

# A Note on Submodular Function Minimization by Chubanov's LP Algorithm

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## Abstract

Recently Dadush, Véggh, and Zambelli (2017) has devised a polynomial submodular function minimization (SFM) algorithm based on their LP algorithm. In the present note we also show a weakly polynomial algorithm for SFM based on the recently developed linear programming feasibility algorithm of Chubanov (2017) to stimulate further research on SFM.

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## 1. Introduction

Since the advent of polynomial combinatorial algorithms for submodular function minimization (SFM) in 1999 [9, 12] substantial improvements over the algorithms have been achieved as an active area of research. From the point of view of the computational complexity the currently best SFM algorithm is obtained by Lee, Sidford, and Wong [10]. On the other hand, important developments in linear programming algorithms have recently been made by Chubanov [2, 3, 4], and Dadush, Véggh, Zambelli [5] among others. Recently Dadush, Véggh, and Zambelli [6] has also devised a polynomial SFM algorithm based on their LP algorithms.

The aim of this note is to give yet another approach to SFM by using Chubanov's LP algorithm [4], which should stimulate further research on SFM and hopefully lead us to some efficient algorithms both practically and theoretically.

## 2. Definitions and Preliminaries

Let  $\mathbb{R}$  be the set of reals,  $\mathbb{Z}$  that of integers,  $\mathbb{Z}_{\geq 0}$  that of nonnegative integers, and  $\mathbb{Z}_{>0}$  that of positive integers.

Let  $V$  be a finite nonempty set of cardinality  $|V| = n$ . For any  $x, y \in \mathbb{R}^V$  define  $\langle x, y \rangle = \sum_{v \in V} x(v)y(v)$ . Consider an arbitrary bounded polyhedron  $Q \subset \mathbb{R}^V$ . Denote by  $\text{Ext}(Q)$  the set of all extreme points of  $Q$  and define

$$C^*(Q) = \{x \in \mathbb{R}^V \mid \langle z, x \rangle \geq 0 \text{ for all } z \in \text{Ext}(Q)\}. \quad (1)$$

Then we have the following.

**Proposition 1.** *For any bounded polyhedron  $Q \subset \mathbb{R}^V$ , the cone  $C^*(Q)$  defined by (1) is full-dimensional, i.e., of dimension  $n = |V|$  if and only if  $\mathbf{0} \notin Q$  or  $\mathbf{0} \in \text{Ext}(Q)$ .*

*Moreover, there exists a non-zero  $x \in C^*(Q)$  if and only if  $\mathbf{0}$  is not in the interior of  $Q$ .*

It should be noted that if  $Q$  is not full-dimensional, then the interior of  $Q$  is empty, so that there always exists a non-zero  $x \in C^*(Q)$ .

Let  $f : 2^V \rightarrow \mathbb{R}$  be a submodular function, i.e.,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad (\forall X, Y \subseteq V), \quad (2)$$

where we assume  $f(\emptyset) = 0$ . (See [7] for more information about relevant definitions and basic results on submodular functions.)

The *submodular polyhedron*  $P(f) \subset \mathbb{R}^V$  associated with the submodular function  $f : 2^V \rightarrow \mathbb{R}$  is given by

$$P(f) = \{x \in \mathbb{R}^V \mid \forall X \subseteq V : x(X) \leq f(X)\} \quad (3)$$

and the *base polyhedron*  $B(f)$  by

$$B(f) = \{x \in \mathbb{R}^V \mid x \in P(f), x(V) = f(V)\}, \quad (4)$$

where for any  $X \subseteq V$  we define  $x(X) = \sum_{v \in X} x(v)$  and  $x(\emptyset) = 0$ .

We consider the membership problem of discerning whether  $\mathbf{0} \in B(f)$  or not and to find a set  $X \subseteq V$  with  $f(X) < 0$  if  $\mathbf{0} \notin B(f)$ , where  $\mathbf{0}$  is the zero vector in  $\mathbb{R}^V$ . Since  $f(V) \neq 0$  implies  $\mathbf{0} \notin B(f)$ , we assume  $f(V) = 0$  when we consider the membership problem, unless otherwise stated. Moreover, we assume without loss of generality

(A\*)  $B(f)$  is full-dimensional, i.e., the dimension of  $B(f)$  is equal to  $|V| - 1 (= n - 1)$ ,

where note that  $B(f)$  lies within the hyperplane  $x(V) = 0$ .<sup>2</sup> This assumption is to guarantee that the maximum dimension of  $W$

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<sup>2</sup>Note that we can find connected components of the submodular system  $(2^V, f)$  in  $O(n^2 \mathbf{EO})$  time by using the algorithm of Bixby, Cunningham, and Topkis [1] and each connected component has a full-dimensional base polyhedron. See [7, Section 3.3]. Here  $\mathbf{EO}$  denotes the time required for the function evaluation of  $f$ .

appearing in (13) is  $n-1$ . We thus consider the relative topology within the hyperplane  $x(V) = 0$ .

For any  $\alpha \geq 0$  define  $f^{(\alpha)} : 2^V \rightarrow \mathbb{R}$  by

$$f^{(\alpha)}(X) = \begin{cases} f(X) + \alpha & \text{if } X \in 2^V \setminus \{\emptyset, V\} \\ f(X) (= 0) & \text{if } X \in \{\emptyset, V\}. \end{cases} \quad (5)$$

We can easily see that  $f^{(\alpha)}$  is a submodular function, which is called the  $\alpha$ -enlargement<sup>3</sup> of  $f$ . It should be noted that  $\mathbf{B}(f^{(\alpha)})$  is full-dimensional, i.e., of dimension  $n-1$ , since  $\mathbf{B}(f^{(\alpha)}) \supseteq \mathbf{B}(f)$  and  $\mathbf{B}(f)$  is full-dimensional under Assumption (A\*). It should be noted that the operation of enlargement is also employed by Dadush, Végh, and Zambelli [6] and plays an important rôle in their algorithm as well.

Define vectors  $\bar{\alpha}, \underline{\alpha} \in \mathbb{R}^V$  by

$$\bar{\alpha}(v) = f(\{v\}), \quad \underline{\alpha}(v) = f(V) - f(V \setminus \{v\}) \quad (\forall v \in V). \quad (6)$$

(See [7, (3.89) and (3.95)]). Then we have

$$\underline{\alpha}(X) \leq x(X) \leq f(X) \leq \bar{\alpha}(X) \quad (\forall x \in \mathbf{B}(f), \forall X \subseteq V). \quad (7)$$

Note that for any base  $b \in \mathbf{B}(f)$  we must have  $\underline{\alpha} \leq b \leq \bar{\alpha}$ , i.e.,  $\mathbf{B}(f) \subseteq [\underline{\alpha}, \bar{\alpha}]$  (a standard box between  $\underline{\alpha}$  and  $\bar{\alpha}$ ).

**Lemma 2.** *We have*

$$0 \geq \min\{f(X) \mid X \subseteq V\} \geq \sum_{v \in V} \min\{0, f(V) - f(V \setminus \{v\})\}. \quad (8)$$

Moreover, for any base  $b \in \mathbf{B}(f)$  we have

$$|b(v)| \leq \max\{f(\{v\}), -f(V) + f(V \setminus \{v\})\} \quad (\forall v \in V). \quad (9)$$

PROOF: The present lemma follows from (6) and (7).  $\square$

The base polyhedron  $\mathbf{B}(f)$  lies in the subspace  $H_0 = \{x \in \mathbb{R}^V \mid x(V) = 0\}$ . Hence, considering the relative topology in the subspace  $H_0$ , we define

$$\mathbf{C}^*(\mathbf{B}(f)) = \{x \in \mathbb{R}^V \mid x(V) = 0, \langle b, x \rangle \geq 0 \text{ for all } b \in \text{Ext}(\mathbf{B}(f))\}. \quad (10)$$

We say  $\mathbf{C}^*(\mathbf{B}(f))$  is full-dimensional if it is of dimension  $n-1$ .

### 3. Membership in Base Polyhedra and Chubanov's LP Algorithm

In the sequel we assume that  $f$  is an integer-valued submodular function. Let us consider the membership problem of  $\mathbf{0} \in \mathbf{B}(f)$ . Applying Chubanov's LP Algorithm [4], we have the following two:

1. When  $\mathbf{C}^*(\mathbf{B}(f))$  is full-dimensional, we can obtain a non-zero vector in  $\mathbf{C}^*(\mathbf{B}(f))$  in polynomial time.
2. If in prescribed polynomial time we cannot obtain such a non-zero vector in  $\mathbf{C}^*(\mathbf{B}(f))$ , then we can conclude that  $\mathbf{C}^*(\mathbf{B}(f))$  is not full-dimensional, which under Assumption (A\*) implies  $\mathbf{0} \in \mathbf{B}(f) \setminus \text{Ext}(\mathbf{B}(f))$  (see Proposition 1).

When we conclude  $\mathbf{0} \in \mathbf{B}(f) \setminus \text{Ext}(\mathbf{B}(f))$ , we see that  $f(X) \geq 0$  for all  $X \subseteq V$ .

On the other hand, when we find a non-zero vector  $w \in \mathbf{C}^*(\mathbf{B}(f))$ , we can get a nonempty proper subset  $Z \subset V$  with  $f(Z) \leq 0$  as follows. This is a standard technique but we show it for completeness (cf. [6, Lemma 3.2]). Letting  $b^*$  be the greedy solution that minimizes  $\langle w, x \rangle$  over  $\mathbf{B}(f)$ , we have  $\langle w, b^* \rangle \geq 0$  since  $w \in \mathbf{C}^*(\mathbf{B}(f))$ . Let  $(v_1, \dots, v_n)$  be a linear ordering of  $V$  that gives the greedy solution  $b^*$ , so that

$$w(v_1) \leq \dots \leq w(v_n). \quad (11)$$

Putting  $S_i = \{v_1, \dots, v_i\}$  for  $i = 1, \dots, n$  and  $S_0 = \emptyset$ , we have

$$\begin{aligned} \langle w, b^* \rangle &= w(v_1)(f(S_1) - f(S_0)) + \dots \\ &\quad + w(v_n)(f(S_n) - f(S_{n-1})) \\ &= (w(v_1) - w(v_2))f(S_1) + \dots \\ &\quad + (w(v_{n-1}) - w(v_n))f(S_{n-1}) \\ &\geq 0. \end{aligned} \quad (12)$$

Since at least one inequality in (11) holds with strict inequality, it follows from (11) and (12) that for some  $i^* \in \{1, \dots, n-1\}$  we have  $f(S_{i^*}) \leq 0$  (with  $w(v_{i^*}) - w(v_{i^*+1}) < 0$ ). We thus obtain a required  $Z = S_{i^*}$ .

Summing up the above arguments, we get the following theorem.

**Theorem 3.** *Under Assumption (A\*) we have a polynomial algorithm that gives us*

- (a) *a nonempty proper subset  $Z \subset V$  such that  $f(Z) \leq 0$  or*
- (b) *a conclusion that  $f(X) \geq 0$  for all  $X \subseteq V$ , i.e.,  $\mathbf{0} \in \mathbf{B}(f)$ .*

We denote the algorithm in Theorem 3 by  $\text{Chubanov}(f)$ .

Let us consider the complexity of  $\text{Chubanov}(f)$ . Define an index  $\nu$  by

$$\nu = \max\{\text{vol}(W) \mid W : \text{a paralleliped formed by } n-1 \text{ vectors } w \in \mathbf{C}^*(\mathbf{B}(f)) \text{ and } \|w\| \leq 1\}, \quad (13)$$

where  $\text{vol}(W)$  is the volume of  $W$  in the Euclidean subspace  $H_0 = \{x \in \mathbb{R}^V \mid x(V) = 0\}$  and  $\|\cdot\|$  denotes the Euclidean norm. We can show that  $\log(\nu^{-1}) = O(n^2 \log nM)$ ,<sup>4</sup> where  $M = \max\{|f(X)| \mid X \subseteq V\}$ . Put  $\nu^* = n^2 \log nM$ .

Due to Proposition 1, adapting Chubanov's theorem [4, Theorem 2.1] to our problem becomes as follows.

**Theorem 4.** *Under Assumption (A\*) Algorithm  $\text{Chubanov}(f)$  runs in  $O((n^5 + n^3T + n^4\mathbf{EO})\nu^*)$  time, where  $\mathbf{EO}$  denotes the time required for the function evaluation of  $f$  and  $T$  denotes the time required for computing a scalar  $\gamma(x)$  such that  $\gamma(x)^2 \|x\|^2 \in [1, \frac{3}{2}]$ , for extreme base  $x \in \mathbf{B}(f)$ .*

<sup>4</sup>Replace the condition  $\|w\| \leq 1$  in (13) by  $\|w\|_1 \equiv \sum_{v \in V} |w(v)| \leq 1$  to get a lower bound of  $\nu$ . Then use a fundamental fact [11, Theorem 3.2] on basic solutions of the associated LP. (A better bound  $O(n \log nM)$  is given by one of the anonymous reviewers with a proof, which improves the value of  $\nu^*$  in Theorem 4.)

<sup>3</sup>See [7, Section 3.1(d)].

It follows from Lemma 2 that  $T = O(\log nM)$ .

Now consider the  $\alpha$ -enlargement  $f^{(\alpha)}$  for  $\alpha = \frac{1}{2}$  and perform  $\text{Chubanov}(f^{(\alpha)})$ , or equivalently  $\text{Chubanov}(2 \cdot f^{(\alpha)})$  for integer-valued submodular function  $2 \cdot f^{(\alpha)}$ . Then it follows from Theorem 3 that we obtain

- (a') a nonempty proper subset  $Z \subset V$  such that  $f(Z) \leq -1$  or
- (b') a conclusion that  $f(X) \geq 0$  for all  $X \subseteq V$ , i.e.,  $\mathbf{0} \in B(f)$ ,

where note that since  $f$  is integer-valued, we have  $f(Z) \leq -\frac{1}{2}$  if and only if  $f(Z) \leq -1$  (and  $f(X) \geq -\frac{1}{2}$  if and only if  $f(X) \geq 0$ ).

This gives us a polynomial algorithm for testing membership  $\mathbf{0} \in B(f)$ , which we denote by  $\text{Membership}(f)$ , and we obtain a set  $X$  with  $f(X) < 0$  when  $\mathbf{0} \notin B(f)$ .

#### 4. Submodular Function Minimization

We have shown how the recent result of Chubanov [4] for linear programming yields a weakly polynomial algorithm  $\text{Membership}(f)$  for testing membership  $\mathbf{0} \in B(f)$ . However, this does not give any minimizer of  $f$  in a direct way. Fujishige and Iwata [8] showed that by invoking  $\text{Membership}(\cdot)$  as a subroutine  $O(n^2)$  times, we can find a minimizer of  $f$ .

In this section we show another way of finding a minimizer of  $f$  by  $O(\log M_*)$  calls for the membership testing algorithm  $\text{Membership}(\cdot)$ , where  $M_*$  is defined by (14).

First we make the following *preprocessing*.

- If  $f(V) > 0$ , then put  $f(V) \leftarrow 0$  for set  $V$  alone. Then, function  $f$  remains submodular and the set of minimizers (except for  $V$ ) remains the same.
- If  $f(V) < 0$ , then for all nonempty  $X \subseteq V$  put  $f(X) \leftarrow f(X) - f(V)$ . Then, function  $f$  remains submodular and the set of minimizers (except for  $\emptyset$ ) remains the same.

Put

$$M_* = - \sum_{v \in V} \min\{0, f(V) - f(V \setminus \{v\})\}. \quad (14)$$

Note that  $0 \geq \min\{f(X) \mid X \subseteq V\} \geq -M_*$  due to Lemma 2. Now we employ Procedure  $\text{Membership}(\cdot)$  as follows.

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#### Algorithm SFM( $f$ )

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**Input:** a submodular function  $f : 2^V \rightarrow \mathbb{Z}$ .

**Output:** a minimizer  $Z$  of  $f$ .

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(0) Let  $\tilde{f}$  be the submodular function after preprocessing  $f$ .

(1) Perform  $\text{Membership}(\tilde{f})$ .

If we decide  $\mathbf{0} \in B(\tilde{f})$ , then

if originally  $f(V) \geq 0$ , then return  $Z = \emptyset$

(since  $\emptyset$  is a minimizer of the original  $f$ ),

if originally  $f(V) < 0$ , then return  $Z = V$

(since  $V$  is a minimizer of the original  $f$ ).

If  $\mathbf{0} \notin B(\tilde{f})$  and we get  $X$  with  $\tilde{f}(X) < 0$ , then

put  $Z \leftarrow X$ ,  $\beta_- \leftarrow 0$ , and  $\beta_+ \leftarrow M_*$  (defined for  $\tilde{f}$ ).

(2) While  $\beta_+ - \beta_- \geq 2$ , do the following (i) and (ii):

(i) Put  $\alpha \leftarrow \lceil (\beta_- + \beta_+)/2 \rceil$ .

(ii) Perform  $\text{Membership}(\tilde{f}^{(\alpha)})$ .

If we decide  $\mathbf{0} \in B(\tilde{f}^{(\alpha)})$ , then put  $\beta_+ \leftarrow \alpha$ .

If  $\mathbf{0} \notin B(\tilde{f}^{(\alpha)})$  and we get  $X$  with  $\tilde{f}^{(\alpha)}(X) < 0$ , then put  $Z \leftarrow X$  and  $\beta_- \leftarrow \alpha$ .

(3) Return  $Z$ .

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**Theorem 5.** Algorithm  $\text{SFM}(f)$  returns a minimizer  $Z$  of  $f$  after repeating Step (2)  $O(\log M_*)$  times.

**PROOF:** The validity of Step (1) follows from the definition of preprocessing. While carrying out Step (2), we keep the property that  $\mathbf{0} \in B(\tilde{f}^{(\beta_+)})$  and  $\mathbf{0} \notin B(\tilde{f}^{(\beta_-)})$ . Hence, when we reach Step (3), we have  $\tilde{f}^{(\beta_-)}(Z) < 0 \leq \tilde{f}^{(\beta_+)}(X)$  ( $\forall X \subseteq V$ ) and  $\beta_+ - \beta_- = 1$ . This implies  $-\beta_+ \leq \tilde{f}(Z) < -\beta_- = -\beta_+ + 1$  and  $-\beta_+ \leq \tilde{f}(X)$  for all  $X \subseteq V$ . It follows that  $Z$  attains the minimum of  $\tilde{f}$  (and hence of  $f$ ), which is equal to  $-\beta_+$ .  $\square$

Combining Fujishige and Iwata's method [8] with  $\text{SFM}(f)$ , we can compute a minimizer of an integer-valued submodular function in  $O((n^5 + n^3T + n^4\mathbf{EO})v^* \min\{n^2, \log M_*\})$  time.

#### 5. Concluding Remarks

It should be noted that when we perform  $\text{Membership}(f)$  and decide  $\mathbf{0} \in B(f)$ , we do not have a certificate of  $\mathbf{0} \in B(f)$  by an expression of  $\mathbf{0}$  as a convex combination of (extreme) bases in  $B(f)$ . Because of this it seems that applying Chubanov's algorithm does not directly give us a dual certificate for a minimizer of  $f$ .

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