H. Paul Williams and J. N. Hooker

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H. P. Williams
London School of Economics
H.P.Williams@lse.ac.uk

J. N. Hooker
Carnegie Mellon University
jh38@andrew.cmu.edu

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Abstract

We generalise polyhedral projection (Fourier-Motzkin elimination) to integer programming (IP) and derive from this an alternative perspective on IP that parallels the classical theory. We first observe that projection of an IP yields an IP augmented with linear congruence relations and finite-domain variables, which we term a *generalised IP*. The projection algorithm can be converted to a branch-and-bound algorithm for generalised IP in which the search tree has bounded depth (as opposed to conventional branching, in which there is no bound). It also leads to valid inequalities that are analogous to Chvátal-Gomory cuts but are derived from congruences rather than rounding, and whose rank is bounded by the number of variables. Finally, projection provides an alternative approach to IP duality. It yields a value function that consists of nested roundings as in the classical case, but in which ordinary rounding is replaced by rounding to the nearest multiple of an appropriate modulus, and the depth of nesting is again bounded by the number of variables.

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1 Introduction

We propose an alternative perspective on integer programming that is based on projection. It begins with the observation that the projection of an integer programming (IP) problem is not an IP problem. More precisely, the projection of an IP problem's feasible set onto a subset of variables is not the feasible set of an IP. It is the feasible set of a system of linear integer inequalities and congruence relations, where the congruence relations define a sublattice of the integer lattice. This suggests that an IP problem can be viewed more generally as an inequality constrained problem over a sublattice of the integer lattice, rather than exclusively over the entire integer lattice as in conventional IP. We will call this a *generalised IP problem*.

The projection problem for generalised IP can be solved by introducing integer auxiliary variables with finite domains, and taking advantage of a generalised Chinese Remainder Theorem. The auxiliary variables are not, generally, the same as slack/surplus variables.

By projecting out all the original variables, the optimization problem can be transformed to one that minimises over a system of congruence relations that involve only the auxiliary variables. A problem of optimising over possibly infinite domains is therefore transformed to one of optimising over finite domains.

This perspective leads to an alternative theory of cutting planes, branching algorithms, and IP duality. The projection algorithm yields valid inequalities that are analogous to Chvátal-Gomory cuts, except that they are derived from congruences rather than rounding, and their rank is bounded by the number of variables. This contrasts with the classical classical Chvátal rank, which has no bound related only to the number of variables [4]. In addition, the projection algorithm can be converted to a branching algorithm that branches on integer auxiliary variables rather than the original integer variables, and in which the possible branches are defined by congruence relations. The depth of the tree is again bounded by the number of variables, whereas a conventional branching tree has unbounded depth. Finally, by applying the projection algorithm to an IP problem with general right-hand sides, one can obtain a value function that is analogous to a Chvátal function [1] in that it contains nesting rounding operations. However, rather than rounding to the nearest integer, one rounds to the nearest multiple of an appropriate modulus. Unlike a Chvátal function, the depth of nesting (which is analogous to cutting plane rank) is bounded by the number of variables, and the function can be obtained by one pass through the model.

We begin with a brief review of projection and duality in linear programming (LP), to clarify how it is generalised for the IP case. We then show by example how to project a generalised IP and prove the correctness of the projection method. We also interpret the projection method as generating cuts analogous to Chvátal-Gomory cuts. We then modify the projection method to produce a branching method that is easily augmented to a branch-and-cut method by solving relaxations. Finally, we show how to construct a value function and prove its correctness.

2 LP Projection

A polyhedron can be projected onto a subspace using Fourier-Motzkin elimination [2, 6]. We will suppose the polyhedron is described by the constraint set of an LP in the following form, where A is an $m \times n$ integral matrix and b is integral:

$$\begin{array}{rl} \min & z \\ \text{subject to} & -cx \geq -z \\ & Ax \geq b \\ & x \in \mathbb{R}^n \end{array} \tag{1}$$

We assume that any nonnegativity constraints on the variables are represented in the above constraints. Fourier-Motzkin elimination relies on the following elementary lemma, which we prove to allow comparison with a parallel result (Theorem 3) that we will prove for IP projection.

Lemma 1 *Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then*

(a) *There exists $x_j \in \mathbb{R}$ such that $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ for all $i \in I$, $k \in K$*

if and only if

(b) $a_{kj}f_i + a_{ij}g_k \leq 0$ *for all $i \in I, k \in K$.*

Proof. (a) $\Rightarrow$ (b). This is obtained by taking a linear combination of each pair of inequalities $a_{ij}x_i \geq f_i$, $-a_{kj}x_i \geq g_k$, using multipliers $1/a_{ij}$ and $1/a_{kj}$, respectively.

(a) $\Leftarrow$ (b). The inequalities in (a) can be written $f_i/a_{ij} \leq x_j \leq -g_k/a_{kj}$ for all i, k. But from (b) we have that $f_i/a_{ij} \leq -g_k/a_{kj}$ for all i, k. We can therefore let $x_j = \max_i\{f_i/a_{ij}\}$ (or $\min_k\{-g_k/a_{kj}\}$), and the inequalities in (a) are satisfied. □

The lemma implies that any variable x_j can be eliminated from (1) by removing each pair of inequalities that have the form $a_{ij}x_j \geq f_i$, $-a_{kj}x_j \geq g_k$ with $a_{ij}, a_{kj} > 0$, and replacing each pair with the inequality $a_{kj}f_i + a_{ij}g_k \leq 0$. The variables x_j can be successively eliminated, in any order, until the constraints of (1) are replaced by inequalities of the form $z \geq \ell$. The minimum value of z can be immediately read from these. It can be shown [3, 7] that after the elimination of r variables, any resulting inequality that depends on more than $r + 1$ of the original inequalities is redundant (implied by the other inequalities).

Note that projecting out any subset of variables from an LP results in another LP. We will see that an analogous property does not hold for integer programming. In general, projecting out variables from an IP results in a *disjunction* of IPs.

We can illustrate projection with a small example (Fig.1).

$$
\begin{array}{rll}
\min & z & \\
\text{subject to} & -x_2 \geq -z & \text{C0} \\
& 2x_1 + x_2 \geq 13 & \text{C1} \\
& -5x_1 - 2x_2 \geq -30 & \text{C2} \\
& -x_1 + x_2 \geq 5 & \text{C3} \\
& x_1, x_2 \in \mathbb{R} &
\end{array}
\tag{2}
$$

The optimal solution is $(x_1, x_2, z) = (2\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3})$, with binding constraints C1 and C3. Eliminating x_1 yields $z \geq x_2$, $x_2 \geq 5$, and $x_2 \geq 7\frac{2}{3}$. Eliminating x_2 from this yields $z \geq 5$ and $z \geq 7\frac{2}{3}$. This confirms the optimal value $7\frac{2}{3}$.

Suppose now that we perturb the right-hand sides of (2) as follows:

$$
\begin{array}{ll}
\min\ z & \\
-x_2 \geq -z & \text{C0} \\
2x_1 + x_2 \geq 13 + \Delta_1 & \text{C1}_\Delta \\
-5x_1 - 2x_2 \geq -30 + \Delta_2 & \text{C2}_\Delta \\
-x_1 + x_2 \geq 5 + \Delta_3 & \text{C3}_\Delta \\
x_1, x_2 \in \mathbb{Z} &
\end{array}
\tag{3}
$$

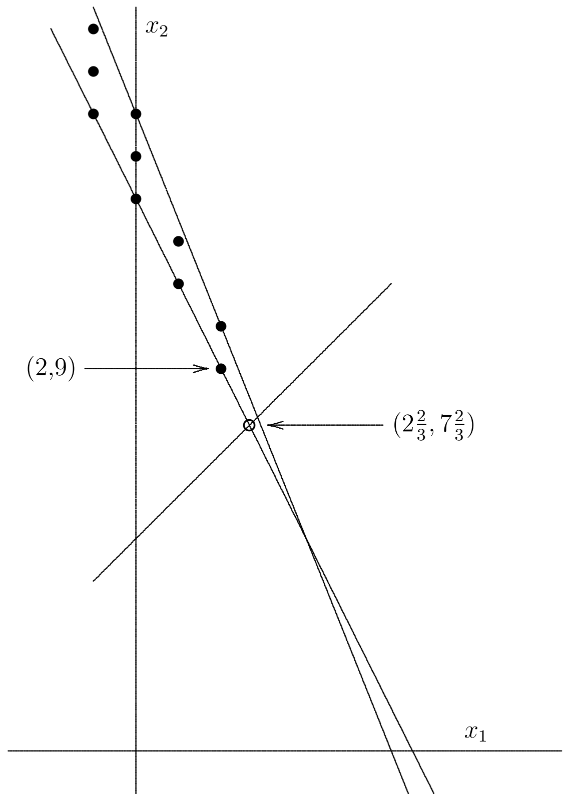

Figure 1: Illustration of a linear (integer) programming problem. Black dots are integer feasible solutions, with $(x_1, x_2) = (2, 9)$ optimal. The small open circle is the optimal solution of the LP.

We can perform the same projection operations while carrying through the perturbations. This yields $z \geq 5 + 5\Delta_1 + 2\Delta_2$ and $z \geq 7\frac{2}{3} + \frac{1}{3}\Delta_1 + \frac{10}{3}\Delta_3$. From this we can write a *value function*

$$v(\Delta_1, \Delta_2, \Delta_3) = \max\left\{5 + 5\Delta_1 + 2\Delta_2,\ 7\tfrac{2}{3} + \tfrac{1}{3}\Delta_1 + \tfrac{2}{3}\Delta_3\right\}$$

that gives the optimal value as a function of the perturbations. The coefficient of each Δ_i in the larger argument of the max when $\Delta_1 = \Delta_2 = \Delta_3 = 0$ is a *dual multiplier* corresponding to constraint i. Because the second term is larger when the $\Delta = 0$, the dual multipliers are $(\frac{1}{3}, 0, \frac{2}{3})$. They can be interpreted as marginal costs or shadow prices.

3 IP Projection

In analogy with the LP case, we consider an IP in the following form:

$$\begin{array}{rl} \min & z \\ \text{subject to} & -cx \geq -z \\ & Ax \geq b \\ & x \in \mathbb{Z}^n \end{array} \tag{4}$$

A generalised IP can be written

$$\begin{array}{rl} \min & z \\ \text{subject to} & -cx - hu \geq -z \\ & Ax + Bu \geq b \\ & r^i x + s^i u \equiv \rho_i \pmod{m_i},\ i \in I \\ & x \in \mathbb{Z}^n \\ & u_j \in D_j \subset \mathbb{Z}_{\geq 0},\ j = 1, \ldots, p \end{array} \tag{5}$$

where $u = (u_1, \ldots, u_p)$ are auxiliary variables restricted to finite domains $D_1, \ldots, D_p$.

When projecting out an integer variable x_j, we can no longer infer $f_i/a_{ij} \leq x_j \leq -g_k/a_{kj}$ as in the proof of Lemma 1. However, we can project out integer variables by strengthening the resultant inequalities. The idea can be illustrated using the example (2) with integer variables x_1, x_2. This is a classical IP with no congruence relations, but we will see that the same method applies to generalised IPs.

Step 1. We first project out x_1. We obtain the following from the constraint pairs shown:

$$\begin{array}{rcll} 5(-x_2 + 13) \leq & 5 \cdot 2x_1 & \leq 2(-2x_2 + 30) & \text{from C1,C2} \\ -x_2 + 13 \leq & 2x_1 & \leq 2(x_2 - 5) & \text{from C1,C3} \end{array} \tag{6}$$

Because the middle term of the first line is divisible by $5 \cdot 2$, we can increase the term $-x_2 + 13$ on the left to the nearest multiple of 2 (unless it is already a multiple of 2) without violating the inequality. We do this by introducing an integer auxiliary variable $u_1 \in \{0, 1\}$. This yields the system on the left below, which implies the system on the right:

$$\begin{array}{lcl} 5(-x_2 + 13 + u_1) \leq 5 \cdot 2x_1 \leq 2(-2x_2 + 30) & \Rightarrow & x_2 \geq 5 + 5u_1 \\ -x_2 + 13 + u_1 \equiv 0 \pmod{2},\ \ u_1 \in \{0, 1\} & & x_2 \equiv u_1 + 1 \pmod{2},\ \ u_1 \in \{0, 1\} \end{array}$$

The congruence relation $-x_2 + 13 + u_1 \equiv 0 \pmod{2}$ reflects the fact that $-x_2 + 13 + u_1$ is a multiple of 2. (We could have just as well introduced a surplus variable on the right.) We similarly strengthen the second line of (6) to obtain:

$$\begin{array}{lcl} -x_2 + 13 + u_1 \leq 2x_1 \leq 2(x_2 - 5) & \Rightarrow & 3x_2 \geq 23 + u_1 \\ -x_2 + 13 + u_1 \equiv 0 \pmod{2},\ \ u_1 \in \{0, 1\} & & x_2 \equiv u_1 + 1 \pmod{2},\ \ u_1 \in \{0, 1\} \end{array}$$

Putting these together, we have the projected system

$$\begin{array}{ll} -x_2 \geq -z & \text{C0} \\ x_2 \geq 5 + 5u_1 & \text{C12} \\ 3x_2 \geq 23 + u_1 & \text{C13} \\ x_2 \equiv u_1 + 1 \pmod 2, \;\; u_1 \in \{0,1\} & \end{array} \tag{7}$$

Step 2. We now wish to project out x_2 from the system (7). The system is now a generalised IP with a congruence relation, which requires an extension of the above idea. We first obtain the following by pairing inequalities, as before:

$$\begin{array}{ll} 5 + 5u_1 \leq x_2 \leq z & \text{from C0, C12} \\ 23 + u_1 \leq 3x_2 \leq 3z & \text{from C0, C13} \end{array} \tag{8}$$

Because $x_2 \equiv u_1 + 1 \pmod 2$, we can increase the left-hand term in the first line until it is congruent to $u_1 + 1 \pmod 2$. Introducing an auxiliary variable u_{12}, we obtain the system on the left below:

$$\begin{array}{lcl} 5 + 5u_1 + u_{12} \leq x_2 \leq z & \Rightarrow & z \geq 5 + 5u_1 + u_{12} \\ 5 + 5u_1 + u_{12} \equiv u_1 + 1 \pmod 2, \;\; u_{12} \in \{0,1\} & & u_{12} \equiv 0 \bmod 2, \;\; u_{12} \in \{0,1\} \end{array}$$

It is clearly desirable that only one congruence in the system (7) contain x_2, so that we can use this kind of reasoning. We indicate below how this can be achieved in general. The second line of (8) gives

$$\begin{array}{lcl} 23 + u_1 + u_{13} \leq 3x_2 \leq 3z & \Rightarrow & z \geq \frac{1}{3}(23 + u_1 + u_{13}) \\ 23 + u_1 + u_{13} \equiv 3u_1 + 3 \pmod 6 & & 4u_1 + u_{13} \equiv 4 \pmod 6, \;\; u_{13} \in \{0,\ldots,5\} \end{array}$$

Note that u_{12} can be fixed to zero and dropped from the problem. We therefore have the projected system

$$\begin{array}{l} z \geq 5 + 5u_1 \\ z \geq \frac{1}{3}(23 + u_1 + u_{13}) \\ 4u_1 + u_{13} \equiv 4 \pmod 6, \;\; u_1 \in \{0,1\}, \; u_{13} \in \{0,\ldots,5\} \end{array} \tag{9}$$

Step 3. We have reduced the original IP to the problem of minimising z subject to a system (9) of inequalities and congruences that involve only z and the auxiliary variables u_1, u_{13}. We can solve the problem, in principle, by enumerating solutions of the congruence in (9), and taking note of the minimum value of z in each. The two solutions are listed in Table 1, where the tightest bound on z in each scenario is shown in boldface. The minimum of these is the optimal value of z, namely $z = 9$, corresponding to $(u_1, u_{13}) = (0, 4)$. Since the bound of 9 comes from C0 and C13, we have $23 + u_1 + u_{13} = 3x_2$ from C13, or $x_2 = 9$. Since C13 comes from C1 and C3, we have $5(-x_2 + 13 + u_1) = 5 \cdot 2x_1$ from C1, or $x_1 = 2$. The optimal solution is therefore $(x_1, x_2, z) = (2, 9, 9)$.

Table 1: Solution of the projected system.

u_1	u_{13}	$5+5u_1$	$\frac{1}{3}(23+u_1+u_{13})$
0	4	5	**9**
1	0	**10**	8

When the variable x_j to be projected out occurs in several congruences, we wish to replace the congruences with an equivalent single congruence containing x_j. This can be accomplished as follows using a generalised Chinese Remainder Theorem (GCRT). Without loss of generality, we suppose the congruences have the form $\alpha x_j \equiv d_s \pmod{m_s}$ for $s \in S$. The GCRT can then be stated as follows.

Theorem 2 (Generalised Chinese Remainder) *Consider a system of congruences $\mathcal{C} = \{\alpha x_j \equiv d_s \pmod{m_s} \mid s \in S\}$, and let $M = lcm\{m_s \,|\, s \in S\}$ and $m'_s = M/m_s$. Then we have: (i) $d_s \equiv d_t \pmod{gcd(m_s, m_t)}$ for all $s, t \in S$, (ii) there is a set of integers λ_s satisfying $\sum_s \lambda_s m'_s = 1$, and (iii) integer x_j solves $\mathcal{C}$ if and only if it solves*

$$\alpha x_j \equiv \sum_{s \in S} \lambda_s m'_s d_s \pmod{M} \tag{10}$$

The multipliers λ_s can be obtained using the well-known Euclidean algorithm.

Proof. Claim (i) can be obtained by subtracting the congruences of $\mathcal{C}$ in pairs. Claim (ii) is a well-known consequence of the Euclidean algorithhm. To show (iii), suppose first that integer x_j satisfies the congruences in $\mathcal{C}$. Taking a linear combination of the congruences in $\mathcal{C}$ with multipliers $\lambda_s m'_s$, we obtain (10). Conversely, suppose x_j satisfies (10). Because $d_s \equiv d_t \pmod{\gcd(m_s, m_t)}$ for all $s, t \in S$, we have

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv \sum_s \lambda_s m'_s d_t \pmod{\gcd_{s \in S}\{\lambda_s m'_s \gcd(m_s, m_t)\}}$$

for any $t \in S$, which implies

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv \sum_s \lambda_s m'_s d_t \pmod{\gcd_{s \in S}\{m'_s \gcd(m_s, m_t)\}} \tag{11}$$

But $\gcd_{s \in S}\{m'_s \gcd(m_s, m_t)\} = m_t$ because $m'_s = M/m_s$. Given this and (ii), (11) simplifies to

$$\sum_{s \in S} \lambda_s m'_s d_s \equiv d_t \pmod{m_t} \tag{12}$$

Also (10) implies

$$\alpha x_j \equiv \sum_{s \in S} \lambda_s m'_s d_s \pmod{m_t}$$

which, together with (12), implies $\alpha x_j \equiv d_t \pmod{m_t}$. Since $t \in S$ is arbitrary, x_j satisfies the congruences in $\mathcal{C}$. □

The general projection method relies on the following theorem.

Theorem 3 *Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then*

(a) *There exists $x_j \in \mathbb{Z}$ such that $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ for all $i \in I$, $k \in K$, and such that $\alpha x_j \equiv d \pmod{m}$,*

if and only if

(b) *$d \equiv 0 \pmod{\beta}$, where $\beta = gcd(\alpha, m)$; there exist $\lambda_\alpha, \lambda_m \in \mathbb{Z}$ satisfying $\lambda_\alpha m + \lambda_m \alpha = \beta$; and there exists $u_i \in \{0, 1, \ldots, a_{ij}m/\beta - 1\}$ such that $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$ for all $i \in I, k \in K$, and $f_i + u_i \equiv \lambda_m a_{ij} d/\beta \pmod{a_{ij}m/\beta}$ for all $i \in I$.*

Proof. (a) $\Rightarrow$ (b). We can write the inequalities in (a) as

$$a_{kj}\alpha f_i \leq a_{ij}a_{kj}\alpha x_j \leq -a_{ij}\alpha g_k \tag{13}$$

for all i, k. From the congruence in (a), $a_{ij}a_{kj}\alpha x_j \equiv a_{ij}a_{kj}d \pmod{a_{ij}a_{kj}m}$. Thus if we let $y = a_{ij}a_{kj}\alpha x_j$, we obtain $y \equiv 0 \pmod{a_{ij}a_{kj}\alpha}$ and $y \equiv a_{ij}a_{kj}d \pmod{a_{ij}a_{kj}m}$. Applying part (i) of the GCRT to these two congruences, we get $d \equiv 0 \pmod{\beta}$. From part (ii), there are integers $\lambda_\alpha, \lambda_s$ for which $\lambda_\alpha \mathrm{lcm}(\alpha, m)/\alpha + \lambda_m \mathrm{lcm}(\alpha, m)/m = 1$. Since $\mathrm{lcm}(\alpha, m)/m = \alpha/\beta$, this is equivalent to $\lambda_\alpha m + \lambda_m \alpha = \beta$, as claimed in (b). From part (iii), we have $y \equiv \lambda_m a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)d/m \pmod{a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)}$, which implies the congruence $y \equiv \lambda_m a_{ij}a_{kj}\alpha d/\beta \pmod{a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)}$, again because $\mathrm{lcm}(\alpha, m)/m = \alpha/\beta$. So from (13) we have

$$a_{kj}\alpha f_i - \lambda_m a_{ij}a_{kj}\alpha d/\beta \leq \gamma \leq -a_{ij}\alpha g_k - \lambda_m a_{ij}a_{kj}\alpha d/\beta \tag{14}$$

where γ is an integer multiple of $a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)$. Since $d \equiv 0 \pmod{\beta}$, β divides d, and the leftmost expression in (14) is an integer multiple of $a_{kj}\alpha$. So we can add $a_{kj}\alpha u_i$ to the left-hand side of (14), and we have

$$a_{kj}\alpha(f_i + u_i) \leq \gamma + \lambda_m a_{ij}a_{kj}\alpha d/\beta \leq -a_{ij}\alpha g_k \tag{15}$$

and

$$a_{kj}\alpha(f_i + u_i) - \lambda_m a_{ij}a_{kj}\alpha d/\beta \equiv 0 \pmod{a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)} \tag{16}$$

Inequality (15) implies the inequality in (b). Congruence (16) simplifies to

$$f_i + u_i - \lambda_m a_{ij}d/\beta \equiv 0 \pmod{a_{ij}m/\beta}$$

which implies the congruence in (b). We can also restrict u_i to $\{0, 1, \ldots, a_{ij}m/\beta - 1\}$. For if u_i were greater than $a_{ij}m/\beta - 1$ then the original inequalities and congruences would still be valid if $a_{ij}m/\beta - 1$ were subtraced from u_i.

(a) $\Leftarrow$ (b). The inequalities in (b) can be written

$$-\frac{g_k}{a_{kj}} \geq \frac{f_i + u_i}{a_{ij}} \tag{17}$$

for all i, k. From (b) we have that $d \equiv 0 \pmod{\beta}$, so that d/β is integral. Also from (b),

$$f_i + u_i \equiv \lambda_m a_{ij} d/\beta \pmod{a_{ij} a_{kj} m/\beta} \tag{18}$$

Because d/β and m/β are integral, this implies $f_i + u_i$ is an integer multiple of a_{ij}. We can therefore let

$$x_j = \max_i \left\{ \frac{f_i + u_i}{a_{ij}} \right\} \tag{19}$$

and x_j is integral. This and (17) imply $-g_k/a_{kj} \geq x_j$, or $-g_k \geq a_{kj} x_j$. To show $a_{ij} x_j \geq f_i$, we note that

$$a_{ij} x_j \geq a_{ij} \frac{f_i + u_i}{a_{ij}} \geq f_i$$

because $u_i \geq 0$. Finally, we show $\alpha x_j \equiv d \pmod{m}$. From (19), we have that $x_j = (f_i + u_i)/a_{ij}$ for some i. So (18) implies that $x_j \equiv \lambda_m d/\beta \pmod{m/\beta}$, and therefore $\alpha x_j \equiv \lambda_m \alpha d/\beta \pmod{m/\beta}$. This implies the following due to $\lambda_\alpha m + \lambda_m \alpha = \beta$ in (b):

$$\alpha x_j \equiv (d - \lambda_\alpha m d/\beta) \pmod{m/\beta}$$

which implies $\alpha x_j \equiv d \pmod{m/\beta}$. But this implies $\alpha x_j \equiv d \pmod{m}$ because it is given in (b) that β divides d. $\square$

We now describe a step of the projection algorithm as it applies to a generalised IP problem. We suppose that the current system $(\mathcal{S}, \mathcal{C})$ consists of a set $\mathcal{S}$ of inequalities and a set $\mathcal{C}$ of congruences in variables z, x_j, and u_i, and finite domains $u \in D$. We then project out variable x_j as follows. We first apply the GCRT to all congruences in $\mathcal{C}$ containing x_j to obtain a single congruence that can be written $\alpha x_j \equiv d \pmod{m}$. We then consider all pairs of inequalities in $\mathcal{S}$ of the form $a_{ij} x_i \geq f_i$ and $-a_{kj} x_j \geq g_k$ for which $a_{ij}, a_{kj} > 0$. We introduce an auxiliary variable u_i for each i, and for each pair we generate the inequality $a_{kj}(f_i + u_i) + a_{ij} g_k \leq 0$ along with the congruence $f_i + u_i \equiv \lambda_m a_{ij} d/\beta \pmod{a_{ij} m/\beta}$ as given in Theorem 3. The multiplier λ_m can be obtained by using the Euclidean algorithm to find multipliers $\lambda_\alpha, \lambda_m$ for which $\lambda_\alpha \mathrm{lcm}(\alpha, m)/\alpha + \lambda_m \mathrm{lcm}(\alpha, m)/m = 1$. Finally, we update the system $(\mathcal{S}, \mathcal{C})$ by removing from $\mathcal{S}$ all inequalities containing x_j, adding to $\mathcal{S}$ all generated inequalities, adding to $\mathcal{C}$ all the associated congruence relations, and adding $u_i \in \{0, \ldots, a_{ij} m/\beta - 1\}$ to the domains.

To solve a generalised IP problem, we suppose the problem is given in the form $(\mathcal{S}, \mathcal{C})$ with domains $u \in D$, as above. It can be viewed as an optimization problem subject to the inequalities $\mathcal{S}$ in variables x_j, over the integer sublattice defined by the congruence relations in $\mathcal{C}$. In a conventional IP problem, the congruences in $\mathcal{C}$ are simply $x_j \equiv 0 \pmod{1}$, which require integrality, and there are no variables u_i. We sequentially project out variables $x_1, \ldots, x_n$, which yields a system $(\mathcal{S}', \mathcal{C}')$ in which $\mathcal{S}'$ contains only z and variables u_i, and $\mathcal{C}'$ contains only u_is. The inequalities in $\mathcal{S}'$ have the form $z \geq v_t(u)$, and the optimal value of the problem is $\min_u \{\max_t \{v_t(u)\} \mid \mathcal{C},\ u \in D\}$. The original problem is therefore transformed to one in which the variables u_i have finite domains.

The above results follow from those of an earlier paper [5], while the results to follow are new.

4 Projection Cuts

Projection supplies the information necessary to derive valid inequalities for IP as it does for LP. Like Chvátal-Gomory cuts, the inequalities can be derived by a linear combination and strengthening operation in which the multipliers are obtained from projection steps. However, the strengthening operation relies on a congruence relation rather than rounding, and the desired congruence relation is likewise obtained from a projection step. We will refer to valid inequalities derived in this fashion as *projection cuts*.

This can be illustrated by the example (2). In step 1 of the projection, C1 and C2 were combined to yield the projection cut C12. This cut can be obtained by a linear combination of these inequalities in which the multipliers (given on the left below) are those used to combine C1 and C2 in (6).

(5)	$2x_1 + x_2 \geq 13 + u_1$	C1′
(2)	$-5x_1 - 2x_2 \geq -30$	C2
	$x_2 \geq 5 + 5u_1$	C12

Before taking the linear combination, C1 is strengthened to obtain C1′ using the same integer auxiliary variable u_1 that was used in computing the projection. The cut is valid when $x_2 \equiv u_1 + 1 \pmod 2$ and $u_1 \in \{0, 1\}$, which are the same conditions under which the auxiliary variable was added in the projection step.

Projection cut C13 is similarly derived from step 1 of the projection. Step 2 of the projection yields two projection cuts from which x_1, x_2 have been eliminated. One cut is $z \geq 5 + 5u_1$, where $u_1 \in \{0, 1\}$, from which we can conclude only that $z \geq 5$. The second is $z \geq \frac{1}{3}(23 + u_1 + u_{13})$, where $4u_1 + u_{13} \equiv 4 \pmod 6$, $u_1 \in \{0, 1\}$, and $u_{13} \in \{0, \ldots, 5\}$. Because the congruence relation has two solutions $(u_1, u_{13}) = (0, 4), (1, 0)$, we can conclude from this cut only that $z \geq 8$.

Thus each cut is associated with a system of congruence relations and a variable domain under which it is valid. The projection algorithm allows one to derive cuts from which all x_js have been eliminated. The optimal value of the original problem is the minimum of z subject to these cuts and congruence relations considered simultaneously. In the example, the two bounds on z yield a bound of 9 when $(u_1, u_{13}) = (0, 4)$ and 10 when $(u_1, u_{13}) = (1, 0)$. The optimal value is therefore 9.

In general, we can define a projection cut as a nonnegative linear combination of two valid inequalities, one of which is strengthened. To make this precise, we define a concept of *rank* analogously with Chvátal-Gomory cuts. Let $\mathcal{S}$ be a system of linear inequalities in variables $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, and let $\mathcal{C}$ be a system of congruences in variables x and $u = (u_1, \ldots, u_t) \in D \subset \mathbb{Z}^t_{\geq 0}$. A *rank 1 projection cut* for $(\mathcal{S}, \mathcal{C})$ is any nonnegative linear combination of $a_{ij}x_j \geq f_i + u_i$ and an inequality in $\mathcal{S}$, where $a_{ij}x_j \geq f_i$ belongs to $\mathcal{S}$ and a congruence of the form $\alpha x_j \equiv d \pmod m$ belongs to $\mathcal{C}$. The rank 1 cut is associated with the congruence relation

$$\alpha(f_i + u_i) \equiv a_{ij}d \pmod{a_{ij}m} \tag{20}$$

and domain $u_i \in \{0, \ldots, a_{ij}m - 1\}$. The cut is *valid* when all (x, u) satisfying $(\mathcal{S}, \mathcal{C})$, congruence (20), $u \in D$, and $u_i \in \{0, \ldots, a_{ij}m - 1\}$ also satisfy the cut.

A *rank k projection cut* for $(\mathcal{S}, \mathcal{C})$ is a rank 1 cut for some system $(\mathcal{S}', \mathcal{C}')$ consisting of cuts of rank $k-1$ or less for $(\mathcal{S}, \mathcal{C})$ and their associated congruences and domains, provided it is not a rank 1 cut for any such system of cuts with rank less than $k-1$. A *projection* cut is any rank k projection cut for finite k.

Theorem 4 *Any projection cut for $(\mathcal{S}, \mathcal{C})$ is valid for $(\mathcal{S}, \mathcal{C})$.*

Proof. It is enough to show that any rank 1 cut for $(\mathcal{S}, \mathcal{C})$ is valid, because then it follows by induction than any rank k cut is valid. Because a nonnegative linear combination of valid inequalities is valid, we can show that a rank 1 cut is valid by showing that $a_{ij}x_j \geq f_i + u_i$ is valid for $(\mathcal{S}, \mathcal{C})$ when $a_{ij}x_j \geq f_i$ is in $\mathcal{S}$, (20) holds, $u \in D$, and $u_i \in \{0, \ldots, a_{ij}m - 1\}$. Equivalently, we wish to show

$$\alpha a_{ij}x_{ij} - a_{ij}d \geq \alpha(f_i + u_i) - a_{ij}d \tag{21}$$

is valid under these conditions. However, we know that $\alpha a_{ij}x_{ij} - a_{ij}d \geq \alpha f_i - a_{ij}d$ is valid, because $a_{ij}x_{ij} \geq f_i$ belongs to $\mathcal{S}$. Also the congruence $\alpha x_j \equiv d \pmod{m}$ implies that the left-hand side of (21) is multiple of $a_{ij}m$. The inequality (21) is therefore valid if u_i is the smallest nonnegative integer for which the right-hand side is a multiple of $a_{ij}m$. For this, it suffices that (20) hold and $u_i \in \{0, \ldots, a_{ij}m - 1\}$. □

We can also show that projection yields projection cuts.

Theorem 5 *Each step of the integer projection method produces rank 1 projection cuts for the system $(\mathcal{S}, \mathcal{C})$ from which the cuts are derived.*

Proof. Each inequality generated by projection has the form

$$a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0 \tag{22}$$

and is derived from $a_{ij}x_j \geq f_i$, $-a_{kj}x_j \geq g_k \in \mathcal{S}$. We wish to show that (22) is a rank 1 projection cut for $(\mathcal{S}, \mathcal{C})$. We first note that (22) is a linear combination of $a_{ij}x_j \geq f_i + u_i$ and $-a_{kj}x_j \geq g_k$, using multipliers $a_{kj}, a_{ij} > 0$, respectively. Because $\alpha x_j \equiv d \pmod{m}$ is in $\mathcal{C}$, it remains only to show that $u_i \in \{0, \ldots, a_{ij}m-1\}$ and that (20) holds. The projection step yields the congruence relation

$$\alpha(f_i + u_i) \equiv \alpha\lambda_m a_{ij}d/\beta \pmod{a_{ij}m/\beta} \tag{23}$$

where $\lambda_\alpha m + \lambda_m \alpha = \beta$. Substituting $\beta - \lambda_\alpha m$ for $\lambda_m \alpha$, this becomes

$$\alpha(f_i + u_i) \equiv a_{ij}d/\beta \pmod{a_{ij}m/\beta}$$

This implies (20) since β divides d. Also, the projection step yields $u_i \in \{0, \ldots, a_{ij}m/\beta - 1\}$, which implies $u_i \in \{0, \ldots, a_{ij}m - 1\}$. □

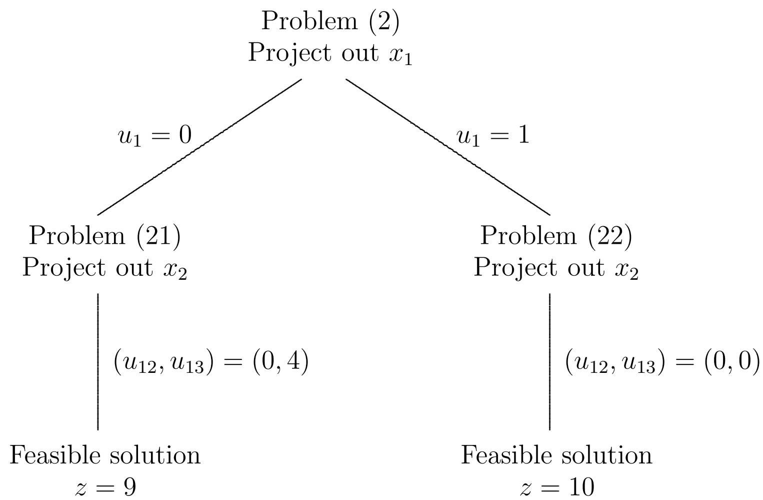

Figure 2: Projection-based branching tree for example (2).

Projection cuts of sufficiently large (but finite) rank can prove optimality, in a manner somewhat parallel to Chvátal-Gomory cuts. Let $\mathcal{S}$ be the set of inequalities in the IP problem (4). Let $(\mathcal{S}', \mathcal{C}')$ be a system of projection cuts for $(\mathcal{S}, \emptyset)$, and $u \in D$ the associated domains. We will say that $(\mathcal{S}', \mathcal{C}')$ and $u \in D$ prove that solution value z^* is optimal for (4) when $\mathcal{S}'$ contains only variables z and u, and z^* is the minimum of z subject to $(\mathcal{S}', \mathcal{C}')$ and $u \in D$. Theorem 3 allows us to conclude that projection cuts of finite rank prove optimality for any given IP problem. In particular,

Corollary 6 *If $\mathcal{S}$ is the constraint set for the IP problem (4), some system of projection cuts for $(\mathcal{S}, \emptyset)$ with rank at most n, together with their associated congruences and domains, proves the optimal value of (4).*

The optimal value 9 of the example (2) is proved by the projection cuts $z \geq 5 + 5u_1$ and $z \geq \frac{1}{3}(23 + u_1 + u_{13})$, together with the congruence $4u_1 + u_{13} \equiv 4 \pmod 6$ and domains $u_1 \in \{0, 1\}$ and $u_{13} \in \{0, \ldots, 5\}$.

5 Solution by Branching

The above analysis of integer projection leads to a branching algorithm for the generalised IP problem $(\mathcal{S}, \mathcal{C})$, $u \in D$. Each time a variable x_j is projected out, we branch on the auxiliary variables u_i created during the projection step. This means that no auxiliary variables appear in the branches. The process is repeated at each branch, until none of the original variables x_j remain. If the original problem contains variables u_i, we branch on them (as well as the auxiliary variables) at the root node.

This can be illustrated using the example (2), for which the branching tree appears in Fig. 2. At the root node of the tree, we carry out step 1 above, which yields the projected system (7). Now, rather than branch on x_1, we branch on $u_1 \in \{0, 1\}$.

Left branch, $u_1 = 0$. Here (7) simplifies to

$$\begin{aligned} &-x_2 \geq -z \\ &x_2 \geq 5 \\ &3x_2 \geq 23 \\ &x_2 \equiv 1 \bmod 2 \end{aligned} \tag{24}$$

We now project out x_2, which yields

$$\begin{array}{lcl} 5 + u_{12} \leq x_2 \leq z & \Rightarrow & z \geq 5 + u_{12} \\ 23 + u_{13} \leq 3x_2 \leq 3z & & z \geq \frac{1}{3}(23 + u_{13}) \\ 5 + u_{12} \equiv 1 \pmod 2, \;\; u_{12} \in \{0, 1\} & & u_{12} \equiv 0 \pmod 2, \;\; u_{12} \in \{0, 1\} \\ 23 + u_{13} \equiv 3 \pmod 6, \;\; u_{13} \in \{0, \ldots, 5\} & & u_{13} \equiv 4 \pmod 6, \;\; u_{13} \in \{0, \ldots, 5\} \end{array}$$

Only one branch $(u_{12}, u_{13}) = (0, 4)$ satifies the congruence. In this branch, the problem is to minimise z subject to $z \geq 5$ and $z \geq 9$, yielding the bound $z \geq 9$.

Right branch, $u_1 = 1$. Here (7) simplifies to

$$\begin{aligned} &-x_2 \geq -z \\ &x_2 \geq 10 \\ &3x_2 \geq 24 \\ &x_2 \equiv 0 \bmod 2 \end{aligned} \tag{25}$$

Projecting out x_2, we get

$$\begin{array}{lcl} 10 + u_{12} \leq x_2 \leq z & \Rightarrow & z \geq 10 + u_{12} \\ 24 + u_{13} \leq 3x_2 \leq 3z & & z \geq 8 + \frac{1}{3}u_{13} \\ 10 + u_{12} \equiv 0 \pmod 2, \;\; u_{12} \in \{0, 1\} & & u_{12} \equiv 0 \pmod 2, \;\; u_{12} \in \{0, 1\} \\ 24 + u_{13} \equiv 0 \pmod 6, \;\; u_{13} \in \{0, \ldots, 5\} & & u_{13} \equiv 0 \pmod 6, \;\; u_{13} \in \{0, \ldots, 5\} \end{array}$$

Only one branch $(u_{12}, u_{13}) = (0, 0)$ is possible, at which the problem is to minimise z subject to $z \geq 10$ and $z \geq 8$, yielding the bound $z \geq 10$.

The optimal solution occurs at the left leaf node, with $z = 9$ and $(u_1, u_{12}, u_{13}) = (0, 0, 4)$.

We can introduce a branch-and-bound mechanism by solving a relaxation at each node. The solution of the relaxation can also indicate how to branch, as in traditional branch and bound, because we can branch on a variable x_j that violates its associated congruence $x_j \equiv d \pmod m$. The simplest relaxation is an LP relaxation obtained by dropping the congruences.

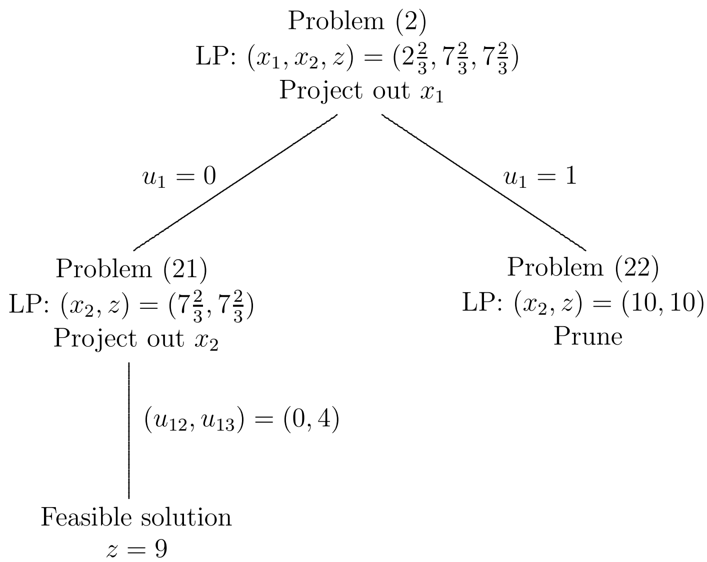

Figure 3: Projection-based branch-and-bound tree for example (2).

For example, the LP relaxation of (2) at the root node has solution $(x_1, x_2, z) = (2\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3})$ (Fig. 3). Because x_1 and x_2 must satisfy the implicit congruence $x_j \equiv 0 \pmod 1$ for $j = 1, 2$, we can project out either variable and branch on the corresponding auxiliary variable. We choose to project out x_1 and branch on u_1. Solving the LP relaxation of (24) in the left branch yields $(x_2, z) = (7\frac{2}{3}, 7\frac{2}{3})$. Because x_2 violates $x_2 \equiv 1 \pmod 2$, we must project out x_2. The LP relaxation of (25) in the right branch has solution $(x_2, z) = (10, 10)$. Because 10 is greater than the incumbent value of 9, it is unnecessary to project out x_2 and branch further. In addition, x_2 satisfies $x_2 \equiv 0 \pmod 2$, which in itself obviates the necessity of further branching.

Note that it may be necessary to branch even when all the variables x_j are integral in the LP solution. The relevant criterion is whether they satisfy their respective congruences.

6 A Value Function and Dual Solution

We can obtain a value function by applying the projection algorithm to inequalities with perturbed right-hand sides. To illustrate the idea, consider the constraint C1 in example (2), which is $2x_1 + x_2 \geq 13$. While projecting out x_1 we used the strengthened inequality

$$-x_2 + 13 + u_1 \leq 2x_1 \tag{26}$$

where

$$-x_2 + 13 + u_1 \equiv 0 \pmod 2 \tag{27}$$

and $u \in \{0, 1\}$. Suppose we now perturb the right-hand side of C1 to obtain the constraint $2x_1 + x_2 \geq 13 + \Delta$, so that (26) becomes $-x_2 + 13 + \Delta + u_1 \leq 2x_1$. This inequality is not generally valid, given congruence (27). However, we can strengthen C1 in a different way by adding $\Delta + \mathrm{mod}_2(u_1 - \Delta)$ rather than u_1:

$$-x_2 + 13 + \Delta + \mathrm{mod}_2(u_1 - \Delta) \leq 2x_1 \tag{28}$$

where $\mathrm{mod}_m(a)$ is the remainder after dividing a by m. This has the same effect as (26) when $\Delta = 0$. To ensure validity, we need the congruence

$$-x_2 + 13 + \Delta + \mathrm{mod}_2(u_1 - \Delta) \equiv 0 \pmod 2 \tag{29}$$

However, this is equivalent to congruence (27), because $u_1 \equiv \Delta + \mathrm{mod}_m(u_1 - \Delta) \pmod m$ due to the obvious fact that $u_1 - \Delta \equiv \mathrm{mod}_m(u_1 - \Delta) \pmod m$. It is easy to show that

$$\Delta + \mathrm{mod}_m(u_1 - \Delta) = u_1 + \lceil \Delta - u_1 \rceil_m \tag{30}$$

where $\lceil a \rceil_m = m\lceil a/m \rceil$ is a rounded up to the nearest multiple of m. So (28) can be written

$$-x_2 + 13 + u_1 + \lceil \Delta - u_1 \rceil_2 \leq 2x_1$$

By incorporating this idea into the projection algorithm, we can derive a value function. Consider again the perturbed example (3).

Step 1. To project out x_1, we combine $\mathrm{C1}_\Delta$ and $\mathrm{C2}_\Delta$ to obtain

$$5(-x_2 + 13 + u_1 + \lceil \Delta_1 - u_1 \rceil_2) \leq 5 \cdot 2x_1 \leq 2(-2x_2 + 30)$$

This yields

$$x_2 \geq 5 + 5u_1 + 5\lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_2 \qquad \mathrm{C12}_\Delta$$

where $x_2 \equiv u_1 + 1 \pmod 2$ as before. We combine $\mathrm{C\Delta 1}$ and $\mathrm{C\Delta 3}$ to obtain

$$3x_2 \geq 23 + u_1 + \lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_3 \qquad \mathrm{C13}_\Delta$$

Step 2. To eliminate x_2, we combine C0 and $\mathrm{C12}_\Delta$ to obtain

$$5 + 5u_1 + u_{12} + 5\left\lceil \lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_2 - u_{12} \right\rceil_2 \leq x_2 \leq z$$

This yields

$$z \geq 5 + 5u_1 + u_{12} + \left\lceil 5\lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_2 - u_{12} \right\rceil_2 \tag{31}$$

where $u_{12} \equiv 0 \pmod 1$ and $x_{12} \in \{0\}$. Note the nesting of functions $\lceil \cdot \rceil_m$, which is analogous to the nesting of rounding operations in a Chvátal function. Because $u_{12} = 0$ and $\lceil \Delta_1 - u_1 \rceil_2$ is even, the bound (31) simplifies to

$$z \geq 5 + 5u_1 + 5\lceil \Delta_1 - u_1 \rceil_2 + \lceil 2\Delta_2 \rceil_2 \qquad \mathrm{C012}_\Delta$$

We similarly combine C0 and $\mathrm{C13}_\Delta$ to obtain

$$3z \geq 23 + u_1 + u_{13} + \left\lceil \lceil \Delta_1 - u_1 \rceil_2 + 2\Delta_3 - u_{13} \right\rceil_6 \qquad \mathrm{C013}_\Delta$$

Table 2: Lower bounds for perturbations in individual constraints i.

		Bound from C012$_\Delta$			Bound from C013$_\Delta$		
u_1	u_{13}	$i=1$	2	3	$i=1$	2	3
0	4	$5+5\lceil\Delta_1\rceil_2$	$5+\lceil 2\Delta_2\rceil_2$	5	$9+\frac{1}{3}\lceil\Delta_1-4\rceil_6$	9	$9+\frac{2}{3}\lceil\Delta_3-2\rceil_3$
1	0	$10+5\lceil\Delta_1-1\rceil_2$	$10+\lceil 2\Delta_2\rceil_2$	10	$8+\frac{1}{3}\lceil\Delta_1-1\rceil_6$	8	$8+\frac{2}{3}\lceil\Delta_3\rceil_3$

where $4u_1+u_{13}\equiv 4 \pmod 6$ and $u_{13}\in\{0,\ldots,5\}$.

Step 3. We now have a value function from C012$_\Delta$ and C013$_\Delta$:

$$v(\Delta_1,\Delta_2,\Delta_3)=\min_{u_1,u_{13}}\left\{\max\left\{\begin{array}{l}5+5u_1+5\lceil\Delta_1-u_1\rceil_2+\lceil 2\Delta_2\rceil_2,\\ \frac{1}{3}(23+u_1+u_{13}+\lceil\lceil\Delta_1-u_1\rceil_2+2\Delta_3-u_{13}\rceil_6\,)\end{array}\right\}\right\}$$

where the minimum is taken over u_1,u_{13} satisfying $u_{12}\equiv 0 \pmod 2$, $4u_1+u_{13}\equiv 4 \pmod 6$, $u_1\in\{0,1\}$, and $u_{13}\in\{0,\ldots,5\}$. In this case, the congruences have only two solutions $(u_1,u_{13})=(0,4),(1,0)$.

The function simplifies when we analyze perturbations of one constraint at a time:

$$v(\Delta_1)=\min_{(u_1,u_{13})=(0,4),(1,0)}\left\{\max\left\{\begin{array}{l}5+5u_1+5\lceil\Delta_1-u_1\rceil_2\,,\\ \frac{1}{3}(23+u_1+u_{13}+\lceil\lceil\Delta_1-u_1\rceil_2-u_{13}\rceil_6)\end{array}\right\}\right\}$$

$$v(\Delta_2)=\min_{(u_1,u_{13})=(0,4),(1,0)}\left\{\max\left\{\begin{array}{l}5+5u_1+\lceil 2\Delta_2\rceil_2,\\ \frac{1}{3}(23+u_1+u_{13})\end{array}\right\}\right\}$$

$$v(\Delta_3)=\min_{(u_1,u_{13})=(0,4),(1,0)}\left\{\max\left\{\begin{array}{l}5+5u_1\\ \frac{1}{3}(23+u_1+u_{13}+\lceil 2\Delta_3-u_{13}\rceil_6)\end{array}\right\}\right\}$$

The resulting bounds in for the two solutions $(u_1,u_{13})=(0,4),(1,0)$ of the congruences appear in Table 2. These bounds are graphed in Figs. 4–6.

As in the case of an LP value function, the coefficient of Δ_i in the term that governs when $\Delta=0$ can be interpreted as a dual multiplier. There are two differences from LP, however. One is that we take a minimum over a maximum rather than simply a maximum. For example, when $\Delta=0$, the expression for $v(\Delta_1)$ becomes $\min\{\max\{5,9\},\max\{10,8\}\}=9$, so that the minimising value of (u_1,u_{13}) is $(0,4)$ and the second term $9+\frac{1}{3}\lceil\Delta_1-4\rceil_6$ of the max governs.

The second difference is that marginal cost can be a discontinuous function of Δ_i. In the LP case, the dual multiplier for Δ_1 is $\frac{1}{3}$ when $\Delta=0$, meaning that optimal cost increases linearly with Δ_1 (at rate $\frac{1}{3}$) in some neighborhood of $\Delta_1=0$. In the IP case, the dual multiplier is again $\frac{1}{3}$ when $\Delta_1=0$, but the change in cost is $\frac{1}{3}\lceil\Delta_1-4\rceil_6$ rather than $\frac{1}{3}\Delta_1$. This means that there is no change until Δ_1-4 reaches a multiple of 6, at which point the cost changes by $\frac{1}{3}\cdot 6$. The dual multiplier can therefore be interpreted as a "jerky" shadow price. It indicates the *average* marginal cost, but the actual cost function is a step function.

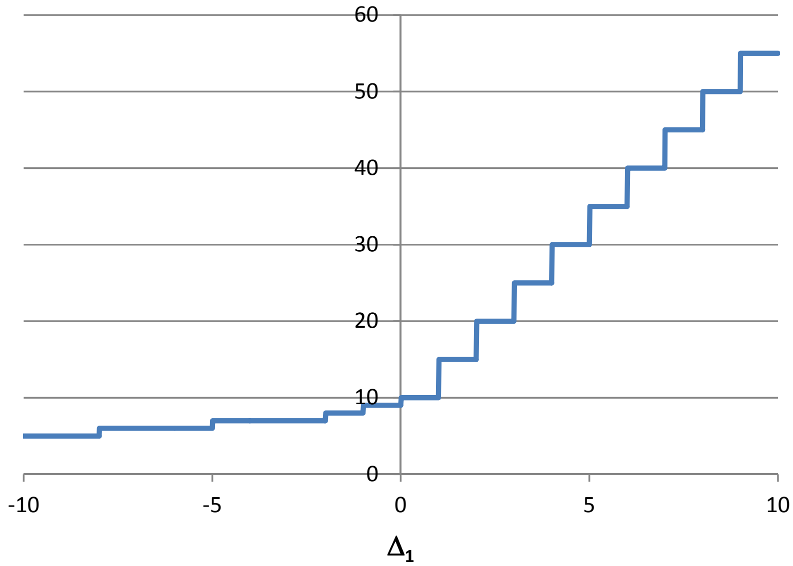

Figure 4: Value function $v(\Delta_1)$ for constraint 1.

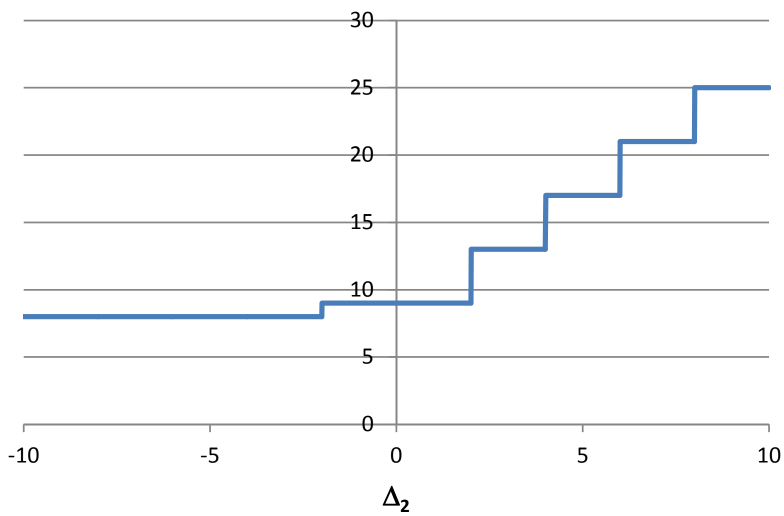

Figure 5: Value function $v(\Delta_2)$ for constraint 2.

Of course, the shadow price changes when a different term of the value function begins to govern, as in the case of LP.

To show that projection creates a value function for an general IP problem, we must extend Theorem 3 to deal with perturbed right-hand sides. Interestingly, the perturbations do not affect the congruences, and the perturbation terms Δ_i appear only in the generated

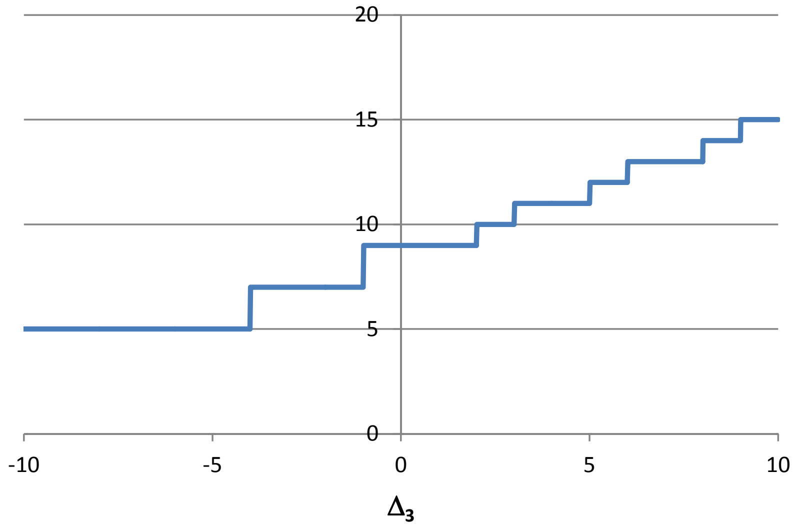

Figure 6: Value function $v(\Delta_3)$ for constraint 3.

inequalities. The inequalities $a_{ij}x_j \geq f_i$ and $-a_{kj}x_j \geq g_k$ in Theorem 3 are replaced with $a_{ij}x_j \geq f_i + \bar{\Delta}_i$ and $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ to account for the effect of perturbations on generated inequalities. Thus $\bar{\Delta}_i = \bar{\Delta}_k = 0$ when all the perturbations are zero.

Theorem 7 *Suppose $a_{ij}, a_{kj} > 0$ for all $i \in I, k \in K$. Then*

(a) *There exists $x_j \in \mathbb{Z}$ such that $a_{ij}x_j \geq f_i + \bar{\Delta}_i$ and $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ for all $i \in I$, $k \in K$, and such that $\alpha x_j \equiv d \pmod{m}$,*

if and only if

(b) *$d \equiv 0 \pmod{\beta}$, where $\beta = gcd(\alpha, m)$; there exist $\lambda_\alpha, \lambda_m \in \mathbb{Z}$ satisfying $\lambda_\alpha m + \lambda_m \alpha = \beta$; and there exists $u_i \in \{0, 1, \ldots, a_{ij}m/\beta - 1\}$ such that*

$$a_{kj}(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) + a_{ij}(g_k + \bar{\Delta}_k) \leq 0 \tag{32}$$

for all $i \in I, k \in K$, and $f_i + u_i \equiv \lambda_m a_{ij} d/\beta \pmod{a_{ij}m/\beta}$ for all $i \in I$.

Furthermore, if $\bar{\Delta}_i = \bar{\Delta}_k = 0$, then inequality (32) reduces to $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$.

Proof. We first note that if $\bar{\Delta}_i = \bar{\Delta}_k = 0$, then in (32) we round $-u_i$ up to the nearest multiple of $a_{ij}m/\beta$, which is zero because $0 \leq u_i < a_{ij}m/\beta$. Thus (32) reduces to $a_{kj}(f_i + u_i) + a_{ij}g_k \leq 0$.

(a) $\Rightarrow$ (b). We can write the inequalities in (a) as

$$a_{kj}\alpha(f_i + \bar{\Delta}_i) \leq a_{ij}a_{kj}\alpha x_j \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) \tag{33}$$

for all i, k. If we let $y = a_{ij}a_{kj}\alpha x_j$, then we can show as in the proof of Theorem 3 that $d \equiv 0 \pmod{\beta}$, $\lambda_\alpha m + \lambda_m \alpha = \beta$ for some $\lambda_\alpha, \lambda_m \in \mathbb{Z}$, and the congruence relation $y \equiv \lambda_m a_{ij}a_{kj}\alpha d/\beta \pmod{a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)}$ holds. From the congruence relation and (33), we have

$$a_{kj}\alpha(f_i + \bar{\Delta}_i) - \lambda_m a_{ij}a_{kj}\alpha d/\beta \leq \gamma \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) - \lambda_m a_{ij}a_{kj}\alpha d/\beta \tag{34}$$

where γ is an integer multiple of $a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)$. Since β divides d, the leftmost expression in (34) is an integer multiple of $a_{kj}\alpha$. So we can add the term

$$a_{kj}\alpha(-\bar{\Delta}_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) \tag{35}$$

to the left-hand side of (34), where the expression $s_i = -\bar{\Delta}_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ takes a value in $\{0, \ldots, a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)/(a_{kj}\alpha) - 1\} = \{0, \ldots, a_{ij}m/\beta - 1\}$. We therefore have

$$a_{kj}\alpha(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) \leq \gamma + \lambda_m a_{ij}a_{kj}\alpha d/\beta \leq -a_{ij}\alpha(g_k + \bar{\Delta}_k) \tag{36}$$

where

$$a_{kj}\alpha(f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}) - \lambda_m a_{ij}a_{kj}\alpha d/\beta \equiv 0 \pmod{a_{ij}a_{kj}\mathrm{lcm}(\alpha, m)} \tag{37}$$

Inequality (36) implies (32). Congruence (37) simplifies to

$$f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta} - \lambda_m a_{ij}d/\beta \equiv 0 \pmod{a_{ij}m/\beta}$$

which implies the congruence in (b) because $\lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ is a multiple of $a_{ij}m/\beta$. Finally, $s_i = \bmod_{a_{ij}m/\beta}(u_i - \bar{\Delta}_i)$ due to (30). Because we need only consider values $0, \ldots, a_{ij}m/\beta - 1$ for s_i, we generate the required values by restricting u_i to $\{0, \ldots, a_{ij}m/\beta - 1\}$.

(a) $\Leftarrow$ (b). The inequalities in (b) can be written

$$-\frac{g_k + \bar{\Delta}_k}{a_{kj}} \geq \frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \tag{38}$$

for all i, k. From (b) we have that $d \equiv 0 \pmod{\beta}$, so that d/β is integral. Also from (b),

$$f_i + u_i \equiv \lambda_m a_{ij}d/\beta \pmod{a_{ij}a_{kj}m/\beta}$$

Because d/β and m/β are integral, this implies $f_i + u_i$ is an integer multiple of a_{ij}. We also have that $\lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}$ is a multiple of $a_{ij}m/\beta$ and therefore a_{ij}. So we can let

$$x_j = \max_i \left\{ \frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \right\}$$

and x_j is integral. This and (38) imply $-(g_k + \bar{\Delta}_k)/a_{kj} \geq x_j$, or $-g_k \geq a_{kj}x_j + \bar{\Delta}_k$. To show that $a_{ij}x_j \geq f_i + \bar{\Delta}_i$, we note that

$$a_{ij}x_j \geq a_{ij}\frac{f_i + u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta}}{a_{ij}} \geq f_i + \bar{\Delta}_i$$

because $u_i + \lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta} = \bar{\Delta}_i + \text{mod}_{a_{ij}m/\beta}(u_i - \bar{\Delta}_i) \geq \bar{\Delta}_i$ due to (30). Finally, it can be shown as in the proof of Theorem 3 that $\alpha x_j \equiv d \pmod{m}$. $\Box$

We can now describe, in general, a step of the projection method applied to an IP (4) with perturbed right-hand sides $b_i + \Delta_i$. If x_j is to be eliminated from the current system $(\mathcal{S}, \mathcal{C})$, we apply the GCRT to the congruences in $\mathcal{C}$ containing x_j to obtain a single congruence $\alpha x_j \equiv d \pmod{m}$. We then consider all pairs of inequalities in $\mathcal{S}$ of the form $a_{ij}x_j \geq f_i + \bar{\Delta}_i$, $-a_{kj}x_j \geq g_k + \bar{\Delta}_k$ for which $a_{ij}, a_{kj} > 0$. When projecting out the first variable, $\bar{\Delta}_i = \Delta_i$ and $\bar{\Delta}_k = \Delta_k$. We generate the inequality (32) and associate it with the congruence $f_i + u_i \equiv \lambda_m a_{ij} d/\beta \pmod{a_{ij}m/\beta}$ and the domain $u_i \in \{0, \ldots, a_{ij}m/d - 1\}$. The multiplier λ_m can be obtained by using the Euclidean algorithm as before. We then update the system $(\mathcal{S}, \mathcal{C})$ by removing from $\mathcal{S}$ all inequalities containing x_j, adding to $\mathcal{S}$ all generated inequalities, adding to $\mathcal{C}$ all the associated congruence relations, and adding $u_i \in \{0, \ldots, a_{ij}m/\beta - 1\}$ to the domains. If x_ℓ is the next variable to be eliminated from a generated inequality (32), we write (32) in the form $a_{ij}x_\ell \geq f_\ell + \bar{\Delta}_\ell$, where $\bar{\Delta}_\ell = a_{kj}\lceil \bar{\Delta}_i - u_i \rceil_{a_{ij}m/\beta} + a_{ij}\bar{\Delta}_k$.

When all variables x_j have been eliminated, the result is a system $(\mathcal{S}', \mathcal{C}')$ and domains $u \in D$ such that $\mathcal{S}$ contains only z and u_is, and $\mathcal{C}$ contains only u_is. The inequalities in $\mathcal{S}$ provide bounds of the form $z \geq v_t(u, \Delta)$. Due to Theorem 7, this describes the projection onto z, and the function

$$v(\Delta) = \min_u \left\{ \max_t \{v_t(u, \Delta)\} \;\middle|\; \mathcal{C}, u \in D \right\}$$

is therefore the optimal value of the perturbed IP problem (4). In other words, $v(\Delta)$ is a value function for (4). It is clear from the form of (32) that $v(\Delta)$ contains nested roundings $\lceil \cdot \rceil_m$. Because n variables are eliminated, the depth of the nesting is at most n.

7 Conclusion

We generalised LP projection (Fourier-Motzkin elimination) to IP projection. This leads to a new branching algorithm in which the depth of the tree is bounded by the number of variables in the IP, in contrast to conventional IP branch-and-bound methods, where there is no bound. It also leads to a complete family of cutting planes where the maximum rank is also bounded by the number of variables in the original IP. Finally, a *value function* for an IP is produced, in which the optimal objective value is given as a function of the right-hand sides. This provides a duality result for IP analogous to that for LP.

Some related results for the more general case of mixed integer/linear programming (MILP) appear in [9]. These results lead to an analytic solution of the MILP when applied only to the constraints binding in the LP relaxation; that is, when applied to an MILP over a cone.

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