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Enumerating branched coverings over surfaces with boundaries

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Abstract

The number of nonisomorphic n-fold branched coverings over a given surface with a boundary is determined by the number of nonisomorphic n-fold graph coverings over a suitable bouquet of circles. A similar enumeration can be done for regular branched coverings. Some explicit formulae for enumerations are also obtained.

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1. Introduction

Throughout this paper, a (closed) surface means a compact connected 2-manifold without a boundary and a compact surface with a boundary will be called a bordered surface. Let us review briefly some known concepts from algebraic topology (cf. [15]).

A continuous mapping $\rho : \mathcal{T} \to \mathcal{S}$ from a surface $\mathcal{T}$ onto another $\mathcal{S}$ is called a *branched covering* of multiplicity n if there exists a finite subset B of points in $\mathcal{S}$ such that the restriction of ρ on $\mathcal{T} - \rho^{-1}(B)$, $\rho|_{\mathcal{T}-\rho^{-1}(B)} : \mathcal{T} - \rho^{-1}(B) \to \mathcal{S} - B$, is an n-fold (n-sheeted) covering projection in the usual sense. The smallest subset B of $\mathcal{T}$ which has this property is called the *branch set* of ρ. A branched covering $\rho : \mathcal{T} \to \mathcal{S}$ is *regular* if there exists a (finite) group $\mathcal{A}$ which acts on $\mathcal{T}$ with at most finitely many fixed points so that the surface $\mathcal{S}$ is homeomorphic to the quotient space $\mathcal{T}/\mathcal{A}$, say by θ, and the quotient

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map $\mathcal{T} \to \mathcal{T}/\mathcal{A}$ is the composition $\theta \circ \rho$ of ρ and θ. We call it simply a *branched $\mathcal{A}$-covering*. In this case, the group $\mathcal{A}$ becomes the covering transformation group of the branched covering $\rho : \mathcal{T} \to \mathcal{S}$. Two branched coverings $\rho : \mathcal{T} \to \mathcal{S}$ and $\rho' : \mathcal{T}' \to \mathcal{S}$ are *isomorphic* (or *equivalent*) if there exists a homeomorphism $\eta : \mathcal{T}' \to \mathcal{T}$ such that $\rho' = \rho \circ \eta$.

Let $\rho : \mathcal{T} \to \mathcal{S}$ be a regular or irregular branched covering of multiplicity n with branch set $B = \{b_1, \ldots, b_r\}$. At the neighborhood of each point $x \in \rho^{-1}(B)$, the projection ρ is topologically equivalent to the complex map $z \mapsto z^m$ with some natural number m. Such an x is called a *ramification point* of ρ, and m is called the *order* of x. Denote by s_m^k the number of ramification points of order m of the mapping ρ in the preimage $\rho^{-1}(b_k)$, where $k = 1, \ldots, r$ and $m = 1, \ldots, n$. The $(r \times n)$-matrix $\sigma = (s_m^k)$ is called the *ramification type* of the covering ρ.

Let $\mathcal{S}$ and σ be as above and let g be the genus of the surface $\mathcal{S}$. Then, the classical Hurwitz enumeration problem can be stated in the following way.

Hurwitz enumeration problem. Determine the number of nonisomorphic coverings of multiplicity n of a surface $\mathcal{S}$ of genus g with a given ramification type σ.

In such a generality, applied both to orientable and nonorientable surfaces, the Hurwitz problem is still open. Hurwitz [4, 5] constructed a generating function for the number of nonequivalent coverings over the sphere having only simple branch points except one specified point and proved that the number of such coverings can be expressed in terms of irreducible characters of the symmetric group. Röhrl [23] obtained upper and lower estimates for the number of nonequivalent coverings with a given ramification type. Some partial solutions of the problem were obtained in [9–18] and [22]. In particular, the number of coverings with a given branch set without restriction on the ramification type were obtained in [11]. The orientable case of the Hurwitz enumeration problem was, in principle, solved completely [19]. The solution is given in terms of irreducible characters of the symmetric group which makes it very complicated. It was known for just a few cases [9, 12, 20, 21] when it is possible to avoid characters of symmetric groups for calculating the number of coverings. A similar work for the nonorientable case with unramified coverings was done in [22]. For other useful information concerning branched coverings over closed surfaces we refer also to the survey [12] and the paper [13].

In this paper, we enumerate the set of nonisomorphic branched coverings (regular or not) over any given bordered surface with a branch set B. The corresponding problem on a closed surface (orientable or not) has been recently solved by Kwak et al. [11]. In our consideration we suppose the branch set B to be prescribed and no restrictions on the ramification type of the covering are given.

2. A classification of branched coverings over bordered surfaces

By the classification theorem of closed surfaces, a closed surface $\mathcal{S}$ is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} \text{the orientable (closed) surface with } k \text{ handles} & \text{if } k \geq 0, \\ \text{the nonorientable (closed) surface with } -k \text{ crosscaps} & \text{if } k < 0. \end{cases}$$

For example, the orientable surfaces $\mathbb{S}_0$ and $\mathbb{S}_1$ are the sphere and the torus, respectively, and the nonorientable surfaces $\mathbb{S}_{-1}$ and $\mathbb{S}_{-2}$ are the projective plane and the Klein bottle, respectively. The number of handles for an orientable surface or the number of crosscaps for a nonorientable surface is called the *genus* of the surface.

Let $\mathcal{D} = \mathcal{D}_r$ denote the family of nonhomeomorphic bordered (orientable or nonorientable) surfaces of Euler characteristic $\chi = 1 - r, r \geq 0$. The simplest one $\mathcal{D}_0$ consists of $\mathbb{S}_0$ with 1 hole, i.e., a 2-disc. The family $\mathcal{D}_1$ consists of $\mathbb{S}_0$ with 2 holes and $\mathbb{S}_{-1}$ with 1 hole; and $\mathcal{D}_2$ consists of $\mathbb{S}_0$ with 3 holes, $\mathbb{S}_{-1}$ with 2 holes and $\mathbb{S}_{-2}$ with 1 hole. Also, note that $\mathcal{S}$ is a bordered surface of Euler characteristic $\chi = 1 - r$ if and only if the fundamental group $\pi_1(\mathcal{S})$ is the free group $\mathcal{F}_r$ of rank r.

For a notational convenience, $\mathcal{D}_r$ also stands for a surface in that family, that is, an orientable surface $\mathbb{S}_k$ with $h = 1+r-2k$ holes as boundary components or a nonorientable surface $\mathbb{S}_k$ with $h = 1+r+k$ holes as boundary components. A regular branched covering with the covering transformation group $\mathcal{A}$ is simply called a branched $\mathcal{A}$-covering in this paper.

The following lemma is trivial.

Lemma 1. *Let two coverings over a closed surface $\mathbb{S}_k$ be given, from which, by cutting off h disjoint disks from $\mathbb{S}_k$ and their fibres, one constructs two coverings over a bordered surface $\mathcal{D}_r = \mathbb{S}_k - \{h$ holes$\}$. Then, two coverings over the closed surface $\mathbb{S}_k$ are isomorphic if and only if two coverings over the bordered surface $\mathcal{D}_r$ are isomorphic.*

A closed surface $\mathbb{S}_k$ (without boundary) can be represented by a $4k$-gon with identification data $\prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1}$ on its boundary if $k > 0$; bi-gon with identification data aa^{-1} on its boundary if $k = 0$; and $-2k$-gon with identification data $\prod_{s=1}^{-k} a_s a_s$ on its boundary if $k < 0$. A similar kind of polygonal representation is also possible for a bordered surface $\mathcal{D}_r$ by cutting off h disjoint disks from the polygonal representation of a surface $\mathbb{S}_k$.

Let $\mathcal{D}_r = \mathbb{S}_k - \{h$ holes$\}$ be a bordered surface, and let B be a finite set of points in the interior $\mathrm{Int}(\mathcal{D}_r)$. Note that the fundamental group $\pi_1(\mathcal{D}_r - B, *)$ of the punctured surface $\mathcal{D}_r - B$ with the base point $* \in \mathcal{D}_r - B$ can be presented as follows:

$$\left\langle a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{|B|}, d_1, \ldots, d_h; \prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1} \prod_{t=1}^{|B|} c_t \prod_{\ell=1}^{h} d_\ell = 1 \right\rangle \quad \text{if } k > 0;$$

$$\left\langle a_1, \ldots, a_{-k}, c_1, \ldots, c_{|B|}, d_1, \ldots, d_h; \prod_{s=1}^{-k} a_s a_s \prod_{t=1}^{|B|} c_t \prod_{\ell=1}^{h} d_\ell = 1 \right\rangle \qquad \text{if } k < 0;$$

$$\left\langle c_1, \ldots, c_{|B|}, d_1, \ldots, d_h; \prod_{t=1}^{|B|} c_t \prod_{\ell=1}^{h} d_\ell = 1 \right\rangle \qquad \text{if } k = 0,$$

where $h = 1+r-2k$ if $k \geq 0$, and $h = 1+r+k$ if $k \leq 0$. For each $t = 1, 2, \ldots, |B|+h$, we take a simple closed curve based at $*$ lying in the face of the surface $\mathcal{D}_r = \mathbb{S}_k - \{h$ holes$\}$ so that it represents the homotopy class of the generators c_t and d_ℓ. For convenience, let $\partial_k = 2k$ if $k \geq 0$, and $\partial_k = -k$ if $k < 0$, so that $\partial_k + h = 1 + r$. Then, it induces a

2-cell embedding of a bouquet of $\partial_k + |B| + h$ circles, say $\mathfrak{B}_{\partial_k+|B|+h}$, into the surface $\mathcal{D}_r - B$. This embedding will be simply denoted by $\mathfrak{B}_{\partial_k+|B|+h} \hookrightarrow \mathcal{D}_r - B$. Note that the fundamental group $\pi_1(\mathcal{D}_r - B)$ of the bordered surface $\mathcal{D}_r - B$ is the free group $\mathcal{F}_{r+|B|}$ of rank $r + |B|$.

Let G be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. We allow loops and multiple edges. Notice that G can be identified with a one-dimensional CW complex in the Euclidean 3-space $\mathbb{R}^3$ so that every graph map is continuous. Every covering over a graph G can be constructed as follows (see [1]). Every edge of G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. Each directed edge e has an initial vertex i_e and a terminal vertex t_e. Following [1], a *permutation voltage assignment* ϕ on a graph G is a map $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$, where S_n is the symmetric group on n elements $\{1, \dots, n\}$. The *permutation derived graph* G^ϕ is defined as follows: $V(G^\phi) = V(G) \times \{1, \dots, n\}$, and for each edge $e \in D(G)$ and $j \in \{1, \dots, n\}$ let there be an edge (e, j) in $D(G^\phi)$ with $i_{(e,j)} = (i_e, j)$ and $t_{(e,j)} = (t_e, \phi(e)j)$. The natural projection $p : G^\phi \to G$ is a covering. Let $\mathcal{A}$ be a finite group. An *ordinary voltage assignment* (or, $\mathcal{A}$*-voltage assignment*) of G is a function $\phi : D(G) \to \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of ϕ are called *voltages*, and $\mathcal{A}$ is called the *voltage group*. The *ordinary derived graph* $G \times_\phi \mathcal{A}$ derived from an ordinary voltage assignment $\phi : D(G) \to \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$, so that an edge (e, g) of $G \times_\phi \mathcal{A}$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = uv \in D(G)$ and $g \in \mathcal{A}$. In the (ordinary) derived graph $G \times_\phi \mathcal{A}$, a vertex (u, g) is denoted by u_g, and an edge (e, g) by e_g. The first coordinate projection $p : G \times_\phi \mathcal{A} \to G$, called the natural projection, commutes with the left multiplication action of the $\phi(e)$ and the right action of $\mathcal{A}$ on the fibers, which is free and transitive, so that p is a regular $|\mathcal{A}|$-fold covering, called simply an $\mathcal{A}$*-covering*. Gross and Tucker [1] showed that every covering (resp. regular covering) over a graph G can be derived from a permutation (resp. ordinary) voltage assignment ϕ.

Let $C^1(\mathfrak{B}_{\partial_k+|B|+h} \hookrightarrow \mathcal{D}_r - B; n)$ (resp. $C^1(\mathfrak{B}_{\partial_k+|B|+h} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$) denote the subset of $(S_n)^{\partial_k+|B|+h}$ (resp. of $(\mathcal{A})^{\partial_k+|B|+h}$) consisting of all $(\partial_k + |B| + h)$-tuples $(\sigma_1, \dots, \sigma_{\partial_k+|B|+h})$ which satisfy the following three conditions:

(C1) The subgroup $\langle \sigma_1, \dots, \sigma_{\partial_k+|B|+h} \rangle$ generated by $\{\sigma_1, \dots, \sigma_{\partial_k+|B|+h}\}$ is transitive on $\{1, 2, \dots, n\}$ (resp. is the full group $\mathcal{A}$), and

(C2) (i) if $k \geq 0$, then $\prod_{i=1}^{k} \sigma_i \sigma_{k+i} \sigma_i^{-1} \sigma_{k+i}^{-1} \prod_{i=1}^{|B|} \sigma_{\partial_k+i} \prod_{j=1}^{h} \sigma_{\partial_k+|B|+j} = 1$,

(ii) if $k < 0$, then $\prod_{i=1}^{-k} \sigma_i \sigma_i \prod_{i=1}^{|B|} \sigma_{\partial_k+i} \prod_{j=1}^{h} \sigma_{\partial_k+|B|+j} = 1$,

(C3) $\sigma_i \neq 1$ for each $i = \partial_k + 1, \dots, \partial_k + |B|$.

Here, a $(\partial_k + |B| + h)$-tuple $(\sigma_1, \dots, \sigma_{\partial_k+|B|+h})$ of permutations in S_n (resp. of elements in $\mathcal{A}$) can be identified with a permutation (resp. ordinary) voltage assignment of the bouquet of circles $\mathfrak{B}_{\partial_k+|B|+h}$. Also, such a voltage assignment φ derives a graph covering over the $\mathfrak{B}_{\partial_k+|B|+h}$, and this covering projection with the embedding $\mathfrak{B}_{\partial_k+|B|+h} \hookrightarrow \mathbb{S}_k$ extends to a branched surface covering over the surface $\mathbb{S}_k$, say $\tilde{p}_\varphi : \mathcal{S}^\varphi \to \mathcal{S}$ (see [1] or [9]). In this case, the condition (C1) guarantees that the covering surface $\mathcal{S}^\varphi$ is connected, and the conditions (C2) and (C3) guarantee that the set B is the same as the branch set of the branched covering $\tilde{p}_\varphi : \mathcal{S}^\varphi \to \mathcal{S}$. By using a similar method as in [9] with Lemma 1, one can obtain the following variant of the Hurwitz existence and classification of branched coverings theorem for bordered surfaces. Recall that $\partial_k + |B| + h = r + |B| + 1$.

Theorem 1. *Every permutation voltage assignment in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; n)$ induces a connected branched n-fold covering over $\mathcal{D}_r$ with branch set B. Conversely, every connected branched n-fold covering over $\mathcal{D}_r$ with branch set B can be derived from a voltage assignment in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; n)$. Moreover, for any given two permutation voltage assignments $\varphi = (\sigma_1, \ldots, \sigma_{r+|B|+1})$ and $\psi = (\tau_1, \ldots, \tau_{r+|B|+1})$ in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; n)$, two branched n-fold surface coverings $\tilde{p}_\varphi : {\mathcal{D}_r}^\varphi \to \mathcal{D}_r$ and $\tilde{p}_\psi : {\mathcal{D}_r}^\psi \to \mathcal{D}_r$ over the bordered surface $\mathcal{D}_r$ are isomorphic if and only if there exists a permutation $\rho \in S_n$ such that*

$$\tau_i = \rho\sigma_i\rho^{-1}$$

for $i = 1, \ldots, r + |B| + 1$. $\square$

Similarly, one can have an analogous theorem for regular branched coverings.

Theorem 2. *Every ordinary voltage assignment in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$ induces a connected branched $\mathcal{A}$-covering over a bordered surface $\mathcal{D}_r$ with branch set B. Conversely, every connected branched $\mathcal{A}$-covering over the bordered surface $\mathcal{D}_r$ with branch set B can be derived from a voltage assignment in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$. Moreover, for any given two voltage assignments $\varphi = (\sigma_1, \ldots, \sigma_{r+|B|+1})$ and $\psi = (\tau_1, \ldots, \tau_{r+|B|+1})$ in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$, two branched $\mathcal{A}$-coverings $\tilde{p}_\varphi : {\mathcal{D}_r}^\varphi \to \mathcal{D}_r$ and $\tilde{p}_\psi : {\mathcal{D}_r}^\psi \to \mathcal{D}_r$ are isomorphic if and only if there exists a group automorphism α of $\mathcal{A}$ such that*

$$\tau_i = \alpha(\sigma_i)$$

for $i = 1, \ldots, r + |B| + 1$. $\square$

3. Computational formulae; a regular case

In this section, we aim to enumerate nonisomorphic *regular* branched coverings over a bordered surface $\mathcal{D}_r$. But, it is sufficient to do it for connected branched $\mathcal{A}$-coverings over the bordered surface $\mathcal{D}_r$ with a given branch set B, where $\mathcal{A}$ is a finite group, because of the regularity of the coverings. To do this, we define an $\mathrm{Aut}(\mathcal{A})$-action on $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$ as follows: For any $\alpha \in \mathrm{Aut}(\mathcal{A})$ and any $(\sigma_1, \ldots, \sigma_{r+|B|+1}) \in C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$, define

$$\alpha \cdot (\sigma_1, \ldots, \sigma_{r+|B|+1}) = (\alpha(\sigma_1), \ldots, \alpha(\sigma_{r+|B|+1})).$$

Then it follows from Theorem 2 that two voltage assignments in $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$ derive isomorphic branched $\mathcal{A}$-coverings over $\mathcal{D}_r$ if and only if they belong to the same orbit under the $\mathrm{Aut}(\mathcal{A})$-action. Notice that this $\mathrm{Aut}(\mathcal{A})$-action on $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$ is free because of the condition (C1). It gives the following lemma.

Lemma 2. *The number* $\mathbf{Isoc}(\mathcal{D}_r, B; \mathcal{A})$ *of nonisomorphic connected branched* $\mathcal{A}$*-coverings over bordered surface* $\mathcal{D}_r$ *with branch set* B *is equal to*

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathcal{A}) = \frac{|C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})|}{|\mathrm{Aut}(\mathcal{A})|}.$$

Let $\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A})$ denote the number of nonisomorphic connected regular graph coverings over the bouquet $\mathfrak{B}_r$ of r circles having $\mathcal{A}$ as the covering transformation group. Since $\pi_1(\mathfrak{B}_r) = \pi_1(\mathcal{D}_r) = \mathcal{F}_r$ the free group of rank r, we have $\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A}) = \mathbf{Isoc}(\mathcal{D}_r, \emptyset; \mathcal{A})$ and so Lemma 2 can be used to estimate the number $\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A})$ as well. Moreover, in the case $B = \emptyset$ there is a one-to-one correspondence between $C^1(\mathfrak{B}_{r+|B|+1} \hookrightarrow \mathcal{D}_r - B; \mathcal{A})$ and a set $\mathfrak{G}(\mathcal{A}, r)$ formed by r-tuples $(\sigma_1, \ldots, \sigma_r)$ of elements in the group $\mathcal{A}$, generating the full group $\mathcal{A}$. Indeed, since $\mathcal{D}_r$ is a bordered surface the number of its holes $h \geq 1$. This means that the element $\sigma_{r+1} = \sigma_{a_k+h}$ in (C2) can be uniquely expressed through elements $\sigma_1, \ldots, \sigma_r$ and (C1) is equivalent to the condition that the group $\mathcal{A}$ is generated by $\sigma_1, \ldots, \sigma_r$. Note that (C3) is redundant in the case $B = \emptyset$.

In turn, the set $\mathfrak{G}(\mathcal{A}, r)$ can be identified with the set $\mathrm{Epi}(\mathcal{F}_r; \mathcal{A})$ of epimorphisms of the free group $\mathcal{F}_r$ onto the group $\mathcal{A}$. Hence, the Lemma 2 gives the following.

Corollary 1. *The number of nonisomorphic connected* $\mathcal{A}$*-coverings over the bouquet* $\mathfrak{B}_r$ *is equal to*

$$\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A}) = \frac{|\mathrm{Epi}(\mathcal{F}_r; \mathcal{A})|}{|\mathrm{Aut}(\mathcal{A})|}.$$

In this section, we introduce a general formula to enumerate $\mathcal{A}$-coverings over a surface $\mathcal{D}_r$ for any finite group $\mathcal{A}$ in terms of the Möbius function defined on the subgroup lattice of $\mathcal{A}$ by Hall [2]. Jones [6, 7] used the Möbius function to find a method for counting normal subgroups of a surface group or a crystallographic group, and applied it to count regular coverings over a surface. Denote by $\mathrm{Hom}(\mathcal{F}_r; \mathcal{A})$ the set of homomorphisms of the free group $\mathcal{F}_r$ into the group $\mathcal{A}$. The set $\mathrm{Hom}(\mathcal{F}_r; \mathcal{A})$ can be naturally identified with the set $\mathcal{A}^r$ of r-tuples of elements of the group $\mathcal{A}$. Hence $|\mathrm{Hom}(\mathcal{F}_r, \mathcal{A})| = |\mathcal{A}|^r$. Also, we have

$$|\mathrm{Hom}(\mathcal{F}_r, \mathcal{A})| = \sum_{K \leq \mathcal{A}} |\mathrm{Epi}(\mathcal{F}_r, K)|,$$

where the sum is taken over all subgroups K of the group $\mathcal{A}$. Now, one can invert the obtained equation to count epimorphisms in terms of homomorphisms, by introducing the *Möbius function* for $\mathcal{A}$. This assigns an integer $\mu(K)$ to each subgroup K of $\mathcal{A}$ by the recursive formula

$$\sum_{H \geq K} \mu(H) = \delta_{K,\mathcal{A}} = \begin{cases} 1 & \text{if } K = \mathcal{A}, \\ 0 & \text{if } K < \mathcal{A}. \end{cases}$$

The equation

$$|\mathrm{Epi}(\mathcal{F}_r, \mathcal{A})| = \sum_{K \leq \mathcal{A}} \mu(K) |\mathrm{Hom}(\mathcal{F}_r, K)|$$

is then easily deduced. Lemma 2 gives

$$\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A}) = \frac{|\mathrm{Epi}(\mathcal{F}_r, \mathcal{A})|}{|\mathrm{Aut}(\mathcal{A})|} = \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{K \le \mathcal{A}} \mu(K)|\mathrm{Hom}(\mathcal{F}_r, K)|$$
$$= \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{K \le \mathcal{A}} \mu(K)|K|^r.$$

As a result, we obtain

Corollary 2. *The number of nonisomorphic connected unbranched $\mathcal{A}$-coverings over the bordered surface $\mathcal{D}_r$ coincides with the number of nonisomorphic connected $\mathcal{A}$-coverings over the bouquet $\mathfrak{B}_r$, and is given by*

$$\mathbf{Isoc}(\mathcal{D}_r; \emptyset, \mathcal{A}) = \mathbf{Isoc}(\mathfrak{B}_r, \mathcal{A}) = \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{H \le \mathcal{A}} \mu(H)|H|^r.$$

When the group $\mathcal{A}$ is Abelian or any dihedral group $\mathbb{D}_n$ of order $2n$, the number $\mathbf{Isoc}(\mathfrak{B}_r; \mathcal{A})$ was explicitly computed in [8] by Burnside's lemma without using the Möbius function μ.

The proof of the following theorem is based on the principle of inclusion and exclusion, and is similar to the proof of Theorem 2 in [11] with Lemma 2.

Theorem 3. *Let B be a b-subset of the interior of a bordered surface $\mathcal{D}_r$. Then, for any finite group $\mathcal{A}$, the number of nonisomorphic connected branched $\mathcal{A}$-coverings over $\mathcal{D}_r$ with branch set B is*

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathcal{A}) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \mathbf{Isoc}(\mathfrak{B}_{r+b-t}; \mathcal{A}).$$

The corresponding enumeration of nonisomorphic connected branched (regular) coverings over a closed surface was formulated in terms of the nonisomorphic unbranched (regular) ones and some nonisomorphic (regular) graph coverings over a suitable bouquet of circles in Theorem 2 in [11]. However, in our bordered case, the former terms have disappeared as shown in Theorem 3 and its difference comes from a fact that the fundamental group of a bordered surface is free.

Corollary 3. *For any finite group $\mathcal{A}$, the number of nonisomorphic connected branched $\mathcal{A}$-coverings over a bordered surface $\mathcal{D}_r$ with branch set B, $|B| = b \ge 0$, is given by formula*

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathcal{A}) = \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{H \le \mathcal{A}} \mu(H)(|H| - 1)^b |H|^r,$$

where $\mu(H)$ is the Möbius function for the group $\mathcal{A}$, and the sum is taken over all subgroups H of the group $\mathcal{A}$.

Proof. By Corollary 2 and Theorem 3, we get

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathcal{A}) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{H \le \mathcal{A}} \mu(H)|H|^{r+b-t}$$

$$= \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{H \le \mathcal{A}} \mu(H) \left(\sum_{t=0}^{b} (-1)^t \binom{b}{t} |H|^{b-t} \right) |H|^r$$
$$= \frac{1}{|\mathrm{Aut}(\mathcal{A})|} \sum_{H \le \mathcal{A}} \mu(H)(|H| - 1)^b |H|^r. \quad \square$$

As an immediate consequence of Corollary 3, we obtain the number of nonisomorphic connected branched $\mathcal{A}$-coverings for the cyclic group $\mathcal{A} = \mathbb{Z}_n$ and the dihedral group $\mathcal{A} = \mathbb{D}_n$ of order $2n$. We note that the cyclic group $\mathcal{A} = \mathbb{Z}_n$ has a unique subgroup $\mathcal{H} = \mathbb{Z}_m$ for each m dividing n, and has no other subgroups. We have $\mu(\mathcal{H}) = \mu(n/m)$ (μ is the Möbius function) and $|\mathrm{Aut}(\mathcal{A})| = \phi(n)$ (ϕ is Euler's totient function).

Corollary 4. *The number of nonisomorphic connected $\mathbb{Z}_n$-coverings over the bordered surface $\mathcal{D}_r$ with branch set B, $|B| = b \ge 0$, is given by*

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathbb{Z}_n) = \frac{1}{\phi(n)} \sum_{m \mid n} \mu\left(\frac{n}{m}\right) (m-1)^b m^r.$$

From here on, we suppose that $0^b = 1$ if $b = 0$.

Corollary 5. *Let $\mathbb{D}_n$ be a dihedral group of order $2n$, $n \ne 2$. Then,*

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathbb{D}_n) = \frac{1}{\phi(n)} \sum_{m|n} \mu\left(\frac{n}{m}\right) [(2m-1)^b 2^r - (m-1)^b] m^{r-1}.$$

In the case $n = 2$ we have

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathbb{D}_2) = \tfrac{1}{6}(3^b \cdot 4^r - 3 \cdot 2^r + 2 \cdot 0^b).$$

Proof. For given $m \mid n$ the group $\mathbb{D}_n$ has exactly one subgroup $\mathbb{Z}_m$ of order m with $\mu(\mathbb{Z}_m) = -(n/m)\mu(n/m)$ and n/m subgroups $\mathbb{D}_m$ with $\mu(\mathbb{D}_m) = \mu(n/m)$. Moreover, $|\mathrm{Aut}(\mathbb{D}_n)| = n\phi(n)$ if $n \ne 2$. See [6] for details. For $n = 2$ the group $\mathbb{D}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and hence $|\mathrm{Aut}(\mathbb{D}_2)| = 6$. The group $\mathbb{D}_2$ contains as a subgroup $\mathbb{D}_2$ itself, three proper subgroups isomorphic to $\mathbb{Z}_2$, and the trivial subgroup $\mathbb{Z}_1$. From the definition of Möbius function we get $\mu(\mathbb{D}_2) = 1$, $\mu(\mathbb{Z}_2) = -1$, and $\mu(\mathbb{Z}_1) = 2$. Hence, by Theorem 2, for $n \ne 2$

$$\mathbf{Isoc}(\mathcal{D}_r, B; \mathbb{D}_n) = \frac{1}{|\mathrm{Aut}(\mathbb{D}_n)|} \sum_{H \le \mathbb{D}_n} \mu(H)(|H| - 1)^b |H|^r$$
$$= \frac{1}{n\phi(n)} \sum_{m \mid n} \left(-\frac{n}{m}\mu\left(\frac{n}{m}\right)(m-1)^b m^r \right.$$
$$\left. + \frac{n}{m}\mu\left(\frac{n}{m}\right)(2m-1)^b (2m)^r \right)$$
$$= \frac{1}{\phi(n)} \sum_{m \mid n} \mu\left(\frac{n}{m}\right) [(2m-1)^b 2^r - (m-1)^b] m^{r-1}.$$

The case $n = 2$ is considered in a similar way. We obtain

$$\begin{aligned}\mathbf{Isoc}(\mathcal{D}_r, B; \mathbb{D}_2) &= \frac{1}{|\mathrm{Aut}(\mathbb{D}_2)|}\sum_{H\le\mathbb{D}_2}\mu(H)(|H|-1)^b|H|^r\\ &= \tfrac{1}{6}(\mu(\mathbb{D}_2)\cdot 3^b\cdot 4^r + 3\mu(\mathbb{Z}_2)\cdot 1^b\cdot 2^b + \mu(\mathbb{Z}_1)\cdot 0^b\cdot 1^r)\\ &= \tfrac{1}{6}(3^b\cdot 4^r - 3\cdot 2^r + 2\cdot 0^b). \quad\square\end{aligned}$$

4. Computational formulae; a general case

Let $\mathbf{Isoc}(\mathfrak{B}_m; n)$ denote the number of nonisomorphic connected n-fold graph coverings over the bouquet $\mathfrak{B}_m$ of m circles. The following theorem is a parallel version of Theorem 3 for general (regular or not) coverings.

Theorem 4. *Let B be a b-subset of the surface $\mathcal{D}_r$. Then the number of nonisomorphic connected n-fold branched coverings over the bordered surface $\mathcal{D}_r$ with branch set B is*

$$\mathbf{Isoc}(\mathcal{D}_r, B; n) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}\mathbf{Isoc}(\mathfrak{B}_{r+b-t}; n).$$

As an application of Theorem 4, we have the following result.

Theorem 5. *Let B be a b-subset of the bordered surface $\mathcal{D}_r$. Then the number of nonisomorphic connected n-fold branched coverings over the surface $\mathcal{D}_r$ with branch set B is*

$$\mathbf{Isoc}(\mathcal{D}_r, B; n) = \frac{1}{n}\sum_{m\,|\,n}\sum_{d\,|\,\frac{n}{m}}\mu\left(\frac{n}{md}\right)d\,T_m(d),$$

where $T_m(d)$ is a polynomial of d defined by

$$T_m(d) = m\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\sum_{\substack{n_1+\cdots+n_k=m\\ n_1,\ldots,n_k\ge 1}}(n_1!\cdots n_k!\,d^m)^{r-1}(n_1!\cdots n_k!\,d^m-1)^b.$$

Proof. By Liskovets' theorem [14] we have

$$\mathbf{Isoc}(\mathfrak{B}_\beta; n) = \frac{1}{n}\sum_{m\,|\,n}S_\beta(m)\sum_{d\,|\,\frac{n}{m}}\mu\left(\frac{n}{md}\right)d^{(\beta-1)m+1},$$

where $S_\beta(m)$ is the number of subgroups of index m in a free group F_β of rank β, determined by Hall [3]. Hence, by Theorem 5,

$$\mathbf{Isoc}(\mathcal{D}_r, B; n) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}\mathbf{Isoc}(\mathfrak{B}_{r+b-t}; n) = \frac{1}{n}\sum_{m\,|\,n}\sum_{d\,|\,\frac{n}{m}}\mu\left(\frac{n}{md}\right)d\,T_m(d),$$

where $T_m(d) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}S_{r+b-t}(m)(d^m)^{r+b-t-1}$.

To find $T_m(d)$, recall [3] that $S_\beta(m)$ satisfies the following formal power series identity

$$\sum_{m\geq 1}\frac{S_\beta(m)}{m}w^m = \log\left(\sum_{n\geq 0}(n!)^{\beta-1}w^n\right).$$

Replacing w by $d^{\beta-1}w$, where $\beta = r+b-t$ we get

$$\sum_{m\geq 1}\frac{S_\beta(m)}{m}d^{m(\beta-1)}w^m = \log\left(\sum_{n\geq 0}(n!\,d^n)^{\beta-1}w^n\right).$$

Applying the linear operator $L_t(f) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}f(t)$ to both sides of the above equality we have

$$\sum_{m\geq 1}\frac{T_m(d)}{m}w^m = L_t\left(\log\left(\sum_{n\geq 0}(n!\,d^n)^{r+b-t-1}w^n\right)\right).$$

Take coefficients of w^m

$$\frac{T_m(d)}{m} = L_t\left(\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\sum_{\substack{n_1+\cdots+n_k=m\\ n_1,\ldots,n_k\geq 1}}(n_1!\,d^{n_1}\cdots n_k!\,d^{n_k})^{r+b-t-1}\right).$$

Taking into account that $L_t(x^{r+b-t-1}) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}x^{b-t}x^{r-1} = x^{r-1}(x-1)^b$ we obtain

$$T_m(d) = m\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\sum_{\substack{n_1+\cdots+n_k=m\\ n_1,\ldots,n_k\geq 1}}(n_1!\cdots n_k!\,d^m)^{r-1}(n_1!\cdots n_k!\,d^m-1)^b. \quad \square$$

Note that $T_m(d)$ is a polynomial on d of degree $m(r-1+b)$. By explicit calculations we get

$$\begin{aligned}
T_1(d) &= d^\nu(d-1)^b,\\
T_2(d) &= 2(2d^2)^\nu(2d^2-1)^b - (d^2)^\nu(d^2-1)^b,\\
T_3(d) &= 3(6d^3)^\nu(6d^3-1)^b - 3(2d^3)^\nu(2d^3-1)^b + (d^3)^\nu(d^3-1)^b,\\
T_4(d) &= 4\cdot(24d^4)^\nu(24d^4-1)^b - (6d^4)^\nu(6d^4-1)^b - 2\cdot(4d^4)^\nu(4d^4-1)^b\\
&\quad + 4\cdot(2d^4)^\nu(2d^4-1)^b - (d^4)^\nu(d^4-1)^b,\\
T_5(d) &= 5\cdot 120_{*^5} - 5\cdot 24_{*^5} - 5\cdot 12_{*^5} + 5\cdot 6_{*^5} + 5\cdot 4_{*^5} - 5\cdot 2_{*^5} + 1_{*^5},\\
T_6(d) &= 6\cdot 720_{*^6} - 6\cdot 120_{*^6} - 6\cdot 48_{*^6} - 3\cdot 36_{*^6} + 6\cdot 24_{*^6} + 12\cdot 12_{*^6}\\
&\quad + 2\cdot 8_{*^6} - 6\cdot 6_{*^6} - 9\cdot 4_{*^6} + 6\cdot 2_{*^6} - 1_{*^6}.
\end{aligned}$$

For simplicity we set $\nu = r-1$ and $N_{*^k} = (Nd^k)^\nu(Nd^k-1)^b$. By applying Theorem 5, we have

Corollary 6. *Let $B, |B| = b \geq 0$ be a branch set of the bordered surface $\mathcal{D}_r$ and $\nu = r - 1 = -\chi(\mathcal{D}_r)$, where $\chi(\mathcal{D}_r)$ is the Euler characteristic of $\mathcal{D}_r$. Then*

$$\mathbf{Isoc}(\mathcal{D}_r, B; 2) = 2 \cdot 2^\nu - 0^b,$$
$$\mathbf{Isoc}(\mathcal{D}_r, B; 3) = 6^\nu \cdot 5^b + 3^\nu \cdot 2^b - 2^\nu,$$
$$\mathbf{Isoc}(\mathcal{D}_r, B; 4) = 24^\nu \cdot 23^b + 8^\nu \cdot 7^b - 6^\nu \cdot 5^b,$$
$$\begin{aligned}\mathbf{Isoc}(\mathcal{D}_r, B; 5) = {} & 120^\nu \cdot 119^b - 24^\nu \cdot 23^b - 12^\nu \cdot 11^b + 6^\nu \cdot 5^b + 5^\nu \cdot 4^b \\ & + 4^\nu \cdot 3^b - 2^\nu,\end{aligned}$$
$$\begin{aligned}\mathbf{Isoc}(\mathcal{D}_r, B; 6) = {} & 720^\nu \cdot 719^b - 120^\nu \cdot 119^b - \tfrac{1}{2} \cdot 36^\nu \cdot 35^b + 24^\nu \cdot 23^b + 18^\nu \cdot 17^b \\ & - 16^\nu \cdot 15^b + 2 \cdot 12^\nu \cdot 11^b - \tfrac{1}{2} \cdot 9^\nu \cdot 8^b + \tfrac{2}{3} \cdot 8^\nu \cdot 7^b \\ & - \tfrac{1}{2} \cdot 6^\nu \cdot 5^b - \tfrac{3}{2} \cdot 4^\nu \cdot 3^b - \tfrac{1}{2} \cdot 3^\nu \cdot 2^b + \tfrac{5}{6} \cdot 2^\nu \cdot 1^b.\end{aligned}$$

Remark. The above formulae are nontrivial even for a disk $\mathcal{D} = \mathcal{D}_0$ without holes. In this case we have $\nu = -1$ and

$$\mathbf{Isoc}(\mathcal{D}, B; 2) = 1 - 0^b,$$
$$\mathbf{Isoc}(\mathcal{D}, B; 3) = \tfrac{1}{6} \cdot 5^b + \tfrac{1}{3} \cdot 2^b - \tfrac{1}{2},$$
$$\mathbf{Isoc}(\mathcal{D}, B; 4) = \tfrac{1}{24} \cdot 23^b + \tfrac{1}{8} \cdot 7^b - \tfrac{1}{6} \cdot 5^b,$$
$$\mathbf{Isoc}(\mathcal{D}, B; 5) = \tfrac{1}{120} \cdot 119^b - \tfrac{1}{24} \cdot 23^b - \tfrac{1}{12} \cdot 11^b + \tfrac{1}{6} \cdot 5^b + \tfrac{1}{5} \cdot 4^b + \tfrac{1}{4} \cdot 3^b - \tfrac{1}{2}.$$

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