# A Greedy Partition Lemma for Directed Domination

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**Abstract**

A directed dominating set in a directed graph $D$ is a set $S$ of vertices of $V$ such that every vertex $u \in V(D) \setminus S$ has an adjacent vertex $v$ in $S$ with $v$ directed to $u$. The directed domination number of $D$, denoted by $\gamma(D)$, is the minimum cardinality of a directed dominating set in $D$. The directed domination number of a graph $G$, denoted $\Gamma_d(G)$, which is the maximum directed domination number $\gamma(D)$ over all orientations $D$ of $G$. The directed domination number of a complete graph was first studied by Erdös [Math. Gaz. 47 (1963), 220–222], albeit in disguised form. In this paper we prove a Greedy Partition Lemma for directed domination in oriented graphs. Applying this lemma, we obtain bounds on the directed domination number. In particular, if $\alpha$ denotes the independence number of a graph $G$, we show that $\alpha \le \Gamma_d(G) \le \alpha(1 + 2\ln(n/\alpha))$.

**Keywords:** directed domination; oriented graph; independence number.
**AMS subject classification: 05C69**

*Research supported in part by the South African National Research Foundation

# 1 Introduction

An *asymmetric digraph* or *oriented graph* $D$ is a digraph that can be obtained from a graph $G$ by assigning a direction to (that is, orienting) each edge of $G$. The resulting digraph $D$ is called an *orientation* of $G$. Thus if $D$ is an oriented graph, then for every pair $u$ and $v$ of distinct vertices of $D$, at most one of $(u, v)$ and $(v, u)$ is an arc of $D$. A *directed dominating set*, abbreviated DDS, in a directed graph $D = (V, A)$ is a set $S$ of vertices of $V$ such that every vertex in $V \setminus S$ is dominated by some vertex of $S$; that is, every vertex $u \in V \setminus S$ has an adjacent vertex $v$ in $S$ with $v$ directed to $u$. Every digraph has a DDS since the entire vertex set of the digraph is such a set.

The *directed domination number* of a *directed graph* $D$, denoted by $\gamma(D)$, is the minimum cardinality of a DDS in $D$. A DDS of $D$ of cardinality $\gamma(D)$ is called a $\gamma(D)$-set. Directed domination in digraphs is well studied (cf. [2, 3, 9, 10, 12, 16, 17, 19, 22, 24]).

The *directed domination number* of a *graph* $G$, denoted $\Gamma_d(G)$, is defined in [7] as the maximum directed domination number $\gamma(D)$ over all orientations $D$ of $G$; that is,

$$\Gamma_d(G) = \max\{\gamma(D) \mid \text{ over all orientations } D \text{ of } G\}.$$

The directed domination number of a complete graph was first studied by Erdös [14] albeit in disguised form. In 1962, Schütte [14] raised the question of given any positive integer $k > 0$, does there exist a tournament $T_{n(k)}$ on $n(k)$ vertices in which for any set $S$ of $k$ vertices, there is a vertex $u$ which dominates all vertices in $S$. Erdös [14] showed, by probabilistic arguments, that such a tournament $T_{n(k)}$ does exist, for every positive integer $k$. The proof of the following bounds on the directed domination number of a complete graph are along identical lines to that presented by Erdös [14]. This result can also be found in [24]. Throughout this paper, log is to the base 2 while ln denotes the logarithm in the natural base $e$.

**Theorem 1** (Erdös [14]) *For* $n \geq 2$, $\log n - 2\log(\log n) \leq \Gamma_d(K_n) \leq \log(n + 1)$.

In [7] this notion of directed domination in a complete graph is extended to directed domination of all graphs.

## 1.1 Notation

For notation and graph theory terminology we in general follow [18]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The *open neighborhood* of $v$ is $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* $v$ is $N_G[v] = \{v\} \cup N_G(v)$. If the graph $G$ is clear from context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. If $A$ and $B$ are subsets of $V(G)$, we let $[A, B]$ denote the set of all edges between $A$ and $B$ in $G$.

We denote the *degree* of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. The average degree in $G$ is denoted by $d_{\rm av}(G)$. The minimum degree among the vertices of $G$ is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$. The parameter $\gamma(G)$ denotes the *domination number* of $G$. The parameters $\alpha(G)$ and $\alpha'(G)$ denote the (vertex) *independence number* and the *matching number*, respectively, of $G$, while the parameters $\chi(G)$ and $\chi'(G)$ denote the *chromatic number* and *edge chromatic number*, respectively, of $G$. The *covering number* of $G$, denoted by $\beta(G)$, is the minimum number vertices that covers all the edges of $G$.

A vertex $v$ in a digraph $D$ *out-dominates*, or simply *dominates*, itself as well as all vertices $u$ such that $(v,u)$ is an arc of $D$. The *out-neighborhood* of $v$, denoted $N^+(v)$, is the set of all vertices $u$ adjacent from $v$ in $D$; that is, $N^+(v) = \{u \mid (v,u) \in A(D)\}$. The *out-degree* of $v$ is given by $d^+(v) = |N^+(v)|$, and the maximum out-degree among the vertices of $D$ is denoted by $\Delta^+(D)$. The *in-neighborhood* of $v$, denoted $N^-(v)$, is the set of all vertices $u$ adjacent to $v$ in $D$; that is, $N^-(v) = \{u \mid (u,v) \in A(D)\}$. The *in-degree* of $v$ is given by $d^-(v) = |N^-(v)|$. The *closed in-neighborhood* of $v$ is the set $N^-[v] = N^-(v) \cup \{v\}$. The maximum in-degree among the vertices of $D$ is denoted by $\Delta^-(D)$.

### 1.2 Known Results

We shall need the following inequality chain established in [7].

**Theorem 2** ([7]) *For every graph $G$ on $n$ vertices, $\gamma(G) \le \alpha(G) \le \Gamma_d(G) \le n - \alpha'(G)$.*

## 2 The Greedy Partition Lemma and its Applications

In this section we present our key lemma, which we call the Greedy Partition Lemma, and its applications. The Greedy Partition Lemma is a generalization of earlier results by Caro [5, 6], Caro and Tuza [8], and Jensen and Toft [20].

First we introduce some additional termininology. Let $G$ be a hypergraph and let $P$ be a hypergraph property. Let $P(G) = \max\{|V(H)| \colon H$ is an induced subhypergraph of $G$ that satisfies property $P\}$. Let $\chi(G,P)$ be the minimum number $q$ such that there exist a partition $V(G) = (V_1, V_2, \ldots, V_q)$ such that $V_i$ induces a subhypergraph having property $P$ for all $i = 1, 2, \ldots, q$. For example, if $P$ is the property of independence, then $P(G) = \alpha(G)$, while $\chi(G,P) = \chi(G)$. If $P$ is the property of edge independence, the $P(G) = \alpha'(G)$, while $\chi(G,P) = \chi'(G)$. If $P$ is the property of being $d$-degenerate (recall that a $d$-degenerate graph is a graph $G$ in which every induced subgraph of $G$ has a vertex with degree at most $d$), then $P(G)$ is the maximum cardinality of a $d$-degenerate subgraph and $\chi(G,P)$ is the minimum partition of $V(G)$ into induced $d$-degenerate graphs. For a subhypergraph $H$ of a hypergraph $G$, we let $G - H$ be the subhypergraph of $G$ with vertex set $V(G) \setminus V(H)$. We are now in a position to state the Greedy Partition Lemma.

**Lemma 3** (**Greedy Partition Lemma**) *Let $\mathcal{H}$ be a class of hypergraphs closed under induced subhypergraphs. Let $t \geq 2$ be an integer and let $f\colon [t, \infty) \to [1, \infty)$ be a positive nondecreasing continuous function. Let $P$ be a hypergraph property such that for every hypergraph $G \in \mathcal{H}$ the following holds.*
(a) *If $|V(G)| \leq t$, then $\chi(G, P) \leq |V(G)|$.*
(b) *If $|V(G)| \geq t$, then $|V(G)| \geq P(G) \geq f(|V(G)|)$.*
*Then for every hypergraph $G \in \mathcal{H}$ of order $n$,*

$$\chi(G, P) \leq t + \int_t^{\max(n,t)} \frac{1}{f(x)}\, dx.$$

**Proof.** We proceed by induction on $n$. We first observe that the value of the given integral is always non-negative. If $n \leq t$, then by condition (a), $\chi(G, P) \leq n \leq t$, and the inequality holds trivially. This establishes the base case. For the inductive hypothesis, assume the inequality holds for every hypergraph in $\mathcal{H}$ with less then $n$ vertices and let $G \in \mathcal{H}$ of order $n$. As observed earlier, if $n \leq t$, then the inequality holds trivially. Hence we may assume that $n > t$. Let $P(G) = z = |V(H)|$ be the cardinality of the largest induced subhypergraph $H$ of $G$ that has property $P$. By condition (b), $z \geq f(n)$. If $z \geq n - t + 1$, then $n - z = |V(G) \setminus V(H)| \leq t - 1$, and so by condition (a), $\chi(G - H, P) \leq t - 1$. Hence, $\chi(G, P) \leq \chi(G - H, P) + 1 \leq t$ and the inequality holds trivially. Therefore we may assume that $z \leq n - t$, and so $|V(G) \setminus V(H)| \geq t$. Thus applying the inductive hypothesis to the induced subhypergraph $G - H \in \mathcal{H}$, and using condition (b), we have that

$$\begin{aligned}
\int_t^n \frac{1}{f(x)}\, dx &= \int_t^{n-z} \frac{1}{f(x)}\, dx + \int_{n-z}^n \frac{1}{f(x)}\, dx \\
&\geq \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(x)}\, dx \\
&\geq \chi(G - H, P) - t + \int_{n-z}^n \frac{1}{f(n)}\, dx \\
&= \chi(G - H, P) - t + z/f(n) \\
&\geq \chi(G, P) - 1 - t + 1 \\
&\geq \chi(G, P) - t,
\end{aligned}$$

which completes the proof of the Greedy Partition Lemma. $\Box$

We next discuss several applications of the Greedy Partition Lemma. For this purpose, we shall need the following lemma. Recall that $d_{\mathrm{av}}(G)$ denotes the average degree in a graph $G$.

**Lemma 4** *For $k \geq 1$ an integer, let $G$ be a graph with $k \geq \alpha(G)$ and let $D$ be an orientation of $G$. Let $H$ be an induced subgraph of $G$ of order $n_H \geq k$ and size $m_H$, and let $D_H$ be the orientation of $H$ induced by $D$. Then the following holds.*
(a) $m_H \geq n_H(n_H - k)/2k$.
(b) $\Delta^+(D_H) \geq (n_H - k)/2k$.

**Proof.** Since $H$ is an induced subgraph of $G$, every independent set in $H$ is an independent set in $G$. In particular, $k \geq \alpha(G) \geq \alpha(H)$. Thus applying the Caro-Wei Theorem (see [4, 25]), we have

$$k \geq \alpha(H) \geq \sum_{v \in V(H)} \frac{1}{d_H(v)+1} \geq \frac{n_H}{d_{\rm av}(H)+1} = \frac{n_H}{(2m_H/n_H)+1} = \frac{n_H^2}{2m_H + n_H},$$

or, equivalently, $m_H \geq n_H(n_H - k)/2k$. This establishes part (a). Part (b) follows readily from Part (a) and the observation that

$$n_H \cdot \Delta^+(D_H) \geq \sum_{v \in V(D_H)} d^+_{D_H}(v) = m_H. \qquad \Box$$

### 2.1 Independence Number

Using the Greedy Partition Lemma we present an upper bound on the directed domination number of a graph in terms of its independence number. First we introduce some additional notation. Let $\alpha \geq 1$ be an integer and let $\mathcal{G}_\alpha$ be the class of all graphs $G$ with $\alpha \geq \alpha(G)$. Since every induced subgraph $F$ of $G \in \mathcal{G}_\alpha$ satisfies $\alpha \geq \alpha(G) \geq \alpha(F)$, the class $\mathcal{G}_\alpha$ of graphs is closed under induced subgraphs.

**Theorem 5** *For $\alpha \geq 1$ an integer, if $G \in \mathcal{G}_\alpha$ has order $n \geq \alpha$, then*

$$\Gamma_d(G) \leq \alpha\left(1 + 2\ln\left(n/\alpha\right)\right).$$

**Proof.** If $\alpha = 1$, then $G = K_n$ and by Theorem 1, $\Gamma_d(G) \leq \log(n+1) \leq 1 + 2\ln n = \alpha\left(1 + 2\ln\left(n/\alpha\right)\right)$. Hence we may assume that $\alpha \geq 2$, for otherwise the desired bound holds. We now apply the Greedy Partition Lemma with $t = \alpha$ and with $f(x)$ the positive nondecreasing continuous function on $[\alpha, \infty)$ defined by $f(x) = (x - \alpha)/2\alpha + 1$ where $x \geq [\alpha, \infty)$. Let $P(G) = 1 + \min\{\Delta^+(D)\}$, where the minimum is taken over all orientations $D$ of $G$. Then, $\Gamma_d(G) \leq \chi(G, P)$. To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph $H \in \mathcal{G}_\alpha$, where $H$ has order $|V(H)| = n_H$. If $|V(H)| \leq \alpha$, then $\Gamma_d(H) \leq \chi(H, P) \leq \alpha$ since in this case $H$ may be the empty graph on $\alpha$ vertices. Thus condition (a) of Lemma 3 holds. If $|V(H)| \geq \alpha$ and $D$ is an arbitrary orientation of $H$, then by Lemma 4, $\Delta^+(D) \geq (n_H - \alpha)/2\alpha$, and so $|V(H)| \geq P(H) \geq$

$(n_H - \alpha)/2\alpha + 1 = f(n_H)$. Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,

$$
\begin{aligned}
\Gamma_d(G) &\leq \alpha + \int_\alpha^n \frac{1}{(x-\alpha)/2\alpha + 1}\, dx \\
&= \alpha + 2\alpha \int_\alpha^n \frac{1}{x+\alpha}\, dx \\
&= \alpha + 2\alpha \ln((n+\alpha)/2\alpha) \\
&\leq \alpha + 2\alpha \ln(n/\alpha) \\
&= \alpha(1 + 2\ln(n/\alpha)). \ \Box
\end{aligned}
$$

Observe that for every graph $G$ of order $n$, we have $\chi(G) \geq n/\alpha(G)$ and $d_{\rm av}(G) + 1 \geq n/\alpha(G)$. Hence as an immediate consequence of Theorem 5, we have the following bounds on the directed domination number of a graph.

**Corollary 1** *Let $G$ be a graph of order $n$. Then the following holds.*
(a) $\Gamma_d(G) \leq \alpha(G)\,(1 + 2\ln(\chi(G)))$.
(b) $\Gamma_d(G) \leq \alpha(G)\,(1 + 2\ln(d_{\rm av}(G) + 1))$.

## 2.2 Degenerate Graphs

A $d$-degenerate graph is a graph $G$ in which every induced subgraph of $G$ has a vertex with degree at most $d$. The property of being $d$-degenerate is a hereditary property that is closed under induced subgraphs, as is the property of the complement of a graph being $d$-degenerate. For $d \geq 1$ an integer, let $\mathcal{F}_d$ be the class of all graphs $G$ whose complement is a $d$-degenerate graph. Thus the class $\mathcal{F}_d$ of graphs is closed under induced subgraphs. We shall need the following lemma.

**Lemma 6** *For $d \geq 1$ an integer, let $G \in \mathcal{F}_d$ and let $H$ be an induced subgraph of $G$ of order $n_H$. If $D$ is an orientation of $G$ and $D_H$ is the orientation of $H$ induced by $D$, then $\Delta^+(D_H) > (n_H - 1)/2 - d$.*

**Proof.** Since $G \in \mathcal{F}_d$, the graph $G$ is the complement of a $d$-degenerate graph $\overline{G}$. Let $G$ have order $n$ and size $m$, and let $\overline{G}$ have size $\overline{m}$. It is a well-known fact that we can label the vertices of the $d$-degenerate graph $\overline{G}$ with vertex labels $1, 2, \ldots, n$ such that each vertex with label $i$ is incident to at most $d$ vertices with label greater than $i$, implying that $\overline{m} \leq dn - d(d+1)/2$. Therefore, $m \geq n(n-1)/2 - dn + d(d+1)/2$. This is true for every graph $G$ whose complement is a $d$-degenerate graph. In particular, this is true for the induced subgraph $H$ of $G$. Therefore if $H$ has size $m_H$, we have $\sum_{v \in V(H)} d^+_{D_H}(v) = m_H \geq n_H(n_H - 1)/2 - dn_H + d(d+1)/2$. Hence, $\Delta^+(D_H) > (n_H - 1)/2 - d$. $\Box$

**Theorem 7** *For $d \geq 1$ an integer, if $G \in \mathcal{F}_d$ has order $n$, then*

$$\Gamma_d(G) \leq 2d + 1 + 2\ln(n - 2d + 1)/2.$$

**Proof.** We apply the Greedy Partition Lemma with $t = 2d + 1$ and with $f(x) = (x - 1)/2 - d + 1$ where $x \geq [2d + 1, \infty)$. Let $P(G) = 1 + \min\{\Delta^+(D)\}$, where the minimum is taken over all orientations $D$ of $G$. Then, $\Gamma_d(G) \leq \chi(G, P)$. To show that the conditions of the Greedy Partition Lemma are satisfied, we consider an arbitrary graph $H \in \mathcal{F}_d$, where $H$ has order $|V(H)| = n_H$. If $|V(H)| \leq 2d + 1$, then $\Gamma_d(H) \leq \chi(H, P) \leq 2d + 1$ since in this case $H$ may be the empty graph on $2d + 1$ vertices. Thus condition (a) of Lemma 3 holds. If $|V(H)| \geq 2d + 1$ and $D$ is an arbitrary orientation of $H$, then by Lemma 6, $\Delta^+(D) \geq (n_H - 1)/2 - d$, and so $|V(H)| \geq P(H) \geq (n_H - 1)/2 - d + 1 = f(n_H)$. Therefore condition (b) of Lemma 3 holds. Hence by the Greedy Partition Lemma,

$$\begin{aligned}
\Gamma_d(G) &\leq 2d + 1 + \int_{2d+1}^{n} \frac{1}{(x-1)/2 - d + 1}\, dx \\
&= 2d + 1 + \int_{2d+1}^{n} \left(\frac{2}{x - 2d + 1}\right) dx \\
&= 2d + 1 + 2\int_{2}^{n-2d+1} \frac{1}{x}\, dx \\
&\leq 2d + 1 + 2\ln(n - 2d + 1)/2. \ \Box
\end{aligned}$$

### 2.3 $K_{1,m}$-Free Graphs

In this section, we establish an upper bound on the directed domination number of a $K_{1,m}$-free graph. We first recall the well-known bound for the usual domination number $\gamma$, which was proved independently by Arnautov in 1974 and in 1975 by Lovász and by Payan.

**Theorem 8** (Arnautov [1], Lovász [21], Payan [23]) *If $G$ is a graph on $n$ vertices with minimum degree $\delta$, then $\gamma(G) \leq n(\log(\delta + 1) + 1)/(\delta + 1)$.*

We show that the above bound on $\gamma$ is nearly preserved by the directed domination number $\Gamma_d$ when we restrict our attention to $K_{1,m}$-free graphs. For this purpose, we shall need the following result due to Faudree et al. [15].

**Theorem 9** ([15]) *If $G$ is a $G$ is a $K_{1,m}$-free graph of order $n$ with $\delta(G) = \delta$ and $\alpha(G) = \alpha$, then $\alpha \leq (m - 1)n/(\delta + m - 1)$.*

We shall prove the following result.

**Theorem 10** *For $m \geq 3$, if $G$ is a $K_{1,m}$-free graph of order $n$ with $\delta(G) = \delta$, then*

$$\Gamma_d(G) < (2(m-1)n\ln(\delta+m-1))/(\delta+m-1).$$

**Proof.** If $\delta < (\sqrt{e}-1)(m-1)$, where $e$ is the base of the natural logarithm, then $\delta < m-1$ and so $(2(m-1)n\ln(\delta+m-1))/(\delta+m-1) > n\ln(\delta+m-1) > n$. Hence we may assume that $\delta \geq (\sqrt{e}-1)(m-1)$, for otherwise the desired upper bound holds trivially. By Theorem 9, $\alpha \leq (m-1)n/(\delta+m-1)$. Substituting $\delta \geq (\sqrt{e}-1)(m-1)$ into this inequality, we get $\alpha \leq (m-1)n/((\sqrt{e}-1)(m-1)+m-1) = (m-1)n/(\sqrt{e}(m-1) = n/\sqrt{e}$. Since the function $x(1+2\ln(n/x))$ is monotone increasing in the interval $[1, n/\sqrt{e}]$, we get, by Theorem 5, that

$$\begin{aligned}
\Gamma_d(G) &\leq \alpha\,(1+2\ln(n/\alpha)) \\
&\leq ((m-1)n/(\delta+m-1))\,(1+2\ln(n(\delta+m-1)/(m-1)n)) \\
&= ((m-1)n/(\delta+m-1))\,(1+2\ln((\delta+m-1)/(m-1))) \\
&= 2(m-1)n(1/2+\ln((\delta+m-1)/(m-1)))/(\delta+m-1) \\
&= 2(m-1)n(\ln\sqrt{e}+\ln((\delta+m-1)/(m-1)))/(\delta+m-1) \\
&< (2(m-1)n\ln(\delta+m-1))/(\delta+m-1),
\end{aligned}$$

as $\sqrt{e} < m-1$. □

We observe that as a special case of Theorem 10, we have that if $G$ is a claw-free graph of order $n$ with $\delta(G) = \delta$, then $\Gamma_d(G) \leq (4n\,(\log(\delta+2)))/(\delta+2)$.

## 2.4 Nordhaus-Gaddum-Type Bounds

In this section we consider Nordhaus-Gaddum-type bounds for the directed domination of a graph. Let $\mathcal{G}_n$ denote the family of all graphs of order $n$. We define

$$\begin{aligned}
\mathrm{NG}_{\min}(n) &= \min\{\Gamma_d(G)+\Gamma_d(\overline{G})\} \\
\mathrm{NG}_{\max}(n) &= \max\{\Gamma_d(G)+\Gamma_d(\overline{G})\}
\end{aligned}$$

where the minimum and maximum are taken over all graphs $G \in \mathcal{G}_n$. Chartrand and Schuster [11] established the following Nordhaus-Gaddum inequalities for the matching number: If $G$ is a graph on $n$ vertices, then $\lfloor n/2 \rfloor \leq \alpha'(G)+\alpha'(\overline{G}) \leq 2\lfloor n/2 \rfloor$.

**Theorem 11** *The following holds.*
(a) $c_1 \log n \leq \mathrm{NG}_{\min}(n) \leq c_2(\log n)^2$ *for some constants* $c_1$ *and* $c_2$.
(b) $n+\log n - 2\log(\log n) \leq \mathrm{NG}_{\max}(n) \leq n+\lceil n/2 \rceil$.

**Proof.** (a) By Ramsey's theory, for all graphs $G \in \mathcal{G}_n$ we have $\max\{\alpha(G), \alpha(\overline{G})\} \geq c\log n$ for some constant $c$. Hence by Theorem 2(a), $\Gamma_d(G)+\Gamma_d(\overline{G}) \geq \alpha(G)+\alpha(\overline{G}) \geq c_1 \log n$

for some constant $c_1$. Further by Ramsey's theory there exists a graph $G \in \mathcal{G}_n$ such that $\max\{\alpha(G), \alpha(\overline{G})\} \le d \log n$ for some constant $d$. Hence by Theorem 5, $\Gamma_d(G) + \Gamma_d(\overline{G}) \le 2d \log n(1 + 2\log(n/d\log n)) \le c_2(\log n)^2$ for some constant $c_2$. This establishes Part (a).

(b) By Theorem 1, $\Gamma_d(K_n) + \Gamma_d(\overline{K}_n) \le n + \log n - 2\log(\log n)$. Hence, $\mathrm{NG}_{\max}(n) \ge n + \log n - 2\log(\log n)$. By Theorem 2(b) and by the Nordhaus–Gaddum inequalities for the matching number, we have that $\Gamma_d(G) + \Gamma_d(\overline{G}) \le 2n - (\alpha'(G) + \alpha'(\overline{G})) \le 2n - \lfloor n/2 \rfloor = n + \lceil n/2 \rceil$. □

# 3 Two Generalizations

In this section, we present two general frameworks of directed domination in graphs.

## 3.1 Directed Multiple Domination

For an integer $r \ge 1$, a *directed $r$-dominating set*, abbreviated DrDS, in a directed graph $D = (V, A)$ is a set $S$ of vertices of $V$ such that for every vertex $u \in V \setminus S$, there are at least $r$ vertices $v$ in $S$ with $v$ directed to $u$. The *directed $r$-domination number* of a *directed graph* $D$, denoted by $\gamma_r(D)$, is the minimum cardinality of a DrDS in $D$. An DrDS of $D$ of cardinality $\gamma_r(D)$ is called a $\gamma_r(D)$-set. The *directed $r$-domination number* of a *graph* $G$, denoted $\Gamma_{d,r}(G)$, is defined as the maximum directed $r$-domination number $\gamma_r(D)$ over all orientations $D$ of $G$; that is, $\Gamma_{d,r}(G) = \max\{\gamma_r(D) \mid \text{ over all orientations } D \text{ of } G\}$. In particular, we note that $\Gamma_d(G) = \Gamma_{d,1}(G)$.

**Theorem 12** *Let $r \ge 1$ be an integer. Let $G$ be a graph of order $n$ with $\alpha(G) = \alpha$. Then the following holds.*
(a) $\Gamma_{d,r}(K_n) \le r\log(n+1)$.
(b) $\Gamma_{d,r}(G) \le r\alpha\,(1 + 2\ln(n/\alpha))$.

**Proof.** (a) By Theorem 1, $\Gamma_d(K_n) \le \log(n+1)$. Let $D_1$ be an orientation of $K_n$ and let $S_1$ be a $\gamma(D_1)$-set. Then, $|S_1| \le \log(n+1)$. We now remove the vertices of the DDS $S_1$ from $D_1$ to produce an orientation $D_2$ of $K_{n_1}$ where $n_1 = n - |S|$. Let $S_2$ be a $\gamma(D_2)$-set. By Theorem 1, $|S_2| \le \log(n_1+1) < \log(n+1)$. We now remove the vertices of the DDS $S_2$ from $D_2$ to produce an orientation $D_3$ of $K_{n_2}$ where $n_3 = n - |S_1| - |S_2|$ and we let $S_3$ be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2 \ldots, S_r$ of sets whose union is a DrDS of $K_n$ of cardinality $\sum_{i=1}^{r} |S_i| \le r\log(n+1)$. This is true for every orientation $D$ of $K_n$. Hence, $\Gamma_{d,r}(K_n) \le r\log(n+1)$. This establishes Part (a).

(b) By Theorem 5, $\Gamma_d(G) \le \alpha\,(1 + 2\ln(n/\alpha))$. We first consider the case when $\alpha \ge n/\sqrt{e}$. Then, $r\alpha\,(1 + 2\ln(n/\alpha)) > n$ for $r = 2$. However the function $x(1 + 2\ln(n/x))$ is monotone increasing in the interval $[1, n/\sqrt{e}\,]$ and we may therefore assume that $\alpha \le n/\sqrt{e}$, for otherwise the desired result holds trivially.

Let $D_1$ be an arbitrary orientation of $G$ and let $S_1$ be a DDS of $G$. We now remove the vertices of $S_1$ from $D_1$ to produce an orientation $D_2$ of the graph $G_1 = G - S_1$ where $G_1$ has order $n_1 = n - |S|$. Let $\alpha(G_1) = \alpha_1$. Since $G_1$ is an induced subgraph of $G$, we have $\alpha_1 \le \alpha$. By Theorem 5, $\Gamma_d(G_1) \le \alpha_1 (1 + 2\ln(n_1/\alpha_1)) < \alpha_1 (1 + 2\ln(n/\alpha_1))$. Since $\alpha_1 \le \alpha \le n/\sqrt{e}$, the monotonicity of the function $x(1 + 2\ln(n/x))$ in the interval $[1, n/\sqrt{e}\,]$ implies that $\alpha_1 (1 + 2\ln(n/\alpha_1)) \le \alpha (1 + 2\ln(n/\alpha))$. Hence, $\Gamma_d(G_1) < \alpha (1 + 2\ln(n/\alpha))$.

Let $S_2$ be a $\gamma(D_2)$-set, and so $|S_2| < \alpha (1 + 2\ln(n/\alpha))$. We now remove the vertices of the DDS $S_2$ from $D_2$ to produce an orientation $D_3$ of $G_2 = G_1 - S_2$ where $n_2 = n - |S_1| - |S_2|$ and we let $S_3$ be a $\gamma(D_3)$-set. Continuing in this way, we produce a sequence $S_1, S_2 \ldots, S_r$ of sets whose union is a DrDS of $G$ of cardinality $\sum_{i=1}^{r} |S_i| \le r\alpha (1 + 2\ln(n/\alpha))$. This is true for every orientation $D$ of $G$. Hence, $\Gamma_{d,r}(G) \le r\alpha (1 + 2\ln(n/\alpha))$. This establishes Part (b). $\square$

### 3.2 Directed Distance Domination

Let $D = (V, A)$ be a directed graph. The distance $d_D(u, v)$ from a vertex $u$ to a vertex $v$ in $D$ is the number of edges on a shortest directed path from $u$ to $v$. For an integer $d \ge 1$, a *directed $d$-distance dominating set*, abbreviated DdDDS, in $D$ is a set $U$ of vertices of $V$ such that for every vertex $v \in V \setminus U$, there is a vertex $u \in U$ with $d_D(u, v) \le d$. The *directed $d$-distance domination number* of a *directed graph $D$*, denoted by $\gamma(D, d)$, is the minimum cardinality of a DdDDS in $D$. The *directed $d$-distance domination number* of a *graph $G$*, denoted $\Gamma_d(G, d)$, is defined as the maximum directed $d$-distance domination number $\gamma_d(D, d)$ over all orientations $D$ of $G$; that is, $\Gamma_d(G, d) = \max\{\gamma(D, d) \mid \text{ over all orientations } D \text{ of } G\}$. In particular, we note that $\Gamma_d(G) = \Gamma_d(G, 1)$.

An independent set $U$ of vertices in $D$ is called a *semi-kernel* of $D$ if for every vertex $v \in V(D) \setminus U$, there is a vertex $u \in U$ such that $d_D(u, v) \le 2$. For the proof of our next result we will use the following theorem due to Chvátal and Lovász [13].

**Theorem 13** (Chvátal, Lovász [13]) *Every directed graph contains a semi-kernel.*

**Theorem 14** *For every integer $d \ge 2$, $\gamma_d(G, d) = \alpha(G)$.*

**Proof.** Let $S$ be a maximum independent set in $G$ and let $D$ be an orientation obtained from $G$ by directing all edges in $[S, V \setminus S]$ from $S$ to $V \setminus S$ and directing all other edges arbitrarily. Every directed $d$-distance dominating set must contain $S$ since no vertex of $S$ is reachable in $D$ from any other vertex of $V(D)$. Hence, $\Gamma_d(G, d) \ge |S| = \alpha(G)$. However if $D^*$ is an arbitrary orientation of the graph $G$, then by Theorem 13 the oriented graph $D^*$ has a semi-kernel $S^*$. Thus, $\gamma(D, d) \le |S^*| \le \alpha(G)$. Since this is true for every orientation of $G$, we have that $\Gamma_d(G, d) \le \alpha(G)$. Consequently, $\gamma_d(G, d) = \alpha(G)$. $\square$

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