $\widehat{H}$, or a positive cycle $C$ of unicoloured edges of $\widehat{G}$ maps to a negative closed walk $f(C)$ of unicoloured edges in $\widehat{H}$. The minimum choices in all lists imply that no vertex $x$ of $C$ can be mapped to an image $y$ with $y<f(x)$. We proceed to modify the images of such cycles $C$ one by one, in the order of increasing smallest vertex in $f(C)$ (in the ordering $<$ ), until we either obtain a homomorphism of signed graphs, or we find that no such homomorphism exists.

Let $w$ be the leaf of the last subtree of type $T_{2}$ rooted at $v_{d}$ (we let $w=v_{d}$ if $v_{d}$ has no subtree of type $T_{2}$ ). We note that if $d<k$, then all edges and loops amongst the vertices that follow $w$ in $<$ are blue, by the properties of a special min ordering. Also note that since the edges of $f(C)$ are unicoloured, they do not include a bicoloured loop on $v_{d}$ (if there is one). We distinguish three possible cases.

- At least one vertex $y$ of $f(C)$ satisfies $y \leq w$ :

The only unicoloured closed walks including $y$ are (red or blue) loops, so $f$ maps the entire cycle $C$ to $y$. As in the reflexive case, we may remove $y$ from all lists of vertices of $C$ and continue seeking a better homomorphism of the underlying graphs ( $G$ to $H$ ).

- All vertices of $f(C)$ except for $v_{d}$ follow $w$ in the order $<$ and $d<k$, or $d=k$ and $v_{d}$ does not have a subtree of Type $T_{3}$ with red loop:
In this case $C$ is a negative cycle of unicoloured edges. The subgraph of $\widehat{H}$ induced by the vertices after $w$ (in the order $<$ ) has only blue edges and loops. Thus there is no homomorphism of signed graphs mapping $\widehat{G} \rightarrow \widehat{H}$.
- All vertices of $f(C)$ except for $v_{d}$ follow $w$ in the order $<, d=k$ and $v_{d}$ has a subtree of Type $T_{3}$ with red loop:
In this case a fairly complex situation may arise because $f(C)$ can be a closed walk using both red and blue loops, along with blue edges; see below.

We now consider the final case in detail. Since $f$ chooses minimum possible values of images (under $<$ ), we could only modify $f$ by mapping some vertices of $C$ that were taken by $f$ to a vertex with a blue loop, to vertex with a red loop instead, if lists allow it. We show how to reduce this problem to solving a system of linear equations modulo two, which can then be solved in polynomial time by (say) Gaussian elimination. We begin by considering the pre-image (under $f$ ) of all vertices in the subtrees of type $T_{3}$ rooted at $v_{d}$. We denote by $P$ the set of vertices $v \in V(G)$ with $f(v)$ equal to a vertex with a blue loop and by $N$ the set of vertices $v \in V(G)$ with $f(v)$ equal to a vertex with a red loop. We say that a vertex $x$ of $G$ is a boundary point if $f(x)=v_{d}$. The set of boundary points is denoted by $B$. Thus the pre-image of the subtrees of type $T_{3}$ rooted at $v_{d}$ is the disjoint union $B \cup P \cup N$. We now focus on the subgraph $\widehat{G}^{\prime}$ of $\widehat{G}$ induced by $B \cup P \cup N$. A region is a connected component of $\widehat{G}^{\prime} \backslash B$ together with all its boundary points, i.e. between any pair of vertices in a region there is a path with no boundary point as an internal vertex.

Given a region $r$ and boundary points $x$ and $y$ (not necessarily distinct), we construct (possibly several) boolean equations on the corresponding variables, using the same symbols $x, y$, and $r$. The variables $x, y$ indicate whether or not the corresponding boundary vertices $x$ and $y$ should be switched before mapping them with $f$ (true corresponds to switching), and the variable $r$ indicates whether the region $r$ will be mapped by $f$ to a blue loop or a red loop (true corresponds to a blue loop). The equations depend of the parity and the sign of walks between the two vertices. If $c$ and $d$ denote parities (even or odd), we say a walk $W$ from $x$ to $y$ in $\widehat{G}^{\prime}$ is a $(c, d)$-walk if it contains no boundary points other than $x$ and $y$, the parity of the number of blue edges in $W$ is $c$, and the parity of the number of red edges in $W$ is $d$. The equations generated by the $(c, d)$-walks are as follows.

- (odd,odd)-walk: We add the equation $x=y+1$. This ensures that exactly one of the boundary vertices has to be switched, in particular $x$ and $y$
must be distinct. The image of the walk must be uni-balanced or anti-uni-balanced (as the whole walk maps to exactly one subtree of type $T_{3}$ ). A walk with an even number of edges but an odd number of red edges is neither. However, if we switch at exactly one of the endpoints, we can freely map all of the non-boundary points to a blue loop or a red loop.
- (even,even)-walk: We add the equation $x=y$. The reasoning is similar to the previous case.
- (odd,even)-walk: We add the equation $x=y+r+1$. The image of the walk is a closed walk with an odd number of edges and positive sign. Thus if both or neither of $x$ and $y$ are switched, then the walk remains positive and $r=1$. Conversely, switching exactly one of $x$ or $y$ makes the walk negative, and $r=0$.
- (even,odd)-walk: We add the equation $x=y+r$. The argument is analogous to the previous case.

It is possible that there are several kinds of walks between the same $x, y$, but we only need to list one of each kind, so the number of equations is polynomial in the size of $G$. A simple labelling procedure can be used for determining which kinds of walks exist, for given boundary points $x$ and $y$ and a region $r$. We start at the vertex $x$, and label its neighbours $n_{x}$ by the appropriate pairs $(c, d)$, determined by the signs of the edges $x n_{x}$. Once a vertex is labelled by a pair $(c, d)$, we correspondingly label its neighbours; a vertex is only given a label $(c, d)$ once even if it is reached with that label several times. Thus a vertex has at most four labels. Any time a vertex receives a new label its neighbours are checked again. The process ends in polynomial time (in the size of the region) as each edge of the region is traversed at most four times. The result is inherent in the labels obtained by $y$.

Finally, for each region we examine the connected component of the nonboundary vertices. Since the arc consistency procedure was done in the first step of the algorithm, all lists of non-boundary points for a given region are the same. Also, by the ordering $<$, these lists must only contain leaves of $v_{d}$. Thus, the non-boundary vertices of the region must map to a single loop. We ensure the choice of the loop is consistent with the lists of each region. If the lists of vertices of some region do not contain a vertex with a red loop, then we add the equation $r=1$ for the region. Similarly, if the lists do not contain a blue loop, then we add the equation $r=0$.

Such a system of boolean linear equations can be solved in polynomial time. Also, the system itself is of polynomial size measured by the size of $\widehat{G}$. This completes the proof.

## 7. General trees

In this section we handle signed trees $\widehat{H}$ in general, i.e., trees in which some vertices have loops while others do not. In homomorphism problems, reflexive
and irreflexive bipartite target graphs $H$ tend to share some similarities, cf. e.g. [1, 10, 11, and also both tend to be simpler. For instance, the general version of list homomorphisms for graphs with possible loops [12] is significantly more involved than both the reflexive and irreflexive bipartite cases [10, 11]. Similarly, considering general signed trees with possible loops introduces an additional level of difficulty.

To simplify the descriptions, we assume, without loss of generality, that all non-loop unicoloured edges are blue, unless noted otherwise. In Figure 9 we introduce our main NP-complete cases.

We first focus on signed trees $\widehat{H}$ without bicoloured non-loop edges. If there are no bicoloured loops either, then Theorem 6 implies that List-S-Hom $(\widehat{H})$ is NP-complete when $\widehat{H}$ has both a red loop and a blue loop, or when the underlying graph is not a bi-arc tree. We now introduce NP-complete cases when bicoloured loops are allowed.

Lemma 20. If $\hat{H}$ contains any of the graphs a)-d) in the family $\mathcal{J}$ in Figure 9 . then the problem List-S- $\operatorname{Hom}(\widehat{H})$ is NP-complete.

Proof. For each of the signed graphs a), b), and c) in family $\mathcal{J}$, we can apply Theorem 8. The figure lists a chain for each of these forbidden subgraphs.

In the final case d), we reduce Not-All-Equal SAT to $\operatorname{List-S-\operatorname {Hom}(H,\pi )}$ where $(H, \pi)$ is the signed graph d) in family $\mathcal{J}$. Let $\left(T^{\prime}, \sigma^{\prime}\right)$ be the signed graph with the list assignments and signature shown in Figure 10 . For each clause $(x, y, z)$ in the instance of Not-All-Equal SAT, we create a copy of $\left(T^{\prime}, \sigma^{\prime}\right)$ identifying the leaves $x, y, z$ in $T^{\prime}$ with the variables in the clause.

We claim that $\left(T^{\prime}, \sigma^{\prime}\right)$ has a list homomorphism to $(H, \pi)$ if and only if we switch at exactly one or two elements of $\{x, y, z\}$. We can then view the switching at one of $\{x, y, z\}$ as setting the variable to true and, conversely, no switching as setting to false.

Consider a mapping of $\left(T^{\prime}, \sigma^{\prime}\right)$ to $(H, \pi)$. It is easy to see that either both $x$ and $m$ are switched or neither is switched. We also observe that if $m$ maps to 1 , then exactly one of $m$ or $y$ must be switched. On the other hand, if $m$ maps to 3 , then neither or both of $m$ and $y$ is switched. (In the first case the image of the $(m, y)$-path is a negative walk, while in the second case it is a positive walk.) Thus, when $m$ maps to 1 , exactly one of $x$ or $y$ is switched, and when $m$ maps to 3 , either both or neither $x$ and $y$ is switched. Finally, if $m$ maps to 1 , then we are free to switch or not switch at $z$. On the other hand, if $m$ maps to 3 , then we must switch at $z$ if and only if we do not switch at $m$. In conclusion, with $m$ mapping to 1 the following truth values are possible for $x, y, z$ respectively: $1,0,0 ; 1,0,1 ; 0,1,0 ; 0,1,1$, and with $m$ mapping to 3 we obtain the possible triples $1,1,0$ and $0,0,1$ for the variables $x, y, z$. These are precisely the not-all-equal values as claimed.

If bicoloured edges are present, we use the following result.
Lemma 21. If $\widehat{H}$ contains any of the graphs e)-n) in family $\mathcal{J}$ in Figure 9 . then the problem List-S- $\operatorname{Hom}(\widehat{H})$ is $N P$-complete.
a)


$$
\begin{aligned}
& U=1-2-\ldots-(k-1)-k-(k-1)-\ldots-2-1-1 \\
& D=1-1-2-\ldots-(k-1)-k-(k-1)-\ldots-2-1 \\
& k \geq 4
\end{aligned}
$$

b)
 $U=3-2-1-2-3-3$
$D=3-3-4-5-4-3$
c)
 $U=6-3-2-1-2-3-6-6$
$D=6-6-3-4-5-4-3-6$
d)

e)


$$
\begin{aligned}
& U=2-3-\ldots-k-(k-1) \\
& D=2-1-2-\ldots-(k-1) \\
& k \geq 4
\end{aligned}
$$

f)

$U=1-2-\ldots-(k-1)-k-(k-1)$
$D=1-1-1-2-\ldots \ldots \ldots-(k-1)$
$k \geq 3$
$k \geq 3$
g)

$U=1-2-\ldots \ldots \ldots \ldots-k-k$
$D=1-1-2-\ldots-(k-1)-k$
$k \geq 2$
h)


$$
\begin{aligned}
& U=3-6-6-3-4-5-4-3 \\
& D=3-2-1-2-3-6-6-3
\end{aligned}
$$

i)
 $\stackrel{Q}{k-1 \quad k}$

$$
\begin{aligned}
& U=2-1-1-2-\ldots \ldots \ldots-(k-1) \\
& D=2-3-\ldots-k-(k-1)-(k-1)
\end{aligned}
$$

bicoloured path (possibly $3=k-1$ with blue loop)
$k \geq 4$, loops at 1 and $k-1$ can be unicoloured by arbitrary colour
j)

disconnected loops
$k \geq 3$
k)


$$
U=3-2-1-2-1-2-3-6-6-3
$$

$$
D=3-6-6-3-4-5-4-5-4-3
$$

1) 


$U=2-2-3-4-3-2-1-2$
$D=2-1-2-3-4-3-2-2$
m)

$U=1-1-2-1$
$D=1-2-1-1$
n)


$$
\begin{aligned}
& U=2-1-1-2-3-4-3-4-3-2 \\
& D=2-3-4-3-4-3-2-1-1-2
\end{aligned}
$$

dotted non-loop edge can be bicoloured or unicoloured
p)

bicoloured path

$$
U=(k-1)-k-(k+1)-k-\ldots-2-1-1-2-\ldots-(k-2)-(k-1)
$$

$$
D=(k-1)-(k-2)-\ldots-2-1-1-2-\ldots-k-(k+1)-k-(k-1)
$$

$$
k \geq 3
$$

q)

$U=1-2-3-2-1-1$
$1 \quad 2 \quad 3$ $D=1-1-2-3-2-1$
r)


$$
\begin{aligned}
& U=2-2-1-2-3-4-5-4-3-2 \\
& D=2-3-4-5-4-3-2-1-2-2
\end{aligned}
$$

Figure 9: The family $\mathcal{J}$. (The dotted loops can be arbitrary or missing, unless stated otherwise.)


Figure 10: The gadget $\left(T^{\prime}, \sigma^{\prime}\right)$ for the case d) in family $\mathcal{J}$.

Proof. For each of the signed graphs e)-n) in family $\mathcal{J}$, except for the case $\mathfrak{j}$ ), we can apply Theorem 8, The figure lists a chain for each of these forbidden subgraphs. The case j) follows from a result in 14 implying that the problem is NP-complete if the vertices with loops of any colour are disconnected. Thus any signed graph $\widehat{H}$ that contains one of the signed graphs in the cases e)n) of the family $\mathcal{J}$ as an induced subgraph has the problem List-S-Hom $(\widehat{H})$ NP-complete.

In cases p), q), and r) in Figure 9 we present three additional NP-complete trees we will use. Note that the case p) is an extension of case q), and the chains are also related. (Note that p) is also similar to a) in family $\mathcal{J}$.) We note that the absence of a loop at 2 is crucial for the chain in the case $p$ ). We also note, for the case r), that the absence of a loop at 4 is crucial, while the edges 12 or 45 could be blue or bicoloured and the given chain would still apply.

Thus we have the following lemma.
Lemma 22. If $\widehat{H}$ contains any of the graphs p), q), r) in family $\mathcal{J}$ in Figure 9 . then the problem List-S- $\operatorname{Hom}(\widehat{H})$ is $N P$-complete.

If $\widehat{H}$ is a signed graph, the bicoloured part of $\widehat{H}$ is the graph $D_{\widehat{H}}$ (with possible loops) consisting of all those edges and loops that occur as bicoloured edges and loops in $\widehat{H}$, and all the vertices they contain. (Thus vertices of $\widehat{H}$ not incident with a bicoloured edge or loop are deleted.) Similarly, the blue part of $\widehat{H}$ is the graph $B_{\widehat{H}}$ with possible loops consisting of all those edges (and loops) that are at least blue in $\widehat{H}$. Since we assume all non-loop edges of $\widehat{H}$ are blue, every vertex of $\widehat{H}$ is included in $B_{\widehat{H}}$. (We may think of $B$ as standing for "blue" and $D$ as standing for "double", in the sense of having both colours.)

We now denote by $\mathcal{T}$ the union of all the NP-complete tree families $\mathcal{F}, \mathcal{G}, \mathcal{J}$. There are further cases that cause the problem to be NP-complete. Theorem 6 implies, in the context of trees, that the problem is NP-complete if there are no bicoloured edges or loops and there is both a red loop and a blue loop. Any signed graph $\widehat{H}$ which is not irreflexive and has a bicoloured edge but no bicoloured loops yields an NP-complete homomorphism (and hence list homomorphism) problem by Theorem 5, since the s-core contains at least one unicoloured loop and one bicoloured edge (counted as two edges). As discussed earlier, if the vertices with loops of any fixed colour induce a disconnected graph, the problem is NP-complete by [14. Finally, as mentioned earlier, if the bicoloured part $D_{\widehat{H}}$ yields an NP-complete list homomorphism problem, then so does $\widehat{H}$,
since for bicoloured inputs, this is the only part of $\widehat{H}$ that can be used. Thus $\operatorname{List-S}-\operatorname{HOM}(\widehat{H})$ is also NP-complete if the unsigned graph $D_{\widehat{H}}$ is not a bi-arc tree, i.e., contains one of the trees in Figures 3 and 4 of 12. Moreover, if $\widehat{H}$ contains no red loops, then it is also true that if the blue part $B_{\widehat{H}}$ yields an NP-complete list homomorphism problem, then so does $\widehat{H}$. Indeed, if there are no red loops (or edges) in $\widehat{H}$, then for an input signed graph $\widehat{G}$ that has only blue edges, there is no cause for switching. In other words a blue input $\widehat{G}$ admits a signed list homomorphism to $\widehat{H}$ if and only if $G$ admits a list homomorphism to $H$. This is a reduction from the list homomorphism problem for $B_{\widehat{H}}$ to the signed list homomorphism problem for $\widehat{H}$.

We say that a signed tree is colour-connected if each of the following subgraphs is connected: the subgraph spanned by non-loop edges that are at least blue, the subgraph spanned by non-loop edges that are at least red, the subgraph spanned by non-loop edges that are bicoloured, the subgraph induced by the vertices with loops that are at least blue, the subgraph induced by the vertices with loops that are at least red, and the subgraph induced by the vertices with loops that are bicoloured.

We call a signed tree $\widehat{H}$ a good signed tree if it satisfies the following conditions.

1. If $\widehat{H}$ has no bicoloured edge, then all the loops are of the same colour (red or blue).
2. If $\widehat{H}$ has a bicoloured non-loop edge, then it also has a bicoloured loop, or it has no loops at all.
3. $\widehat{H}$ is colour-connected.
4. The blue part $B_{\widehat{H}}$ is a bi-arc tree.
5. $\widehat{H}$ contains no signed tree from the family $\mathcal{T}$.

### 7.1. Assuming no red loops

In this subsection, we assume that $\widehat{H}$ has no red loops. It follows from the previous section, that if such $\widehat{H}$ is not good, then List-S-Hom $(\widehat{H})$ is NPcomplete. In particular, $\widehat{H}$ is colour-connected, since (as observed before), 14 implies that the problem is NP-complete if the vertices with loops of any colour are disconnected, and the family e) in $\mathcal{J}$ implies that the problem is NP-complete if the subgraph spanned by non-loop edges that are bicoloured is not connected. Also recall that all unicoloured non-loop edges are assumed to be blue, and thus all non-loop edges that are at least red are in fact bicoloured. In the next subsection, we prove this fact (that signed trees that are not good have NP-complete problems) is true if we allow red loops as well.

We first analyze the structure of good signed trees without red loops.
Let $\widehat{H}$ be a good signed tree with no red loops and at least one bicoloured loop. Since the blue part $B_{\widehat{H}}$ is a bi-arc tree, we can use the results of 12 and [13], which together characterize bi-arc trees as trees in which vertices with loops induce a connected subgraph, and which are either obtained from a reflexive caterpillar by deleting the loops at a (possibly empty) subset of leaves
(illustrated in Figure 11, repeated from Figure 5 of [12]), or obtained from an irreflexive 2-caterpillar in one of the following ways: (1) (possibly) adding a loop at a good vertex $v$, or (2) adding a loop at a good vertex $v$ and on one neighbour $w$ of $v$ which has the property that each neighbour of $w$ other than $v$ is a leaf, or (3) adding a loop at a good vertex $v$ and on a (possibly empty) set of neighbours of $v$ that are leaves. Here a good vertex is a vertex $v$ for which there does not exist a path $P$ with seven vertices, with the middle vertex $u$ connected to $v$ by a path (possibly with zero edges) which is disjoint from $P$. It is easy to see that if $v$ is a good vertex, then there exists a spine in which $v$ is the first vertex, $v=v_{1}$ (and, in case (2), the vertex $w$ is a child of $v$, not on the spine; similarly in case (3) the leaves of $v$ to which loops have been added are children of $v$ not on the spine). These cases are illustrated in Figure 12, repeated here from Figure 6 in 12 . The two 2 -caterpillars in that figure will be called Type (a) and Type (b), as shown.

Proposition 23. Let $\widehat{H}$ be a good signed tree without red loops but with at least one bicoloured loop.

Then $\widehat{H}$ is either

- obtained from a good reflexive caterpillar (with spine $v_{1}, v_{2}, \ldots, v_{k}$ ) by
- removing loops at a subset $S$ of leaves, and
- optionally replacing any bicoloured edges $v_{i} u$ by blue edges for these leaves $u \in S$, or
- is a signed 2-caterpillar (with spine $v_{1}, v_{2}, \ldots, v_{k}$ ) obtained from a bi-arc tree by
- replacing each edge and loop by a bicoloured edge and loop (respectively),
- optionally, for Type (b) 2-caterpillars, adding a blue loop at a leaf adjacent to $v_{1}$, and
- optionally adding, at a spine vertex $v_{i}$ or at a loopless child of a $v_{i}$, a blue edge leading to a new (loopless) leaf.


Figure 11: Bi-arc caterpillars from 12.
Proof. Assume first that the blue part $B_{\widehat{H}}$ is a bi-arc tree of the first type, in other words, a caterpillar with loops on all vertices on the spine and possibly some leaves. We now proceed analogously to the proof of Lemma 17 , using the


Figure 12: Bi-arc 2-caterpillars from 12.
absence of signed trees from the family $\mathcal{J}$ instead of those from the family $\mathcal{G}$. We sketch the analogy, and leave the detailed proof to the reader. For this, it helps to refer to the annotations in Figure 6 relating the cases of the family $\mathcal{G}$ to the more general trees in the family $\mathcal{J}$. It is also helpful to point out that the case h ) of the family $\mathcal{G}$ is closely related to the case i) of the family $\mathcal{J}$ (as well as to b) of the family $\mathcal{F}$ ). There is a common generalization to all three, but it has a technical formulation we chose to omit, because other cases of the family $\mathcal{J}$ cover the same situations; in particular the reader should note the case n ), which is also helpful in the omitted proof. The principal difference from the proof in the reflexive case is caused by the requirement that certain loops in cases h) and l) in $\mathcal{G}$ have to remain present in the corresponding cases in family $\mathcal{J}$. This results in the fact that some vertices $v_{1}, v_{2}, \ldots, v_{d-1}$ can have incident blue edges off the spine, as long as they lead to vertices without loops, as enforced by the absence of the signed trees from the family $\mathcal{J}$.

Thus $\widehat{H}$ is indeed a caterpillar obtained from a reflexive signed caterpillar by removing loops at some leaves, and optionally replacing the bicoloured edges by blue edges to some of those leaves.

In the remaining cases, the blue part $B_{\widehat{H}}$ is a 2-caterpillar obtained by adding suitable loops to an irreflexive bi-arc tree, cf. the two bi-arc trees in Figure 12 Specifically, there are two cases to consider.

In the first case, the blue part $B_{\widehat{H}}$ has two loops, at least one of which is bicoloured. According to [13], we may choose the spine so that one loop of $B_{\widehat{H}}$ is at $v_{1}$ and the other at its child $u$. If the loop at $v_{1}$ is bicoloured in $\widehat{H}$, then the absence of p ) and $q$ ) in family $\mathcal{J}$ implies that we may assume the spine consists of bicoloured edges only, and if a vertex $v_{i}$ on the spine has a (necessarily loopless) neighbour $w$ that is not a leaf, then the edge $v_{i} w$ is bicoloured. The neighbour $u$ has a loop and needs to be considered separately. We first claim that the edge $u v_{1}$ must be bicoloured, else $\widehat{H}$ contains the subtree n) from the family $\mathcal{J}$ (with 2 corresponding to $v_{1}$ ), or g ) from the family $\mathcal{J}$ (with $k=2$ ).

Moreover, if the loop at $u$ is unicoloured, then $u$ must be a leaf, otherwise $\widehat{H}$ would contain r ) from the family $\mathcal{J}$. If the loop at $v_{1}$ is unicoloured, then the loop at $u$ must be bicoloured (we assumed that a bicoloured loop exists). Now, unless $\widehat{H}$ arose from a reflexive caterpillar, it must contain a) from the family $\mathcal{J}$ (if $u v_{1}$ is blue), or l) from family $\mathcal{J}$ (if $u v_{1}$ is bicoloured). (Note that in both cases, the chain applies even if the edges 23,34 are bicoloured.) In conclusion, in this case we either have both loops at $v_{1}$ and $u$ (as well as the edge joining them) bicoloured, or the loop at $v_{1}$ and the edge $u v_{1}$ is bicoloured, the loop at $u$ is blue and a leaf. The former situation is depicted on the left of Figure 14 ( $u$ is depicted as the child of $v_{1}$ in $T_{1}$ ), and the latter situation is a special case of the tree on the right, with only one child $(u)$ of $v_{1}$ having a (blue) loop. Thus going from $D_{\widehat{H}}$ to $\widehat{H}$ we only added a blue loop on a leaf $u$ adjacent to $v_{1}$, and then added some blue edges leading to leaves from any spine vertex $v_{i}$, or from any child of $v_{2}, v_{3}, \ldots, v_{k}$, or from any child of $v_{1}$ other than $u$.

In the second case, the blue part $B_{\widehat{H}}$ has one loop at $v_{1}$ and possibly several other loops at leaf children of $v_{1}$. If the loop at $v_{1}$ is bicoloured, and possibly some of the loops at its children are also bicoloured, then the proof proceeds exactly as in the previous case, concluding that any edge joining two vertices with loops must be bicoloured (else there would be a copy of the subtree n ) from the family $\mathcal{J}$ ) and the children of $v_{1}$ with loops are leaves. If, say, leaf $u$ has a bicoloured loop and all other loops, including the loop at $v_{1}$, are blue in $\widehat{H}$, we again obtain a contradiction to the absence of a) from family $\mathcal{J}$ or l) from family $\mathcal{J}$, unless $\widehat{H}$ arose from a reflexive caterpillar. In conclusion, in this case, going from $D_{\widehat{H}}$ to $\widehat{H}$ involved only the addition of blue loops on leaves adjacent to $v_{1}$, and a possible addition of some blue edges from spine vertices or from non-loop children of spine vertices, leading to leaves as described.

### 7.2. Allowing red loops

We now consider signed graphs $\widehat{H}$ in which red loops are allowed. We denote by $\widehat{H^{\prime}}$ the signed tree obtained from $\widehat{H}$ by deleting all vertices with red loops. We focus on the blue part $B_{\widehat{H^{\prime}}}$ instead of $B_{\widehat{H}}$ because $\widehat{H^{\prime}}$ has no red loops and satisfies the assumptions of Proposition 23 .


Figure 13: Good signed trees obtained from a good reflexive tree by deleting loops.

Proposition 24. Let $\widehat{H}$ be a good signed tree with at least one bicoloured loop. Then $\widehat{H}$ is either

- obtained from a good reflexive caterpillar (with spine $v_{1}, v_{2}, \ldots, v_{k}$ ) by
- removing loops at a subset $S$ of leaves, and


Figure 14: Good signed trees obtained from a bi-arc tree as described in Proposition 24.

- optionally replacing any bicoloured edges $v_{i} u$ by blue edges for these leaves $u \in S$, or
- is a signed 2 -caterpillar (with spine $v_{1}, v_{2}, \ldots, v_{k}$ ) obtained from a bi-arc tree by
- replacing each edge and loop by bicoloured edge and loop (respectively),
- optionally, for Type (b) 2-caterpillars, adding a unicoloured loop at any leaf adjacent to $v_{1}$, and
- optionally adding, at a spine vertex $v_{i}$ or at a loopless child of a $v_{i}$, a blue edge leading to a new (loopless) leaf.

Proof. Since $\widehat{H^{\prime}}$ (defined above) satisfies the assumptions of Proposition 23 the tree $\widehat{H^{\prime}}$ is described by the proposition, and we now consider where can the vertices of $B_{\widehat{H}}-B_{\widehat{H^{\prime}}}$ be added, without violating any of the assumptions on $\widehat{H}$. Since the vertices with red loops must form a connected subgraph, they must be adjacent to each other and then to vertices with bicoloured loops. We now take in turn each case in the previous proof.

Consider the first case, when $\widehat{H^{\prime}}$ is a caterpillar with reflexive spine vertices, and suppose $x$ is a vertex in $\widehat{H}-\widehat{H^{\prime}}$ with a red loop. If $x$ is adjacent to a vertex on the spine of $\widehat{H^{\prime}}$, then $\widehat{H}$ is indeed a good reflexive caterpillar with some loops on leaves removed. If $x$ is adjacent to a leaf of $\widehat{H^{\prime}}$ with a red loop, then either $\widehat{H}$ contains a copy of $F_{2}$, or $\widehat{H}$ is another good caterpillar with a different spine, and possibly different preferred colour, from which some loops at leaves have been removed.

In the second case, let $x$ be again a vertex of $\widehat{H}-\widehat{H^{\prime}}$ with a red loop. We claim it is adjacent to $v_{1}$ by a bicoloured edge; indeed it cannot be adjacent to a child $u$ of $v_{1}$, then $u$ would have to have a bicoloured loop or a red loop. In this case, $\widehat{H}$ is either is a caterpillar with reflexive spine and we are in the previous case, or we would have, in red, a path with three loops followed by two non-loops, which is NP-complete according to Theorem 5.1 of [12], see Figure 3 in that paper.

In both cases of the proof above, we note that when the red loops are deleted (without deleting their vertices), we obtain a signed graph which also satisfies
the assumptions of Proposition 23. Moreover, if the red loops are all changed to be blue, the same conclusion holds. These observations justify the following corollary.

Corollary 25. Suppose $\widehat{H}$ is a signed tree. If the blue part $B_{\widehat{H}}$ is not a bi-arc tree, then List-S- $\operatorname{Hom}(\widehat{H})$ is NP-complete. If the underlying unsigned tree is not a bi-arc tree, then List-S-Hom $(\widehat{H})$ is $N P$-complete.

It follows from the first statement of Corollary 25 that if a signed tree is not good then List-S-Hom $(\widehat{H})$ is NP-complete even if there are red loops in $\widehat{H}$.

We now state our main theorem of this section.
Theorem 26. If $\widehat{H}$ is a good signed tree, then List-S-Hom $(\widehat{H})$ is polynomialtime solvable.

We can explicitly state the dichotomy classification as follows.
Corollary 27. Let $\widehat{H}$ be a signed tree.
If any of the following conditions apply, then LisT-S-Hom $(\widehat{H})$ is NP-complete.

1. $\widehat{H}$ has no bicoloured loop, but there is a bicoloured (non-loop) edge and a unicoloured loop.
2. $\widehat{H}$ has no bicoloured edge, but there is a red loop and a blue loop.
3. The bicoloured part $D_{\widehat{H}}$ is not a bi-arc tree, i.e., contains a subgraph from Figures 3 or 4 of [12].
4. The blue part $B_{\widehat{H}}$ is not a bi-arc tree, i.e., contains a subgraph from Figures 3 or 4 of [12].
5. $\widehat{H}$ contains a signed tree from the family $\mathcal{T}$.
6. The set of vertices of $\widehat{H}$ with red (respectively blue, or at least blue, or bicoloured) loops induces a disconnected graph.

If none of the conditions apply, then List-S-Hom $(\widehat{H})$ polynomial-time solvable.

We also state the result in the more usual complementary way, where the polynomial cases are enumerated first. Note that here all the conditions are required to be satisfied to yield a polynomial case.

Corollary 28. Let $\widehat{H}$ be a signed tree. If all of the following conditions apply, then LIST-S-Hom $(\widehat{H})$ is polynomial-time solvable.

1. If $\widehat{H}$ has a bicoloured non-loop edge, then it has a bicoloured loop, or it has no loops at all.
2. If $\widehat{H}$ has no bicoloured edge, then all unicoloured loops are of the same colour.
3. The bicoloured part $D_{\widehat{H}}$ is a bi-arc tree.
4. The blue part $B_{\widehat{H}}$ is a bi-arc tree.
5. $\widehat{H}$ contains no signed tree from the family $\mathcal{T}$.
6. The vertices with red (respectively blue, respectively bicoloured) loops induce a connected subgraph of $\widehat{H}$.
If at least one of the conditions fails, then List-S-HOM $(\widehat{H})$ is NP-complete.
We now return to the proof of Theorem 26
Proof. We show that for a good signed tree $\widehat{H}$, the problem List-S-Hom $(\widehat{H})$ is polynomial-time solvable. We may assume there is a bicoloured loop, else the result follows from Theorems 6 and 12 . By Propositon 24 we distinguish two cases.

For the first case, let $\widehat{H}$ be a good signed tree obtained from a good reflexive caterpillar (with spine $v_{1}, v_{2}, \ldots, v_{k}$ ) by removing loops at a subset $S$ of leaves, and optionally replacing any bicoloured edges $v_{i} u$ by blue edges for the leaves $u \in S$. As in the case of reflexive trees, we use a special min ordering of $\widehat{H}$. This means that if a vertex $v_{i}$ (with $1 \leq i \leq d-1$ ) has a non-loop neighbour $u$ connected by unicoloured edge, then $u$ is ordered to come after $v_{i+1}$ in the special min ordering. Now we can use our algorithm for reflexive trees, with the observation that if there is a negative cycle $C$ mapped to a unicoloured edge $v_{i} u$, then we can remove $u$ from lists of all vertices in $C$ and continue in modifying the images of such cycles.

For the second case, let $\widehat{H}$ be a good signed 2-caterpillar (having spine $v_{1}, v_{2}, \ldots, v_{k}$ ), obtained from a bi-arc tree by replacing edges and loops by bicoloured edges and loops (respectively), optionally adding unicoloured loops at leaves of $v_{1}$, and then adding blue edges from the spine or children of the spine to loopless leaves. We set $T=V(\widehat{H})$ and $T^{\prime}=V\left(D_{\widehat{H}}\right)$; moreover, we set $L=T \backslash T^{\prime}$. It follows from Corollary 27 that all vertices of $L$ are loopless leaves in $T$ incident with exactly one blue edge. We also note that if two distinct vertices $a$ and $b$ in $T^{\prime}$ are adjacent in $\widehat{H}$, then they are adjacent by a bicoloured edge.

To prove List-S-Hom $(\widehat{H})$ is polynomial-time solvable, we shall construct a suitable majority polymorphism. Recall that for a signed graph $\widehat{H}$, the switching graph $S(\widehat{H})$ is constructed as follows. We represent $\widehat{H}$ as $(H, \pi)$ where the signature $\pi$ has all unicoloured non-loop edges blue (positive), and define $S(\widehat{H})$ to be the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$in which each vertex $x$ of $H$ gives rise to two vertices $x, x^{\prime}$ of $H^{+}$and each edge $x y$ of $H$ gives rise to edges $x y, x^{\prime} y^{\prime}$ of the colour $\pi(x y)$ in $H^{+}$and edges $x y^{\prime}, x^{\prime} y$ of the opposite colour; this definition also applies to loops, by letting $x=y$. For any vertex $x$ of $H$, we shall denote by $x^{*}$ one of $x, x^{\prime}$, and by $s\left(x^{*}\right)$ the other one of $x, x^{\prime}$. A majority polymorphism of $\left(H^{+}, \pi^{+}\right)$is a ternary mapping $F$ on the vertices of $H^{+}$such that $F\left(x^{*}, y^{*}, z^{*}\right)$ is adjacent to $F\left(u^{*}, v^{*}, w^{*}\right)$ in blue (red) provided $x^{*}$ is adjacent to $u^{*}$ in blue (red), $y^{*}$ is adjacent to $v^{*}$ in blue (red), and $z^{*}$ is adjacent to $w^{*}$ in blue (red, respectively), and such that if two arguments from $x^{*}, y^{*}, z^{*}$ are equal, then the assigned value $F\left(x^{*}, y^{*}, z^{*}\right)$ is also equal to it. A semi-conservative majority polymorphism assigns $F\left(x^{*}, y^{*}, z^{*}\right)$ to be one
of the values $x^{*}, y^{*}, z^{*}, s\left(x^{*}\right), s\left(y^{*}\right), s\left(z^{*}\right)$, and a conservative majority polymorphism assigns $F\left(x^{*}, y^{*}, z^{*}\right)$ to be one of the values $x^{*}, y^{*}, z^{*}$. As outlined in Section 3, if the edge-coloured graph $\left(H^{+}, \pi^{+}\right)$admits a semi-conservative majority polymorphism, then the signed list homomorphism problem for $(H, \pi)$ is polynomial-time solvable. We shall in fact construct a conservative majority polymorphism of $\left(H^{+}, \pi^{+}\right)$.

To construct a conservative majority polymorphism $F\left(x^{*}, y^{*}, z^{*}\right)$ for triples $\left(x^{*}, y^{*}, z^{*}\right)$ from $V\left(H^{+}\right)$, we will of course define values of triples with repetition to be the repeated value,

$$
F\left(x^{*}, y^{*}, y^{*}\right)=F\left(y^{*}, x^{*}, y^{*}\right)=F\left(y^{*}, y^{*}, x^{*}\right)=y^{*}
$$

Now we partition the triples $\left(x^{*}, y^{*}, z^{*}\right)$ of distinct vertices of $V\left(H^{+}\right)$into two sets $R_{1}$ and $R_{2}$, where $R_{1}$ consists of those triples $\left(x^{*}, y^{*}, z^{*}\right)$ for which at most one of $x, y, z$ is in $L$, and $R_{2}$ consists of triples that have at least two of $x, y, z$ in $L$. (The vertices $x, y, z$ of $\widehat{H}$ need not be distinct, as long as $x^{*}, y^{*}, z^{*}$ are distinct.) Note that two triples $\left(x_{1}^{*}, y_{1}^{*}, z_{1}^{*}\right),\left(x_{2}^{*}, y_{2}^{*}, z_{2}^{*}\right)$ that are coordinatewise adjacent in $\widehat{H}$ cannot both be in $R_{2}$, and if they are both in $R_{1}$, then there is a coordinate $t \in\{x, y, z\}$ such that $t_{1}=t_{2}$, or the edge $t_{1} t_{2}$ is bicoloured in $\widehat{H}$.

The definition of $F\left(x^{*}, y^{*}, z^{*}\right)$ will differ for triples with $(x, y, z) \in R_{1}$, where we explicitly describe the value $F\left(x^{*}, y^{*}, z^{*}\right)$, and for triples with $(x, y, z) \in R_{2}$, where we merely prove that a suitable value $F\left(x^{*}, y^{*}, z^{*}\right)$ exists.

First we consider the underlying unsigned tree of $\widehat{H}$. It clearly contains all edges of $B_{\widehat{H}}$, but it also contains loops that are red in $\widehat{H}$. By Corollary 25 this tree (with vertex set $T$ ), which we also denote by $T$, is a bi-arc tree, and hence has a majority polymorphism $f$ [12, 13].

We now describe the polymorphism $f$ from [12], assuming, as above, that the bi-arc tree $T$ is one of the trees in Figure 12 .

In both cases, $T$ is a 2 -caterpillar with spine $v_{1}, v_{2}, \ldots, v_{k}$, on which only $v_{1}$ has a loop, and either there is only one additional loop on a child of $v_{1}$ (which may have children), or any number of loops on children of $v_{1}$ which must be leaves. We denote by $T_{i}$ the subtree rooted at the vertex $v_{i}$ of the spine as shown in Figure 14. We note that in this notation we include the spine vertex $v_{i}$ in the tree $T_{i}$, unlike the convention in [12], where the spine vertex is explicitly excluded. (Compare Figure 12 and 14 ) To be able to conveniently apply the results of [12], we refer to the vertices in $T_{i}$ other than the root $v_{i}$ as being inside $T_{i}$. We also remark that in Section 5 we used yet another convention for naming subtrees $T_{i}$ - they were rooted subtrees at individual children of spine vertices, cf. Figure 4. In any event, in the current context each subtree $T_{i}$ is ordered by depth first search (in the case of $T_{1}$ giving higher priority to vertices with loops), and a total ordering of $T$ is obtained by concatenating these DFS orderings from $T_{1}$ to $T_{2}$ and so on. We also colour the vertices of $T$ by two colours, in a proper colouring ignoring the self-adjacencies due to the loops. The value $f(x, y, z)$ is defined as the majority of $x, y, z$ if two of the arguments $x, y, z$ are equal, and otherwise it is defined according to the following rules.

Rule (A) Assume $x, y, z$ are distinct and in the same colour class. Let $r(x)$, $r(y), r(z)$ be the (not necessarily distinct) roots of the trees containing $x, y, z$ respectively, and let $v_{m}$ be the median of these vertices on the spine. Then $f(x, y, z)$ is the vertex from amongst $x, y, z$ in the tree $T_{m}$, and if there are more than one in $T_{m}$, it is the first vertex in the DFS ordering unless one of the following occurs, in which case it is the second vertex in the DFS ordering.

- All three vertices $x, y, z$ lie inside $T_{m}$ with $m \geq 2$;
- all three vertices $x, y, z$ lie inside $T_{1}$ and at most one of them has a loop;
- exactly two of $x, y, z$ lie inside $T_{1}$ and exactly one of them has a loop;
- exactly two of $x, y, z$ lie inside $T_{1}$, neither has a loop, exactly one of them is adjacent to the unique neighbour of $v_{1}$ with a loop, and the third vertex of $x, y, z$ is not $v_{1}$.

Rule (B) Assume $x, y, z$ are distinct but not all in the same colour class.
Then $f(x, y, z)$ is the first vertex in the DFS ordering of the two vertices in the same colour class, except when $\{x, y, z\}$ contains $v_{1}$ and at least one of its leaf neighbours with a loop, in which case $f(x, y, z)=v_{1}$.

We now use the above conservative majority $f$ on $T$ to define a conservative majority $F$ on triples in $R_{1}$. We say that a vertex $y$ dominates a vertex $x$ in $\widehat{H}$ if any blue (or red) neighbour of $x$ is also blue (red respectively) neighbour of $y$.

Rule (1) Assume that at least two of $x^{*}, y^{*}, z^{*}$ are equal, say $y^{*}=z^{*}$. As mentioned earlier, we define $F\left(x^{*}, y^{*}, y^{*}\right)$ to be the repeated value,

$$
F\left(x^{*}, y^{*}, y^{*}\right)=F\left(y^{*}, x^{*}, y^{*}\right)=F\left(y^{*}, y^{*}, x^{*}\right)=y^{*}
$$

Rule (2) Assume that $x^{*}, y^{*}, z^{*}$ are distinct but two of $x, y, z$ are equal. Then for triples $\left(x^{*}, y^{*}, z^{*}\right)$ we define the value $F$ to be the first version of the repeated vertex, i.e.,

$$
F\left(x^{*}, s\left(x^{*}\right), y^{*}\right)=F\left(x^{*}, y^{*}, s\left(x^{*}\right)\right)=F\left(y^{*}, x^{*}, s\left(x^{*}\right)\right)=x^{*}
$$

unless $x \in L$ or $x$ has a unicoloured loop in $\widehat{H}$ and $y$ dominates $x$ in $\widehat{H}$, in which case

$$
F\left(x^{*}, s\left(x^{*}\right), y^{*}\right)=F\left(x^{*}, y^{*}, s\left(x^{*}\right)\right)=F\left(y^{*}, x^{*}, s\left(x^{*}\right)\right)=y^{*}
$$

(For example $F\left(x, x^{\prime}, y\right)=x$ and $F\left(x^{\prime}, x, y\right)=x^{\prime}$, but $F\left(x, x^{\prime}, y\right)=F\left(x^{\prime}, x, y\right)=$ $y$ if $y$ dominates $x$ and $x$ has a unicoloured loop or is in L.)

Rule (3) Assume $x^{*}, y^{*}, z^{*}$ are distinct and also $x, y, z$ are distinct. For triples $\left(x^{*}, y^{*}, z^{*}\right)$, we define $F\left(x^{*}, y^{*}, z^{*}\right)$ to be the argument in the same coordinate as $f(x, y, z)$, except if $f(x, y, z) \in L$ and another vertex $t \in\{x, y, z\}$ dominates $f(x, y, z)$, in which case we define $F\left(x^{*}, y^{*}, z^{*}\right)$ to be the argument in the same coordinate as $t$.

It is easy to check that if two triples $\left(x^{*}, y^{*}, z^{*}\right)$ and $\left(u^{*}, v^{*}, w^{*}\right)$ are coordinatewise adjacent in blue (red) in $S(\widehat{H})$, then $(x, y, z)$ is adjacent to $(u, v, w)$ in $T$ and hence $f(x, y, z)$ is adjacent to $f(u, v, w)$ in $T$. We now check that we can also conclude that $F\left(x^{*}, y^{*}, z^{*}\right)$ is adjacent to $F\left(u^{*}, v^{*}, w^{*}\right)$ in blue (red respectively).

Case 1. $x, y, z$ are distinct and $u, v, w$ are distinct.
If $f(x, y, z)$ and $f(u, v, w)$ choose the same coordinate, then $F\left(x^{*}, y^{*}, z^{*}\right)$ and $F\left(u^{*}, v^{*}, w^{*}\right)$ also choose the same coordinate, and hence the values are adjacent in the right colour. (This remains true even if one or both of the choices $F\left(x^{*}, y^{*}, z^{*}\right)$ and $F\left(u^{*}, v^{*}, w^{*}\right)$ were modified by domination.) Otherwise, suppose without loss of generality that $f(x, y, z)=x$ and $f(u, v, w)=v$. Then the vertex $x$ is adjacent in $T$ to both $u$ and $v$ and hence is not a loop-free leaf, and similarly $v$ is not a loop-free leaf. If $x \neq v$, this means that the edge $x v$ is bicoloured in $\widehat{H}$ and hence $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ is adjacent to $F\left(u^{*}, v^{*}, w^{*}\right)=v^{*}$ in both colours. If $x=v$, the same argument applies if the loop $x v$ is bicoloured, so let us assume it is unicoloured. In this situation, Proposition 24 implies that the vertex $x=v$ must be a leaf child of $v_{1}$ in $T$, and $u=y=v_{1}$. This is governed by the special case of Rule (B) in the definition of the majority polymorphism $f$, which implies that we would have $f(x, y, z)=v_{1}$ contradicting $f(x, y, z)=x$, so this case does not occur.

Case 2. $u, v, w$ are distinct but $x, y, z$ are not distinct, say $x=y$ (but perhaps $x^{*} \neq y^{*}$ ).

This means that $f(u, v, w)$ is adjacent to $x=f(x, y, z)$ in $T$, and $x$ is not a loop-free leaf since it is adjacent to both $u$ and $v$. We now observe that if $F\left(u^{*}, v^{*}, w^{*}\right)$ was chosen in the same coordinate as $f(u, v, w)$, then $f(u, v, w) \in$ $T^{\prime}$ and otherwise $F\left(u^{*}, v^{*}, w^{*}\right)$ was chosen in the same coordinate as some $t \in T^{\prime}$ by Rule (3). Recall $(u, v, w)$ is in $R_{1}$ so at most one of $u$ and $v$ belongs to $L$ and both are adjacent to $x$. If $f(u, v, w) \in L$, then the other coordinate ( $u$ or $v$ ) is the dominating vertex $t$. Hence, $f(u, v, w)$ or $t$ is adjacent to $x$ by a bicoloured edge in $\widehat{H}$, and $F\left(u^{*}, v^{*}, w^{*}\right)$ is adjacent to $x^{*}$ and to $s\left(x^{*}\right)$ in both colours. (Note that $F\left(x^{*}, y^{*}, z^{*}\right)$ is $x^{*}$ regardless of whether $y^{*}=x^{*}$ or $y^{*}=s\left(x^{*}\right)$.)

Case 3. Each triple $x, y, z$ and $u, v, w$ has exactly one repetition.
Suppose first that the repetition is in different positions, say $x=y$ and $v=w$.

Then $f(x, y, z)=x$ is adjacent to $f(u, v, w)=v$ in $T$. If $x \neq v$, then the edge $x v$ is bicoloured in $\widehat{H}$, and $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ or $F\left(x^{*}, y^{*}, z^{*}\right)=s\left(x^{*}\right)$ and $F\left(u^{*}, v^{*}, w^{*}\right)=v^{*}$ or $F\left(u^{*}, v^{*}, w^{*}\right)=s\left(v^{*}\right)$ are adjacent in both colours. If $x=v$, then the same argument applies if the loop is bicoloured, and if it is unicoloured, then Proposition 24 implies that $u=z$ and $u$ has a bicoloured loop and dominates $x$, whence $F\left(x^{*}, y^{*}, z^{*}\right)=z^{*}$ and $F\left(u^{*}, v^{*}, w^{*}\right)=u^{*}$ and the adjacency is correct. On the other hand, if the repetition is in the same positions, say $x=y, u=v$, then $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ is adjacent to $F\left(u^{*}, v^{*}, w^{*}\right)=u^{*}$ by the definition of $F$, and hence the edge has the correct colour.

Case 4. One triple has all vertices the same, say, $x=y=z$ (but possibly $\left.x^{*} \neq y^{*}\right)$.

If we also have $u=v=w$, then by the pigeon principle some coordinate contains both $F\left(x^{*}, y^{*}, z^{*}\right)$ in $\left(x^{*}, y^{*}, z^{*}\right)$ and $F\left(u^{*}, v^{*}, w^{*}\right)$ in $\left(u^{*}, v^{*}, w^{*}\right)$, and so we have the correct adjacency. If $x$ is joined to $u$ by a bicoloured edge in $\widehat{H}$, then $F\left(x^{*}, y^{*}, z^{*}\right)$ is joined to $F\left(u^{*}, v^{*}, w^{*}\right)$ with the correct adjacency (even in the domination case of Rule 3 ). As both $x, u \notin L$, the only way for the edge joining them to be unicoloured, is $x=u$ and the edge is a unicoloured loop. In this case by Proposition 24, $w$ dominates $u$ and is joined to $u=x$ with a bicoloured edge in $\widehat{H}$, again ensuring the right adjacency.

Now we prove that one can extend the definition of $F$ to $R_{2}$ so that it remains a polymorphism. (It is of course possible to define each $F\left(u^{*}, v^{*}, w^{*}\right)$ for $\left(u^{*}, v^{*}, w^{*}\right) \in R_{2}$ directly, but we found the arguments become more transparent if we only verify that a suitable choice for $F\left(u^{*}, v^{*}, w^{*}\right)$ is always possible.)

Consider first values $F\left(x^{*}, y^{*}, z^{*}\right)$ with all three vertices $x, y, z$ in $L$. This means that $x$ is incident in $\widehat{H}$ with only one (necessarily blue) edge, say $x x_{1}$, and similarly for blue edges $y y_{1}, z z_{1}$. Thus in the switching graph $S(\widehat{H})$ the vertex $x$ is incident with only one blue edge, namely $x x_{1}$, and one red edge, namely $x x_{1}^{\prime}$, and similarly for $x^{\prime}$ and for $y, y^{\prime}, z, z^{\prime}$. Note that $\left(x_{1}, y_{1}, z_{1}\right) \in R_{1}$ because two vertices of $L$ are never adjacent. To choose the value of $F\left(x^{*}, y^{*}, z^{*}\right)$, we only need to take into account the existing values of $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{1}^{* *}\right)$, where $x_{1}^{* *}$ is also either $x_{1}$ or $x_{1}^{\prime}$, and similarly for $y_{1}^{* *}, z_{1}^{* *}$. For example, $\left(x, y^{\prime}, z^{\prime}\right)$ is coordinatewise adjacent in blue only to $\left(x_{1}, y_{1}^{\prime}, z_{1}^{\prime}\right)$ and in red only to $\left(x_{1}^{\prime}, y_{1}, z_{1}\right)$, and the choices of $F\left(x_{1}, y_{1}^{\prime}, z_{1}^{\prime}\right)$ and $F\left(x_{1}^{\prime}, y_{1}, z_{1}\right)$ occur in the same coordinate, by the definition of $F$ on $R_{1}$; if, say, $F\left(x_{1}, y_{1}^{\prime}, z_{1}^{\prime}\right)=x_{1}$ and $F\left(x_{1}^{\prime}, y_{1}, z_{1}\right)=x_{1}^{\prime}$, then setting $F\left(x, y^{\prime}, z^{\prime}\right)=x$ ensures that $F\left(x, y^{\prime}, z^{\prime}\right)=x$ is adjacent to $F\left(x_{1}, y_{1}^{\prime}, z_{1}^{\prime}\right)=$ $x_{1}$ in blue and to $F\left(x_{1}^{\prime}, y_{1}, z_{1}\right)=x_{1}^{\prime}$ in red, as required. Thus in general we can choose the value $F\left(x^{*}, y^{*}, z^{*}\right)$ in the same coordinate as $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{1}^{* *}\right)$, and satisfy the polymorphism property. (Note that this argument applies even if the vertices $x, y, z$ are not distinct.)

It remains to consider the case when exactly two of $x, y, z$ belong to $L$, say $x \in L$ and $y \in L$, with unique (blue) neighbours $x_{1}$ and $y_{1}$ in $\widehat{H}$, and $z \notin L$, with neighbours $z_{1}, \ldots, z_{p}$. We want to show that there is a suitable value for each $F\left(x^{*}, y^{*}, z^{*}\right)$ that maintains the polymorphism property. In the proofs below, we use the fact that $\left(x^{*}, y^{*}, z^{*}\right)$ is coordinate-wise adjacent in at least blue to each $\left(x_{1}^{*}, y_{1}^{*}, z_{i}^{*}\right)$ and possibly also $\left(x_{1}^{*}, y_{1}^{*}, s\left(z_{i}^{*}\right)\right)$ (if the edge $z z_{i}$ is bicoloured), and adjacent in at least red to each $\left(s\left(x_{1}^{*}\right), s\left(y_{1}^{*}\right), s\left(z_{i}^{*}\right)\right)$ and possibly also $\left(s\left(x_{1}^{*}\right), s\left(y_{1}^{*}\right), z_{i}^{*}\right)$ (if the edge $z z_{i}$ is bicoloured). In any event, we again denote the relevant triples by $\left(x_{1}^{* *}, y_{1}^{* *}, z_{1}^{* *}\right)$.

Suppose that $x, y, z$ are of the same colour. We observe that $x$ and $y$ cannot lie on the spine, since they are in $L$.

Consider first the case that $x, y$ lie inside the same tree $T_{r}$. Recall that we say "inside" to mean $x$ and $y$ are not on the spine; thus vertices $x_{1}$ and $y_{1}$ also belong to $T_{r}$, and so $T_{r}$ is the median tree. If $z$ also lies inside $T_{r}$, then $x, y, z$ are all children of $v_{r}$ or all are grandchildren of $v_{r}$. The argument is similar to the case where $F\left(x^{*}, y^{*}, z^{*}\right)$ is chosen according to the unique neighbours of $x, y, z$. In the former case $\left(x^{*}, y^{*}, z^{*}\right)$ is adjacent to $\left(v_{r}^{*}, v_{r}^{*}, z_{i}^{*}\right)$ and in the
latter $x, y, z$ are leaves that do have unique neighbours. If no neighbour $z_{i}$ of $z$ is in $T_{r}$, then each value $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{i}^{* *}\right)$ is either $x_{1}^{* *}$ or $y_{1}^{* *}$ independently of the location of $z_{i}$, and hence choosing correspondingly $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ or $=y^{*}$ will ensure the polymorphism property. If some neighbour $z_{i}$ of $z$ lies in $T_{r}, r>1$, then $z$ is the root of $T_{r-1}$, or of $T_{r}$, or of $T_{r+1}$, and in this case, we can choose $F\left(x^{*}, y^{*}, z^{*}\right)=z^{*}$. Indeed, in this case, $z z_{i}$ is bicoloured, and $z_{i}=x_{1}=y_{1}$ (if $z$ is in $T_{r-1}$ or $T_{r+1}$ ) or $z x_{1}, z y_{1}$ are also bicoloured (if $z$ is in $T_{r}$ ). If $r=1$, then in addition to the previous case the vertices $z, z_{1}, \ldots, z_{p}$ can have loops (the vertices $x, y, x_{1}, y_{1}$ do not have loops). Since $x_{1}$ and $y_{1}$ have the same colour, if the colour of $z_{i}$ is different (when $z=z_{i}$ ), we have the value of $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{i}^{* *}\right)$ equal to the first or second coordinate, and we can choose $F\left(x^{*}, y^{*}, z^{*}\right)$ accordingly. Note that these arguments apply also when $y^{*}=s\left(x^{*}\right)$, i.e., $x=y$.

If $x \in T_{r}, y \in T_{s}$ with $r \neq s$, the arguments are similar. If no neighbour $z_{j}$ of $z$ lies in $T_{r}$ or $T_{s}$, then we can choose $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ or $=y^{*}$ or $z^{*}$, depending on which is the median tree, and if some $z_{j}$ lies in, say, $T_{r}$, then we can choose $F\left(x^{*}, y^{*}, z^{*}\right)=x^{*}$ or $=z^{*}$ as above.

Next we consider the case when $x, y, z$ do not have the same colour. We may assume that $x, y$ have different colours, since if $x, y$ have the same colour and $z$ has a different colour then any relevant $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{i}^{* *}\right)$ is $x_{1}^{* *}$ or $y_{1}^{* *}$ regardless of $z_{i}$ and we can choose $F\left(x^{*}, y^{*}, z^{*}\right)$ accordingly. However, if $z$ has a loop, then we need to consider also $F\left(x_{1}^{* *}, y_{1}^{* *}, z^{* *}\right)$, which could be $z^{* *}$ or $s\left(z^{* *}\right)$ if $z$ is the vertex $v_{1}$; in that case we can set $F\left(x^{*}, y^{*}, z^{*}\right)=z^{*}$.

Thus assume without loss of generality that $x, z$ have the same colour, but the colour of $y$ is different. Then unless $z$ has a loop, $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{i}^{* *}\right)$ is $x_{1}^{* *}$ or $z_{i}^{* *}$, depending on whether $z_{i}$ precedes or follows $x_{1}$ in the DFS ordering. If all neighbours $z_{i}$ precede $x_{1}$, or all follow $x_{1}$, the uniform choice of $F\left(x_{1}^{* *}, y_{1}^{* *}, z_{i}^{* *}\right)$ allows one to choose $F\left(x^{*}, y^{*}, z^{*}\right)$ accordingly. The only situation when some $z_{i}$ precedes $x_{1}$ and another $z_{j}$ follows $x_{1}$ occurs when $z$ is the root of the tree $T_{r}$ containing $x$. It is easy to see that in that case we can set $F\left(x^{*}, y^{*}, z^{*}\right)=z^{*}$. Finally, when $z$ has a loop, then we also need to consider $F\left(x_{1}^{* *}, y_{1}^{* *}, z^{* *}\right)=z^{* *}$ and we set $F\left(x^{*}, y^{*}, z^{*}\right)=z^{*}$.

## 8. Conclusions

It seems difficult to give a full combinatorial classification of the complexity of list homomorphism problems for general signed graphs. We have accomplished this for signed trees (with possible loops). The polynomial algorithms rely on min ordering or on majority polymorphisms, and neither of the methods alone is sufficient.

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