# Affine Hecke algebras and the Schubert calculus 

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## 0. Introduction

Using a combinatorial approach which avoids geometry, this paper studies the ring structure of $K_{T}(G / B)$, the $T$-equivariant $K$-theory of the (generalized) flag variety $G / B$. Here, the data $G \supseteq B \supseteq T$ is a complex reductive algebraic group (or symmetrizable Kac-Moody group) $G$, a Borel subgroup $B$, and a maximal torus $T$, and $K_{T}(G / B)$ is the Grothendieck group of $T$-equivariant coherent sheaves on $G / B$. Because of the $T$-equivariance the ring $K_{T}(G / B)$ is an $R$-algebra, where $R$ is the representation ring of $T$. As explained by Grothendieck [Gd] (in the non Kac-Moody case) and Kostant and Kumar [KK] (in the general Kac-Moody case), the ring $K_{T}(G / B)$ has a natural $R$-basis $\left\{\left[\mathcal{O}_{X_{w}}\right] \mid w \in W\right\}$, where $W$ is the Weyl group and $\mathcal{O}_{X_{w}}$ is the structure sheaf of the Schubert variety $X_{w} \subseteq G / B$. One of the main problems in the field is to understand the structure constants of the ring $K_{T}(G / B)$ with this basis, that is, the coeffients $c_{w v}^{z}$ in the equations

$$
\begin{equation*}
\left[\mathcal{O}_{X_{w}}\right]\left[\mathcal{O}_{X_{v}}\right]=\sum_{z \in W} c_{w v}^{z}\left[\mathcal{O}_{X_{z}}\right] \tag{0.1}
\end{equation*}
$$

Our approach is to work completely combinatorially and define $K_{T}(G / B)$ as a quotient of the affine nil-Hecke algebra. The fact that the combinatorial approach coincides with the geometric one is a consequence of the results of Kostant and Kumar [KK] and Demazure [D]. In the combinatorial literature the elements $\left[\mathcal{O}_{X_{w}}\right]$ are often called (double) Grothendieck polynomials.

[^0]Let $P$ be the weight lattice of $G$ and, for $\lambda \in P$, let $\left[X^{\lambda}\right]$ be the homogeneous line bundle on $G / B$ corresponding to the character of $T$ indexed by $\lambda$. The theorem of Pittie $[\mathrm{P}]$ says that the ring $K_{T}(G / B)$ is generated by the $\left[X^{\lambda}\right], \lambda \in P$. Steinberg [St] strengthened this result by displaying specific $\left[X^{-\lambda_{w}}\right], w \in W$, which form an $R$-basis of $K_{T}(G / B)$. These results are often collectively known as the "Pittie-Steinberg theorem".

The theorems which we prove in Section 2 are simply different points of view on the PittieSteinberg theorem. Though we are not aware of any reference which states these theorems in the generality which we consider, these theorems should be considered well known.

Let $s_{1}, \ldots, s_{n}$ be the simple reflections in $W$ (determined by the data $(G \supseteq B \supseteq T)$ ), let $w_{0}$ be the longest element of $W$ and let $P^{+}$be the set of dominant weights in $P$. The Schubert varieties $X_{w_{0} s_{i}}$ are the codimension one Schubert varieties in $G / B$. In section 3 we prove "Pieri-Chevalley" formulas for the products

$$
\begin{equation*}
\left[X^{\lambda}\right]\left[\mathcal{O}_{X_{w}}\right], \quad\left[X^{-\lambda}\right]\left[\mathcal{O}_{X_{w}}\right], \quad\left[X^{w_{0} \lambda}\right]\left[\mathcal{O}_{X_{w}}\right], \quad \text { and } \quad\left[\mathcal{O}_{X_{w_{0} s_{i}}}\right]\left[\mathcal{O}_{X_{w}}\right], \tag{0.2}
\end{equation*}
$$

for $\lambda \in P^{+}, w \in W$ and $1 \leq i \leq n$. All of these Pieri-Chevalley formulas are given in terms of the combinatorics of the Littelmann path model [Li1-3]. The formula which we give for the first product in (0.2) is due to Pittie and Ram [PR1]. In this paper we provide more details of proof than appeared in [PR1]. The other formulas for the products in (0.2) follow by applying the duality theorem of Brion [Br, Theorem 4] to the first formula. However, here we give an independent, combinatorial, proof and deduce Brion's result as a consequence. The last formula is a consequence of the nice formula

$$
\begin{equation*}
\left[\mathcal{O}_{X_{w_{0} s_{i}}}\right]=1-e^{w_{0} \omega_{i}}\left[X^{-\omega_{i}}\right], \tag{0.3}
\end{equation*}
$$

which is an easy consequence of the first two Pieri-Chevalley rules.
It is not difficult to "specialize" product formulas for $K_{T}(G / B)$ to corresponding product formulas for $K(G / B), H_{T}^{*}(G / B)$, and $H^{*}(G / B)$ (by using the Chern character and comparing lowest degree terms, and ignoring the $T$-action). Thus the products which are computed in this paper also give results for ordinary Grothendieck polynomials, double Schubert polynomials, and ordinary Schubert polynomials. In section 4 we explain how to do these conversions. For most of these cases the specialized versions of our Pieri-Chevalley rules are already very well known (see, for example, [Ch]).

In Section 5 we give explicitly
(a) two different kinds of formulas for $\left[\mathcal{O}_{X_{w}}\right]$ in terms of $X^{\lambda}$, and
(b) complete computations of the products in (0.1)
for the rank two root systems. This data allows us to make a "positivity conjecture" for the coefficients $c_{w v}^{z}$ in (0.1). This conjecture generalizes the theorems of Brion [ Br , formula before Theorem 1] and Graham [Gr, Corollary 4.1], which treat the cases $K(G / B)$ and $H_{T}^{*}(G / B)$, respectively.

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## 1. Preliminaries

Fix the following data and notation:

| $\mathfrak{h}^{*}$ | is a real vector space of dimension $n$, |
| :--- | :--- |
| $R$ | is a reduced irreducible root system in $\mathfrak{h}^{*}$, |
| $R^{+}$ | is a set of positive roots in $R$, |
| $W$ | is the Weyl group of $R$, |
| $s_{1}, \ldots, s_{n}$ | are the simple reflections in $W$, |
| $m_{i j},\left\{\begin{array}{l}\text { is }\end{array}\right.$ |  |
| $R(w)=\left\{\alpha \in R^{+} \mid w \alpha \notin R^{+}\right\}$ | is the order of $s_{i} s_{j}$ in $W, i \neq j$, |
| $\ell(w)=\operatorname{Card}(R(w))$ | is the length of $w \in W$, |
| $\leq$ | is the Bruhat-Chevalley order on $W$, |
| $\alpha_{1}, \ldots, \alpha_{n}$ | are the simple roots in $R^{+}$, |
| $\omega_{1}, \ldots, \omega_{n}$ | are the fundamental weights, |
| $P=\sum_{i=1}^{n} \mathbb{Z} \omega_{i}$ | is the weight lattice, |
| $P^{+}=\sum_{i=1}^{n} \mathbb{Z}_{\geq 0} \omega_{i}$ | is the set of dominant integral weights. |

For a brief, easy, introduction to root systems with lots of pictures for visualization see [NR]. By [Bou VI §1 no. 6 Cor. 2 to Prop. 17], if $w=s_{i_{1}} \cdots s_{i_{p}}$ be a reduced word for $w$, then

$$
\begin{equation*}
R(w)=\left\{\alpha_{i_{p}}, s_{i_{p}} \alpha_{i_{p-1}}, \ldots, s_{i_{p}} \cdots s_{i_{2}} \alpha_{i_{1}}\right\} \tag{1.1}
\end{equation*}
$$

The affine nil-Hecke algebra is the algebra $\tilde{H}$ given by generators $T_{1}, \ldots, T_{n}$ and $X^{\lambda}, \lambda \in P$, with relations

$$
\begin{equation*}
T_{i}^{2}=T_{i}, \quad \underbrace{T_{i} T_{j} T_{i} \cdots}_{m_{i j} \text { factors }}=\underbrace{T_{j} T_{i} T_{j} \cdots}_{m_{i j} \text { factors }}, \quad X^{\lambda} X^{\mu}=X^{\lambda+\mu}, \tag{1.2}
\end{equation*}
$$

and

$$
\begin{equation*}
X^{\lambda} T_{i}=T_{i} X^{s_{i} \lambda}+\frac{X^{\lambda}-X^{s_{i} \lambda}}{1-X^{-\alpha_{i}}} \tag{1.3}
\end{equation*}
$$

Let $T_{w}=T_{i_{1}} \cdots T_{i_{p}}$ for a reduced word $w=s_{i_{1}} \cdots s_{i_{p}}$. Then

$$
\begin{equation*}
\left\{X^{\lambda} T_{w} \mid w \in W, \lambda \in P\right\} \quad \text { and } \quad\left\{T_{w} X^{\lambda} \mid w \in W, \lambda \in P\right\} \tag{1.4}
\end{equation*}
$$

are bases of $\tilde{H}$.
Both the nil-Hecke algebra,

$$
\begin{equation*}
H=\mathbb{Z} \text {-span }\left\{T_{w} \mid w \in W\right\}, \quad \text { and } \quad \mathbb{Z}[X]=\mathbb{Z} \text {-span }\left\{X^{\lambda} \mid \lambda \in P\right\} \tag{1.5}
\end{equation*}
$$

are subalgebras of $\tilde{H}$. The action of $W$ on $\mathbb{Z}[X]$ is given by defining

$$
\begin{equation*}
w X^{\lambda}=X^{w \lambda}, \quad \text { for } w \in W, \lambda \in P, \tag{1.6}
\end{equation*}
$$

and extending linearly. The proof of the following theorem is given in [ R , Theorem 1.13 and Theorem 1.17]. The first statement of the theorem is due to Bernstein, Zelevinsky, and Lusztig [ $\mathrm{Lu}, 8.1]$ and the second statement is due to Steinberg [St] and is known as the Pittie-Steinberg theorem.

Theorem 1.7. Define

$$
\begin{equation*}
\lambda_{w}=w^{-1} \sum_{s_{i} w<w} \omega_{i}, \quad \text { for } w \in W . \tag{1.8}
\end{equation*}
$$

The center of $\tilde{H}$ is $Z(\tilde{H})=\mathbb{Z}[X]^{W}$ and each element $f \in \mathbb{Z}[X]$ has a unique expansion

$$
\begin{equation*}
f=\sum_{w \in W} f_{w} X^{-\lambda_{w}}, \quad \text { with } f_{w} \in \mathbb{Z}[X]^{W} . \tag{1.9}
\end{equation*}
$$

Let $\varepsilon_{i}=1-T_{i}$ and let $\varepsilon_{w}=\varepsilon_{i_{1}} \cdots \varepsilon_{i_{p}}$ for a reduced word $w=s_{i_{1}} \cdots s_{i_{p}}$. Then $\varepsilon_{w}$ is well defined and independent of the reduced word for $w$ since

$$
\begin{equation*}
\varepsilon_{i}^{2}=\varepsilon_{i}, \quad \text { and } \quad \underbrace{\varepsilon_{i} \varepsilon_{j} \varepsilon_{i} \cdots}_{m_{i j} \text { factors }}=\underbrace{\varepsilon_{j} \varepsilon_{i} \varepsilon_{j} \cdots}_{m_{i j} \text { factors }} . \tag{1.10}
\end{equation*}
$$

The second equality is a consequence of the formulas

$$
\begin{equation*}
\varepsilon_{w}=\sum_{v \leq w}(-1)^{\ell(v)} T_{v} \quad \text { and } \quad T_{w}=\sum_{v \leq w}(-1)^{\ell(v)} \varepsilon_{v} \tag{1.11}
\end{equation*}
$$

which are straightforward to verify by induction on the length of $w$.

## 2. The ring $K_{T}(G / B)$

Let $H$ and $\mathbb{Z}[X]$ be as in (1.5). The trivial representation of $H$ is defined by the homomorphism $\mathbf{1}: H \rightarrow \mathbb{Z}$ given by $\mathbf{1}\left(T_{i}\right)=1$. The first of the maps

$$
\begin{array}{rlll}
\mathbb{Z}[X] & \xrightarrow{\longrightarrow} \tilde{H} T_{w_{0}} & \xrightarrow{\sim} \tilde{H} \otimes_{H} \mathbf{1} \\
f & \longmapsto f T_{w_{0}} & \longmapsto & f \otimes \mathbf{1}
\end{array}
$$

is an $\tilde{H}$-module isomorphism if the action of $\tilde{H}$ on $\mathbb{Z}[X]$ is given by

$$
\begin{equation*}
T_{i} \cdot f=\frac{X^{\alpha_{i}} f-s_{i} f}{X^{\alpha_{i}}-1}, \quad \text { for } f \in \mathbb{Z}[X] . \tag{2.1}
\end{equation*}
$$

The group algebra of $P$ is

$$
\begin{equation*}
R=\mathbb{Z} \text {-span }\left\{e^{\lambda} \mid \lambda \in P\right\} \quad \text { with } \quad e^{\lambda} e^{\mu}=e^{\lambda+\mu} \tag{2.2}
\end{equation*}
$$

for $\lambda, \mu \in P$. Extend coefficients to $R$ so that $\tilde{H}_{R}=R \otimes_{\mathbb{Z}} \tilde{H}$ and $R[X]=R \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ are $R$-algebras. Define $K_{T}(G / B)$ to be the $\tilde{H}_{R}$-module

$$
\begin{equation*}
K_{T}(G / B)=R-\operatorname{span}\left\{\left[\mathcal{O}_{X_{w}}\right] \mid w \in W\right\}, \tag{2.3}
\end{equation*}
$$

so that the $\left[\mathcal{O}_{X_{w}}\right], w \in W$, are an $R$-basis of $K_{T}(G / B)$, with $\tilde{H}_{R}$-action given by

$$
X^{\lambda}\left[\mathcal{O}_{X_{1}}\right]=e^{\lambda}\left[\mathcal{O}_{X_{1}}\right], \quad \text { and } \quad T_{i}\left[\mathcal{O}_{X_{w}}\right]= \begin{cases}{\left[\mathcal{O}_{X_{w s}}\right],} & \text { if } w s_{i}>w,  \tag{2.4}\\ {\left[\mathcal{O}_{X_{w}}\right],} & \text { if } w s_{i}<w .\end{cases}
$$

If $R$ is an $R[X]$-module via the $R$-algebra homomorphism given by
then, as $\tilde{H}_{R}$-modules, $K_{T}(G / B) \cong \tilde{H}_{R} \otimes_{R[X]} R_{e}$, where $R_{e}$ is the $R$-rank $1 R[X]$-module determined by the homomorphism $e$.

Let $Q$ be the field of fractions of $R$ and let $\bar{Q}$ be the algebraic closure of $Q$. For $w \in W$ let

$$
\begin{equation*}
b_{w} \quad \text { in } \bar{Q} \otimes_{R} K_{T}(G / B) \quad \text { be determined by } \quad X^{\lambda} b_{w}=e^{w \lambda} b_{w}, \quad \text { for } \lambda \in P . \tag{2.6}
\end{equation*}
$$

If the $b_{w}$ exist, then they are a $\bar{Q}$-basis of $\bar{Q} \otimes_{R} K_{T}(G / B)$ since they are eigenvectors with distinct eigenvalues. If $\tau_{i}, 1 \leq i \leq n$, are the operators on $\bar{Q} \otimes_{R} K_{T}(G / B)$ given by

$$
\begin{equation*}
\tau_{i}=T_{i}-\frac{1}{1-X^{-\alpha_{i}}}, \quad \text { then } \quad b_{1}=\left[\mathcal{O}_{X_{1}}\right] \quad \text { and } \quad \tau_{i} b_{w}=b_{w s_{i}}, \quad \text { for } w s_{i}>w \tag{2.7}
\end{equation*}
$$

because, a direct computation with relation (1.3) gives that $X^{\lambda} \tau_{i} b_{w}=\tau_{i} X^{s_{i} \lambda} b_{w}=\tau_{i} e^{w s_{i} \lambda} b_{w}=$ $e^{w s_{i} \lambda} b_{w s_{i}}$. Thus the $b_{w}, w \in W$, exist and the form of the $\tau$-operators shows that, in fact, they form a $Q$-basis of $Q \otimes_{R} K_{T}(G / B)$ (it was not really necessary to extend coefficients all the way to $\bar{Q})$. Equations (2.6) and (2.7) force

$$
\underbrace{\tau_{i} \tau_{j} \tau_{i} \cdots}_{m_{i j} \text { factors }}=\underbrace{\tau_{j} \tau_{i} \tau_{j} \cdots,}_{m_{i j} \text { factors }} \quad \text { and the equality } \quad \tau_{i}^{2}=\frac{1}{\left(X^{\alpha_{i}}-1\right)\left(X^{-\alpha_{i}}-1\right)}
$$

is checked by direct computation using (1.3). Let $\tau_{w}=\tau_{i_{1}} \cdots \tau_{i_{p}}$ for a reduced word $w=s_{i_{1}} \cdots s_{i_{p}}$. Then, for $w \in W$,

$$
\begin{equation*}
b_{w}=\tau_{w^{-1}} b_{1}, \quad\left[\mathcal{O}_{X_{w}}\right]=T_{w^{-1}}\left[\mathcal{O}_{X_{1}}\right] \quad \text { and we define } \quad\left[\mathcal{I}_{X_{w}}\right]=\varepsilon_{w^{-1}}\left[\mathcal{O}_{X_{1}}\right] \tag{2.8}
\end{equation*}
$$

where $\varepsilon_{w}$ is as in (1.11). In terms of geometry, $\left[\mathcal{O}_{X_{w}}\right]$ is the class of the structure sheaf of the Schubert variety $X_{w}$ in $G / B$ and, up to a sign, $\left[\mathcal{I}_{X_{w}}\right]$ is class of the sheaf $\mathcal{I}_{X_{w}}$ determined by the exact sequence $0 \rightarrow \mathcal{I}_{X_{w}} \rightarrow \mathcal{O}_{X_{w}} \rightarrow \mathcal{O}_{\partial X_{w}} \rightarrow 0$, where $\partial X_{w}=\bigsqcup_{v<w} B v B$ (see [Ma, Theorem $2.1(i i)]$ and [LS, equation (4)]. We are not aware of a good geometric characterization of the basis $\left\{\left[X^{-\lambda_{w}}\right] \mid w \in W\right\}$ of $K_{T}(G / B)$ which appears in the following theorem.

Theorem 2.9. Let $\lambda_{w}, w \in W$, be as defined in Theorem 2.9 and let $\left[X^{\lambda}\right]=X^{\lambda}\left[\mathcal{O}_{X_{w_{0}}}\right]=$ $X^{\lambda} T_{w_{0}}\left[\mathcal{O}_{X_{1}}\right]$ for $\lambda \in P$. Then the $\left[X^{-\lambda_{w}}\right], w \in W$, form an $R$-basis of $K_{T}(G / B)$.

Proof. Up to constant multiples, $\left[\mathcal{O}_{X_{w_{0}}}\right]=T_{w_{0}}\left[\mathcal{O}_{X_{1}}\right]$ is determined by the property

$$
\begin{equation*}
T_{i}\left[\mathcal{O}_{X_{w_{0}}}\right]=\left[\mathcal{O}_{X_{w_{0}}}\right], \quad \text { for all } 1 \leq i \leq n . \tag{2.10}
\end{equation*}
$$

If constants $c_{w} \in Q$ are given by

$$
\left[\mathcal{O}_{X_{w_{0}}}\right]=\sum_{w \in W} c_{w} b_{w},
$$

then comparing coefficients of $b_{w s_{i}}$, for $w s_{i}>w$, on each side of (2.10) yields a recurrence relation for the $c_{w}$,

$$
\begin{equation*}
c_{w}=c_{w s_{i}}\left(\frac{1}{1-e^{-w \alpha_{i}}}\right) \quad \text { for } w s_{i}>w, \quad \text { which implies } \quad c_{w_{0} v^{-1}}=\prod_{\alpha \in R(v)} \frac{1}{1-e^{w_{0} \alpha}} \tag{2.11}
\end{equation*}
$$

via (1.1) and the fact that $c_{w_{0}}=1$. Thus,

$$
\left[X^{-\lambda_{v}}\right]=X^{-\lambda_{v}}\left[\mathcal{O}_{X_{w_{0}}}\right]=\sum_{w \in W} c_{w} e^{-w \lambda_{v}} b_{w},
$$

and if $C, M$ and $A$ are the $|W| \times|W|$ matrices given by

$$
C=\operatorname{diag}\left(c_{w}\right), \quad M=\left(e^{-w \lambda_{v}}\right), \quad \text { and } \quad A=\left(a_{z w}\right), \quad \text { where } \quad b_{w}=\sum_{z \in W} a_{z w}\left[\mathcal{O}_{X_{z}}\right],
$$

then the transition matrix between the $X^{-\lambda_{v}}$ and the $\left[\mathcal{O}_{X_{z}}\right]$ is the product $A C M$. By (2.8) and the definition of the $\tau_{i}$, the matrix $A$ has determinant 1. Using the method of Steinberg [ St ] and subtracting row $e^{-s_{\alpha} w \lambda_{v}}$ from row $e^{-w \lambda_{v}}$ in the matrix $M$ allows one to conclude that $\operatorname{det}(M)$ is divisible by

$$
\prod_{\alpha \in R^{+}}\left(1-e^{-\alpha}\right)^{|W| / 2} \quad \text { and identifying } \quad \prod_{w \in W} e^{-w \lambda_{w}}=\prod_{i=1}^{n} \prod_{s_{i} w<w} e^{-\omega_{i}}=\left(e^{-\rho}\right)^{|W| / 2}
$$

as the lowest degree term determines $\operatorname{det}(M)$ exactly. Thus,

$$
\operatorname{det}(A C M)=1 \cdot\left(\prod_{w \in W} \prod_{\alpha \in R(w)} \frac{1}{1-e^{-\alpha}}\right)\left(e^{\rho} \prod_{\alpha \in R^{+}}\left(1-e^{-\alpha}\right)\right)^{|W| / 2}=\left(e^{\rho}\right)^{|W| / 2}
$$

Since this is a unit in $R$, the transition matrix between the $\left[\mathcal{O}_{X_{w}}\right]$ and the $X^{-\lambda_{v}}$ is invertible.

Theorem 2.12. The composite map

$$
\begin{array}{rlll}
\Phi: \quad R[X] & \longrightarrow & \tilde{H}_{R} T_{w_{0}} & \hookrightarrow \\
f & \longmapsto & \tilde{H}_{R} & \longrightarrow \\
& \longmapsto T_{w_{0}} & & h
\end{array} l K_{T}(G / B)
$$

is surjective with kernel

$$
\operatorname{ker} \Phi=\left\langle f-e(f) \mid f \in R[X]^{W}\right\rangle
$$

the ideal of the ring $R[X]$ generated by the elements $f-e(f)$ for $f \in R[X]^{W}$. Hence

$$
K_{T}(G / B) \cong \frac{R[X]}{\left\langle f-e(f) \mid f \in R[X]^{W}\right\rangle}
$$

has the structure of a ring.
Proof. Since $\Phi\left(X^{\lambda}\right)=X^{\lambda} T_{w_{0}}\left[\mathcal{O}_{X_{1}}\right]=X^{\lambda}\left[\mathcal{O}_{X_{w_{0}}}\right]$, it follows from Theorem 2.9 that $\Phi$ surjective. Thus $K_{T}(G / B) \cong R[X] / \operatorname{ker} \Phi$. Let $I=\left\langle f-e(f) \mid f \in R[X]^{W}\right\rangle$. If $f \in R[X]^{W}$ then, for all $\lambda \in P$,

$$
\begin{aligned}
\Phi\left(X^{\lambda}(f-e(f))\right) & =X^{\lambda}(f-e(f)) T_{w_{0}}\left[\mathcal{O}_{X_{1}}\right]=X^{\lambda} T_{w_{0}}(f-e(f))\left[\mathcal{O}_{X_{1}}\right] \\
& =X^{\lambda} T_{w_{0}}(e(f)-e(f))\left[\mathcal{O}_{X_{1}}\right]=0,
\end{aligned}
$$

since $f-e(f) \in Z\left(\tilde{H}_{R}\right)$. Thus $I \subseteq \operatorname{ker} \Phi$. The ring $K_{T}(G / B)=R[X] / \operatorname{ker} \Phi$ is a free $R$-module of rank $|W|$ and, by Theorem 1.7, so is $R[X] / I$. Thus $\operatorname{ker} \Phi=I$.

## 3. Pieri-Chevalley formulas

Recall that both

$$
\left\{X^{\lambda} T_{w^{-1}} \mid \lambda \in P, w \in W\right\} \quad \text { and } \quad\left\{T_{z^{-1}} X^{\mu} \mid \mu \in P, z \in W\right\} \quad \text { are bases of } \tilde{H}
$$

If $c_{w, \lambda}^{\mu, z} \in \mathbb{Z}$ are the entries of the transition matrix between these two bases,

$$
\begin{equation*}
X^{\lambda} T_{w^{-1}}=\sum_{z \in W, \mu \in P} c_{w, \lambda}^{\mu, z} T_{z^{-1}} X^{\mu} \tag{3.1}
\end{equation*}
$$

then applying each side of (3.1) to $\left[\mathcal{O}_{X_{1}}\right]$ gives that

$$
\left[X^{\lambda}\right]\left[\mathcal{O}_{X_{w}}\right]=\sum_{z \in W, \mu \in P} c_{w, \lambda}^{\mu, z} e^{\mu}\left[\mathcal{O}_{X_{z}}\right], \quad \text { in } K_{T}(G / B)
$$

This is the most general form of "Pieri-Chevalley rule". The problem is to determine the coefficients $c_{w, \lambda}^{\mu, z}$.

## The path model

A path in $\mathfrak{h}^{*}$ is a piecewise linear map $p:[0,1] \rightarrow \mathfrak{h}^{*}$ such that $p(0)=0$. For each $1 \leq i \leq n$ there are root operators $e_{i}$ and $f_{i}$ (see [L3] Definitions 2.1 and 2.2) which act on the paths. If $\lambda \in P^{+}$the path model for $\lambda$ is

$$
\mathcal{T}^{\lambda}=\left\{f_{i_{1}} f_{i_{2}} \cdots f_{i_{l}} p_{\lambda}\right\},
$$

the set of all paths obtained by applying the root operators to $p_{\lambda}$, where $p_{\lambda}$ is the straight path from 0 to $\lambda$, that is, $p_{\lambda}(t)=t \lambda, 0 \leq t \leq 1$. Each path $p$ in $\mathcal{T}^{\lambda}$ is a concatenation of segments

$$
\begin{equation*}
p=p_{w_{1} \lambda}^{a_{1}} \otimes p_{w_{2} \lambda}^{a_{2}} \otimes \cdots \otimes p_{w_{r} \lambda}^{a_{r}} \quad \text { with } \quad w_{1} \geq w_{2} \geq \cdots \geq w_{r} \quad \text { and } \quad a_{1}+a_{2}+\cdots+a_{r}=1 \tag{3.2}
\end{equation*}
$$

where, for $v \in W$ and $a \in(0,1], p_{v \lambda}^{a}$ is a piece of length $a$ from the straight line path $p_{v \lambda}=v p_{\lambda}$. If $W_{\lambda}=\operatorname{Stab}(\lambda)$ then the $w_{j}$ should be viewed as cosets in $W / W_{\lambda}$ and $\geq$ denotes the order on $W / W_{\lambda}$ inherited from the Bruhat-Chevalley order on $W$. The total length of $p$ is the same as the total length of $p_{\lambda}$ which is assumed (or normalized) to be 1 . For $p \in \mathcal{T}^{\lambda}$ let

$$
\begin{aligned}
p(1) & =\sum_{i=1}^{r} a_{i} w_{i} \lambda \quad \text { be the endpoint of } p, \\
\iota(p) & =w_{1}, \quad \text { the initial direction of } p, \quad \text { and } \\
\phi(p) & =w_{r}, \quad \text { the final direction of } p .
\end{aligned}
$$

If $h \in \mathcal{T}^{\lambda}$ is such that $e_{i}(h)=0$ then $h$ is the head of its $i$-string

$$
S_{i}^{\lambda}(h)=\left\{h, f_{i} h, \ldots, f_{i}^{m} h\right\},
$$

where $m$ is the smallest positive integer such that $f_{i}^{m} h \neq 0$ and $f_{i}^{m+1} h=0$. The full path model $\mathcal{T}^{\lambda}$ is the union of its $i$-strings. The endpoints and the inital and final directions of the paths in the $i$-string $S_{i}^{\lambda}(h)$ have the following properties:

$$
\begin{array}{cl}
\left(f_{i}^{k} h\right)(1) & =h(1)-k \alpha_{i}, \quad \text { for } 0 \leq k \leq m, \\
\text { either } & \iota(h)=\iota\left(f_{i} h\right)=\cdots=\iota\left(f_{i}^{m} h\right)<s_{i} \iota(h) \\
\text { or } \quad & \iota(h)<\iota\left(f_{i} h\right)=\cdots=\iota\left(f_{i}^{m} h\right)=s_{i} \iota(h), \quad \text { and }  \tag{3.3}\\
\text { either } & s_{i} \phi\left(f_{i}^{m} h\right)<\phi(h)=\cdots=\phi\left(f_{i}^{m-1} h\right)=\phi\left(f_{i}^{m} h\right) \\
\text { or } & s_{i} \phi\left(f_{i}^{m} h\right)=\phi(h)=\cdots=\phi\left(f_{i}^{m-1} h\right)<\phi\left(f_{i}^{m} h\right) .
\end{array}
$$

The first property is [L2] Lemma 2.1a, the second is is [L1] Lemma 5.3, and the last is a result of applying [L2] Lemma 2.1e to [L1] Lemma 5.3. All of these facts are really coming from the explicit form of the action of the root operators on the paths in $\mathcal{T}^{\lambda}$ which is given in [L1] Proposition 4.2.

Let $\lambda \in P^{+}, w \in W$ and $z \in W / W_{\lambda}$, and let $p \in \mathcal{T}^{\lambda}$ be such that $\iota(p) \leq w W_{\lambda}$ and $\phi(p) \geq z$. Write $p$ in the form (3.2) and let $\tilde{w}_{1}, \ldots, \tilde{w}_{r}, \tilde{z}$ be the maximal (in Bruhat order) coset representatives of the cosets $w_{1}, \ldots, w_{r}, z$ such that

$$
\begin{equation*}
w \geq \tilde{w}_{1} \geq \tilde{w}_{2} \geq \cdots \geq \tilde{w}_{r} \geq \tilde{z} . \tag{3.4}
\end{equation*}
$$

Theorem 3.5. Recall the notation $\varepsilon_{v}$ from (1.11). Let $\lambda \in P^{+}$and let $W_{\lambda}=\operatorname{Stab}(\lambda)$. Let $w \in W$. Then, in the affine nil-Hecke algebra $\tilde{H}$,

$$
X^{\lambda} T_{w^{-1}}=\sum_{\substack{p \in \mathcal{T} \\ \iota(p) \leq w W_{\lambda}}} T_{\phi(p)^{-1}} X^{p(1)} \quad \text { and } \quad X^{\lambda} \varepsilon_{w^{-1}}=\sum_{\substack{p \in \mathcal{\lambda} \\ \iota(p)=w}} \sum_{\substack{z \in W / W_{\lambda} \\ z \leq \phi(p)}}(-1)^{\ell(w)+\ell(z)} \varepsilon_{\tilde{z}^{-1}} X^{p(1)},
$$

where, if $W_{\lambda} \neq\{1\}$ then $T_{\phi(p)^{-1}}=T_{\tilde{w}_{r}^{-1}}$ and $\varepsilon_{z^{-1}}=\varepsilon_{\tilde{z}^{-1}}$ with $\tilde{w}_{r}$ and $\tilde{z}$ as in (3.4).
Proof. (a) The proof is by induction on $\ell(w)$. Let $w=s_{i} v$ where $s_{i} v>v$. Define

$$
\mathcal{T}_{\leq w}^{\lambda}=\left\{p \in \mathcal{T}^{\lambda} \mid \iota(p) \leq w W_{\lambda}\right\} .
$$

Assume $w=s_{i} v>v$. Then the facts in (3.3) imply that
(1) $\mathcal{T}_{\leq w}^{\lambda}$ is a union of the strings $S_{i}(h)$ such that $h \in \mathcal{T}_{\leq v}^{\lambda}$, and
(2) If $h \in \mathcal{T}_{\leq v}^{\lambda}$ then either $S_{i}(h) \subseteq \mathcal{T}_{\leq v}^{\lambda}$ or $S_{i}(h) \cap \mathcal{T}_{\leq v}^{\lambda}=\{h\}$.

Using the facts in (3.3), a direct computation with the relation (1.3) establishes that, if $h \in \mathcal{T}_{\leq v}^{\lambda}$ then

$$
\begin{aligned}
\sum_{p \in S_{i}(h)} T_{\phi(p)^{-1}} X^{\eta(1)}=T_{\phi(h)^{-1} X^{h(1)} T_{i},} \quad \text { and } \\
\sum_{p \in S_{i}(h)} T_{\phi(p)^{-1}} X^{\eta(1)}=\left\{\begin{array}{ll}
T_{\phi(h)^{-1}} X^{h(1)} T_{i}, & \text { if } S_{i}(h) \subseteq \mathcal{T}_{\leq v}^{\lambda}, \\
T_{\phi(h)^{-1} X^{h(1)} T_{i},}, & \text { if } S_{i}(h) \cap \mathcal{T}_{\leq v}^{\lambda}
\end{array}=\{h\} .\right.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& X^{\lambda} T_{w^{-1}}=X^{\lambda} T_{v^{-1}} T_{i}=\left(\sum_{p \in \mathcal{T}_{\underline{v}}^{\lambda}} T_{\phi(p)^{-1} X^{p(1)}}\right) T_{i} \quad \text { (by induction) } \\
& =\sum_{\substack{h \in \mathcal{T}_{\begin{subarray}{c}{\lambda} }}} \\
{e_{i}(h)=0}\end{subarray}}\left(\sum_{\substack{ \\
S_{i}(h) \subseteq \mathcal{T}_{\underline{\imath}} \leq v}} \sum_{p \in S_{i}(h)} T_{\phi(p)^{-1} X^{p(1)}}+\sum_{S_{i}(h) \cap \mathcal{T}_{\underline{\lambda}}=\{h\}} T_{\phi(h)^{-1} X^{h(1)}}\right) T_{i} \\
& =\sum_{\substack{h \in \mathcal{T} \backslash w \\
e_{i}(h)=0}}\left(\sum_{S_{i}(h) \subseteq \mathcal{T}_{\leq v}} T_{\phi(h)^{-1}} X^{h(1)} T_{i}+\sum_{S_{i}(h) \cap \mathcal{T}_{\leq v}=\{h\}} T_{\phi(h)^{-1}} X^{h(1)}\right) T_{i} \\
& =\sum_{\substack{h \in \mathcal{T}_{\begin{subarray}{c}{2} }}} \\
{e_{i}(h)=0}\end{subarray}}\left(\sum_{S_{i}(h) \subseteq \mathcal{T}_{\leqq v}} T_{\phi(h)^{-1}} X^{h(1)} T_{i}+\sum_{S_{i}(h) \cap \mathcal{T}_{\leq v}=\{h\}} \sum_{p \in S_{i}(h)} T_{\phi(p)^{-1}} X^{p(1)}\right) \\
& =\sum_{p \in \mathcal{T}_{\underline{\prime}}} T_{\phi(p)^{-1}} X^{p(1)} .
\end{aligned}
$$

(b) The proof is similar to case (a). For $w \in W$ let

$$
\mathcal{T}_{=w}^{\lambda}=\left\{p \in \mathcal{T}^{\lambda} \mid \iota(p)=w W_{\lambda}\right\}
$$

Assume $w=s_{i} v>v$. Then the facts in (3.3) imply that
(1) $\mathcal{T}_{=w}^{\lambda}$ is a union of the strings $S_{i}(h)$ such that $h \in \mathcal{T}_{=h}^{\lambda}$, and
(2) If $h \in \mathcal{T}_{=v}^{\lambda}$ then either $S_{i}(h) \subseteq \mathcal{T}_{=v}^{\lambda}$ or $S_{i}(h) \cap \mathcal{T}_{=v}^{\lambda}=\{h\}$.

Let

$$
\begin{equation*}
\mathcal{E}_{\phi(p)}=\sum_{\substack{z \in W / W_{\lambda} \\ z \leq \phi(p)}}(-1)^{\ell(z)} \varepsilon_{\tilde{z}-1} \tag{3.6}
\end{equation*}
$$

Using (3.3), a direct computation with the relation (1.3) establishes that, if $h \in \mathcal{T}_{=v}^{\lambda}$ with $e_{i} h=0$ then

$$
\sum_{p \in S_{i}(h)} \mathcal{E}_{\phi(p)} X^{p(1)} T_{i}=0, \quad \text { and } \quad \mathcal{E}_{\phi(h)} X^{h(1)} T_{i}=-\sum_{p \in S_{i}(h)-\{h\}} \mathcal{E}_{\phi(p)} X^{p(1)}
$$

Thus

$$
\begin{aligned}
X^{\lambda} \varepsilon_{w^{-1}} & =X^{\lambda} \varepsilon_{v^{-1}} \varepsilon_{i}=(-1)^{\ell(v)}\left(\sum_{p \in \mathcal{T}_{\underline{\lambda}}} \mathcal{E}_{\phi(p)} X^{p(1)}\right) T_{i} \\
& =(-1)^{\ell(v)}\left(\sum_{S_{i}(h) \subseteq \mathcal{T}_{\underline{\lambda}}} \sum_{p \in S_{i}(h)} \mathcal{E}_{\phi(p)} X^{p(1)}+\sum_{S_{i}(h) \cap \mathcal{T}_{\underline{\lambda}}=\{h\}} \mathcal{E}_{\phi(h)} X^{h(1)}\right) T_{i} \\
& =(-1)^{\ell(v)}\left(0-\sum_{S_{i}(h) \cap \mathcal{T}_{\underline{\lambda}}=\{h\}} \sum_{p \in S_{i}(h)-\{h\}} \mathcal{E}_{\phi(p)} X^{p(1)}\right) \\
& =(-1)^{\ell(w)}\left(\sum_{p \in \mathcal{T}_{\underline{\mathcal{\lambda}}}} \mathcal{E}_{\phi(p)} X^{p(1)}\right) .
\end{aligned}
$$

Corollary 3.7. Let $\lambda, \mu \in P^{+}$and let $w \in W$. Then, in the affine nil-Hecke algebra $\tilde{H}$,

$$
\left.\begin{array}{rl}
X^{-\lambda} T_{w^{-1}} & =\sum_{\substack{p \in \mathcal{T}-w_{0} \lambda \\
\phi(p)=w w_{0}}} \sum_{z \in W / W_{-} w_{0} \lambda}^{z w_{0} \geq \iota(p)}<
\end{array}(-1)^{\ell(w)+\ell(z)} T_{\tilde{z}^{-1}} X^{p(1)} \quad \text { and }\right)
$$

Proof. The second identity is a restatement of the first with a change of variable $\mu=-w_{0} \lambda$. The first identity is obtained by applying the algebra involution

$$
\begin{array}{rlcccc}
\tilde{H} & \longrightarrow & \tilde{H} \\
T_{w} & \longmapsto & \varepsilon_{w} \\
X^{\lambda} & \longmapsto & X^{-\lambda}
\end{array} \quad \text { and the bijection } \quad \mathcal{T}^{\lambda} \quad \longrightarrow \mathcal{T}^{-w_{0} \lambda}
$$

where $p^{*}$ is the same path as $p$ except translated so that its endpoint is at the origin. Representation theoretically, this bijection corresponds to the fact that $L(\lambda)^{*} \cong L\left(-w_{0} \lambda\right)$, if $L(\lambda)$ is the simple $G$-module of highest weight $\lambda$. Note that $p^{*}(1)=-p(1), \iota\left(p^{*}\right)=\phi(p) w_{0}$, and $\phi\left(p^{*}\right)=\iota(p) w_{0}$.

Applying the identities from Theorem 3.5 and Corollary 3.7 to $\left[\mathcal{O}_{X_{1}}\right]$ yields the following product formulas in $K_{T}(G / B)$. In particular, this gives a combinatorial proof of the ( $T$-equivariant extension) of the duality theorem of Brion [Br, Theorem 4]. For $\lambda \in P$ and $w \in W$ let $\left[X^{\lambda}\right]=$ $X^{\lambda}\left[\mathcal{O}_{X_{w_{0}}}\right]=X^{\lambda} T_{w_{0}}\left[\mathcal{O}_{X_{1}}\right]$ and let $c_{\lambda, w}^{z}$ be given by

$$
\begin{equation*}
\left[X^{\lambda}\right]\left[\mathcal{O}_{X_{w}}\right]=\sum_{z \in W} c_{\lambda, w}^{z}\left[\mathcal{O}_{X_{z}}\right], \tag{3.8}
\end{equation*}
$$

Corollary 3.9. Let $\lambda \in P^{+}, w \in W$ and $W_{\lambda}=\operatorname{Stab}(\lambda)$. Then, with notation as in (3.8),

$$
\begin{gathered}
c_{\lambda, w}^{z}=\sum_{\substack{p \in \mathcal{T} \\
w W_{\lambda} \geq(p) \geq \phi(p)=z W_{\lambda}}} e^{p(1)}, \\
c_{w_{0} \lambda, w}^{z}=(-1)^{\ell(w)+\ell(z)} c_{\lambda, z w_{0}}^{w w_{0}}, \quad \text { and } \quad c_{-\lambda, w}^{z}=(-1)^{\ell(w)+\ell(z)} c_{-w_{0} \lambda, z w_{0}}^{w w_{0}} .
\end{gathered}
$$

Proposition 3.10. For $1 \leq i \leq n, \quad\left[\mathcal{O}_{X_{w_{0} s_{i}}}\right]=1-e^{w_{0} \omega_{i}}\left[X^{-\omega_{i}}\right]$.
Proof. We shall show that

$$
\begin{equation*}
X^{-\omega_{i}}\left[\mathcal{O}_{X_{w_{0}}}\right]=e^{-w_{0} \omega_{i}}\left(\left[\mathcal{O}_{X_{w_{0}}}\right]-\left[\mathcal{O}_{X_{w_{0} s_{i}}}\right]\right) \tag{3.11}
\end{equation*}
$$

and the result will follow by solving for $\left[\mathcal{O}_{X_{s_{i} w_{0}}}\right]$. Let $\omega_{j}=-w_{0} \omega_{i}$. By Corollary 3.9,

$$
c_{-\omega_{i}, w_{0}}^{z}=(-1)^{\ell\left(w_{0}\right)+\ell(z)} c_{\omega_{j}, z w_{0}}^{1}=(-1)^{\ell\left(w_{0}\right)+\ell(z)} \sum_{\substack{p \in \mathcal{T}_{j} \omega_{j} \\ z w_{0} \geq \iota(p) \geq \phi(p)=1}} e^{p(1)} .
$$

The straight line path to $\omega_{j}, p_{\omega_{j}}$, has $\iota_{z w_{0}}\left(p_{\omega_{j}}\right)=\phi_{z w_{0}}\left(\omega_{j}\right)$ and is the unique path in $\mathcal{T}^{\omega_{j}}$ which may have final direction 1. Suppose $\phi_{z w_{0}}\left(p_{\omega_{j}}\right)=1$. Then, since $s_{j}$ is the only simple reflection which is not in $\operatorname{Stab}\left(\omega_{j}\right)$, it must be that $z w_{0} \nsupseteq s_{k}$ for all $k \neq j$. Thus $z w_{0}=1$ or $z w_{0}=s_{j}$ and so $c_{-\omega_{i}, w_{0}}^{z} \neq 0$ only if $z=w_{0}$ or $z=s_{j} w_{0}=w_{0} s_{i}$. Now (3.11) follows since $p_{\omega_{j}}$ has endpoint $\omega_{j}=-w_{0} \omega_{i}$.

Corollary 3.12. Let $c_{w v}^{z}$ be as in (3.8). Then, for $1 \leq i \leq n, c_{w_{0} s_{i}, w}^{w}=-\left(e^{-\left(w \omega_{i}-w_{0} w_{i}\right)}-1\right)$, and

$$
c_{w_{0} s_{i}, w}^{z}=(-1)^{\ell(w)+\ell(z)+1} \sum_{\substack{p \mathcal{T}-w_{0} w_{i} \\ z w_{0} \geq(p) \geq \phi(p)=w w_{0}}} e^{w_{0} \omega_{i}+p(1)}, \quad \text { for } z \neq w .
$$

Proof. This follows from Proposition 3.10 and Corollary 3.9 and the fact that, in the case when $z=w$, there is a unique path $p$ with $w w_{0}=\iota(p)=\phi(p)=w w_{0}$ and endpoint $p(1)=w w_{0}\left(-w_{0} \omega_{i}\right)=$ $-w \omega_{i}$.

## 4. Converting to $H_{T}^{*}(G / B)$

The graded nil-Hecke algebra is the algebra $H_{\mathrm{gr}}$ given by generators $t_{1}, \ldots, t_{n}$ and $x_{\lambda}, \lambda \in P$, with relations

$$
\begin{equation*}
t_{i}^{2}=0, \quad \underbrace{t_{i} t_{j} t_{i} \cdots}_{m_{i j} \text { factors }}=\underbrace{t_{j} t_{i} t_{j} \cdots}_{m_{i j} \text { factors }}, \quad x_{\lambda+\mu}=x_{\lambda}+x_{\mu}, \quad \text { and } \quad x_{\lambda} t_{i}=t_{i} x_{s_{i} \lambda}+\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle . \tag{4.1}
\end{equation*}
$$

The subalgebra of $H_{\mathrm{gr}}$ generated by the $x_{\lambda}$ is the polynomial ring $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$, where $x_{i}=x_{\omega_{i}}$, and $W$ acts on $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ by

$$
w x_{\lambda}=x_{w \lambda} \quad \text { and } \quad w(f g)=(w f)(w g), \quad \text { for } w \in W, \lambda \in P, f, g \in \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right] .
$$

Then the last formula in (4.1) generalizes to

$$
f t_{i}=t_{i}\left(s_{i} f\right)+\frac{f-s_{i} f}{\alpha_{i}}, \quad \text { for } f \in \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]
$$

Let $t_{w}=t_{i_{1}} \cdots t_{i_{p}}$ for a reduced word $w=s_{i_{1}} \cdots s_{i_{p}}$ and let $\mathbb{Z} W^{*}$ be the subalgebra of $H_{\mathrm{gr}}$ spanned by the $t_{w}, w \in W$. Then

$$
\left\{x_{1}^{m_{1}} \cdots x_{n}^{m_{n}} t_{w} \mid w \in W, \quad m_{i} \in \mathbb{Z}_{\geq 0}\right\} \quad \text { and } \quad\left\{t_{w} x_{1}^{m_{1}} \cdots x_{n}^{m_{n}} \mid w \in W, \quad m_{i} \in \mathbb{Z}_{\geq 0}\right\}
$$

are bases of $H_{\mathrm{gr}}$.
Let $S=\mathbb{Z}\left[y_{1}, \ldots, y_{n}\right]$ and extend coefficients to $S$ so that $H_{\mathrm{gr}, S}=S \otimes_{\mathbb{Z}} H_{\mathrm{gr}}$ and $S\left[x_{1}, \ldots, x_{n}\right]=$ $S \otimes_{\mathbb{Z}} \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ are $S$-algebras. Define $H_{T}^{*}(G / B)$ to be the $H_{\mathrm{gr}, S}$ module

$$
\begin{equation*}
H_{T}^{*}(G / B)=S-\operatorname{span}\left\{\left[X_{w}\right] \mid w \in W\right\}, \tag{4.2}
\end{equation*}
$$

so that the $\left[X_{w}\right], w \in W$, are an $S$-basis of $K_{T}(G / B)$, with $H_{\mathrm{gr}, S}$-action given by

$$
x_{i}\left[X_{1}\right]=y_{i}\left[X_{1}\right], \quad \text { and } \quad t_{i}\left[X_{w}\right]= \begin{cases}{\left[X_{w s_{i}}\right],} & \text { if } w s_{i}>w,  \tag{4.3}\\ 0, & \text { if } w s_{i}<w,\end{cases}
$$

Let $y$ be the $S$-algebra homomorphism given by

$$
\begin{aligned}
y: S\left[x_{1}, \ldots, x_{n}\right] & \longrightarrow S \\
x_{i} & \longmapsto y_{i}
\end{aligned}
$$

so that $H_{T}^{*}(G / B) \cong H_{\mathrm{gr}, S} \otimes_{S\left[x_{1}, \ldots, x_{n}\right]} y$ as $H_{\mathrm{gr}, S \text {-modules Then, using analogous methods to the }}$ $K_{T}(G / B)$ case proves the following theorem, which gives the ring structure of $H^{*} T(G / B)$ (see also the proof of [KR, Prop. 2.9] for the same argument with (non-nil) graded Hecke algebras).

Theorem 4.4. The composite map

$$
\begin{aligned}
\Phi: \quad S\left[x_{1}, \ldots, x_{n}\right] & \longrightarrow H_{\mathrm{gr}, S} t_{w_{0}} \\
f & \longmapsto
\end{aligned} H_{\mathrm{gr}, S} \quad \longrightarrow \quad H_{T}^{*}(G / B)
$$

is surjective with kernel

$$
\operatorname{ker} \Phi=\left\langle f-y(f) \mid f \in S\left[x_{1}, \ldots, x_{n}\right]^{W}\right\rangle
$$

the ideal of the ring $\left.S_{[ } x_{1}, \ldots, x_{n}\right]$ generated by the elements $f-y(f)$ for $f \in S\left[x_{1}, \ldots, x_{n}\right]^{W}$. Hence

$$
H_{T}^{*}(G / B) \cong \frac{\mathbb{Z}\left[y_{1}, \ldots, y_{n}, x_{1}, \ldots, x_{n}\right]}{\left\langle f-y(f) \mid f \in S\left[x_{1}, \ldots, x_{n}\right]^{W}\right\rangle}
$$

has the structure of a ring.
As a vector space $H_{\mathrm{gr}}=\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right] \otimes \mathbb{Z} W_{\mathrm{gr}}$. Let $\widehat{H_{\mathrm{gr}}}=\mathbb{Q}\left[\left[x_{1}, \ldots, x_{n}\right]\right] \otimes \mathbb{Q} W_{g r}$ with multiplication determined by the relations in (4.1). Then $\widehat{H_{\mathrm{gr}}}$ is a completion of $H_{\mathrm{gr}}$ (this simply allows us to write infinite sums) and the elements of $\widehat{H_{\mathrm{gr}}}$ given by

$$
\begin{equation*}
\operatorname{ch}\left(X^{\lambda}\right)=\sum_{r \geq 0} \frac{1}{r!} x_{\lambda}^{r} \quad \text { and } \quad \operatorname{ch}\left(T_{i}\right)=t_{i} \cdot \frac{x_{\alpha_{i}}}{1-\operatorname{ch}\left(X^{\alpha_{i}}\right)} \tag{4.5}
\end{equation*}
$$

satisfy the relations of $\tilde{H}$ and thus ch extends to a ring homomorphism ch: $\tilde{H} \longrightarrow \widehat{H_{\mathrm{gr}}}$. It is this fact that really makes possible the transfer from $K$-theory to cohomomology possible. Though is it not difficult to check that the elements in (3.5) satisfy the defining relations of $\tilde{H}$ it is helpful to realize that these formulas come from geometry. As explained in [PR2], the action of $T_{i}$ on $K_{T}(G / B)$ and the action of $t_{i}$ on $H_{T}^{*}(G / B)$ are, respectively, the push-pull operators $\pi_{i}^{*}\left(\pi_{i}\right)_{!}$and $\pi_{i}^{*}\left(\pi_{i}\right)_{*}$, where if $P_{i}$ is a minimal parabolic subgroup of $G$ then $\pi_{i}: G / P_{i} \rightarrow G / B$ is the natural surjection. Then the first formula in (3.5) is the definition of the Chern character, and the second formula is the Grothedieck-Riemann-Roch theorem applied to the map $\pi_{i}$. The factor $\alpha_{i} /\left(1-\operatorname{ch}\left(X^{\alpha_{i}}\right)\right)$ is the Todd class of the bundle of tangents along the fibers of $\pi_{i}$ (see [Hz, page 91]).

Then $\widehat{H_{T}^{*}}(G / B)_{\mathbb{Q}}=\mathbb{Q}\left[\left[y_{1}, \ldots, y_{n}\right]\right] \otimes_{\mathbb{Z}\left[y_{1}, \ldots, y_{n}\right]} H_{T}^{*}(G / B)$ is the appropriate completion of $H_{T}^{*}(G / B)$ to use to transfer the ring homomorphism ch: $\tilde{H}_{R} \rightarrow \widehat{H_{\mathrm{gr}}}$ to a ring homomorphism

$$
\begin{equation*}
\text { ch: } K_{T}(G / B) \longrightarrow \widehat{H_{T}^{*}}(G / B)_{\mathbb{Q}} \quad \text { by setting } \quad \operatorname{ch}\left(h\left[\mathcal{O}_{X_{1}}\right]\right)=\operatorname{ch}(h)\left[X_{1}\right], \quad \text { for } h \in \tilde{H}_{R} \tag{4.6}
\end{equation*}
$$

The ring $\widehat{H_{T}^{*}}(G / B)_{\mathbb{Q}}$ is a graded ring with

$$
\begin{equation*}
\operatorname{deg}\left(y_{i}\right)=1 \quad \text { and } \quad \operatorname{deg}\left(\left[X_{w}\right]\right)=\ell\left(w_{0}\right)-\ell(w) \tag{4.7}
\end{equation*}
$$

and, for $w \in W, \quad \operatorname{ch}\left(\left[\mathcal{O}_{X_{w}}\right]\right)=\left[X_{w}\right]+$ higher degree terms.
In summary, if $e_{i}=e^{\omega_{i}}, X_{i}=X^{\omega_{i}}, y_{i}=y_{\omega_{i}}, x_{i}=x_{\omega_{i}}$,

$$
\begin{aligned}
R[X] & =\mathbb{Z}\left[e_{1}^{ \pm 1}, \ldots, e_{n}^{ \pm 1}, X_{1}^{ \pm 1}, \ldots, X_{n}^{ \pm 1}\right], \quad \text { and } \quad \widehat{S}\left[x_{1}, \ldots, x_{n}\right]=\mathbb{Q}\left[\left[y_{1}, \ldots, y_{n}\right]\right]\left[x_{1}, \ldots, x_{n}\right] \\
\mathbb{Z}[X] & =\mathbb{Z}\left[X_{1}^{ \pm 1}, \ldots, X_{n}^{ \pm 1}\right],
\end{aligned}
$$

then there is a commutative diagram of ring homomorphisms

$$
\begin{aligned}
K_{T}(G / B) & =\frac{R[X]}{\left\langle f-e(f) \mid f \in R[X]^{W}\right\rangle} \quad \xrightarrow{\mathrm{ch}} \quad H_{T}^{*}(G / B)_{\mathbb{Q}}
\end{aligned}=\frac{\widehat{S}\left[x_{1}, \ldots, x_{n}\right]}{\left\langle f-y(f) \mid f \in \widehat{S}\left[x_{1}, \ldots, x_{n}\right]^{W}\right\rangle} .
$$

## 5. Rank two and a positivity conjecture

In this section we will give explicit formulas for the rank two root systems. The data supports the following positivity conjecture which generalizes the theorems of Brion [Br, formula before Theorem 1] and Graham [Gr, Corollary 4.1].

Conjecture 5.1. For $\beta \in R^{+}$let $y_{\beta}=e^{-\beta}$ and $a_{\beta}=e^{-\beta}-1$ and let $d(w)=\ell\left(w_{0}\right)-\ell(w)$ for $w \in W$. Let $c_{w v}^{z}$ be the structure constants of $K_{T}(G / B)$ with respect to the basis $\left\{\left[\mathcal{O}_{X_{w}}\right] \mid w \in W\right\}$ as defined in (0.1). Then

$$
c_{w v}^{z}=(-1)^{d(w)+d(v)-d(z)} f(\alpha, y), \quad \text { where } \quad f(\alpha, y) \in \mathbb{Z}_{\geq 0}\left[\alpha_{\beta}, y_{\beta} \mid \beta \in R^{+}\right]
$$

that is, $f(\alpha, y)$ is a polynomial in the variables $\alpha_{\beta}$ and $y_{\beta}, \beta \in R^{+}$, which has nonnegative integral coefficients.

In the following, for brevity, use the following notations:

$$
\begin{array}{llll}
\text { in } K_{T}(G / B), & {[w]=\left[\mathcal{O}_{X_{w}}\right],} & \alpha_{r s}=e^{-\left(r \alpha_{1}+s \alpha_{2}\right)}-1, & \text { and } \\
\begin{array}{lll}
\text { in } K(G / B), & {[w]=\left[\mathcal{O}_{w}\right],} & \alpha_{r s}=0, \\
\text { in } H_{T}^{*}(G / B), & {[w]=\left[X_{w}\right],} & \alpha_{r s}=r \alpha_{1}+s \alpha_{2},
\end{array} & \text { and } & y_{r s}=1, \\
\text { in } H^{*}(G / B), & {[w]=\left[X_{w}+s \alpha_{2}\right)} \\
\text { ind } & \alpha_{r s}=0, & y_{r s}=1, \\
\text { and } & y_{r s}=1,
\end{array}
$$

and in $H_{T}^{*}(G / B)$ and in $H^{*}(G / B)$ the terms in $\}$ brackets do not appear.
Type $A_{2}$. For the root system $R$ of type $A_{2}$

$$
\begin{array}{llll}
\alpha_{1}=-\omega_{1}+2 \omega_{2}, & \lambda_{1}=\rho, & \lambda_{s_{1}}=\omega_{2}=\frac{1}{3} \alpha_{1}+\frac{2}{3} \alpha_{2}, & \lambda_{s_{2} s_{1}}=s_{2} \omega_{2}=\frac{1}{3} \alpha_{1}-\frac{1}{3} \alpha_{2}, \\
\alpha_{2}=2 \omega_{1}-\omega_{2}, & \lambda_{w_{0}}=0, & \lambda_{s_{2}}=\omega_{1}=\frac{2}{3} \alpha_{1}+\frac{1}{3} \alpha_{2}, & \lambda_{s_{1} s_{2}}=s_{1} \omega_{1}=-\frac{1}{3} \alpha_{1}+\frac{1}{3} \alpha_{2} .
\end{array}
$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$
\begin{array}{ll}
{\left[s_{1} s_{2} s_{1}\right]=1,} & {[1]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2}\right],} \\
{\left[s_{2} s_{1}\right]=1-e^{-\omega_{1}} X^{-\omega_{2}},} & {\left[s_{1} s_{2}\right]=1-e^{-\omega_{2}} X^{-\omega_{1}}} \\
{\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2} s_{1}\right],} & {\left[s_{2}\right]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1} s_{2}\right],}
\end{array}
$$

and

$$
\begin{aligned}
{\left[s_{1} s_{2} s_{1}\right] } & =1, \quad\left[s_{1} s_{2}\right]=1-e^{-\omega_{2}} X^{-\omega_{1}}, \quad\left[s_{2} s_{1}\right]=1-e^{-\omega_{1}} X^{-\omega_{2}} \\
{\left[s_{1}\right] } & =1-e^{-\omega_{2}} X^{-s_{1} \omega_{1}}-e^{-\omega_{2}} X^{-\omega_{1}}+e^{-2 \omega_{2}} X^{-\omega_{2}} \\
{\left[s_{2}\right] } & =1-e^{-\omega_{1}} X^{-s_{2} \omega_{2}}-e^{-\omega_{1}} X^{-\omega_{2}}+e^{-2 \omega_{1}} X^{-\omega_{1}} \\
{[1] } & =1-e^{-\omega_{2}} X^{-s_{1} \omega_{1}}-e^{-\omega_{1}} X^{-s_{2} \omega_{2}}+e^{-2 \omega_{1}} X^{-\omega_{1}}+e^{-2 \omega_{2}} X^{-\omega_{2}}-e^{-\rho} X^{-\rho} .
\end{aligned}
$$

The multiplication of the Schubert classes is given by

$$
\begin{aligned}
{[1]^{2} } & =-\alpha_{10} \alpha_{01} \alpha_{11}[1], & {\left[s_{1}\right]^{2} } & =\alpha_{01} \alpha_{11}\left[s_{1}\right], \\
{[1]\left[s_{1}\right] } & =\alpha_{01} \alpha_{11}[1], & {\left[s_{1}\right]\left[s_{2}\right] } & =-\alpha_{11}[1], \\
{[1]\left[s_{2}\right] } & =\alpha_{10} \alpha_{11}[1], & {\left[s_{1}\right]\left[s_{1} s_{2}\right] } & =y_{01}[1]-\alpha_{01}\left[s_{1}\right], \\
{[1]\left[s_{1} s_{2}\right] } & =-\alpha_{11}[1], & {\left[s_{1}\right]\left[s_{2} s_{1}\right] } & =-\alpha_{11}\left[s_{1}\right], \\
{[1]\left[s_{2} s_{1}\right] } & =-\alpha_{11}[1], & &
\end{aligned}
$$

$$
\begin{array}{rlrl}
{\left[s_{1} s_{2}\right]^{2}} & =y_{01}\left[s_{2}\right]-\alpha_{01}\left[s_{1} s_{2}\right], & {\left[s_{2} s_{1}\right]^{2}=y_{10}\left[s_{1}\right]-\alpha_{10}\left[s_{2} s_{1}\right] .} \\
{\left[s_{1} s_{2}\right]\left[s_{2} s_{1}\right]} & =\{-[1]\}+\left[s_{1}\right]+\left[s_{2}\right], & &
\end{array}
$$

Type $B_{2}$. For the root system $R$ of type $B_{2}$

$$
\begin{array}{rlrl}
\alpha_{1}=2 \omega_{1}-\omega_{2}, & \lambda_{1}=\rho=2 \alpha_{1}+\frac{3}{2} \alpha_{2}, & \lambda_{s_{1}}=\omega_{2}=\alpha_{1}+\alpha_{2}, \\
\alpha_{2}=-2 \omega_{1}+2 \omega_{2}, & \lambda_{w_{0}}=0, & \lambda_{s_{2}}=\omega_{1}=\alpha_{1}+\frac{1}{2} \alpha_{2}, \\
\lambda_{s_{2} s_{1}}=s_{2} \omega_{2}=\alpha_{1}, & \lambda_{s_{1} s_{2} s_{1}}=s_{1} s_{2} \omega_{2}=-\alpha_{1}, \\
\lambda_{s_{1} s_{2}}=s_{1} \omega_{1}=\frac{1}{2} \alpha_{2}, & \lambda_{s_{2} s_{1} s_{2}}=s_{2} s_{1} \omega_{1}=-\frac{1}{2} \alpha_{2} .
\end{array}
$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$
\begin{array}{ll}
{\left[s_{1} s_{2} s_{1} s_{2}\right]=1,} & {[1]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2}\right],} \\
{\left[s_{1} s_{2} s_{1}\right]=1-e^{-\omega_{2}} X^{-\omega_{2}},} & {\left[s_{2} s_{1} s_{2}\right]=1-e^{-\omega_{1}} X^{-\omega_{1}},} \\
{\left[s_{2} s_{1}\right]=\left(1-e^{-\omega_{1}} X^{-s_{1} \omega_{1}}\right)\left[s_{2} s_{1} s_{2}\right],} & {\left[s_{1} s_{2}\right]=\left(1-e^{s_{2} s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{2} s_{1} s_{2}\right],} \\
{\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2} s_{1}\right],} & {\left[s_{2}\right]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1} s_{2}\right],}
\end{array}
$$

and

$$
\begin{aligned}
& {\left[s_{1} s_{2} s_{1} s_{2}\right]=} 1, \quad\left[s_{1} s_{2} s_{1}\right]=1-e^{-\omega_{2}} X^{-\omega_{2}}, \quad\left[s_{2} s_{1} s_{2}\right]=1-e^{-\omega_{1}} X^{-\omega_{1}}, \\
& {\left[s_{1} s_{2}\right]=}\left(1-e^{-\omega_{2}}\right)-e^{-\omega_{2}} X^{-\omega_{2}}-e^{-\omega_{2}} X^{-s_{2} \omega_{2}}+\left(e^{-\rho}+e^{-s_{1} \rho}\right) X^{-\omega_{1}}, \\
& {\left[s_{2} s_{1}\right]=} 1-e^{-\omega_{1}} X^{-\omega_{1}}-e^{-\omega_{1}} X^{-s_{1} \omega_{1}}+e^{-2 \omega_{1}} X^{-\omega_{2}}, \\
& {\left[s_{1}\right]=}\left(1-e^{-\omega_{2}}\right)+\left(e^{-\rho}+e^{-s_{1} \rho}\right) X^{-s_{1} \omega_{1}}+\left(e^{-\rho}+e^{-s_{1} \rho}\right) X^{-\omega_{1}} \\
& \quad-e^{-\omega_{2}} X^{-s_{1} s_{2} \omega_{2}}-e^{-\omega_{2}} X^{-s_{2} \omega_{2}}-\left(e^{-2 \omega_{2}}+e^{-\omega_{2}}\right) X^{-\omega_{2}}, \\
& {\left[s_{2}\right]=}\left(1+e^{-2 \omega_{1}}\right)+e^{-2 \omega_{1}} X^{-s_{2} \omega_{2}}+e^{-2 \omega_{1}} X^{-\omega_{2}} \\
& \quad-e^{-\omega_{1}} X^{-s_{2} s_{1} \omega_{1}}-e^{-\omega_{1}} X^{-s_{1} \omega_{1}}-\left(e^{-3 \omega_{1}}+e^{-\omega_{1}}\right) X^{-\omega_{1}}, \\
& {[1]=\left(1+e^{-2 \omega_{1}}\right)-e^{-\omega_{1}} X^{-s_{2} s_{1} \omega_{1}}+\left(e^{-\rho}+e^{-s_{1} \rho}\right) X^{-s_{1} \omega_{1}}-\left(e^{-3 \omega_{1}}+e^{-\omega_{1}}\right) X^{-\omega_{1}} } \\
& \quad-e^{-\omega_{2}} X^{-s_{1} s_{2} \omega_{2}}+e^{-2 \omega_{1}} X^{-s_{2} \omega_{2}}-\left(e^{-2 \omega_{2}}+e^{-\omega_{2}}\right) X^{-\omega_{2}}+e^{-\rho} X^{-\rho} .
\end{aligned}
$$

The multiplication of the Schubert classes is given by

$$
\begin{array}{rlrl}
{[1]^{2}} & =\alpha_{10} \alpha_{01} \alpha_{11} \alpha_{21}[1], & {\left[s_{1} s_{2} s_{1}\right]^{2}=\left\{-y_{11}\left[s_{1}\right]\right\}+\left(y_{01}+y_{11}\right)\left[s_{2} s_{1}\right]-\alpha_{01}\left[s_{1} s_{2} s_{1}\right],} \\
{[1]\left[s_{1}\right]} & =-\alpha_{01} \alpha_{11} \alpha_{21}[1], & {\left[s_{1} s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2}\right]=\left\{[1]-\left[s_{1}\right]-\left[s_{2}\right]\right\}+\left[s_{1} s_{2}\right]+\left[s_{2} s_{1}\right],} \\
{[1]\left[s_{2}\right]} & =-\alpha_{10} \alpha_{11} \alpha_{21}[1], & & {\left[s_{2} s_{1} s_{2}\right]^{2}=y_{10}\left[s_{1} s_{2}\right]-\alpha_{10}\left[s_{2} s_{1} s_{2}\right],} \\
{[1]\left[s_{1} s_{2}\right]} & =\alpha_{11} \alpha_{21}[1], & {\left[s_{2} s_{1}\right]^{2}=-\alpha_{21} y_{10}\left[s_{1}\right]+\alpha_{10} \alpha_{21}\left[s_{2} s_{1}\right],} \\
{[1]\left[s_{2} s_{1}\right]} & =\alpha_{11} \alpha_{21}[1], & & \\
{[1]\left[s_{1} s_{2} s_{1}\right]} & =-\alpha_{11}\left(1+y_{11}\right)[1], & {\left[s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1}\right]=y_{21}\left[s_{1}\right]-\alpha_{21}\left[s_{2} s_{1}\right],} \\
{[1]\left[s_{2} s_{1} s_{2}\right]} & =-\alpha_{21}[1], & {\left[s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2}\right]} & =\left\{-y_{10}[1]\right\}+y_{10}\left[s_{1}\right]+y_{10}\left[s_{2}\right]-\alpha_{10}\left[s_{2} s_{1}\right], \\
{\left[s_{1}\right]^{2}} & =-\alpha_{01} \alpha_{11} \alpha_{21}\left[s_{1}\right], & {\left[s_{2}\right]^{2}=-\alpha_{10} \alpha_{11} \alpha_{21}\left[s_{2}\right],} \\
{\left[s_{1}\right]\left[s_{2}\right]} & =\alpha_{11} \alpha_{21}[1], & {\left[s_{2}\right]\left[s_{1} s_{2}\right]} & =\alpha_{11} \alpha_{21}\left[s_{2}\right], \\
{\left[s_{1}\right]\left[s_{1} s_{2}\right]} & =-\alpha_{11}\left(y_{01}+y_{11}\right)[1]+\alpha_{01} \alpha_{11}\left[s_{1}\right], & {\left[s_{2}\right]\left[s_{2} s_{1}\right]} & =-\alpha_{21} y_{10}[1]+\alpha_{10} \alpha_{21}\left[s_{2}\right], \\
{\left[s_{1}\right]\left[s_{2} s_{1}\right]} & =\alpha_{11} \alpha_{21}\left[s_{1}\right], & {\left[s_{2}\right]\left[s_{1} s_{2} s_{1}\right]} & =y_{21}[1]-\alpha_{21}\left[s_{2}\right], \\
{\left[s_{1}\right]\left[s_{1} s_{2} s_{1}\right]} & =-\alpha_{11}\left(1+y_{11}\right)\left[s_{1}\right], & {\left[s_{2}\right]\left[s_{2} s_{1} s_{2}\right]} & =-\alpha_{21}\left[s_{2}\right],
\end{array}
$$

$$
\begin{aligned}
{\left[s_{1} s_{2}\right]^{2} } & =-\alpha_{11}\left(y_{01}+y_{11}\right)\left[s_{2}\right]+\alpha_{01} \alpha_{11}\left[s_{1} s_{2}\right], \\
{\left[s_{1} s_{2}\right]\left[s_{2} s_{1}\right] } & =\left(\left\{\alpha_{11}\right\}+y_{21}\right)[1]-\alpha_{11}\left[s_{1}\right]-\alpha_{21}\left[s_{2}\right], \\
{\left[s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1}\right] } & =\left\{-\left(y_{01}+y_{11}\right)[1]\right\}+y_{01}\left[s_{1}\right]+\left(y_{11}+y_{12}\right)\left[s_{2}\right]-\alpha_{01}\left[s_{1} s_{2}\right], \\
{\left[s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2}\right] } & =y_{11}\left[s_{2}\right]-\alpha_{11}\left[s_{1} s_{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
{\left[s_{2} s_{1}\right]^{2} } & =-\alpha_{21} y_{10}\left[s_{1}\right]+\alpha_{10} \alpha_{21}\left[s_{2} s_{1}\right], \\
{\left[s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1}\right] } & =y_{21}\left[s_{1}\right]-\alpha_{21}\left[s_{2} s_{1}\right], \\
{\left[s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2}\right] } & =\left\{-y_{10}[1]\right\}+y_{10}\left[s_{1}\right]+y_{10}\left[s_{2}\right]-\alpha_{10}\left[s_{2} s_{1}\right],
\end{aligned}
$$

Type $G_{2}$. For the root system $R$ of type $G_{2}$

$$
\begin{array}{ll}
\lambda_{1}=\rho=5 \alpha+3 \alpha_{2}, & \lambda_{s_{1} s_{2} s_{1}}=s_{1} s_{2} \omega_{2}=\alpha_{2}, \\
\lambda_{s_{1}}=\omega_{2}=3 \alpha_{1}+2 \alpha_{2}, & \lambda_{s_{2} s_{1} s_{2} s_{2}}, s_{2} s_{1} s_{2} \omega_{2}=-\alpha_{2}, \\
\lambda_{s_{2}}=\omega_{1}=2 \alpha_{1}+\alpha_{2}, & \lambda_{s_{1} s_{2} s_{1} s_{2}}, s_{1} s_{2} s_{1} \omega_{1}=-\alpha_{1}, \\
\lambda_{s_{2} s_{1}}=s_{2} \omega_{2} 3 \alpha_{1}+\alpha_{2}, & \lambda_{s_{1} s_{2} s_{1} s_{2} s_{1}} s_{1} s_{2} s_{1} s_{2} \omega_{2}=-3 \alpha_{1}-\alpha_{2}, \\
\lambda_{s_{1} s_{2}}=s_{1} \omega_{1}=\alpha_{1}+\alpha_{2}, & \lambda_{s_{2} s_{1} s_{2} s_{1} s_{2}}=s_{2} s_{1} s_{2} s_{1} \omega_{1}=-\alpha_{1}-\alpha_{2}, \\
\lambda_{s_{2} s_{1} s_{2}}=s_{2} s_{1} \omega_{1}=\alpha_{1}, & \lambda_{w_{0}}=0 .
\end{array}
$$

Formulas for the Schubert classes in terms of homogeneous line bundles can be given by

$$
\begin{array}{ll}
{\left[s_{1} s_{2} s_{1} s_{2} s_{1} s_{2}\right]=1,} & {[1]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2}\right],} \\
{\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=1-e^{-\omega_{2}} X^{-\omega_{2}},} & {\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=1-e^{-\omega_{1}} X^{-\omega_{1}},} \\
{\left[s_{2} s_{1} s_{2} s_{1}\right]=\left(1-e^{-\omega_{1}} X^{-s_{1} \omega_{1}}\right)\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right],} & {\left[s_{1} s_{2} s_{1} s_{2}\right]\left(1-e^{-s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right],} \\
{\left[s_{1} s_{2} s_{1}\right]=\text { see below, }} & {\left[s_{2} s_{1} s_{2}\right]=\frac{e^{-s_{2} s_{2} \omega_{1}} X^{-\omega_{1}}}{1+X^{-\omega_{1}}}\left[s_{1} s_{2} s_{1} s_{2}\right],} \\
{\left[s_{2} s_{1}\right]=\left(1-e^{-\omega_{1}} X^{-s_{1} s_{2} s_{1} \omega_{1}}\right)\left[s_{2} s_{1} s_{2}\right],} & {\left[s_{1} s_{2}\right]\left(1-e^{s_{2} s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1} s_{2}\right],} \\
{\left[s_{1}\right]=\left(1-e^{s_{2} \omega_{2}} X^{-\omega_{2}}\right)\left[s_{2} s_{1}\right],} & {\left[s_{2}\right]=\left(1-e^{s_{1} \omega_{1}} X^{-\omega_{1}}\right)\left[s_{1} s_{2}\right],}
\end{array}
$$

$$
\left[s_{1} s_{2} s_{1}\right]=\frac{\left(1-e^{-\alpha_{2}} X^{-\omega_{2}}\right)\left[s_{2} s_{1} s_{2} s_{1}\right]+e^{-\alpha_{2}}\left(1+e^{\omega_{1}} X^{-\omega_{2}}\right)\left[s_{2} s_{1}\right]}{1+e^{-\alpha_{2}}},
$$

and

The multiplication of the Schubert classes is given by

$$
\begin{aligned}
{[1]^{2} } & =\alpha_{10} \alpha_{01} \alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1], \\
{[1]\left[s_{1}\right] } & =-\alpha_{01} \alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1], \\
{[1]\left[s_{2}\right] } & =-\alpha_{10} \alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1], \\
{[1]\left[s_{1} s_{2}\right] } & =\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1], \\
{[1]\left[s_{2} s_{1}\right] } & =\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1] \\
{[1]\left[s_{1} s_{2} s_{1}\right] } & =-\alpha_{11} \alpha_{21} \alpha_{32}\left(1+y_{11}+y_{21}\right)[1],
\end{aligned}
$$

$$
[1]\left[s_{2} s_{1} s_{2}\right]=-\alpha_{21} \alpha_{31} \alpha_{32}[1]
$$

$$
[1]\left[s_{1} s_{2} s_{1} s_{2}\right]=\alpha_{21} \alpha_{32}\left(1+y_{21}\right)[1]
$$

$$
[1]\left[s_{2} s_{1} s_{2} s_{1}\right]=\alpha_{21} \alpha_{32}\left(1+y_{21}\right)[1]
$$

$$
[1]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=-\alpha_{32}\left(1+y_{32}\right)[1]
$$

$$
[1]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=-\alpha_{21}\left(1+y_{21}\right)[1]
$$

$$
\begin{aligned}
& {\left[w_{0}\right]=1, \quad\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=1-y_{21} X^{-\omega_{1}}, \quad\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=1-y_{32} X^{-\omega_{2}},} \\
& {\left[s_{2} s_{1} s_{2} s_{1}\right]=1-y_{21} X^{-\omega_{1}}-y_{21} X^{-s_{1} \omega_{1}}+y_{42} X^{-\omega_{2}} \text {, }} \\
& {\left[s_{1} s_{2} s_{1} s_{2}\right]=\left(1-y_{32}\right)+\left(y_{22}+y_{42}+y_{43}+y_{53}\right) X^{-\omega_{1}}-y_{32} X^{-s_{1} \omega_{1}}-y_{32} X^{-s_{2} s_{1} \omega_{1}}} \\
& -y_{32} X^{-\omega_{2}}-y_{32} X^{-s_{2} \omega_{2}}, \\
& {\left[s_{2} s_{1} s_{2}\right]=\left(1-y_{21}+y_{42}\right)+\left(y_{42}-y_{21}-y_{52}-y_{53}-y_{63}\right) X^{-\omega_{1}}+\left(y_{42}-y_{21}\right) X^{-s_{1} \omega_{1}}} \\
& +\left(y_{42}-y_{21}\right) X^{-s_{2} s_{1} \omega_{1}}+y_{42} X^{-\omega_{2}}+y_{42} X^{-s_{2} \omega_{2}}, \\
& {\left[s_{1} s_{2} s_{1}\right]=\left(1-2 y_{32}\right)+\left(y_{22}+y_{42}+y_{43}+y_{53}\right) X^{-\omega_{1}}+\left(y_{22}+y_{42}+y_{43}+y_{53}\right) X^{-s_{1} \omega_{1}}} \\
& -y_{32} X^{-s_{2} s_{1} \omega_{1}}-y_{32} X^{-s_{1} s_{2} s_{1} \omega_{1}} \\
& -\left(y_{32}+y_{43}+y_{53}\right) X^{-\omega_{2}}-y_{32} X^{-s_{2} \omega_{2}}-y_{32} X^{-s_{1} s_{2} \omega_{2}}, \\
& {\left[s_{2} s_{1}\right]=\left(1-y_{21}+2 y_{42}\right)+\left(y_{42}-y_{21}-y_{52}-y_{53}-y_{63}\right) X^{-\omega_{1}}} \\
& +\left(y_{42}-y_{21}-y_{32}-y_{53}-y_{63}\right) X^{-s_{1} \omega_{1}}+\left(y_{42}-y_{21}\right) X^{-s_{2} s_{1} \omega_{1}} \\
& +\left(y_{42}-y_{21}\right) X^{-s_{1} s_{2} s_{1} \omega_{1}}+\left(y_{42}+y_{63}\right) X^{-\omega_{2}}+y_{42} X^{-s_{2} \omega_{2}}+y_{42} X^{-s_{1} s_{2} \omega_{2}}, \\
& {\left[s_{1} s_{2}\right]=1-y_{11}-y_{21}-y_{32}-y_{43}-y_{53}+\left(y_{22}+y_{32}\right)\left(1+y_{10}+y_{20}\right) X^{-\omega_{1}}} \\
& +\left(y_{22}+y_{32}+y_{42}\right) X^{-s_{1} \omega_{1}}+\left(y_{22}+y_{32}+y_{42}\right) X^{-s_{2} s_{1} \omega_{1}} \\
& -\left(y_{32}+y_{43}+y_{53}\right) X^{-\omega_{2}}-\left(y_{32}+y_{43}+y_{53}\right) X^{-s_{2} \omega_{2}}-y_{32} X^{-s_{1} s_{2} \omega_{2}}-y_{32} X^{-s_{2} s_{1} s_{2} \omega_{2}}, \\
& {\left[s_{2}\right]=\left(1+y_{31}+y_{32}+2 y_{42}+y_{63}\right)-\left(y_{21}+y_{52}+y_{53}+y_{84}\right) X^{-\omega_{1}}-\left(y_{21}+y_{52}+y_{53}\right) X^{-s_{1} \omega_{1}}} \\
& -\left(y_{21}+y_{52}+y_{53}\right) X^{-s_{2} s_{1} \omega_{1}}-y_{21} X^{-s_{1} s_{2} s_{1} \omega_{1}}-y_{21} X^{-s_{2} s_{1} s_{2} s_{1} \omega_{1}} \\
& +\left(y_{42}+y_{63}\right) X^{-\omega_{2}}+\left(y_{42}+y_{63}\right) X^{-s_{2} \omega_{2}}+y_{42} X^{-s_{1} s_{2} \omega_{2}}+y_{42} X^{-s_{2} s_{1} s_{2} \omega_{2}}, \\
& {\left[s_{1}\right]=1-\left(y_{11}+y_{21}+y_{32}+2 y_{43}+2 y_{53}\right)+\left(y_{22}+y_{54}\right)\left(1+y_{10}+y_{20}\right) X^{-\omega_{1}}} \\
& +\left(y_{22}+y_{54}\right)\left(1+y_{10}+y_{20}\right) X^{-s_{1} \omega_{1}}+\left(y_{22}+y_{32}+y_{42}\right) X^{-s_{2} s_{1} \omega_{1}} \\
& +\left(y_{22}+y_{32}+y_{42}\right) X^{-s_{1} s_{2} s_{1} \omega_{1}}-\left(y_{32}+y_{43}+y_{53}+y_{64}\right) X^{-\omega_{2}}-\left(y_{32}+y_{43}+y_{53}\right) X^{-s_{2} \omega_{2}} \\
& -\left(y_{32}+y_{43}+y_{53}\right) X^{-s_{1} s_{2} \omega_{2}}-y_{32} X^{-s_{2} s_{1} s_{2} \omega_{2}}-y_{32} X^{-s_{1} s_{2} s_{1} s_{2} \omega_{2}}, \\
& {[1]=\left(1+y_{31}+y_{42}+y_{63}-y_{53}-y_{43}\right)-y_{21}\left(1+y_{32}\right)^{2} X^{-\omega_{1}}} \\
& +y_{22}\left(1+y_{10}+y_{20}\right)\left(1+y_{21}+y_{31}\right) X^{-s_{1} \omega_{1}}-\left(y_{21}+y_{52}+y_{53}\right) X^{-s_{2} s_{1} \omega_{1}} \\
& +y_{22} X^{-s_{1} s_{2} s_{1} \omega_{1}}-y_{21} X^{-s_{2} s_{1} s_{2} s_{1} \omega_{1}}-y_{32}\left(1+y_{11}\right)\left(1+y_{21}\right) X^{-\omega_{2}}+\left(y_{42}+y_{63}\right) X^{-s_{2} \omega_{2}} \\
& -\left(y_{32}+y_{43}+y_{53}\right) X^{-s_{1} s_{2} \omega_{2}}+y_{42} X^{-s_{2} s_{1} s_{2} \omega_{2}}-y_{32} X^{-s_{1} s_{2} s_{1} s_{2} \omega_{2}}+y_{53} X^{-\rho} .
\end{aligned}
$$

$$
\begin{aligned}
{\left[s_{1}\right]^{2} } & =-\alpha_{01} \alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{2}\right] } & =\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}[1] \\
{\left[s_{1}\right]\left[s_{1} s_{2}\right] } & =-\alpha_{11} \alpha_{21} \alpha_{32}\left(y_{01}+y_{11}+y_{21}\right)[1]+\alpha_{01} \alpha_{11} \alpha_{21} \alpha_{32}\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{2} s_{1}\right] } & =\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{1} s_{2} s_{1}\right] } & =-\alpha_{11} \alpha_{21} \alpha_{32}\left(1+y_{11}+y_{21}\right)\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{2} s_{1} s_{2}\right] } & =\alpha_{21} \alpha_{32}\left(y_{11}+y_{21}\right)[1]-\alpha_{11} \alpha_{21} \alpha_{32}\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2}\right] } & =-\alpha_{32}\left(y_{22}+y_{32}\right)[1]+\alpha_{11} \alpha_{32}\left(1+y_{11}\right)\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1}\right] } & =\alpha_{21} \alpha_{32}\left(1+y_{21}\right)\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right] } & =-\alpha_{32}\left(1+y_{32}\right)\left[s_{1}\right] \\
{\left[s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right] } & =y_{32}[1]-\alpha_{32}\left[s_{1}\right]
\end{aligned}
$$

$$
\begin{aligned}
{\left[s_{2}\right]^{2} } & =-\alpha_{10} \alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{1} s_{2}\right] } & =\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{2} s_{1}\right] } & =-\alpha_{21} \alpha_{31} \alpha_{32} y_{10}[1]+\alpha_{10} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{1} s_{2} s_{1}\right] } & =\alpha_{21} \alpha_{32}\left(y_{21}+y_{31}\right)[1]-\alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{2} s_{1} s_{2}\right] } & =-\alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2}\right] } & =\alpha_{21} \alpha_{32}\left(1+y_{21}\right)\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1}\right] } & =-\alpha_{21}\left(y_{31}+y_{52}\right)[1]+\alpha_{21} \alpha_{31}\left(1+y_{21}\right)\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right] } & =y_{63}[1]-\alpha_{21}\left(1+y_{21}+y_{42}\right)\left[s_{2}\right] \\
{\left[s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right] } & =-\alpha_{21}\left(1+y_{21}\right)\left[s_{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
& {\left[s_{1} s_{2}\right]^{2}=}-\alpha_{11} \alpha_{21} \alpha_{32}\left(y_{01}+y_{11}+y_{21}\right)\left[s_{2}\right]+\alpha_{01} \alpha_{11} \alpha_{21} \alpha_{32}\left[s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{2} s_{1}\right]=} \alpha_{21} \alpha_{32}\left(y_{11}+y_{21}+\alpha_{31}\right)[1]-\alpha_{11} \alpha_{21} \alpha_{32}\left[s_{1}\right]-\alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1}\right]=}-\alpha_{32}\left(y_{32}+y_{42}\left\{+\alpha_{11}\left(y_{01}+2 y_{11}+y_{21}\right)\right\}\right)[1]+\alpha_{11} \alpha_{32}\left(y_{01}+y_{11}\right)\left[s_{1}\right] \\
&+\left(\alpha_{31} \alpha_{32} y_{11}+\alpha_{11} \alpha_{32}\left(y_{01}+y_{11}+y_{21}\right)\right)\left[s_{2}\right]-\alpha_{01} \alpha_{11} \alpha_{32}\left[s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2}\right]=} \alpha_{21} \alpha_{32}\left(y_{11}+y_{21}\right)\left[s_{2}\right]-\alpha_{11} \alpha_{21} \alpha_{32}\left[s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2}\right]=}-\alpha_{32}\left(y_{22}+y_{32}\right)\left[s_{2}\right]+\alpha_{11} \alpha_{32}\left(1+y_{11}\right)\left[s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1}\right]=}\left(y_{63}\left\{+\alpha_{32}\left(y_{11}+y_{21}\right)\right\}\right)[1]-\alpha_{32} y_{11}\left[s_{1}\right]-\left(\alpha_{32}\left(y_{11}+y_{21}\right)+\alpha_{31} y_{32}\right)\left[s_{2}\right] \\
& \quad+\alpha_{11} \alpha_{32}\left[s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=\left\{\begin{aligned}
& \left.-\left(y_{33}+y_{43}+y_{53}\right)[1]\right\}+y_{33}\left[s_{1}\right]+\left(y_{33}+y_{43}+y_{53}\right)\left[s_{2}\right] \\
& -\alpha_{11}\left(1+y_{11}+y_{22}\right)\left[s_{1} s_{2}\right] \\
{\left[s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=} & y_{32}\left[s_{2}\right]-\alpha_{32}\left[s_{1} s_{2}\right]
\end{aligned}\right.}
\end{aligned}
$$

$$
\begin{aligned}
& {\left[s_{2} s_{1}\right]^{2}=}-\alpha_{21} \alpha_{31} \alpha_{32} y_{10}\left[s_{1}\right]+\alpha_{10} \alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1}\right]=} \alpha_{21} \alpha_{31}\left(y_{21}+y_{31}\right)\left[s_{1}\right]-\alpha_{21} \alpha_{31} \alpha_{32}\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2}\right]=}-\alpha_{21}\left(y_{51}+y_{52}\left\{+\alpha_{31} y_{10}\right\}\right)[1]+\alpha_{21}\left(\alpha_{10} y_{31}+\alpha_{32} y_{10}\right)\left[s_{1}\right] \\
&+\alpha_{21} \alpha_{31}\left(y_{10}+y_{21}\right)\left[s_{2}\right]-\alpha_{10} \alpha_{21} \alpha_{31}\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2}\right]=}\left(y_{62}\left\{+\alpha_{31}\left(y_{21}+y_{31}\right)\right\}\right)[1]-\left(\alpha_{31} y_{21}+\alpha_{10}\left(y_{31}+y_{41}\right)\right)\left[s_{1}\right] \\
&-\left(\alpha_{31} y_{21}+\alpha_{32} y_{31}\right)\left[s_{2}\right]+\alpha_{21} \alpha_{31}\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1}\right]=}-\alpha_{21}\left(y_{31}+y_{52}\right)\left[s_{1}\right]+\alpha_{21} \alpha_{31}\left(1+y_{21}\right)\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=} y_{63}\left[s_{1}\right]-\alpha_{21}\left(1+y_{21}+y_{42}\right)\left[s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=\left\{-y_{31}[1]\right\}+y_{31}\left[s_{1}\right]+y_{31}\left[s_{2}\right]-\alpha_{31}\left[s_{2} s_{1}\right] }
\end{aligned}
$$

$$
\begin{aligned}
& {\left[s_{1} s_{2} s_{1}\right]^{2}=-\alpha_{32}\left(y_{32}+y_{42}\left\{+\alpha_{11}\left(y_{11}+y_{21}\right)\right\}\right)\left[s_{1}\right]} \\
& +\left(\alpha_{11} \alpha_{32}\left(y_{01}+y_{11}+y_{21}\right)+\alpha_{31} \alpha_{32} y_{11}\right)\left[s_{2} s_{1}\right]-\alpha_{01} \alpha_{11} \alpha_{32}\left[s_{1} s_{2} s_{1}\right] \\
& {\left[s_{1} s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2}\right]=\left(1\left\{+\alpha_{11}\left(y_{11}+y_{22}+y_{33}+y_{31}+y_{42}\right)+\alpha_{31}\left(y_{21}+y_{32}\right)+\alpha_{32} y_{21}\right\}\right)[1]} \\
& -\left(\alpha_{11}\left(y_{21}+\alpha_{32}\right)+\alpha_{10}\left(y_{31}+y_{41}+y_{32}+y_{42}\right)\right)\left[s_{1}\right] \\
& -\left(\alpha_{31}\left(y_{21}+y_{32}\right)+\alpha_{11}\left(y_{21}+y_{32}+y_{31}+\alpha_{42}\right)\left[s_{2}\right]\right. \\
& +\alpha_{11} \alpha_{32}\left[s_{1} s_{2}\right]+\alpha_{21} \alpha_{31}\left[s_{2} s_{1}\right] \\
& {\left[s_{1} s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2}\right]=\left\{-\left(y_{33}+2 y_{43}+y_{53}+\alpha_{11}\left(y_{01}+y_{11}\right)+\alpha_{21}\left(y_{11}+y_{21}\right)\right)[1]\right\}} \\
& +\left(y_{33}+y_{43}\left\{+\alpha_{11}\left(y_{01}+y_{11}\right)+\alpha_{21}\left(y_{11}+y_{21}\right)\right\}\right)\left[s_{1}\right] \\
& \left(\left(y_{33}+y_{43}+y_{53}\right)\left\{+\alpha_{11}\left(y_{01}+y_{11}\right)+\alpha_{21}\left(y_{11}+y_{21}\right)\right\}\right)\left[s_{2}\right] \\
& -\alpha_{11}\left(y_{01}+y_{11}+y_{22}\right)\left[s_{1} s_{2}\right]-\left(\alpha_{11}\left(y_{01}+y_{11}\right)+\alpha_{21}\left(y_{11}+y_{21}\right)\right)\left[s_{2} s_{1}\right] \\
& +\alpha_{01} \alpha_{11}\left[s_{1} s_{2} s_{1}\right] \\
& {\left[s_{1} s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1}\right]=\left(y_{62}\left\{+\alpha_{32} y_{21}\right\}\right)\left[s_{1}\right]-\left(\alpha_{31} y_{32}+\alpha_{32}\left(y_{11}+y_{21}\right)\right)\left[s_{2} s_{1}\right]+\alpha_{11} \alpha_{32}\left[s_{1} s_{2} s_{1}\right]} \\
& {\left[s_{1} s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=\left\{-\left(y_{43}+y_{53}\right)\left[s_{1}\right]\right\}+\left(y_{33}+y_{43}+y_{53}\right)\left[s_{2} s_{1}\right]-\alpha_{11}\left(1+y_{11}+y_{22}\right)\left[s_{1} s_{2} s_{1}\right]} \\
& {\left[s_{1} s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=\left\{\left(y_{11}+y_{21}\right)[1]-\left(y_{11}+y_{21}\right)\left[s_{1}\right]-\left(y_{11}+y_{21}\right)\left[s_{2}\right]\right\}} \\
& +y_{11}\left[s_{1} s_{2}\right]+\left(y_{11}+y_{21}\right)\left[s_{2} s_{1}\right]-\alpha_{11}\left[s_{1} s_{2} s_{1}\right] \\
& {\left[s_{2} s_{1} s_{2}\right]^{2}=-\alpha_{21}\left(y_{21}+y_{42}\right)\left[s_{2}\right]+\left(\alpha_{11} \alpha_{21} y_{31}+\alpha_{21} \alpha_{31} y_{10}\right)\left[s_{1} s_{2}\right]-\alpha_{10} \alpha_{21} \alpha_{31}\left[s_{2} s_{1} s_{2}\right]} \\
& {\left[s_{2} s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2}\right]=y_{53}\left[s_{2}\right]-\left(\alpha_{21} y_{31}+\alpha_{11} \alpha_{21} \alpha_{32} y_{21}\right)\left[s_{1} s_{2}\right]+\alpha_{21} \alpha_{31}\left[s_{2} s_{1} s_{2}\right]} \\
& {\left[s_{2} s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1}\right]=\left\{-\left(y_{51}+y_{52}+\alpha_{31} y_{10}\right)[1]\right\}+\left(y_{41}\left\{+\alpha_{31} y_{10}\right\}\right)\left[s_{1}\right]+\left(y_{42}+y_{52}\left\{+\alpha_{31} y_{10}\right\}\right)\left[s_{2}\right]} \\
& -\left(\alpha_{11} y_{31}+\alpha_{31} y_{10}\right)\left[s_{1} s_{2}\right]-\alpha_{31} y_{10}\left[s_{2} s_{1}\right]+\alpha_{10} \alpha_{31}\left[s_{2} s_{1} s_{2}\right] \\
& {\left[s_{2} s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=\left\{\left(y_{31}+y_{32}+y_{42}\right)[1]-\left(y_{31}+y_{32}\right)\left[s_{1}\right]-\left(y_{31}+y_{32}+y_{42}\right)\left[s_{2}\right]\right\}} \\
& +\left(y_{31}+y_{32}\right)\left[s_{1} s_{2}\right]+y_{31}\left[s_{2} s_{1}\right]-\alpha_{31}\left[s_{2} s_{1} s_{2}\right] \\
& {\left[s_{2} s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=y_{31}\left[s_{1} s_{2}\right]-\alpha_{31}\left[s_{2} s_{1} s_{2}\right]}
\end{aligned}
$$

$$
\left.\begin{array}{rl}
{\left[s_{1} s_{2} s_{1} s_{2}\right]^{2}=\{ } & \left.-y_{43}\left[s_{2}\right]\right\}+\left(y_{32}+y_{42}\left\{+\alpha_{01} y_{21}+\alpha_{32} y_{11}\right\}\right)\left[s_{1} s_{2}\right] \\
& -\left(\alpha_{01}\left(y_{11}+y_{21}\right)+\alpha_{31}\left(y_{01}+y_{11}\right)\right)\left[s_{2} s_{1} s_{2}\right]+\alpha_{01} \alpha_{11}\left[s_{1} s_{2} s_{1} s_{2}\right] \\
{\left[s_{1} s_{2} s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1}\right]=\{ } & \left(y_{21}+y_{31}+y_{32}+y_{42}+\alpha_{11}\right)[1] \\
& \left.-\left(y_{21}+y_{31}+y_{32}+\alpha_{11}\right)\left[s_{1}\right]-\left(y_{21}+y_{31}+y_{32}+y_{42}+\alpha_{11}\right)\left[s_{2}\right]\right\} \\
& +\left(y_{31}+y_{42}\left\{,+\alpha_{11}\right\}\right)\left[s_{1} s_{2}\right]+\left(y_{21}+y_{31}\left\{+\alpha_{11}\right\}\right)\left[s_{2} s_{1}\right] \\
& -\alpha_{11}\left[s_{1} s_{2} s_{1}\right]-\alpha_{31}\left[s_{2} s_{1} s_{2}\right]
\end{array}\right\} \begin{aligned}
& \\
& {\left[s_{1} s_{2} s_{1} s_{2}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=\{ }-\left(y_{01}+y_{11}+y_{21}+y_{22}+y_{32}\right)[1] \\
&+\left(y_{01}+y_{11}+y_{21}+y_{22}\right)\left[s_{1}\right]+\left(y_{01}+y_{11}+y_{21}+y_{22}+y_{32}\right)\left[s_{2}\right] \\
&\left.\quad-\left(y_{01}+y_{11}+y_{21}+y_{22}\right)\left[s_{1} s_{2}\right]-\left(y_{01}+y_{11}+y_{21}\right)\left[s_{2} s_{1}\right]\right\} \\
&+y_{01}\left[s_{1} s_{2} s_{1}\right]+\left(y_{01}+y_{11}+y_{21}\right)\left[s_{2} s_{1} s_{2}\right]-\alpha_{01}\left[s_{1} s_{2} s_{1} s_{2}\right] \\
& {\left[s_{1} s_{2} s_{1} s_{2}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=\left\{-y_{21}\left[s_{1} s_{2}\right]\right\}+\left(y_{11}+y_{21}\right)\left[s_{2} s_{1} s_{2}\right]-\alpha_{11}\left[s_{1} s_{2} s_{1} s_{2}\right] } \\
& {\left[s_{2} s_{1} s_{2} s_{1}\right]^{2}=\left\{-y_{52}\left[s_{1}\right]+\left(y_{42}+y_{52}\right)\left[s_{2} s_{1}\right]\right\}-\left(\alpha_{11} y_{31}+\alpha_{31} y_{10}\right)\left[s_{1} s_{2} s_{1}\right]+\alpha_{10} \alpha_{31}\left[s_{2} s_{1} s_{2} s_{1}\right] } \\
& {\left[s_{2} s_{1} s_{2} s_{1}\right]\left[s_{1} s_{2} s_{1} s_{2} s_{1}\right]=\left\{y_{42}\left[s_{1}\right]-\left(y_{31}+y_{41}\right)\left[s_{2} s_{1}\right]\right\}+\left(y_{31}+y_{32}\right)\left[s_{1} s_{2} s_{1}\right]-\alpha_{31}\left[s_{2} s_{1} s_{2} s_{1}\right] } \\
& {\left[s_{2} s_{1} s_{2} s_{1}\right]\left[s_{2} s_{1} s_{2} s_{1} s_{2}\right]=\left\{-y_{10}[1]+y_{10}\left[s_{1}\right]+y_{10}\left[s_{2}\right]-y_{10}\left[s_{1} s_{2}\right]-y_{10}\left[s_{2} s_{1}\right]\right\} } \\
&+y_{10}\left[s_{1} s_{2} s_{1}\right]+y_{10}\left[s_{2} s_{1} s_{2}\right]-\alpha_{10}\left[s_{2} s_{1} s_{2} s_{1}\right]
\end{aligned}
$$

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