Modular Irreducible Representations of the Symmetric Group as Linear Codes

Adalbert Kerber, Axel Kohnert
Lehrstuhl II für Mathematik
University of Bayreuth
D-95440 Bayreuth, Germany

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Abstract

We describe a particularly easy way of evaluating the modular irreducible matrix representations of the symmetric group. It shows that Specht's approach to the ordinary irreducible representations, along *Specht polynomials,* can be unified with Clausen's approach to the modular irreducible representations using *symmetrized standard bideterminants.* The unified method, using *symmetrized Specht polynomials* is very easy to explain, and it follows directly from Clausen's theorem by replacing the indeterminate x_{ij} of the letter place algebra by ${x_j}^i$.

Our approach is implemented in SYMMETRICA. It was used in order to obtain computational results on code theoretic properties of the p-modular irreducible representation $[\lambda]_p$ corresponding to a p-regular partition λ via embedding it into representation spaces obtained from ordinary irreducible representations. The first embedding is into the permutation representation induced from the column group of a standard Young tableau of shape λ. The second embedding is the embedding of $[\lambda]_p$ into the space of $\overline{[\lambda]}$, the p-modular representation obtained from the ordinary irreducible representation $[\lambda]$ by reducing the coefficients modulo p.

We include a few tables with dimensions and minimum distances of these codes, others can be found via our home page.

1 Introduction

The ordinary representation theory of symmetric groups is well established, and there are also many results known on the modular representation theory of this class of groups. A breakthrough was the definition of the modular irreducible representations by G. D. James ([9]) as factor modules. The problem with this approach is, that we do not obtain an explicit basis this way. Hence we better use M. Clausen's approach —

another breakthrough ([2],[3],[4]) — using standard bideterminants. It has the advantage that the modular irreducibles show up as submodules and not as factor modules. We therefore can get a explicit basis. Clausen's method was successfully implemented by A. Golembiowski ([8]) using Capelli operators.

Our approach, using multivariate polynomials, is very natural since when we act upon a multivariate polynomial corresponding to a standard tableau we obtain a multivariate polynomial that does not correspond to a standard tableau, in general, but the point is that in this case we know what it means that a nonstandard multivariate polynomial is an integral linear combination of standard ones, while we do not know at all why, and how, a nonstandard Young tableau should be an integral linear combination of standard ones.

Multivariate polynomials were first used by W. Specht ([14]) in order to evaluate linear representations of the symmetric group. Astonishingly enough he never tried to apply them for the modular case, too, although he had all the necessary tools in his hand. The purpose of the present paper is to show, that we can easily combine his methods with Clausen's, as it will be described in the next sections.

2 Specht's Polynomials

When W. Specht — a student of I. Schur — entered the representation theory of the symmetric group it was well known that the standard Young tableaux of a given shape λ (a partition of n) form a basis of an ordinary irreducible representation space of S_n. Here is, for example, one of the 16 standard Young tableaux of shape $\lambda = (3, 2, 1)$:

$$\begin{array}{ccc} 0 & 1 & 5 \\ 2 & 3 & \\ 4 & & \end{array} .$$

Standard means that the entries are weakly increasing in the rows from left to right, and that they are strictly increasing in the columns from top to bottom, while *Young* tableau means that the n entries are the different elements of the set $n := \{0, \ldots, n-1\}$, if λ is a partition of n. In the literature, the entries are usually taken from $\{1, \ldots, n\}$, but for technical reasons (see below) it is better to number from 0 onwards.
The problem is, that the symmetric group acts in a natural way on tableaux, but the result of the application of a permutation to a standard tableau can be a nonstandard tableau, and it is by no means clear how a nonstandard tableau can be written as a linear combination of standard ones.
For this reason W. Specht introduced *polynomials* corresponding to the tableaux, and it is obvious how a given polynomial can be written as a linear combination of other polynomials.
The polynomials in question are nowadays called *Specht polynomials,* they are products of the Vandermonde determinants corresponding to the rows of the tableau in question. (Warning: Specht used the Vandermonde determinants corresponding to the

columns, we use the rows instead, in order to obtain later on the modular irreducible that is usually associated to λ and *not* the representation mostly associated to λ'. The disadvantage is that we this way obtain in the ordinary case the associated one.) Here is an example. The above tableau gives rise to the product of Vandermonde determinants

$$\text{Spe}\begin{pmatrix} 0 & 1 & 5 \\ 2 & 3 & \\ 4 & & \end{pmatrix} = V(0,1,5)\cdot V(2,3)\cdot V(4),$$

where the entries in the rows correspond to the indices of the variables in the Vandermonde determinant, i.e.

$$V(0,1,5)\cdot V(2,3)\cdot V(4) = \det\begin{pmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_5 \\ x_0^2 & x_1^2 & x_5^2 \end{pmatrix}\cdot\det\begin{pmatrix} 1 & 1 \\ x_2 & x_3 \end{pmatrix}\cdot\det(1)$$

$$= x_1x_3x_5^2 - x_1x_2x_5^2 - x_1^2x_3x_5 + x_1^2x_2x_5 - x_0x_3x_5^2 + x_0x_2x_5^2$$
$$+x_0x_1^2x_3 - x_0x_1^2x_2 + x_0^2x_3x_5 - x_0^2x_2x_5 - x_0^2x_1x_3 + x_0^2x_1x_2.$$

The symmetric group S_6 (on $6 = \{0,\ldots,5\}$) acts upon this polynomial by permuting the indices i of the indeterminates x_i. For example, the image under the permutation $\pi := (2,3,4) = (2,\pi(2),\pi^2(2))$ is the polynomial

$$x_1x_4x_5^2 - x_1x_3x_5^2 - x_1^2x_4x_5 + x_1^2x_3x_5 - x_0x_4x_5^2 + x_0x_3x_5^2$$
$$+x_0x_1^2x_4 - x_0x_1^2x_3 + x_0^2x_4x_5 - x_0^2x_3x_5 - x_0^2x_1x_4 + x_0^2x_1x_3.$$

(Note that we interpret $\pi(i) = j$ as: Replace i by j under the action of π.) The point is that we can reconstruct the tableau from the Specht polynomial: The product of the main terms in the evaluation of the Vandermonde determinants is the monomial summand

$$x_0^{a_0}\cdots x_{n-1}^{a_{n-1}}$$

of the Specht polynomial with lexicographically smallest sequence $(a_0,\ldots,a_{n-1})$ of exponents, it is called the *leading monomial.* It contains the variables in the columns except the leftmost column, the elements of which are not represented in the main terms! The variable x_i occurs with an exponent a_i that is the number of the column (when the leftmost column is of number 0). In our example the leading monomial is $x_1x_3x_5^2$, and so, since $n = 5$, the leftmost column contains the entries $0, 2, 4$, while 1 and 3 are contained in the column with number 1, and 5 is the entry in the column of number 2.

An immediate consequence is that the Specht polynomials corresponding to the standard Young tableaux of shape λ are *linearly independent.* Moreover, they generate *a subspace invariant under the action of the symmetric group* on the indices of the variables. This follows from the fact that the action of a permutation on the lexicographically smallest monomial gives a monomial with a lexicographically bigger sequence of exponents, the leading monomial in a nonzero element always comes from a standard tableau. Even more, since the symmetric group action is transitive on the minimal monomials, this module is the space of an *ordinary irreducible* representation. This gives (for details see e.g. [14], [10], sections 7.1 and 7.2, or [13], section 5.6) Specht's main result, the

2.1 Theorem *The Specht polynomials*

$$Spe(t_i^\lambda)$$

corresponding to the f^λ different standard Young tableaux t_i^λ of shape λ are a basis of an ordinary irreducible representation module $S^{\lambda'}$, the Specht module *corresponding to λ'. This representation is usually denoted by $[\lambda']$.*

Here is an example in detail: We want to compute a matrix corresponding to the irreducible module of S_5 labelled by the partition $\lambda := (2^2, 1)$. We start with the 5 standard Young tableaux of shape $\lambda = (2^2, 1)$:

$$t_0^{(2^2,1)} = \begin{matrix} 0 & 3 \\ 1 & 4 \\ 2 & \end{matrix}, \; t_1^{(2^2,1)} = \begin{matrix} 0 & 2 \\ 1 & 4 \\ 3 & \end{matrix}, \; t_2^{(2^2,1)} = \begin{matrix} 0 & 2 \\ 1 & 3 \\ 4 & \end{matrix}, \; t_3^{(2^2,1)} = \begin{matrix} 0 & 1 \\ 2 & 4 \\ 3 & \end{matrix}, \; t_4^{(2^2,1)} = \begin{matrix} 0 & 1 \\ 2 & 3 \\ 4 & \end{matrix}.$$

The corresponding Specht polynomials are

$$\text{Spe}(t_0^{(2^2,1)}) = (x_3 - x_0)(x_4 - x_1) = x_3x_4 - x_1x_3 - x_0x_4 + x_0x_1,$$

$$\text{Spe}(t_1^{(2^2,1)}) = (x_2 - x_0)(x_4 - x_1) = x_2x_4 - x_1x_2 - x_0x_4 + x_0x_1,$$

$$\text{Spe}(t_2^{(2^2,1)}) = (x_2 - x_0)(x_3 - x_1) = x_2x_3 - x_1x_2 - x_0x_3 + x_0x_1,$$

$$\text{Spe}(t_3^{(2^2,1)}) = (x_1 - x_0)(x_4 - x_2) = x_1x_4 - x_1x_2 - x_0x_4 + x_0x_2,$$

$$\text{Spe}(t_4^{(2^2,1)}) = (x_1 - x_0)(x_3 - x_2) = x_1x_3 - x_1x_2 - x_0x_3 + x_0x_2.$$

In order to compute the representing matrix for the elementary transposition (0,1), we let it act on the Specht polynomials, obtaining

$$(0,1)\text{Spe}(t_0^{(2^2,1)}) = x_3x_4 - x_1x_4 - x_0x_3 + x_0x_1,$$

where the leading monomial is x_3x_4 so that we continue in the following way:

$$\begin{aligned} &= \text{Spe}(t_0^{(2^2,1)}) - x_1x_4 + x_1x_3 + x_0x_4 - x_0x_3 \\ &= \text{Spe}(t_0^{(2^2,1)}) - \text{Spe}(t_3^{(2^2,1)}) + x_1x_3 - x_1x_2 - x_0x_3 + x_0x_2 \\ &= \text{Spe}(t_0^{(2^2,1)}) - \text{Spe}(t_3^{(2^2,1)}) + \text{Spe}(t_4^{(2^2,1)}). \end{aligned}$$

Analogously we evaluate that

$$\begin{aligned}(0,1)\mathrm{Spe}(t_1^{(2^2,1)}) &= \mathrm{Spe}(t_1^{(2^2,1)}) - \mathrm{Spe}(t_3^{(2^2,1)}),\\ (0,1)\mathrm{Spe}(t_2^{(2^2,1)}) &= \mathrm{Spe}(t_2^{(2^2,1)}) - \mathrm{Spe}(t_4^{(2^2,1)}),\\ (0,1)\mathrm{Spe}(t_3^{(2^2,1)}) &= -\mathrm{Spe}(t_3^{(2^2,1)}),\\ (0,1)\mathrm{Spe}(t_4^{(2^2,1)}) &= -\mathrm{Spe}(t_4^{(2^2,1)}),\end{aligned}$$

which gives the representing matrix

$$D^{(2^2,1)}((0,1)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ -1 & -1 & 0 & -1 & 0\\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

The experienced reader sees from the trace of this matrix, that this ordinary irreducible representation is the representation usually denoted by $[3,2]$. As we already said, we associate it to the partition $(3,2)' = (2^2,1)$, since later on we want to obtain directly from it the 3-modular irreducible representation that corresponds to the partition $(2^2,1)$.

3 Clausen's Symmetrized Bideterminants

A breakthrough towards constructive modular representation theory of the symmetric groups is due to M. Clausen, who gave generators for the modular irreducible representation spaces, *symmetrized bideterminants.* We should like briefly to describe this. Clausen used bideterminants, multivariate polynomials introduced in invariant theory in particular by Turnbull and Rota (see e.g. [6]). They are associated with *bitableaux,* consisting of two tableaux of the same shape λ. The bitableau is called a *standard* bitableau, if both tableaux are standard. Here is a standard bitableau of shape $\lambda = (2^2,1)$:

$$\left(\begin{array}{ll} 0 & 3\\ 1 & 3\\ 2 & \end{array}, \begin{array}{ll} 0 & 2\\ 1 & 4\\ 3 & \end{array}\right).$$

The corresponding *bideterminant* is the product of determinants, formed from corresponding rows in the two tableaux which form the bitableau:

$$\mathrm{Bid}\left(\begin{array}{ll} 0 & 3\\ 1 & 4\\ 2 & \end{array}, \begin{array}{ll} 0 & 2\\ 1 & 4\\ 3 & \end{array}\right) := \left(\begin{array}{ll|ll} 0 & 3 & 0 & 2\end{array}\right)\cdot\left(\begin{array}{ll|ll} 1 & 4 & 1 & 4\end{array}\right)\cdot\left(\begin{array}{l|l} 2 & 3\end{array}\right),$$

where

$$\left(\begin{array}{lll|lll} i_0 & \dots & i_{m-1} & j_0 & \dots & j_{m-1}\end{array}\right) := \det\begin{pmatrix} x_{i_0j_0} & \dots & x_{i_0j_{m-1}}\\ \vdots & & \vdots\\ x_{i_{m-1}j_0} & \dots & x_{i_{m-1}j_{m-1}}\end{pmatrix} \in \mathbb{Z}[x_{i,j}].$$

For example

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \end{array}\right) := \det \begin{pmatrix} x_{11} & x_{10} \\ x_{31} & x_{30} \end{pmatrix}.$$

The next notion that we have to introduce is *symmetrization.* It can be applied to one of the two tableaux in a bitableau and it can be done with respect to rows or columns. Here we shall use symmetrization according to columns, and of the right hand tableau, and we shall abbreviate it by putting a *square* bracket. The left tableau is (if $\lambda = (\lambda_0, \ldots, \lambda_{h-1})$)

$$T_\lambda := \begin{array}{llllll} 0 & 1 & \ldots & \ldots & \ldots & \lambda_0 - 1 \\ 0 & 1 & \ldots & \ldots & \lambda_1 - 1 & \\ \vdots & & & & & \\ 0 & 1 & \ldots & \lambda_{h-1} - 1 & & \end{array},$$

while the right component is a standard Young tableau t_i^λ of shape λ. It is clear what $\pi(t_i^\lambda)$ means, for π in the *column group* $C(t_i^\lambda)$ of t_i^λ, i.e. the group of permutations π that leave each entry in its column. Clearly

$$C(t_i^\lambda) = \times_i S_{\lambda_i'},$$

where λ_i' denotes the length of the i-th column of the Young tableau t_i^λ and $S_{\lambda_i'}$ the symmetric group on the set of entries in this column.

We can now define what we mean by the symmetrized bideterminant $\mathrm{Bid}(T_\lambda, t_i^\lambda)$, namely

$$\mathrm{Bid}(T_\lambda, t_i^\lambda] := \sum_{\pi \in C(t_i^\lambda)} \mathrm{Bid}(T_\lambda, \pi(t_i^\lambda)).$$

For example, the symmetrized bideterminant

$$\mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 0 & 3 \\ 1 & 4 \\ 2 & \end{array}\right]$$

turns out to be the following sum of bideterminants:

$$\mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 0 & 3 \\ 1 & 4 \\ 2 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 0 & 3 \\ 2 & 4 \\ 1 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 1 & 3 \\ 0 & 4 \\ 2 & \end{array}\right)$$

$$+\mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 0 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 2 & 3 \\ 0 & 4 \\ 1 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 2 & 3 \\ 1 & 4 \\ 0 & \end{array}\right)$$

$$+\mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 0 & 4 \\ 1 & 3 \\ 2 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 0 & 4 \\ 2 & 3 \\ 1 & \end{array}\right) + \mathrm{Bid}\left(T_{(2^2,1)}, \begin{array}{ll} 1 & 4 \\ 0 & 3 \\ 2 & \end{array}\right)$$

$$+\mathrm{Bid}\left(T_{(2^2,1)},\ \begin{matrix}1&4\\2&3\\0&\end{matrix}\right)+\mathrm{Bid}\left(T_{(2^2,1)},\ \begin{matrix}2&4\\0&3\\1&\end{matrix}\right)+\mathrm{Bid}\left(T_{(2^2,1)},\ \begin{matrix}2&4\\1&3\\0&\end{matrix}\right).$$

These symmetrized bideterminants are polynomials over the integers, and so they can easily be reduced modulo a prime number p, obtaining a *p-reduced bideterminant*

$$\mathrm{Bid}_p\left(T_{(2^2,1)},\ \begin{matrix}0&3\\1&4\\2&\end{matrix}\right),$$

for example. Correspondingly, there are the *p-reduced symmetrized bideterminants* $\mathrm{Bid}_p\left(T_\lambda, t_i^\lambda\right]$ arising from the symmetrized bideterminant by reducing the coefficients modulo p. The definition of bideterminants shows that these polynomials are multivariate polynomials over the integers and in the indeterminates $x_{i,j}$ upon which the elements of the symmetric group act as follows:

$$\pi(x_{i,j}) := x_{i,\pi(j)}.$$

The main result on the reduced symmetrized bideterminants is due to Clausen:

3.1 Theorem *Assume a prime number p and a p-regular partition λ of n (which means a partition that contains no p parts λ_i of equal length). Then the p-reduced column symmetrized bideterminants*

$$\mathit{Bid}_p\left(T_\lambda, t_i^\lambda\right]$$

form a system of generators for the p-modular irreducible representation corresponding to λ, if t_i^λ runs through the standard Young tableaux of shape λ.

4 Symmetrized Specht Polynomials

The crucial point is, that symmetrized bideterminants

$$\mathrm{Bid}\left(T_\lambda, t_i^\lambda\right]$$

are sums of products of determinants of the following particular form, since the left component is the tableau T_λ,

$$\left(\begin{array}{ccc|ccc} 0 & \dots & m & j_0 & \dots & j_m \end{array}\right) := \det\begin{pmatrix} x_{0,j_0} & \dots & x_{0,j_m}\\ \vdots & & \vdots \\ x_{m,j_0} & \dots & x_{m,j_m}\end{pmatrix}.$$

This determinant is transformed into a Vandermonde determinant, *a Specht polynomial,* by applying the transformation $x_{ij} \longmapsto x_j^i$:

$$\left(\begin{array}{ccc|ccc} 0 & \dots & m & j_0 & \dots & j_m \end{array}\right) \longmapsto \det\begin{pmatrix} 1 & \dots & 1\\ x_{j_0} & \dots & x_{j_m}\\ \vdots & & \vdots \\ x_{j_0}^m & \dots & x_{j_m}^m\end{pmatrix} = V(j_0,\dots,j_m).$$

For this reason we introduced the notation $\text{Spe}(t_i^\lambda)$ for the Specht polynomial of t_i^λ, the product of the Vandermonde determinants of the rows of t_λ^i. We indicate by

$$\text{Spe}[t_i^\lambda] = \sum_{\pi \in C(t_i^\lambda)} \text{Spe}(\pi t_i^\lambda)$$

the column symmetrized version. Finally we define by

$$\text{Spe}_p[t_i^\lambda]$$

the polynomial obtained by reducing the coefficients in $\text{Spe}[t_i^\lambda]$ modulo p. The main consequence is the following result obtained from Clausen's theorem by the substitution mentioned above:

4.1 Theorem *Assume a prime number p and a p-regular partition λ of n. Then the p-reduced symmetrized Specht polynomials*

$$\mathit{Spe}_p[t_i^\lambda]$$

generate the space of the p-modular irreducible representation $[\lambda]_p$ corresponding to λ, if t_i^λ runs through the standard Young tableaux of shape λ.

The reason is that the transformation $x_{ij} \mapsto x_j^i$ is a ring homomorphism $\mathbb{F}_p[x_{ij}] \to \mathbb{F}_p[x_j]$ that commutes with the action of the symmetric group. If we apply it to an irreducible representation space we either get zero or an irreducible module. In our case here it is clearly not the zero mapping.

4.2 Example We should like to evaluate generators of the 3-modular irreducible representation corresponding to the 3-regular partition $(2^2, 1)$. The Specht polynomials

$$\text{Spe}(t_0^{(2^2,1)}), \ldots, \text{Spe}(t_4^{(2^2,1)})$$

were already evaluated. The symmetrized Specht polynomials turn out to be, for example,

$$\text{Spe}[t_0^{(2^2,1)}] = 4x_0x_1 + 4x_0x_2 - 4x_0x_3 - 4x_0x_4 + 4x_1x_2 - 4x_1x_3 - 4x_1x_4$$
$$-4x_2x_3 - 4x_2x_4 + 12x_3x_4.$$

The others can be obtained by performing suitable permutations of the variables x_i. Reduction modulo 2 gives the zero polynomial in accordance with the fact that the partition $(2^2, 1)$ is *not* 2-regular, while reduction modulo 3 gives

$$\text{Spe}_3[t_0^{(2^2,1)}] = x_0x_1 + x_0x_2 + 2x_0x_3 + 2x_0x_4 + x_1x_2 + 2x_1x_3 + 2x_1x_4$$
$$+2x_2x_3 + 2x_2x_4 + 0x_3x_4.$$

The sequence of coefficients of the monomials $x_i x_k$ (numbered lexicographically) is $1, 1, 2, 2, 1, 2, 2, 2, 2, 0$. The corresponding sequences of all the five symmetrized Specht polynomials turn out to be, after reduction of the coefficients modulo 3, the rows of the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 2 \\ 2 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 1 & 2 \end{pmatrix}.$$

According to the above theorem, the rows of this matrix describe generators of the 3-modular irreducible submodule corresponding to the partition $(2^2, 1)$. As the 3-rank of this matrix is 4, this 3-modular irreducible representation is of dimension 4. Elementary row transformations give the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

which shows that its first four rows form the basis $b_0, \ldots, b_3$, where

$$b_0 = x_0x_1 + x_0x_2 + 2x_0x_3 + 2x_0x_4 + x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4,$$

$$b_1 = x_0x_2 + 2x_0x_3 + x_1x_2 + 2x_1x_3 + x_2x_4 + 2x_3x_4,$$

$$b_2 = 2x_0x_3 + x_0x_4 + x_1x_2 + 2x_1x_4 + 2x_2x_4 + x_3x_4,$$

$$b_3 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

We want to compute the representing matrix for the elementary transposition (0,1). Applying this transposition to the indices of the variables we obtain that

$$(0,1)b_0 = b_0,\ (0,1)b_1 = b_1,\ (0,1)b_2 = b_1 + 2b_2,\ (0,1)b_3 = b_1 + b_2 + b_3.$$

This gives the matrix representing $(0,1)$ in the 3-modular irreducible representation $[2^2, 1]_3$ over $GF(3)$ corresponding to the partition $(2^2, 1)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\diamond$

A program that evaluates such representing matrices is implemented in SYMMETRICA ([11]), a software package for the representation theory of symmetric groups and related classes of groups. It can be downloaded from http://www.symmetrica.de

and it can also be used online. The problem is the big number of monomials occurring in the generators. Of course, the evaluation of a particular representing matrix makes use of leading monomials.

5 Modular Irreducibles as Codes

We are now in a position to consider modular irreducible representations as linear codes. Many interesting results on codes obtained from *reducible* modular representations of symmetric groups can be found in [15] and [12]. The authors, K.-H. Zimmermann and R. A. Liebler, consider, among various other situations, modular representations obtained from ordinary ones by reduction modulo the prime characteristic, embedded into permutation representations. In contrast to this we consider embeddings of modular *irreducibles* here.

We have seen hat the reduced symmetrized Specht polynomials corresponding to the standard Young tableaux generate the representation space. We can therefore easily obtain a *generator matrix* of this space, for example, the matrix

$$\Gamma = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

mentioned above. Its rows form a basis, and they are elements of the vector space $GF(3)^{10}$. Hence the generated vector space is a subspace of the space that carries the permutation representation induced from the column group of a standard Young tableau of shape $(2^2, 1)$.

We can therefore consider this four-dimensional subspace as a ternary linear $(10, 4)$-code and can try to find out, for example, its minimum distance d_3, which is the minimal number of nonzero coordinates in a nonzero vector of this space of dimension 4 over $GF(3)$. This linear code, formed by this particular 3-modular irreducible representation of S_5 has the block length 10, the dimension 4, the minimal distance $d_3 = 6$ and the order of the prime field is $p = 3$, for short: it is a $(10, 4, 6, 3)$-code.

Systematic computer calculations using SYMMETRICA gave tables for various degrees n of small symmetric groups S_n. They contain in their first column the partitions λ of n. The row labelled by λ contains informations about linear codes obtained from modular irreducible representations corresponding to this partition, their dimensions and their minimal distances, for various small primes. The second column contains the dimensions m of the corresponding permutation representations, i.e. the dimensions of the representations induced by the identity representation of the column group of a standard tableau of shape λ,

$$m = \binom{n}{\lambda'_0, \lambda'_1, \ldots}.$$

The entry at the intersection of the row labelled by λ and the column labelled by the prime p contains the dimension of the p-modular irreducible representation corresponding to λ, if λ is p-regular (otherwise the entry is 0).
The following column contains the minimum distance d_p of the representation space considered as a subspace of the permutation module corresponding to λ. The values were obtained using A. Wassermann's implementation of the LLL-algorithm ([7]) for the evaluation of short vectors (cases $p = 2, 3$) and another software package due to K.-J. Zimmermann using iterated Gaussian algorithm.
We consider the smallest primes $p = 2, 3, 5, \ldots$ and show small tables while further tables can be found via internet under the address

http://www.mathe2.uni-bayreuth.de/axel/codes/modinperm/

Here is a numerical example, the case $n = 5$:

λ	m	2	d_2	3	d_3	5	d_5
(5)	120	1	120	1	120	1	120
$(4,1)$	60	4	24	4	24	3	42
$(3,2)$	30	4	16	1	30	5	12
$(3,1^2)$	20	0		6	6	3	14
$(2^2,1)$	10	0		4	6	5	4
$(2,1^3)$	5	0		0		1	5
(1^5)	1	0		0		0	

The entries $10, 4, 6$ in the row of $\lambda = (2^2, 1)$ and the columns of $m, 3, d_3$ mean that the 3-modular irreducible representation corresponding to the 3-regular partition $(2^2, 1)$, embedded into the permutation module of the column group is of block length 10, of dimension 4 and it has the minimum distance $d_3 = 6$, so that this embedding gives a linear $(10, 4, 6, 3)$-code.
The table shows — if we compare it with known results on optimal codes — that for the following entries the corresponding codes are optimal in the sense that the minimum distance is maximal for the blocklength and the dimension in question.

$$(60, 4, 24, 2)$$
$$(30, 4, 16, 2)$$
$$(10, 4, 6, 3)$$

There is another embedding of the modular irreducibles which is more interesting, since the blocklength is much smaller. We can embed the modular irreducible $[\lambda]_p$ corresponding to λ into the representation space of $\overline{[\lambda]}$, the representation obtained from the ordinary irreducible $[\lambda]$ by writing it over the ring of integers and then reducing the entries modulo p. In order to obtain this embedding we use the fact that the p-reduced symmetrized Specht polynomials form a generating system and that, before

p-reduction, we can replace a symmetrized Specht polynomial by a uniquely defined linear combination of standard Specht polynomials:

$$\mathrm{Spe}[t_i^\lambda] = \sum_j \alpha_j \mathrm{Spe}(t_j^\lambda),$$

from which it follows that we can express the generators

$$\mathrm{Spe}_p[t_i^\lambda] = \sum_j (\alpha_j)_p \mathrm{Spe}_p(t_j^\lambda)$$

in terms of reduced Specht polynomials, obtaining this way a new generator matrix of the code in question. For example, the coefficients of the monomial summands in the symmetrized Specht polynomials corresponding to the standard Young tableaux of shape $(2^2, 1)$ are listed in the following array:

	x_0x_1	x_0x_2	x_0x_3	x_0x_4	x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4
$\mathrm{Spe}[t_0^{(2^2,1)}]$	4	4	−4	−4	4	−4	−4	−4	−4	12
$\mathrm{Spe}[t_1^{(2^2,1)}]$	4	−4	4	−4	−4	4	−4	−4	12	−4
$\mathrm{Spe}[t_2^{(2^2,1)}]$	4	−4	−4	4	−4	−4	4	12	−4	−4
$\mathrm{Spe}[t_3^{(2^2,1)}]$	−4	4	4	−4	−4	−4	12	4	−4	−4
$\mathrm{Spe}[t_4^{(2^2,1)}]$	−4	4	−4	4	−4	12	−4	−4	4	−4

Hence we can easily obtain the symmetrized Specht polynomials as linear combinations of the standard Specht polynomials, using the array of the standard Specht polynomials evaluated above:

	x_0x_1	x_0x_2	x_0x_3	x_0x_4	x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4
$\mathrm{Spe}(t_0^{(2^2,1)})$	1	0	0	−1	0	−1	0	0	0	1
$\mathrm{Spe}(t_1^{(2^2,1)})$	1	0	0	−1	−1	0	0	0	1	0
$\mathrm{Spe}(t_2^{(2^2,1)})$	1	0	−1	0	−1	0	0	1	0	0
$\mathrm{Spe}(t_3^{(2^2,1)})$	0	1	0	−1	−1	0	1	0	0	0
$\mathrm{Spe}(t_4^{(2^2,1)})$	0	1	−1	0	−1	1	0	0	0	0

.

For example, $\mathrm{Spe}[t_0^{(2^2,1)}]$ is equal to the linear combination

$$12\mathrm{Spe}(t_0^{(2^2,1)}) - 4\mathrm{Spe}(t_1^{(2^2,1)}) - 4\mathrm{Spe}(t_2^{(2^2,1)}) - 4\mathrm{Spe}(t_3^{(2^2,1)}) + 8\mathrm{Spe}(t_4^{(2^2,1)}),$$

so that, by reduction modulo 3, we get

$$\mathrm{Spe}_3[t_0^{(2^2,1)}] \quad = \quad 2\mathrm{Spe}(t_1^{(2^2,1)}) + 2\mathrm{Spe}(t_2^{(2^2,1)}) + 2\mathrm{Spe}(t_3^{(2^2,1)}) + 2\mathrm{Spe}(t_4^{(2^2,1)}).$$

To get a basis we can take the first 4 of the 5 symmetrized Specht polynomials. The next 3 turn out to be

$$\begin{aligned}
\mathrm{Spe}_3[t_1^{(2^2,1)}] &= 2\mathrm{Spe}(t_0^{(2^2,1)}) + 2\mathrm{Spe}(t_2^{(2^2,1)}) + 2\mathrm{Spe}(t_3^{(2^2,1)}).\\
\mathrm{Spe}_3[t_2^{(2^2,1)}] &= 2\mathrm{Spe}(t_0^{(2^2,1)}) + 2\mathrm{Spe}(t_1^{(2^2,1)}) + \mathrm{Spe}(t_3^{(2^2,1)}) + \mathrm{Spe}(t_4^{(2^2,1)}).\\
\mathrm{Spe}_3[t_3^{(2^2,1)}] &= 2\mathrm{Spe}(t_0^{(2^2,1)}) + 2\mathrm{Spe}(t_1^{(2^2,1)}) + \mathrm{Spe}(t_2^{(2^2,1)}) + \mathrm{Spe}(t_4^{(2^2,1)}).
\end{aligned}$$

and so we find the following matrix of generators of the code in question, where the columns correspond to the standard Young tableaux

$$t_0^{(2^2,1)} = \begin{array}{cc} 0 & 3 \\ 1 & 4 \\ 2 & \end{array},\ t_1^{(2^2,1)} = \begin{array}{cc} 0 & 2 \\ 1 & 4 \\ 3 & \end{array},\ t_2^{(2^2,1)} = \begin{array}{cc} 0 & 2 \\ 1 & 3 \\ 4 & \end{array},\ t_3^{(2^2,1)} = \begin{array}{cc} 0 & 1 \\ 2 & 4 \\ 3 & \end{array},\ t_4^{(2^2,1)} = \begin{array}{cc} 0 & 1 \\ 2 & 3 \\ 4 & \end{array}.$$

from which we finally obtain the desired *generator matrix*

$$\Gamma = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \end{pmatrix}$$

of the representation space of $[2^2,1]_3$ embedded into the representation space of the 3-modular representation $\overline{[2^2,1]}$ obtained from the ordinary irreducible representation $[2^2,1]$ by reduction modulo 3.

Various further computational results are gathered in the following tables (additional ones can be found on our home page the address of which was given already): The leftmost column gives a partition, the next column shows the dimension f^λ of the corresponding ordinary irreducible representation, i.e. it gives the block lengths of the code. The other columns provide the dimensions of the corresponding codes, depending on the characteristic $p = 2,3,5,7$ of the prime field, and in most cases (some dimensions were too big) there are also the minimum distances d_p given. In the *non-trivial* cases when they are maximal with respect to blocklength and dimension, they are put in frameboxes in order to emphasize that the corresponding code is nontrivial and *distance optimal:*

	f^λ	2	d_2	3	d_3
(3)	1	1	1	1	1
$(2,1)$	2	2	1	1	2
$(1,1,1)$	1	0		0	

	f^λ	2	d_2	3	d_3
(4)	1	1	1	1	1
$(3,1)$	3	2	2	3	1
$(2,2)$	2	0		1	2
$(2,1,1)$	3	0		3	1
$(1,1,1,1)$	1	0		0	

	f^λ	2	d_2	3	d_3	5	d_5
(5)	1	1	1	1	1	1	1
$(4,1)$	4	4	1	4	1	3	2
$(3,2)$	5	4	2	1	4	5	1
$(3,1,1)$	6	0		6	1	3	3
$(2,2,1)$	5	0		4	2	5	1
$(2,1,1,1)$	4	0		0		1	4
$(1,1,1,1,1)$	1	0		0		0	

	f^λ	2	d_2	3	d_3	5	d_5
(6)	1	1	1	1	1	1	1
$(5,1)$	5	4	2	4	2	5	1
$(4,2)$	9	4	$\boxed{4}$	9	1	8	2
$(4,1,1)$	10	0		6	3	10	1
$(3,3)$	5	0		1	4	5	1
$(3,2,1)$	16	16	1	4	6	8	5
$(3,1,1,1)$	10	0		0		10	1
$(2,2,2)$	5	0		0		5	1
$(2,2,1,1)$	9	0		9	1	1	9
$(2,1,1,1,1)$	5	0		0		5	1
$(1,1,1,1,1,1)$	1	0		0		0	

	f^λ	2	d_2	3	d_3	5	d_5	7	d_7
(7)	1	1	1	1	1	1	1	1	1
$(6,1)$	6	6	1	6	1	6	1	5	2
$(5,2)$	14	14	1	13	2	8	4	14	1
$(5,1,1)$	15	0		15	1	15	1	10	3
$(4,3)$	14	8	$\boxed{4}$	1	8	13	2	14	1
$(4,2,1)$	35	20	4	20	2	35	1	35	1
$(4,1,1,1)$	20	0		0		20	1	10	4
$(3,3,1)$	21	0		6	4	8	6	21	1
$(3,2,2)$	21	0		15	2	13	3	21	1
$(3,2,1,1)$	35	0		13	6	35	1	35	1
$(3,1,1,1,1)$	15	0		0		15	1	5	5
$(2,2,2,1)$	14	0		0		1	14	14	1
$(2,2,1,1,1)$	14	0		0		6	4	14	1
$(2,1,1,1,1,1)$	6	0		0		0		1	6
$(1,1,1,1,1,1,1)$	1	0		0		0		0	

	f^λ	2	d_2	3	d_3	5	d_5	7	d_7
(8)	1	1	1	1	1	1	1	1	1
$(7,1)$	7	6	2	7	1	7	1	7	1
$(6,2)$	20	14	3	13	4	20	1	19	
$(6,1,1)$	21	0		21	1	21	1	21	1
$(5,3)$	28	8	8	28	1	21		28	1
$(5,2,1)$	64	64	1	35	2	43		45	
$(5,1,1,1)$	35	0		0		35	1	35	1
$(4,4)$	14	0		1	8	13	2	14	1
$(4,3,1)$	70	40	8	7	8	70	1	70	1
$(4,2,2)$	56	0		35	2	13		56	1
$(4,2,1,1)$	90	0		90	1	90	1	45	
$(4,1,1,1,1)$	35	0		0		35	1	35	1
$(3,3,2)$	42	0		21	4	21		42	1
$(3,3,1,1)$	56	0		13	12	43		56	1
$(3,2,2,1)$	70	0		28	8	70	1	70	1
$(3,2,1,1,1)$	64	0		0		21		19	
$(3,1,1,1,1,1)$	21	0		0		0		21	1
$(2,2,2,2)$	14	0		0		1	1	14	1
$(2,2,2,1,1)$	28	0		0		7		28	1
$(2,2,1,1,1,1)$	20	0		0		20	1	1	
$(2,1,1,1,1,1,1)$	7	0		0		0		7	1
$(1,1,1,1,1,1,1,1)$	1	0		0		0		0	

Among these codes arising from 2-modular irreducible representations of symmetric groups, there are — besides the trivially optimal codes and the codes for which the minimal distance is not yet available for reasons of computer power — the distance optimal binary codes with the parameter triples (n, k, d) equal to

$$(9, 4, 4) \text{ and } (14, 8, 4).$$

A. Betten, see his home page

http://www.math.colostate.edu/ betten

found out that there are exactly 4 isometry classes of $(9, 4, 4)$-codes and exactly 48 isometry classes of $(14, 8, 4)$-codes over $\mathbb{F}_2$ among the altogether 134 isometry classes of binary $(9, 4)$-codes and the 102.445 classes of binary $(18, 4)$-codes without columns containing zeros only. The total numbers of isometry classes were obtained by H. Fripertinger, see

http://www.mathe2.uni-bayreuth.de/frib/codes/tables.html

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