An inequality for regular near polygons

Paul Terwilliger * Chih-wen Weng [†]

November 25, 2003

Abstract

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ . We show

$$\theta_1 \le \frac{k - a_1 - c_2}{c_2 - 1}.$$

We show the following (i)–(iii) are equivalent. (i) Equality is attained above; (ii) Γ is *Q*-polynomial with respect to θ_1 ; (iii) Γ is a dual polar graph or a Hamming graph.

Keywords: near polygon, distance-regular graph, *Q*-polynomial, dual polar graph, Hamming graph.

AMS Subject Classification: 05E30.

1 Introduction

Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$ (see Section 2 for formal definitions). Suppose the intersection numbers $a_1 > 0$ and $c_2 > 1$. It was shown by Brouwer, Cohen and Neumaier that if Γ has classical parameters $(d, q, 0, \beta)$ then Γ is a Hamming graph or a dual polar graph [2, Theorem 9.4.4]. The same conclusion was obtained by the second author under the assumption that Γ is Q-polynomial and has diameter

^{*}Department of Mathematics, University of Wisconsin-Madison, USA

[†]Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C.

 $d \geq 4$ [11, Corollary 5.7]. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of Γ . It is known that $\theta_0 = k$, where k denotes the valency of Γ . By [2, Proposition 4.4.6(i)],

$$\theta_d \ge -\frac{k}{a_1+1},$$

with equality if and only if Γ is a near 2*d*-gon. We now state our result.

Theorem 1.1. Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k, and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ . Then

$$\theta_1 \le \frac{k - a_1 - c_2}{c_2 - 1}.\tag{1.1}$$

Moreover, the following (i)-(iii) are equivalent.

- (i) Equality is attained in (1.1);
- (ii) Γ is Q-polynomial with respect to θ_1 ;
- (iii) Γ is a dual polar graph or a Hamming graph.

2 Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [2] for more background information.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂ and diameter $d := \max\{\partial(x, y) | x, y \in X\}$. For $x \in X$ and for all integers i, set

$$\Gamma_i(x) := \{ y | y \in X, \partial(x, y) = i \}.$$

Let k denote a nonnegative integer. We say Γ is regular with valency k whenever $|\Gamma_1(x)| = k$ for all $x \in X$. Pick an integer $i \ (0 \le i \le d)$. For $x \in X$ and for $y \in \Gamma_i(x)$, set

$$B(x,y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \qquad (2.1)$$

$$A(x,y) := \Gamma_1(x) \cap \Gamma_i(y), \qquad (2.2)$$

$$C(x,y) := \Gamma_1(x) \cap \Gamma_{i-1}(y).$$
(2.3)

The graph Γ is said to be *distance-regular* whenever for all integers $i \ (0 \le i \le d)$, and for all $x, y \in X$ with $\partial(x, y) = i$, the numbers

$$c_i := |C(x,y)|, \qquad a_i := |A(x,y)|, \qquad b_i := |B(x,y)|$$
(2.4)

are independent of x and y. We call the c_i , a_i , b_i the *intersection numbers* of Γ . We observe $c_0 = 0$, $a_0 = 0$, $b_d = 0$ and $c_1 = 1$. For the rest of this paper we assume Γ is distance-regular with diameter $d \geq 3$. We observe Γ is regular with valence $k = b_0$ and that

$$c_i + a_i + b_i = k \quad (0 \le i \le d) \tag{2.5}$$

[2, p. 126].

We recall the Bose-Mesner algebra of Γ . For $0 \leq i \leq d$ let A_i denote the matrix in $Mat_X(\mathbb{R})$ which has xy entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the *i*th distance matrix of Γ . Observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{d} A_i = J$; (aiii) $A_i^t = A_i \ (0 \le i \le d)$, (aiv) $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \ (0 \le i, j \le d)$, where I denotes the identity matrix and J denotes the all ones matrix. We abbreviate $A := A_1$ and call this the adjacency matrix of Γ . Let \mathbf{M} denote the subalgebra of $\operatorname{Mat}_X(\mathbb{R})$ generated by A. Using (ai)–(aiv) we find A_0, A_1, \cdots, A_d form a basis of \mathbf{M} . We call \mathbf{M} the Bose-Mesner algebra of Γ . By [1, p. 59, p. 64], \mathbf{M} has a second basis E_0, E_1, \cdots, E_d such that (ei) $E_0 = |X|^{-1}J$; (eii) $\sum_{i=0}^{d} E_i = I$; (eiii) $E_i^t = E_i \ (0 \le i \le d)$; (eiv) $E_i E_j = \delta_{ij} E_i \ (0 \le i, j \le d)$. We call E_0, E_1, \cdots, E_d the primitive idempotents for Γ . Since E_0, E_1, \cdots, E_d form a basis for \mathbf{M} there exist real scalars $\theta_0, \theta_1, \cdots, \theta_d$ such that $A = \sum_{i=0}^{d} \theta_i E_i$. By this and (eiv) we find $A E_i = \theta_i E_i \ (0 \le i \le d)$. Observe $\theta_0, \theta_1, \cdots, \theta_d$ are mutually distinct since A generates \mathbf{M} . We assume the E_i are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_d$. We call θ_i the eigenvalue of Γ corresponding to E_i . By [1, p. 197] we have $\theta_0 = k$ and $-k \le \theta_i \le k$ $(0 \le i \le d)$. We call θ_0 the trivial eigenvalue.

Let θ denote an eigenvalue of Γ and let E denote the corresponding primitive idempotent. Since $E \in \mathbf{M}$, there exist real numbers $\sigma_0, \sigma_1, \cdots, \sigma_d$ such that

$$E = m|X|^{-1} \sum_{i=0}^{d} \sigma_i A_i,$$
(2.6)

where $m = \operatorname{rank} E$. We have $\sigma_0 = 1$ and

$$c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \qquad (0 \le i \le d), \tag{2.7}$$

where σ_{-1} , σ_{d+1} denote indeterminates [1, p. 191]. The sequence σ_0 , σ_1 , \cdots , σ_d is called the *cosine sequence* associated with θ . Let σ_0 , σ_1 , \cdots , σ_d denote the cosine sequence associated with the eigenvalue k. Comparing (2.5) and (2.7) we find $\sigma_i = 1$ ($0 \le i \le d$). By the *trivial cosine sequence* of Γ we mean the cosine sequence associated with k. Let θ denote an eigenvalue of Γ and let $\sigma_0, \sigma_1, \cdots, \sigma_d$ denote the corresponding cosine sequence. By (2.7),

$$\sigma_1 = \theta k^{-1}, \tag{2.8}$$

$$\sigma_2 = \frac{\theta^2 - a_1 \theta - k}{k b_1}.$$
(2.9)

Combining (2.8) and (2.9) we find

$$(\sigma_1 - \sigma_2)b_1 = (\theta + 1)(\sigma_0 - \sigma_1).$$
(2.10)

Set $V = \mathbb{R}^X$ (column vectors). We define the inner product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V).$$

For each $x \in X$ set

$$\hat{x} = (0, 0, \cdots, 1, 0, \cdots, 0)^t,$$

where the 1 is in coordinate x. We observe $\{\hat{x} | x \in X\}$ is an orthonormal basis for V. By (2.6), for $x, y \in X$ we have

$$\langle E\hat{x}, E\hat{y} \rangle = m|X|^{-1}\sigma_i, \qquad (2.11)$$

where $i = \partial(x, y)$.

By a *clique* in Γ we mean a nonempty set consisting of mutually adjacent vertices of Γ . A given clique in Γ is said to be *maximal* whenever it is not properly contained in a clique. The graph Γ is said to be a *near polygon* whenever

(i) Each maximal clique has cardinality $a_1 + 2$;

- (ii) For all maximal cliques ℓ and for all $x \in X$, either
 - (iia) $\partial(x, y) = d$ for all $y \in \ell$, or
 - (iib) there exists an integer i $(0 \le i \le d-1)$ and a unique $z \in \ell$ such that $\partial(x, z) = i$ and $\partial(x, y) = i + 1$ for all $y \in \ell \{z\}$.

We give an alternate description of a near polygon. Let $K_{1,2,1}$ denote the graph with 4 vertices s, x, y, s' such that $\partial(s, x) = \partial(s, y) = \partial(x, y) = \partial(x, s') = \partial(y, s') = 1$ and $\partial(s, s') = 2$. Then by [2, Theorem 6.4.1] Γ is a near polygon if and only if both the following (i')-(ii') hold.

- (i') Γ does not contain an induced $K_{1,2,1}$ subgraph;
- (ii')

$$a_i = a_1 c_i \qquad (0 \le i \le d - 1).$$
 (2.12)

Assume Γ is a near polygon. Then

$$a_d \ge a_1 c_d. \tag{2.13}$$

Moreover $a_d = a_1c_d$ if and only if no maximal clique satisfies (iia) above [2, Theorem 6.4.1]. In this case we call Γ a *near* 2d-gon. Otherwise we call Γ a *near* (2d + 1)-gon. Assume Γ is a near polygon. The Hoffman bound states that

$$\theta_d \ge -\frac{k}{a_1+1},\tag{2.14}$$

with equality if and only if Γ is a near 2*d*-gon [2, Proposition 4.4.6(i)].

Definition 2.1. Let Γ denote a distance-regular graph with diameter $d \geq 3$. We say Γ has *classical parameters* (d, q, α, β) whenever the intersection numbers are given by

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i - 1 \\ 1 \end{bmatrix} \right) \qquad (0 \le i \le d), \tag{2.15}$$

$$b_i = \left(\begin{bmatrix} D\\1 \end{bmatrix} - \begin{bmatrix} i\\1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i\\1 \end{bmatrix} \right) \qquad (0 \le i \le d), \qquad (2.16)$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \dots + q^{j-1}.$$
 (2.17)

We give two examples of near polygon distance-regular graphs with classical parameters (d, q, α, β) .

Example 2.2. The Hamming graph H(d, n) $(n \ge 2)$ [4], [5], [6], [8].

$$\begin{aligned} X &= \text{all } d\text{-tuples of elements from the set } \{1, 2, \cdots, n\}, \\ xy &\in R \text{ iff } x, y \text{ differ in exactly 1 coordinate } (x, y \in X), \\ q &= 1, \quad \alpha = 0, \quad \beta = n - 1, \\ c_i &= i, \quad b_i = (d - i)(n - 1), \quad a_i = (n - 2)i \quad (0 \leq i \leq d), \\ \theta_i &= (d - i)(n - 1) - i \quad (0 \leq i \leq d). \end{aligned}$$

Example 2.3. The Dual polar graphs [3], [7].

Let U denote a finite vector space with one of the following non-degenerate forms:

name	$\dim(U)$	field	form	ϵ
$B_d(p^n)$	2d + 1	$GF(p^n)$	quadratic	1
$C_d(p^n)$	2d	$GF(p^n)$	symplectic	1
$D_d(p^n)$	2d	$GF(p^n)$	$\frac{\text{quadratic}}{(\text{Witt index } d)}$	0
${}^{2}D_{d+1}(p^{n})$	2d + 2	$GF(p^n)$	$\frac{\text{quadratic}}{(\text{Witt index } d)}$	2
$^{2}A_{2d}(p^{n})$	2d + 1	$GF(p^{2n})$	Hermitean	$\frac{3}{2}$
$^{2}A_{2d-1}(p^{n})$	2d	$GF(p^{2n})$	Hermitean	$\frac{1}{2}$

A subspace of U is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is d.
$$\begin{split} X &= \text{set all maximal isotropic subspaces of } U, \\ xy &\in R \text{ iff } \dim(x \cap y) = d - 1 \quad (x, y \in X), \\ \alpha &= 0, \quad \beta = q^{\epsilon}, \\ c_i &= \frac{q^i - 1}{q - 1}, \quad a_i = \frac{q^{i + \epsilon} - q^i - q^{\epsilon} + 1}{q - 1} \quad (0 \leq i \leq d), \\ b_i &= \frac{q^{i + \epsilon} (q^{d - i} - 1)}{q - 1} \quad (0 \leq i \leq d - 1), \\ \theta_i &= \frac{q^{d + \epsilon - i} - q^{\epsilon} - q^i + 1}{q - 1} \quad (0 \leq i \leq d), \end{split}$$

where

$$q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n}$$
 respectively.

The following three theorems will be used in the proof of our results.

Theorem 2.4. ([9, Theorem 4.1]) Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let q denote a real number at least 1. Then the following conditions (i), (ii) are equivalent.

- (i) Γ has a nontrivial cosine sequence $\sigma_0, \sigma_1, \cdots, \sigma_d$ such that $\sigma_{i-1} q\sigma_i$ is independent of $i \quad (1 \le i \le d)$.
- (ii) The intersection numbers of Γ are such that $qc_i b_i q(qc_{i-1} b_{i-1})$ is independent of i $(1 \le i \le d)$.

Furthermore, if (i), (ii) hold, then

$$c_3 \ge (c_2 - q)(1 + q + q^2).$$
 (2.18)

Theorem 2.5. ([9, Theorem 4.2]) Let Γ denote a distance-regular graph with diameter $d \geq 3$, and let q denote a real number at least 1. Then the following conditions (i), (ii) are equivalent.

(i) Statements (i), (ii) hold in Theorem 2.4, and $c_3 = (c_2 - q)(1 + q + q^2)$.

(ii) There exists $\alpha, \beta \in \mathbb{R}$ such that Γ has classical parameters (d, q, α, β) .

Theorem 2.6. ([2, Theorem 9.4.4]) Let Γ denote a distance-regular graph with diameter $d \geq 3$ with classical parameters $(d, q, 0, \beta)$. Assume the intersection numbers $a_1 > 0$ and $c_2 > 1$. Suppose Γ is a near polygon. Then Γ is a dual polar graph or a Hamming graph.

3 The inequality

In this section we obtain the inequality in Theorem 1.1.

Lemma 3.1. Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency k, and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ . Then

$$\theta_1 \le \frac{k - a_1 - c_2}{c_2 - 1}.\tag{3.1}$$

Proof. Abbreviate $E = E_1$. Let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the cosine sequence associated with θ_1 . Fix any two vertices $x, y \in X$ with $\partial(x, y) = 2$. We consider the vectors

$$u = \sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w}, \qquad (3.2)$$

$$v = E\hat{x} - E\hat{y}. \tag{3.3}$$

By the Cauchy-Schwartz inequality,

$$|u||^2 ||v||^2 \ge \langle u, v \rangle^2.$$
 (3.4)

We compute the terms in (3.4). Using (2.11), (3.2), (3.3) we find

$$||v||^2 = 2m|X|^{-1}(\sigma_0 - \sigma_2), \qquad (3.5)$$

$$\langle u, v \rangle = 2ma_2 |X|^{-1} (\sigma_1 - \sigma_2).$$
 (3.6)

We now compute $||u||^2$. To do this we first discuss the distances between vertices in A(x, y) and vertices in A(y, x). We claim that for all $z \in A(x, y)$, z is adjacent to $c_2 - 1$ vertices in A(y, x) and is at distance 2 from the remaining $a_2 - c_2 + 1$ vertices in A(y, x). To see this fix $z \in A(x, y)$. Then $\ell :=$ $A(x, z) \cup \{x, z\}$ is a maximal clique; hence there exists a unique vertex $s \in \ell$ with $\partial(s, y) = 1$. That is $s \in C(x, y) \cap C(z, y)$. Observe $|C(x, y) \cap C(z, y)| = 1$, since any other $s' \in C(x, y) \cap C(z, y)$ will cause either xss'y or sxzs' to be a $K_{1,2,1}$ subgraph. Hence there are $c_2 - 1$ vertices in $C(z, y) \cap A(y, x)$. Observe for $w \in A(y, x)$ we have $\partial(w, x) = 2$ and $\partial(w, s) \leq 2$ so $\partial(w, z) \leq 2$. We have now proved the claim. Using the claim and applying (2.11) we find

$$\|u\|^{2} = \|\sum_{z \in A(x,y)} E\hat{z}\|^{2} + \|\sum_{w \in A(y,x)} E\hat{w}\|^{2} - 2\langle \sum_{z \in A(x,y)} E\hat{z}, \sum_{w \in A(y,x)} E\hat{w} \rangle$$

$$= 2ma_{2}|X|^{-1}(\sigma_{0} + (a_{1} - c_{2})\sigma_{1} + (c_{2} - a_{1} - 1)\sigma_{2}).$$
(3.7)

Evaluating (3.4) using (3.5)–(3.7) we routinely find

$$(\sigma_0 + (a_1 - c_2)\sigma_1 + (c_2 - a_1 - 1)\sigma_2)(\sigma_0 - \sigma_2) \ge a_2(\sigma_1 - \sigma_2)^2.$$
(3.8)

Evaluating (3.8) using (2.8), (2.9), (2.12) we obtain

$$(\theta_1 - k)^2 (\theta_1(a_1 + 1) + k)(k - \theta_1(c_2 - 1) - a_1 - c_2) \ge 0.$$
(3.9)

Clearly $(\theta_1 - k)^2 > 0$. By (2.14) and since $\theta_1 > \theta_d$ we find $\theta_1(a_1 + 1) + k > 0$. Evaluating (3.9) using these comments we find

$$k - \theta_1(c_2 - 1) - a_1 - c_2 \ge 0$$

and (3.1) follows.

Remark 3.2. Referring to Example 2.2 and Example 2.3, the eigenvalue θ_1 satisfies (3.1) with equality.

We comment on the proof of Lemma 3.1.

Lemma 3.3. With the notation of Lemma 3.1, the following (i)-(iii) are equivalent.

- (i) Equality is attained in (3.1).
- (ii) For all $x, y \in X$ such that $\partial(x, y) = 2$,

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} \in \operatorname{Span}(E\hat{x} - E\hat{y}).$$
(3.10)

(iii) There exist $x, y \in X$ such that $\partial(x, y) = 2$ and

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} \in \operatorname{Span}(E\hat{x} - E\hat{y}).$$
(3.11)

Here $E = E_1$.

Proof. Observe from the proof of Lemma 3.1 that equality is attained in (3.1) if and only if equality is attained in (3.4). We claim $v \neq 0$. This will follow from (3.5) provided we can show $\sigma_0 \neq \sigma_2$. Suppose $\sigma_0 = \sigma_2$. Setting $\theta = \theta_1$ and $\sigma_2 = \sigma_0$ in (2.10) and simplifying the result we find $\theta_1 = -b_1 - 1$. This is inconsistent with (2.14) and $\theta_1 > \theta_d$. We have now shown $\sigma_0 \neq \sigma_2$ and it follows $v \neq 0$. We now see that equality is attained in (3.4) if and only if $u \in \text{Span}(v)$. The result follows.

4 The case of equality

In this section we consider the case of equality in (3.1).

Lemma 4.1. Let Γ denote a near polygon distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ and let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the corresponding cosine sequence. Suppose equality holds in (3.1). Then $\sigma_{i-1} - q\sigma_i$ is independent of $i \ (1 \leq i \leq d)$, where $q = c_2 - 1$.

Proof. Setting $c_2 = q + 1$ in (3.1) and using $k - a_1 - 1 = b_1$ we find $\theta_1 + 1 = b_1 q^{-1}$. In particular $\theta_1 \neq -1$. Observe $\sigma_1 \neq \sigma_2$; otherwise $\sigma_0 = \sigma_1$ by (2.10) forcing $\theta_1 = k$ by (2.8), a contradiction. Evaluating (2.10) using $\theta_1 + 1 = b_1 q^{-1}$ we find

$$\frac{\sigma_0 - \sigma_1}{\sigma_1 - \sigma_2} = q. \tag{4.1}$$

Fix two vertices $x, y \in X$ with $\partial(x, y) = 2$. Abbreviate $E = E_1$. By Lemma 3.3 there exists $\lambda \in \mathbb{R}$ such that

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} = \lambda(E\hat{x} - E\hat{y}).$$

$$(4.2)$$

Fix an integer i $(1 \le i \le d-1)$ and pick $u \in X$ with $\partial(u, x) = i-1$ and $\partial(u, y) = i+1$. Taking the inner product of $E\hat{u}$ with both sides of (4.2),

$$a_2(\sigma_i - \sigma_{i+1}) = \lambda(\sigma_{i-1} - \sigma_{i+1}).$$
(4.3)

Setting i = 1 in (4.3) we find $a_2(\sigma_1 - \sigma_2) = \lambda(\sigma_0 - \sigma_2)$. From (4.1) we find $\sigma_0 - \sigma_2 = (\sigma_1 - \sigma_2)(1 + q)$. By these comments $\lambda = a_2/(q + 1)$. Evaluating (4.3) using this we find

$$\sigma_{i-1} - q\sigma_i = \sigma_i - q\sigma_{i+1} \quad (1 \le i \le d-1).$$

From this we find $\sigma_{i-1} - q\sigma_i$ is independent of *i* for $1 \le i \le d$.

Lemma 4.2. Let Γ denote a near polygon distance-regular graph with $d \geq 3$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let θ_1 denote the second largest eigenvalue of Γ and assume equality holds in (3.1). Then Γ has classical parameters $(d, q, 0, \beta)$.

Proof. Let the scalar q be as in Lemma 4.1. By Lemma 4.1 we have Theorem 2.4(i) and hence Theorem 2.4(ii). Applying Theorem 2.4(ii) with i = 2, 3 we find

$$qc_2 - b_2 - q(qc_1 - b_1) = qc_3 - b_3 - q(qc_2 - b_2).$$
(4.4)

Simplifying (4.4) using (2.5) and $c_2 = q + 1$, $a_2 = a_1c_2$ we obtain

$$(a_1 + 1 + q)(1 + q + q^2 - c_3) = a_3 - a_1c_3.$$
(4.5)

By (2.12) we have $a_3 = a_1c_3$ if d > 3, and by (2.13) we have $a_3 \ge a_1c_3$ if d = 3. In any case $a_3 \ge a_1c_3$ so the right-hand side of (4.5) is nonnegative. Also $a_1 + 1 + q > 0$ since $q = c_2 - 1$. Evaluating (4.5) using these comments we find

$$c_3 \le 1 + q + q^2. \tag{4.6}$$

By (2.18) and using $c_2 = 1 + q$ we find $c_3 \ge 1 + q + q^2$. Now apparently $c_3 = 1 + q + q^2$. We can now check that the assumption $c_3 = (c_2 - q)(1 + q + q^2)$ in Theorem 2.5(i) holds. Applying Theorem 2.5 we find there exist real numbers α, β such that Γ has classical parameters (d, q, α, β) . By (2.15) we find $c_2 = (1 + q)(1 + \alpha)$. By the construction $c_2 = q + 1$. Comparing these equations we find $\alpha = 0$.

Proof of Theorem 1.1. The inequality (1.1) is from (3.1).

(i) \Longrightarrow (iii). By Lemma 4.2, Γ has classical parameters $(d, q, 0, \beta)$. By this and Theorem 2.6 we find Γ is a dual polar graph or a Hamming graph.

 $(iii) \Longrightarrow (ii)$ This is immediate from [2, Corollary 8.5.3].

 $(ii) \Longrightarrow (i)$ Lemma 3.3(ii) holds by [9, Theorem 3.3], so Lemma 3.3(i) holds and the result follows.

References

- [1] E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin/Cummings, London, 1984.
- [2] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin, 1989.
- [3] P. Cameron. Dual polar spaces. *Geom. Dedicata*, 12(1982), 75–85.
- [4] Y. Egawa. Characterization of H(n,q) by the parameters. J. Combin. Theory Ser. A, 31(1981), 108–125.
- [5] A. Neumaier. Characterization of a class of distance-regular graphs. J. Reine Angew. Math., 357(1985), 182–192.
- [6] N. Sloane. An introduction to association schemes and coding theory; in Theory and Application of Special functions (R. Askey, Ed.). Academic Press, New York, 1975.
- [7] D. Stanton. Some q-Krawtchouk polynomials on Chevalley groups. Amer. J. Math., 102(4)(1980), 625–662.
- [8] P. Terwilliger. Root systems and the Johnson and Hamming graphs. European J. Combin., 8(1987), 73–102.
- [9] P. Terwilliger. A new inequality for distance-regular graphs. *Discrete Math.*, 137(1995), 319–332.
- [10] C. Weng. Kite-Free P- and Q-Polynomial Schemes Graphs and Combinatorics, 11(1995), 201-207.
- [11] C. Weng. D-bounded Distance-regular Graphs. Europ. J. Combinatorics, 18(1997), 211–229.

Paul Terwilliger Department of Mathematics University of Wisconsin-Madison Van Vleck Hall 480 Lincoln Drive Madison, WI 53706-1388 USA email: terwilli@math.wisc.edu

Chih-wen Weng Department of Applied Mathematics National Chiao Tung University 1001 Ta Hsueh Road, Hsinchu Taiwan 30050 email: weng@math.nctu.edu.tw