THE RATIO OF THE NUMBERS OF ODD AND EVEN CYCLES IN OUTERPLANAR GRAPHS

AKIHIRO HIGASHITANI AND NAOKI MATSUMOTO

ABSTRACT. In this paper, we investigate the ratio of the numbers of odd and even cycles in outerplanar graphs. We verify that the ratio generally diverges to infinity as the order of a graph diverges to infinity. We also give sharp estimations of the ratio for several classes of outerplanar graphs, and obtain a constant upper bound of the ratio for some of them. Furthermore, we consider similar problems in graphs with some pairs of forbidden subgraphs/minors, and propose a challenging problem concerning claw-free graphs.

1. INTRODUCTION

How different is the number of odd cycles and that of even cycles in a graph? (The number of subgraphs means that of *distinct* subgraphs; see Subsection 1.3 for details.) This is a very natural and fundamental question in graph theory, but, as far as we know, there is no serious study for this question. Thus, we investigate the ratio of the number of odd cycles and that of even cycles in graphs, in particular, in outerplanar graphs.

In the introduction, we first provide a short survey for the study on graph polynomials some of which can be applied to evaluate the number of odd cycles and that of even cycles in outerplanar graphs. Next we summarize main previous studies on the number of cycles in graphs, and then we describe our results and the organization of this paper. For fundamental terminologies and notations undefined in this paper, we refer the reader to [5].

1.1. Graph polynomial. The chromatic polynomial is the most classical graph polynomial, introduced by Birkhoff (cf. [4]), which counts the number of proper colorings of a graph with a given number of colors. This polynomial can count the number of acyclic orientations of a graph by assigning -1 to the (unique) variable of the polynomial. Tutte [27] developed a more general graph polynomial, namely the *Tutte polynomial*. This polynomial can count forests and spanning subgraphs (or spanning forests) by suitably setting values of two variables, and specializes to the chromatic polynomial by assigning some constant to one of two variables.

²⁰²⁰ Mathematics Subject Classification. Primary: 05C30; Secondary: 05C31.

Key words and phrases. Enumeration, Outerplanar graph, Cycle, Subtree, Claw-free.

The first named author is supported by JSPS Grant-in-Aid for Scientific Research (C) 20K03513. The second named author is supported by JSPS Grant-in-Aid for Early-Career Scientists 19K14583.

For counting of spanning subgraphs of a graph, Farrell [12] introduced a *family* polynomial (or F-polynomial), which counts the number of spanning subgraphs of a graph with each component belonging to the family.

Jamison [16] developed a *subtree polynomial* which counts all subtrees of a tree, and well investigated the coefficients of the polynomial [17, 18, 19]. In particular, he discovered the following interesting fact on the difference between the number of odd subtrees and that of even subtrees, which can be used to evaluate the ratio of the number of odd cycles and that of even cycles in some family of outerplanar graphs, where a graph is *odd* (resp. *even*) if the order of the graph is odd (resp. even).

Theorem 1.1 ([18]). For any tree T, the number of odd subtrees minus that of even subtrees in T is equal to the independence number of T.

The research has been continuing on subtree polynomials; for example, see [7, 29]. A few years ago, Dod et al. [10] introduced a *bipartition polynomial* which is a common generalization of several polynomials, namely, the domination polynomial, the Ising polynomial, the matching polynomial and the cut polynomial. Note that each cycle in a planar graph G drawn on the plane without edge crossings corresponds to a cut (i.e., a set of edges which makes a graph disconnected) in the dual graph of G, where the *dual graph* of a graph G drawn on the plane is a plane graph which has a vertex for each face of G and an edge for each pair of faces in G sharing an edge of G. Thus, the cut polynomial can evaluate the number of cycles of a given planar graph even if we use this polynomial; in fact, this counting problem is #P-complete [28]. For other polynomials and related topics, see a book [25]. (We can also define the cycle polynomial whose coefficients are the number of cycles. However, this is just an analogy of other polynomials, and hence, there seems no serious study on this polynomial.¹)

1.2. The number of cycles. For a graph G, the classical estimation of the number of cycles $\nu(G)$ was given as follows:

$$\mu(G) \le \nu(G) \le 2^{\mu(G)} - 1,$$

where $\mu(G)$ is the circuit rank (or the cyclomatic number) of G. Volkmann [30] gave another lower bound to $\nu(G)$ using the minimum degree of G, and some authors study graphs G with $\nu(G) = 2^{\mu(G)} - 1$; for example, see [3, 24]. Counting cycles in graphs has probably begun in 1960s by several groups, e.g., [8, 14]. Khomenko and Golovko [21] gave a formula counting the number of cycles of a given length using the adjacency matrix. However, described as in the previous subsection, since

¹For several graphs, the cycle polynomial is determined; see https://mathworld.wolfram.com/CyclePolynomial.html

counting cycles in a given graph is hard in general, there are many studies on the number of cycles for particular graph classes; for example, see [2].

It is deeply and widely studied to count the number of hamiltonian cycles. It is well known that determining whether a graph has a hamiltonian cycle is NPhard [20], and hence, many authors count the number of hamiltonian cycles in prescribed graph classes, e.g., bipartite graphs [26], regular graphs [15] and planar triangulations [6]. For related studies and other topics, see surveys [13, 23], and for the directed version of such problems, see another survey [22]. Moreover, we refer the reader to a more recent paper [1] which investigates the number of (hamiltonian) cycles in planar graphs with prescribed connectivity.

Despite of those several kinds of results on the number of cycles, there seems no result on the difference between the number of odd cycles and that of even cycles as far as we know.

1.3. Contribution. We first introduce terminologies and notations to mention our results. All graphs considered in this paper are finite simple undirected graphs. For a graph G, we denote by V(G), E(G) and F(G) the set of vertices, edges and faces, respectively. Note that we define F(G) only if G is embedded on some surface. In this paper, the number of subgraphs means the number of distinct subgraphs, where two subgraphs H_1 and H_2 of G are *distinct* if $V(H_1) \neq V(H_2)$ or $E(H_1) \neq E(H_2)$. Let $c_o(G)$ and $c_e(G)$ denote the number of odd cycles and that of even cycles, respectively.

Let G be an outerplanar graph embedded on the plane so that all vertices lie on the boundary walk of the infinite face, where a *finite* (resp. *infinite*) face of G is a face of G with its boundary walk bounding a finite (resp. infinite) region. In what follows, an outerplanar graph means one embedded on the plane. Note that there is only one infinite face for any outerplanar graph. The boundary walk of the infinite face of G is denoted by ∂G and an edge e of G is *diagonal* if e does not belong to ∂G . We say that ∂G is odd (resp. even) if the length of ∂G is odd (resp. even). When ∂G is an even cycle, we always color ∂G by two colors, black and white. In this case, an *odd* (resp. *even*) *chord* of G is a diagonal of G joining two vertices with the same color (resp. distinct colors). When ∂G is a cycle, the *dual tree* of G, denoted by T_G , is the graph obtained from the dual graph of G by deleting the vertex corresponding to the infinite face of G.

In this paper, for an outerplanar graph G where ∂G is a cycle, i.e., G is 2connected, we have the following results on $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$:

- Both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ diverge to infinity as $|V(G)| \to \infty$ in general, not depending on the parity of ∂G (Theorem 2.4).
- If the size of each finite face of an outerplanar graph G is odd and $|F(G)| \ge 3$, then $\frac{|F(G)|-1}{2c_e(G)} + 1 \le c_o(G)/c_e(G) \le \frac{|F(G)|-2}{c_e(G)} + 1$ (Theorem 3.1).

- If ∂G is even, then $c_o(G)/c_e(G) \leq k$, where k is the number of odd chords, and this bound is sharp (Theorem 3.2).
- If ∂G is even and the dual tree T_G is a path, then $c_o(G)/c_e(G) \leq 2$, and this bound is sharp (Proposition 3.3).
- If the dual tree T_G is a star and there is at least one odd face corresponding to a leaf of T_G , then both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ converge to 1 as $|F(G)| \to \infty$ (Proposition 3.4).

Note that if a graph G is bipartite, then $c_o(G) = 0$ always holds. Thus, we deal with only non-bipartite graphs in what follows. Moreover, if an outerplanar graph G is not 2-connected, then we can obtain similar results by applying the above results to each block of G (see Section 5).

In Section 4, we consider conditions for a graph G concerning forbidden subgraphs/minors that $c_o(G)/c_e(G)$ or $c_e(G)/c_o(G)$ is bounded by some constant. Recall that every outerplanar graph is characterized as a $K_{2,3}$ -minor-free and K_4 -minor-free graph [5, Exercise 10.5.12]. Our first result (Theorem 2.4) implies that the $K_{2,3}$ minor-free and K_4 -minor-free condition is not sufficient to bound $c_o(G)/c_e(G)$ or $c_e(G)/c_o(G)$ by a constant. Moreover, by application of Wagner's proof [31], every 2-connected graph is $K_{2,3}$ -minor-free if and only if it is either isomorphic to K_4 or outerplanar (cf. [11]). Hence, replacing $K_{2,3}$ -minor-free with $K_{1,3}$ -free (or, popularly, claw-free), we show that every $K_{1,3}$ -free and K_4 -minor-free graph is outerplanar, and we completely characterize the structure of such outerplanar graphs (Theorem 4.1). As a corollary, for any $K_{1,3}$ -free and K_4 -minor-free graph G, both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ are bounded by a constant unless G is a cycle (Corollary 4.2).

1.4. Organization of the paper. In the next section, we construct 2-connected outerplanar graphs G such that $c_o(G)/c_e(G)$ or $c_e(G)/c_o(G)$ diverges to infinity. In Section 3, we show upper/lower bounds of the ratio of the number of odd cycles and that of even cycles in 2-connected outerplanar graphs. In Section 4, we consider the ratio of the number of odd cycles and that of even cycles in graphs characterized by forbidden subgraphs/minors regarding to outerplanar graphs. In the final section, we give concluding remarks and future perspectives.

2. Outerplanar graphs with many odd/even cycles

We first introduce two particular 2-connected outerplanar graphs G such that ∂G is odd.

The graph \mathcal{R}_k : Given an odd number k with $k \geq 3$, prepare a cycle $C_k = v_0 v_1 v_2 \dots v_{k-1} v_0$ of length k. Prepare k copies of the 4-cycle denoted by D_0, D_1, \dots, D_{k-1} . For each $i \in \{0, 1, \dots, k-1\}$, identify an edge of D_i and $v_i v_{i+1}$ where the subscripts are modulo k. The resulting 2-connected outerplanar graph is denoted by \mathcal{R}_k ; see Figure 1.

The graph \mathcal{L}_m : Given an odd number m with $m \geq 3$, prepare a path $P_m = v_0 v_1 v_2 \dots v_{m-1}$ of order m. For each $i \in \{0, \dots, (m-3)/2\}$, join v_i and v_{m-1-i} . The resulting 2-connected outerplanar graph is denoted by $\mathcal{L}_{(m-1)/2}$; see Figure 2.

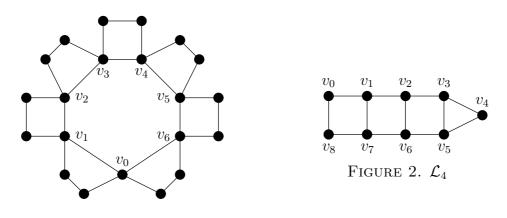


FIGURE 1. \mathcal{R}_7

Proposition 2.1.

(1) $c_o(\mathcal{R}_k)/c_e(\mathcal{R}_k)$ diverges to infinity as $k \to \infty$. (2) $c_e(\mathcal{L}_m)/c_o(\mathcal{L}_m)$ diverges to infinity as $m \to \infty$.

Proof. (1) The number of even cycles in \mathcal{R}_k is exactly k (i.e., equal to the number of 4-cycles). On the other hand, since whether an odd cycle passes an edge shared by C_k and a 4-cycle does not change the parity of the length of the cycle, we see that the number of odd cycles is equal to 2^k . Hence, $c_o(\mathcal{R}_k)/c_e(\mathcal{R}_k) = 2^k/k \to \infty$ as $k \to \infty$.

(2) Since every odd cycle passes $v_{m-1}v_mv_{m+1}$, the number of odd cycles in \mathcal{L}_m is equal to m (i.e., equal to the number of added edges in the construction). On the other hand, since every even cycle passes exactly two added edges, the number of even cycles is equal to $\binom{m}{2}$. Hence, $c_e(\mathcal{L}_m)/c_o(\mathcal{L}_m) = \binom{m}{2}/m \to \infty$ as $m \to \infty$. \Box

Next we show a similar result as above for 2-connected outerplanar graphs G such that ∂G is even. Let $q \equiv 0 \pmod{4}$ be a positive integer. Let \mathcal{H}_q be the graph obtained from two copies of $\mathcal{L}_{q/2}$ by identifying two $v_{q/2-1}v_{q/2}$'s of them, where the labels are as in the construction of $\mathcal{L}_{q/2}$; see Figure 3. Let \mathcal{T}_n be the graph obtained from \mathcal{R}_n and \mathcal{L}_n for some odd $n \geq 3$ by identifying an edge u_2u_3 of a 4-cycle $u_0u_1u_2u_3u_0$ in \mathcal{R}_n with $u_0u_1 \in E(C_n)$ and an edge v_0v_{2n} , where the labels are as in the construction of \mathcal{L}_n ; see Figure 4. Note that \mathcal{H}_q and \mathcal{T}_n are 2-connected non-bipartite outerplanar graphs and the boundary cycle of each infinite face is even. Moreover, \mathcal{H}_q has exactly one odd chord and \mathcal{L}_n has exactly (n + 1) odd chords.

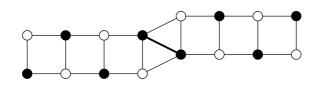


FIGURE 3. \mathcal{H}_8 ; a bold line denotes the identified edge.

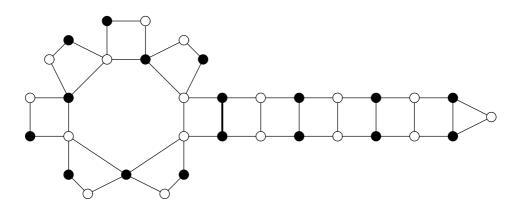


FIGURE 4. \mathcal{T}_7 ; a bold line denotes the identified edge.

To show the desired claim, we prepare the following useful observation. For a graph G and an edge $f \in E(G)$, let $c_o(G, f)$ and $c_e(G, f)$ denote the number of odd cycles and that of even cycles passing f in G, respectively.

Observation 2.2. Let G_1 and G_2 be graphs, and let e_1 and e_2 be edges of G_1 and G_2 , respectively. Let G be the graph obtained from G_1 and G_2 by identifying e_1 and e_2 . Then the following equalities hold:

(1)
$$c_o(G) = c_o(G_1) + c_o(G_2) + c_o(G_1, e_1)c_e(G_2, e_2) + c_e(G_1, e_1)c_o(G_2, e_2)$$

(2)
$$c_e(G) = c_e(G_1) + c_e(G_2) + c_o(G_1, e_1)c_o(G_2, e_2) + c_e(G_1, e_1)c_e(G_2, e_2)$$

Proposition 2.3.

(1) $c_o(\mathcal{T}_n)/c_e(\mathcal{T}_n)$ diverges to infinity as $n \to \infty$. (2) $c_e(\mathcal{H}_q)/c_o(\mathcal{H}_q)$ diverges to infinity as $q \to \infty$.

Proof. By the proof of Proposition 2.1, we have

$$c_o(\mathcal{R}_n) = 2^n$$
, $c_e(\mathcal{R}_n) = n$, $c_o(\mathcal{L}_n) = n$ and $c_e(\mathcal{L}_n) = \binom{n}{2}$.

Let e be an edge of a 4-cycle in \mathcal{R}_n with each end not lying on C_n . Let $f_1 = v_0 v_{2n}$ and $f_2 = v_{n-1} v_n$ be the edges of \mathcal{L}_n where the labels of vertices are as in its

construction. Then we have

$$c_o(\mathcal{R}_n, e) = 2^{n-1} \quad \text{and} \quad c_e(\mathcal{R}_n, e) = 1,$$

$$c_o(\mathcal{L}_n, f_1) = 1 \quad \text{and} \quad c_e(\mathcal{L}_n, f_1) = n - 1$$

$$c_o(\mathcal{L}_n, f_2) = n \quad \text{and} \quad c_e(\mathcal{L}_n, f_2) = 0.$$

Thus, by Observation 2.2, we have

$$\frac{c_o(\mathcal{T}_n)}{c_e(\mathcal{T}_n)} = \frac{c_o(\mathcal{R}_n) + c_o(\mathcal{L}_n) + c_o(\mathcal{R}_n, e)c_e(\mathcal{L}_n, f_1) + c_e(\mathcal{R}_n, e)c_o(\mathcal{L}_n, f_1)}{c_e(\mathcal{R}_n) + c_e(\mathcal{L}_n) + c_o(\mathcal{R}_n, e)c_o(\mathcal{L}_n, f_1) + c_e(\mathcal{R}_n, e)c_e(\mathcal{L}_n, f_1)} \\ = \frac{2^n + n + 2^{n-1}(n-1) + 1}{n + \binom{n}{2} + 2^{n-1} + n - 1} \\ = \frac{n+1 + (n+1)/2^{n-1}}{1 + (n^2 + 3n - 2)/2^n} \to \infty \quad \text{as } n \to \infty,$$

and

$$\frac{c_e(\mathcal{H}_q)}{c_o(\mathcal{H}_q)} = \frac{2c_e(\mathcal{L}_q) + c_o(\mathcal{L}_q, f_2)^2 + c_e(\mathcal{L}_q, f_2)^2}{2c_o(\mathcal{L}_q) + 2c_o(\mathcal{L}_q, f_2)c_e(\mathcal{L}_q, f_2)} = q - \frac{1}{2} \to \infty \quad \text{as } q \to \infty.$$

Therefore, the proposition holds.

The following theorem is a direct consequence of Propositions 2.1 and 2.3.

Theorem 2.4. There are 2-connected outerplanar graphs G such that for any $r \in \{c_o(G)/c_e(G), c_e(G)/c_o(G)\}$, r diverges to infinity as $|V(G)| \to \infty$, not depending on the parity of ∂G .

3. Bounds

We first consider outerplanar graphs all of whose finite faces are odd, where a finite face f is odd (resp. even) if the length of the boundary walk of f is odd (resp. even). In this case, we can directly apply Theorem 1.1 to evaluate the number of odd/even cycles.

Theorem 3.1. Let G be a 2-connected outerplanar graph with each finite face odd and $|F(G)| \geq 3$. Then $\frac{|F(G)|-1}{2c_e(G)} + 1 \leq c_o(G)/c_e(G) \leq \frac{|F(G)|-2}{c_e(G)} + 1$. In particular, $1 \leq c_o(G)/c_e(G) \leq 2$.

Proof. By the assumption, for any subtree H of the dual tree T_G , the parity of the order of H is the same as that of the cycle in G bounding a union of faces corresponding to vertices of H. Thus, Theorem 1.1 implies that $c_o(G) - c_e(G) =$ $\alpha(T_G)$, where $\alpha(T_G)$ denotes the independence number of T_G . Since $\frac{|F(G)|-1}{2} \leq$ $\alpha(T_G) \leq (|F(G)| - 1) - 1$, we obtain that $1 \leq \frac{|F(G)|-1}{2c_e(G)} + 1 \leq c_o(G)/c_e(G) \leq$ $\frac{|F(G)|-2}{c_e(G)} + 1$. Since $c_e(G)$ is equal to the number of even subtrees of T_G , one has $c_e(G) \geq |E(T_G)| = |F(G)| - 2 \geq 1$ by $|F(G)| \geq 3$. Hence, $c_o(G)/c_e(G) \leq 2$ also holds. Next, we give a general upper bound of $c_o(G)/c_e(G)$ for a 2-connected outerplanar graph G with even ∂G as follows.

Theorem 3.2. Let G be a 2-connected outerplanar graph with even ∂G and let $k \ge 0$ be the number of odd chords of G. Suppose that G has at least one even chord. Then $c_o(G)/c_e(G) \le k$. Furthermore, this bound is sharp.

Proof. We prove the theorem by induction on k. If k = 0, then $c_o(G)/c_e(G) = 0$, so we are done. Assume that $k \ge 1$ and the theorem holds for any non-negative integer smaller than k.

Let e be an odd chord of G and let G' be the 2-connected outerplanar graph obtained from G by removing e. Since G' has k-1 odd chords and at least one even chord, $c_o(G')/c_e(G') \leq k-1$ by induction hypothesis. On the other hand, Gcan be obtained from two outerplanar graphs G_1 and G_2 by identifying e, where e lies on both ∂G_1 and ∂G_2 . Thus, $c_e(G') = c_o(G_1, e)c_o(G_2, e) + \alpha$, where $\alpha \geq 1$ since G' has at least one even chord. By $c_o(G')/c_e(G') \leq k-1$, we know $c_o(G') \leq (k-1) (c_o(G_1, e)c_o(G_2, e) + \alpha)$. Note that $c_e(G) \geq c_e(G')$. Therefore, we have

$$\frac{c_o(G)}{c_e(G)} \le \frac{(k-1)\left(c_o(G_1, e)c_o(G_2, e) + \alpha\right) + c_o(G_1, e) + c_o(G_2, e)}{c_o(G_1, e)c_o(G_2, e) + \alpha} \le (k-1) + \frac{c_o(G_1, e) + c_o(G_2, e)}{c_o(G_1, e)c_o(G_2, e) + \alpha} \le k,$$

where the final inequality holds by $\alpha \geq 1$ and $c_o(G_1, e), c_o(G_2, e) \geq 1$.

Regarding the sharpness of $c_o(G)/c_e(G) \leq k$, since \mathcal{T}_k has (k+1) odd chords and at least one even chord when $k \geq 3$, we see that $c_o(\mathcal{T}_k)/c_e(\mathcal{T}_k) = \frac{k+1+(k+1)/2^{k-1}}{1+(k^2+3k-2)/2^k} \geq k$ if $k \geq 11$. Therefore, this upper bound is the best possible. \Box

In the remaining part of this section, we show upper/lower bounds of the ratio for outerplanar graphs with prescribed dual trees, namely a path and a star.

Proposition 3.3. Let G be a 2-connected outerplanar graph with even ∂G and assume that the dual tree T_G of G is a path. Then $c_o(G)/c_e(G) \leq 2$. Furthermore, this bound is sharp.

Proof. Let a and b be the number of odd and even chords of G, respectively. Since T_G is a path, we have

$$c_o(G) = a \cdot b + 2a = a(b+2)$$
 and $c_e(G) = {b \choose 2} + {a \choose 2} + 2b + 1$

We suppose to the contrary that $c_o(G)/c_e(G) > 2$. This together with the above two equations leads to the following:

$$\frac{c_o(G)}{c_e(G)} = \frac{2a(b+2)}{a(a-1) + (b+1)(b+2)} > 2 \iff a(b+2) > a(a-1) + (b+1)(b+2).$$

By this inequality, we see that a(b-a+3) > (b+1)(b+2) > 0, so $b-a+3 \ge 1$, i.e., $a \le b+2$. We also see that $(a-b-1)(b+2) > a(a-1) \ge 0$, so $a \ge b+1$. Thus, we obtain that a = b+1 or b+2. However, for each of those two cases, we have $c_o(G)/c_e(G) = (b+2)/(b+1) \le 2$, a contradiction.

Regarding the sharpness of this bound, every outerplanar graph G obtained from an even cycle by adding exactly one odd chord attains the equality.

Proposition 3.4. Let G be a 2-connected outerplanar graph and assume that the dual tree T_G of G is a star. If there is at least one odd face corresponding to a leaf of T_G , then both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ converge to 1 as $|F(G)| \to \infty$.

Proof. First suppose that ∂G is even. Let a and b be the number of odd chords and that of even chords of G, respectively; note that $a \geq 1$ by the assumption. Let f be the face in G corresponding to the center vertex of T_G . Since T_G is a star and ∂G is even, the parity of the number of odd chords lying on ∂f is the same as that of the length of ∂f . If a is even, then we have

$$c_o(G) = 2^b \sum_{k=0}^{a/2-1} {a \choose 2k+1} + a = 2^{a+b-1} + a, \text{ and}$$
$$c_e(G) = 2^b \sum_{k=0}^{a/2} {a \choose 2k} + b = 2^{a+b-1} + b.$$

If a is odd, then we have

$$c_o(G) = 2^b \sum_{k=0}^{(a-1)/2} {a \choose 2k} + a = 2^{a+b-1} + a, \text{ and}$$
$$c_e(G) = 2^b \sum_{k=0}^{(a-1)/2} {a \choose 2k+1} + b = 2^{a+b-1} + b.$$

Thus, regardless of the parity of a, $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ converge to 1 as $|F(G)| \to \infty$ (i.e., $a + b \to \infty$) since $2^{a+b-1} \gg a + b$.

Next suppose that ∂G is odd. Let f be the face in G corresponding to the center vertex of T_G . Let a (resp. b) be the number of chords shared by f and an odd (resp. even) face corresponding to a leaf of T_G . Since T_G is a star and ∂G is odd, the parity of the length of ∂f is opposite to the parity of the number of odd chords lying on ∂f . Thus, the number of odd and even cycles can be calculated similarly to the first case depending on the opposite parity of a.

4. FORBIDDEN SUBGRAPHS/MINORS CONDITION

A graph G is *H*-free (resp. *H*-minor-free) if G contains no H as its induced subgraph (resp. as a minor). Recall that Theorem 2.4 implying that the K_4 -minor-free and $K_{2,3}$ -minor-free condition is not sufficient to bound $c_o(G)/c_e(G)$ or $c_e(G)/c_o(G)$ by a constant for a 2-connected outerplanar graph G. Thus, we consider another forbidden subgraphs/minors condition which bounds $c_o(G)/c_e(G)$ or $c_e(G)/c_o(G)$ by a constant.

It is easy to see that for any $k \geq 3$, there is a 2-connected C_k -free outerplanar graph such that neither $c_o(G)/c_e(G)$ nor $c_e(G)/c_o(G)$ can be bounded by a constant since we can make each cycle in the constructions of \mathcal{R} and \mathcal{L} be arbitrarily long. Of course, no 2-connected graph of order at least 3 is C_3 -minor-free, and every H-minor-free graph is H-free. Therefore, we focus on a 2-connected $K_{1,3}$ -free (or claw-free) and K_4 -minor-free graphs.

We here introduce two particular outerplanar graphs. Let $C_t = v_0 v_1 \dots v_{t-1}$ be a cycle with $t \geq 3$. For several indices $i \in \{0, 1, \dots, t-1\}$, we add a vertex r_i to make a triangle $v_i v_{i+1} r_i$, where subscripts are modulo t (see Figure 5). The set of 2^t resulting graphs constructed above is denoted by \mathbb{S}_t . Let \mathbb{Z}_d be the outerplanar graph obtained from \mathcal{L}_d (where recall that $d \geq 1$) by joining v_i and v_{2d-1-i} for each $i \in \{0, \dots, d-2\}$ (see Figure 6), and let \mathbb{Z}_d^* be the graph obtained from \mathbb{Z}_d by removing v_d . Observe that $\mathbb{Z}_d, \mathbb{Z}_d^*$ and any graph in \mathbb{S}_t are 2-connected outerplanar graphs and that they are $K_{1,3}$ -free.

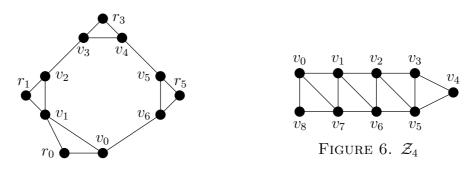


FIGURE 5. A graph in S_7

Theorem 4.1. If a 2-connected graph G is $K_{1,3}$ -free and K_4 -minor-free, then G is outerplanar. Furthermore, G is isomorphic to $\mathcal{Z}_d, \mathcal{Z}_d^*$ or a graph in \mathbb{S}_t .

Proof. Let G be a 2-connected $K_{1,3}$ -free and K_4 -minor-free graph. Note that G has no vertex of degree at least 5; otherwise, G has an induced $K_{1,3}$ or a K_4 -minor consisting of the vertex and its neighbors. Thus, $\deg_G(x) \leq 4$ for every vertex x of G, where $\deg_G(x)$ denotes the degree of a vertex x in G.

It is well known that every K_4 -minor-free graph is a subgraph of a 2-tree [9], where a 2-tree is a graph obtained from a triangle by repeatedly adding vertices in such a way that each added vertex has two adjacent neighbors (i.e., those three vertices induce a triangle). Hence, since every 2-tree is planar, we embed G into the plane in such a way that the length of the boundary cycle of the outer face is the longest among all planar embeddings of G. Let $C = u_0 u_1 \dots u_{m-1}$ for some $m \ge 3$ be the outer cycle of G. If all vertices of G lie on C, then G is outerplanar. Thus, we may suppose that $u_i \in V(C)$ has degree 3 or 4 and to the contrary that u_i has a neighbor not lying on C. Let $K_4(a, b, c, d)$ denote a K_4 -minor in G consisting of four vertices a, b, c, d and internally disjoint paths between them, where two paths are *internally disjoint* if they do not share vertices except their end vertices.

First suppose $\deg_G(u_i) = 3$. Let a be a unique neighbor of u_i which does not lie on C. A path P between two vertices x and y in which any vertex of P does not lie on C except x, y is called an *inner* (x, y)-*path*. Since G is $K_{1,3}$ -free, we may assume by symmetry that $u_{i-1}a \in E(G)$ or $u_{i-1}u_{i+1} \in E(G)$. Moreover, since G is K_4 -minor-free, there is at most one of an inner (a, u_{i-1}) -path and an inner (a, u_{i+1}) -path, say the former.

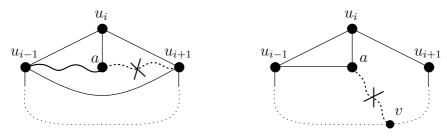


FIGURE 7. The case when $\deg_G(u_i) = 3$

See Figure 7. The left depicts the case when $u_{i-1}u_{i+1} \in E(G)$ and there is an inner (a, u_{i-1}) -path. In this case, if there is also an inner (a, u_{i+1}) -path, then G has a K_4 -minor $K_4(u_{i-1}, u_i, u_{i+1}, a)$, a contradiction. The right depicts the case when $u_{i-1}a \in E(G)$ but $u_{i-1}u_{i+1} \notin E(G)$. In this case, if there is an inner (a, v)-path for some $v \in V(C) \setminus \{u_{i-1}, u_i\}$, then G has a K_4 -minor $K_4(u_{i-1}, u_i, v, a)$. Thus, in either case, we can obtain another outerplanar embedding of G with outer cycle longer than C, by making the outer cycle pass $u_i a$ and an inner (a, u_{i-1}) -path instead of $u_{i-1}u_i$. Intuitively, we can obtain this embedding by applying a "jump" of the edge $u_{i-1}u_i$ over a as shown in Figure 8. This contradicts that C is the longest outer cycle.

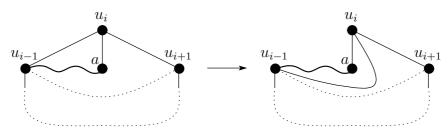


FIGURE 8. A "jump" of $u_{i-1}u_i$

Next suppose $\deg_G(u_i) = 4$. Let a, b be neighbors of u_i with $a, b \notin \{u_{i-1}, u_i\}$. By symmetry, we first consider the case when $b \in V(C)$ (see Figure 9). Since four vertices u_{i-1}, u_i, u_{i+1}, a do not induce a $K_{1,3}, u_{i-1}a \in E(G)$. Thus, G has no inner (a, v)-path for any $v \in V(C) \setminus \{u_{i-1}, u_i\}$; otherwise G has a K_4 -minor $K_4(u_{i-1}, u_i, a, v)$. Therefore, similarly to the case when $\deg_G(u_i) = 3$, we can obtain another outerplanar embedding of G with outer cycle longer than C by applying a "jump" of the edge $u_{i-1}u_i$ over a.

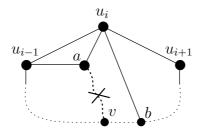


FIGURE 9. The case when $\deg_G(u_i) = 4$ and $b \in V(C)$

We next consider the case when neither a nor b lies on C. Any four vertices in $\{u_{i-1}, u_i, u_{i+1}, a, b\}$ do not induce a $K_{1,3}$, we have one of the following configurations by symmetry: (1) $u_{i-1}u_{i+1}$, $ab \in E(G)$, (2) $u_{i-1}a, ab \in E(G)$ but $u_{i-1}u_{i+1} \notin E(G)$, and (3) $u_{i-1}a, bu_{i+1} \in E(G)$ (see Figure 10). In the case (1), there is at most one of an inner (a, u_{i-1}) -path and an inner (b, u_{i+1}) -path, and so we may assume that there is the former one (see the left of Figure 10).

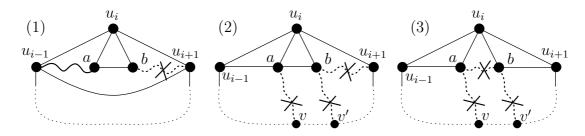


FIGURE 10. The case when $\deg_G(u_i) = 4$

By similar arguments in the case when $\deg_G(u_i) = 3$, as depicted in Figure 10, (1) there is no inner (b, u_{i+1}) -path, (2) there is neither an inner (a, v)-path, an inner (b, v')-path nor an inner (b, u_{i+1}) -path, (3) there is neither an inner (a, v)-path, an inner (b, v')-path nor an inner (a, b)-path, where $v, v' \in V(C) \setminus \{u_{i-1}, u_i, u_{i+1}\}$. Thus, similarly to the above cases, we can obtain another outerplanar embedding of G with outer cycle longer than C, by applying a "jump" of the edge $u_{i-1}u_i$ over a, b in cases (1) and (2) and over only a in case (3).

Therefore, we can conclude that G is outerplanar.

Next, we show that G is isomorphic to $\mathcal{Z}_d, \mathcal{Z}_d^*$ or a graph in \mathbb{S}_t . If $|V(G)| \leq 4$, then we can easily verify that the theorem holds, and hence, we assume that $|V(G)| \geq 5$.

Note that every vertex of degree at least 3 in a $K_{1,3}$ -free graphs belongs to at least one triangle. Thus, we divide the proof into the following two cases.

Case 1. There is a pair of triangles sharing an edge.

Let xuv and uvy be two triangles sharing an edge uv. If $\deg_G(u) = \deg_G(v) = 3$, then $|V(G)| \leq 4$ since G is 2-connected and outerplanar, a contradiction. Thus, at least one of u and v is of degree 4, say u. Let w_1 be a neighbor of u other than v, x, y. Since G is $K_{1,3}$ -free, w_1 is adjacent to y by symmetry. If $\deg_G(v) = \deg_G(y) = 3$, then $G \cong \mathbb{Z}_2$ and if there is a triangle vyz, then $G \in S_3$. Thus, $\deg_G(v) = 4$ or $\deg_G(y) = 4$ but there is no triangle sharing vy with uvy.

We consider only the case when $\deg_G(v) = 3$ and $\deg_G(y) = 4$, since two other cases (1) $\deg_G(v) = \deg_G(y) = 4$ and (2) $\deg_G(v) = 4$ and $\deg_G(y) = 3$ may be similarly proved. Let w_2 be the fourth neighbor of y (other than u, v, w_1). Since Gis $K_{1,3}$ -free and has no triangle sharing vy with uvy, G has a triangle w_1w_2y ; observe that w_2 can be adjacent to neither x nor v by the outerplanarity of G. Then we next consider whether $\deg(w_1) = 4$. By repeating this argument, we can conclude that $G \cong \mathbb{Z}_d$ or $G \cong \mathbb{Z}_d^*$ for some $d \ge 2$.

Case 2. Otherwise.

We may assume that any two triangles of G do not share an edge. In this case, we show that $G \in \mathbb{S}_t$ for some t. Let $\partial G = u_0 u_1 \dots u_{n-1}$. Since it follows from what G is $K_{1,3}$ -free that any chord of G is contained in a triangle, it suffices to show that every triangle of G is $u_{i-1}u_iu_{i+1}$, that is, the three vertices are consecutive on ∂G , where the subscripts are modulo n. Suppose to the contrary that there is a triangle $u_iu_ju_k$ with $|j - i| \geq 2$, $j \neq i - 2$ and i < j < k. In this case, we have $\deg_G(u_i) = 4$ or $\deg_G(u_j) = 4$, and hence, without loss of generality, we suppose $\deg_G(u_i) = 4$. Similarly to the proof of Case 1, u_j (or u_k) has to adjacent to u_{i+1} (or u_{i-1}), which contradicts that any two triangles do not share an edge. Therefore, the three vertices of each triangle of G are consecutive on ∂G , which implies that $G \in \mathbb{S}_t$ for some t, since any two triangles share at most one vertex.

By Theorem 4.1, we have the following corollary.

Corollary 4.2. If a 2-connected graph G is $K_{1,3}$ -free and K_4 -minor-free, then both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ are bounded by a constant unless G is a cycle.

Proof. Let G be a 2-connected $K_{1,3}$ -free and K_4 -minor-free graph. By Theorem 4.1, G is isomorphic to $\mathcal{Z}_d, \mathcal{Z}_d^*$ or a graph in \mathbb{S}_t . If G is isomorphic to \mathcal{Z}_d or \mathcal{Z}_d^* , then we are done by Theorem 3.1 since every finite face of G is triangular. If $G \in \mathbb{S}_t$ and G is not a cycle, then we are also done by Proposition 3.4 since there is at least one triangular face corresponding to a leaf of a dual tree.

As noted in the proof of Theorem 4.1, every vertex of degree at least 3 in a $K_{1,3}$ free graph belongs to a triangle. Intuitively, if there is a cycle C passing exactly one

edge of a triangle xyz, say xy, then there is another cycle C' passing xzy (ignoring the details whether C passes z). It is notable that the parity of the lengths of Cand that of C' are different, that is, we guess that the number of odd cycles in a 2-connected $K_{1,3}$ -free graph is almost the same as that of even cycles. Therefore, we conclude this section proposing the following challenging conjecture. (Note that we can easily construct infinitely many non-2-connected $K_{1,3}$ -free graphs G such that $c_o(G)/c_e(G) \to \infty$ as $|V(G)| \to \infty$.)

Conjecture 4.3. For any 2-connected $K_{1,3}$ -free graph G, both $c_o(G)/c_e(G)$ and $c_e(G)/c_o(G)$ are bounded by a constant unless G is a cycle.

5. Concluding Remarks

Throughout this paper, we addressed only 2-connected outerplanar graphs. If an outerplanar graph G is not 2-connected, then G consists of several blocks such that each two blocks share at most one vertex, where a *block* of G is a maximal 2-connected subgraph of G. Note that each cycle in G consists of edges in exactly one block. Therefore, by applying our results to each block, we can obtain the corresponding results for any non-2-connected outerplanar graphs. (We do not specifically write the statements of the corresponding results since it is just a tedious routine.)

For several 2-connected outerplanar graphs G with prescribed dual trees, a path, a star and a broom (the dual tree of \mathcal{T}_n), we can show the results on the sharpness of the ratio of the numbers of odd and even cycles. It is a natural open problem to evaluate the ratio for 2-connected outerplanar graphs with other dual trees. In particular, it is of interest to evaluate the ratio using the number of leaves of a dual tree.

For other particular graph classes, e.g., planar graphs, the analysis of the ratios seems more difficult and complicated even if the graph is a planar triangulation. Thus, we need to find some reasonable assumption or forbidden subgraphs/minors conditions for such graph classes, and we believe that the study on this problem will give a huge contribution to the graph theory.

References

- A. Alahmadi, R.E.L. Aldred and C. Thomassen, Cycles in 5-connected triangulations, J. Combin. Theory Ser. B 140 (2020), 27–44.
- [2] B.F. AlBdaiwi, On the number of cycles in a graph, Math. Slovaca 68 (2018), 1–10.
- [3] A. Arman, D.S. Gunderson and S. Tsaturian, Triangle-free graphs with the maximum number of cycles, *Discrete Math.* 339 (2016), 699–711.
- [4] G.D. Birkhoff and D.C. Lewis, Chromatic polynomials, Trans. Amer. Math. Soc. 60 (1946), 355-451.
- [5] J.A. Bondy and U.S.R. Murty Graph Theory, Grad. Texts in Math. 244 Springer (2008).

- [6] G. Brinkmann, J. Souffriau, N. Van Cleemput, On the number of hamiltonian cycles in triangulations with few separating triangles, J. Graph Theory 87 (2018), 164–175.
- [7] J.I. Brown and L. Mol, On the roots of the subtree polynomial, European J. Combin. 89 (2020), #103181.
- [8] D. Cartwright and T.C. Gleason, The number of paths and cycles in a digraph, *Psychometrika* 31 (1966), 179–199.
- [9] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952), 85–92.
- [10] M. Dod, T. Kotek, J. Preen and P. Tittmann, Bipartition polynomials, the Ising model and domination in graphs, *Discuss. Math. Graph Theory* 35 (2015), 335–353.
- [11] M.N. Ellingham, E.A. Marshall, K. Ozeki and S. Tsuchiya, A characterization of K_{2,4}-minorfree graphs, SIAM J. Discrete Math. **30** (2016), 955–975.
- [12] E.J. Farrell, On a general class of graph polynomials, J. Combin. Theory Ser. B 26 (1979), 111–122.
- [13] R.J. Gould Advances on the Hamiltonian problem: a survey Graphs Combin. 19 (2003), 7–52.
- [14] F. Harary and B. Manvel, On the number of cycles in a graph, Matematický časopis 21 (1971), 55–63.
- [15] M. Haythorpe, On the minimum number of hamiltonian cycles in regular graphs, *Exp. Math.* 27 (2018), 426–430.
- [16] R.E. Jamison, On the average number of nodes in a subtree of a tree, J. Combin. Theory Ser. B 35 (1983), 207–223.
- [17] R.E. Jamison, Monotonicity of the mean order of subtrees, J. Combin. Theory Ser. B 37 (1984), 70–78.
- [18] R.E. Jamison, Alternating whitney sums and matchings in trees, part 1, Discrete Math. 67 (1987), 177–189.
- [19] R.E. Jamison, Alternating whitney sums and matchings in trees, part 2, Discrete Math. 79 (1990), 177–189.
- [20] R.M. Karp, Reducibility among combinatorial problems, Complexity of computer computations, Springer (1972), 85–103.
- [21] N.P. Khomenko and L.D. Golovko, Identifying certain types of parts of a graph and computing their number, Ukrainian Math. J. 24 (1972), 313–321.
- [22] D. Kühn and D. Osthus, A survey on Hamilton cycles in directed graphs, *European J. Comb.* 33 (2012), 750–766.
- [23] K. Ozeki, N. Van Cleemput, C.T. Zamfirescu, Hamiltonian properties of polyhedra with few 3-cuts – A survey, *Discrete Math.* **341** (2018), 2646–2660.
- [24] D. Rautenbach and I. Stella, On the maximum number of cycles in a Hamiltonian graph, Discrete Math. 304 (2005), 101–107.
- [25] Y. Shi, M. Dehmer, X. Li, I. Gutman, Graph Polynomials, CRC Press (2016).
- [26] C. Thomassen, On the number of Hamiltonian cycles in bipartite graphs, Combin. Probab. Comput. 5 (1996), 437–442.
- [27] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954), 80–91.
- [28] T.C. Vijayaraghavan, Checking equality of matroid linear representations and the cycle matching problem, *Electron. Colloquium Comput. Complex.* 16 (2009), #9.
- [29] A. Vince and H. Wang, The average order of a subtree of a tree, J. Combin. Theory Ser. B 100 (2010), 161–170.

[30] L. Volkmann, Estimations for the number of cycles in a graph, *Period. Math. Hungar.* **33** (1996), 153–161.

[31] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937), 570–590.

(A Higashitani) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

Email address: higashitani@ist.osaka-u.ac.jp

(N. Matsumoto) Research Institute for Digital Media and Content, Keio University, Yokohama, Kanagawa 232-0062, Japan

Email address: naoki.matsumo10@gmail.com