# An inequality for regular near polygons 

Paul Terwilliger * Chih-wen Weng ${ }^{\dagger}$

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#### Abstract

Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency $k$ and intersection numbers $a_{1}>0, c_{2}>1$. Let $\theta_{1}$ denote the second largest eigenvalue of $\Gamma$. We show $$
\theta_{1} \leq \frac{k-a_{1}-c_{2}}{c_{2}-1}
$$

We show the following (i)-(iii) are equivalent. (i) Equality is attained above; (ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_{1}$; (iii) $\Gamma$ is a dual polar graph or a Hamming graph.

Keywords: near polygon, distance-regular graph, $Q$-polynomial, dual polar graph, Hamming graph.

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## 1 Introduction

Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$ (see Section 2 for formal definitions). Suppose the intersection numbers $a_{1}>$ 0 and $c_{2}>1$. It was shown by Brouwer, Cohen and Neumaier that if $\Gamma$ has classical parameters $(d, q, 0, \beta)$ then $\Gamma$ is a Hamming graph or a dual polar graph [2, Theorem 9.4.4]. The same conclusion was obtained by the second author under the assumption that $\Gamma$ is $Q$-polynomial and has diameter

[^0]$d \geq 4$ [11, Corollary 5.7]. Let $\theta_{0}>\theta_{1}>\cdots>\theta_{d}$ denote the eigenvalues of $\Gamma$. It is known that $\theta_{0}=k$, where $k$ denotes the valency of $\Gamma$. By [2, Proposition 4.4.6(i)],
$$
\theta_{d} \geq-\frac{k}{a_{1}+1}
$$
with equality if and only if $\Gamma$ is a near $2 d$-gon. We now state our result.
Theorem 1.1. Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency $k$, and intersection numbers $a_{1}>0, c_{2}>1$. Let $\theta_{1}$ denote the second largest eigenvalue of $\Gamma$. Then
\[

$$
\begin{equation*}
\theta_{1} \leq \frac{k-a_{1}-c_{2}}{c_{2}-1} \tag{1.1}
\end{equation*}
$$

\]

Moreover, the following (i)-(iii) are equivalent.
(i) Equality is attained in (1.1);
(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_{1}$;
(iii) $\Gamma$ is a dual polar graph or a Hamming graph.

## 2 Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [2] for more background information.

Let $\Gamma=(X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set $X$, edge set $R$, path-length distance function $\partial$ and diameter $d:=\max \{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and for all integers $i$, set

$$
\Gamma_{i}(x):=\{y \mid y \in X, \partial(x, y)=i\}
$$

Let $k$ denote a nonnegative integer. We say $\Gamma$ is regular with valency $k$ whenever $\left|\Gamma_{1}(x)\right|=k$ for all $x \in X$. Pick an integer $i(0 \leq i \leq d)$. For $x \in X$ and for $y \in \Gamma_{i}(x)$, set

$$
\begin{align*}
B(x, y) & :=\Gamma_{1}(x) \cap \Gamma_{i+1}(y)  \tag{2.1}\\
A(x, y) & :=\Gamma_{1}(x) \cap \Gamma_{i}(y)  \tag{2.2}\\
C(x, y) & :=\Gamma_{1}(x) \cap \Gamma_{i-1}(y) . \tag{2.3}
\end{align*}
$$

The graph $\Gamma$ is said to be distance-regular whenever for all integers $i(0 \leq$ $i \leq d)$, and for all $x, y \in X$ with $\partial(x, y)=i$, the numbers

$$
\begin{equation*}
c_{i}:=|C(x, y)|, \quad a_{i}:=|A(x, y)|, \quad b_{i}:=|B(x, y)| \tag{2.4}
\end{equation*}
$$

are independent of $x$ and $y$. We call the $c_{i}, a_{i}, b_{i}$ the intersection numbers of $\Gamma$. We observe $c_{0}=0, a_{0}=0, b_{d}=0$ and $c_{1}=1$. For the rest of this paper we assume $\Gamma$ is distance-regular with diameter $d \geq 3$. We observe $\Gamma$ is regular with valence $k=b_{0}$ and that

$$
\begin{equation*}
c_{i}+a_{i}+b_{i}=k \quad(0 \leq i \leq d) \tag{2.5}
\end{equation*}
$$

[2, p. 126].
We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq d$ let $A_{i}$ denote the matrix in $\operatorname{Mat}_{X}(\mathbb{R})$ which has $x y$ entry

$$
\left(A_{i}\right)_{x y}=\left\{\begin{array}{ll}
1 & \text { if } \partial(x, y)=i \\
0 & \text { if } \partial(x, y) \neq i
\end{array} \quad(x, y \in X)\right.
$$

We call $A_{i}$ the $i$ th distance matrix of $\Gamma$. Observe (ai) $A_{0}=I$; (aii) $\sum_{i=0}^{d} A_{i}=$ $J ;$ (aiii) $A_{i}^{t}=A_{i}(0 \leq i \leq d)$, (aiv) $A_{i} A_{j}=\sum_{h=0}^{d} p_{i j}^{h} A_{h}(0 \leq i, j \leq d)$, where $I$ denotes the identity matrix and $J$ denotes the all ones matrix. We abbreviate $A:=A_{1}$ and call this the adjacency matrix of $\Gamma$. Let $\mathbf{M}$ denote the subalgebra of $\operatorname{Mat}_{X}(\mathbb{R})$ generated by $A$. Using (ai)-(aiv) we find $A_{0}, A_{1}, \cdots, A_{d}$ form a basis of $\mathbf{M}$. We call $\mathbf{M}$ the Bose-Mesner algebra of $\Gamma$. Вy [1] p. 59, p. 64], $\mathbf{M}$ has a second basis $E_{0}, E_{1}, \cdots, E_{d}$ such that (ei) $E_{0}=|X|^{-1} J$; (eii) $\sum_{i=0}^{d} E_{i}=I$; (eiii) $E_{i}^{t}=E_{i}(0 \leq i \leq d) ;($ eiv $) E_{i} E_{j}=\delta_{i j} E_{i}(0 \leq i, j \leq d)$. We call $E_{0}, E_{1}, \cdots, E_{d}$ the primitive idempotents for $\Gamma$. Since $E_{0}, E_{1}$, $\cdots, E_{d}$ form a basis for $\mathbf{M}$ there exist real scalars $\theta_{0}, \theta_{1}, \cdots, \theta_{d}$ such that $A=\sum_{i=0}^{d} \theta_{i} E_{i}$. By this and (eiv) we find $A E_{i}=\theta_{i} E_{i}(0 \leq i \leq d)$. Observe $\theta_{0}, \theta_{1}, \cdots, \theta_{d}$ are mutually distinct since $A$ generates $\mathbf{M}$. We assume the $E_{i}$ are indexed so that $\theta_{0}>\theta_{1}>\cdots>\theta_{d}$. We call $\theta_{i}$ the eigenvalue of $\Gamma$ corresponding to $E_{i}$. By [1] p. 197] we have $\theta_{0}=k$ and $-k \leq \theta_{i} \leq k$ $(0 \leq i \leq d)$. We call $\theta_{0}$ the trivial eigenvalue.

Let $\theta$ denote an eigenvalue of $\Gamma$ and let $E$ denote the corresponding primitive idempotent. Since $E \in \mathbf{M}$, there exist real numbers $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ such that

$$
\begin{equation*}
E=m|X|^{-1} \sum_{i=0}^{d} \sigma_{i} A_{i} \tag{2.6}
\end{equation*}
$$

where $m=\operatorname{rank} E$. We have $\sigma_{0}=1$ and

$$
\begin{equation*}
c_{i} \sigma_{i-1}+a_{i} \sigma_{i}+b_{i} \sigma_{i+1}=\theta \sigma_{i} \quad(0 \leq i \leq d) \tag{2.7}
\end{equation*}
$$

where $\sigma_{-1}, \sigma_{d+1}$ denote indeterminates [1, p. 191]. The sequence $\sigma_{0}, \sigma_{1}, \cdots$, $\sigma_{d}$ is called the cosine sequence associated with $\theta$. Let $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ denote the cosine sequence associated with the eigenvalue $k$. Comparing (2.5) and (2.7) we find $\sigma_{i}=1(0 \leq i \leq d)$. By the trivial cosine sequence of $\Gamma$ we mean the cosine sequence associated with $k$. Let $\theta$ denote an eigenvalue of $\Gamma$ and let $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ denote the corresponding cosine sequence. By (2.7),

$$
\begin{align*}
\sigma_{1} & =\theta k^{-1}  \tag{2.8}\\
\sigma_{2} & =\frac{\theta^{2}-a_{1} \theta-k}{k b_{1}} \tag{2.9}
\end{align*}
$$

Combining (2.8) and (2.9) we find

$$
\begin{equation*}
\left(\sigma_{1}-\sigma_{2}\right) b_{1}=(\theta+1)\left(\sigma_{0}-\sigma_{1}\right) \tag{2.10}
\end{equation*}
$$

Set $V=\mathbb{R}^{X}$ (column vectors). We define the inner product

$$
\langle u, v\rangle=u^{t} v \quad(u, v \in V)
$$

For each $x \in X$ set

$$
\hat{x}=(0,0, \cdots, 1,0, \cdots, 0)^{t},
$$

where the 1 is in coordinate $x$. We observe $\{\hat{x} \mid x \in X\}$ is an orthonormal basis for $V$. By (2.6), for $x, y \in X$ we have

$$
\begin{equation*}
\langle E \hat{x}, E \hat{y}\rangle=m|X|^{-1} \sigma_{i}, \tag{2.11}
\end{equation*}
$$

where $i=\partial(x, y)$.
By a clique in $\Gamma$ we mean a nonempty set consisting of mutually adjacent vertices of $\Gamma$. A given clique in $\Gamma$ is said to be maximal whenever it is not properly contained in a clique. The graph $\Gamma$ is said to be a near polygon whenever
(i) Each maximal clique has cardinality $a_{1}+2$;
(ii) For all maximal cliques $\ell$ and for all $x \in X$, either
(iia) $\partial(x, y)=d$ for all $y \in \ell$, or
(iib) there exists an integer $i(0 \leq i \leq d-1)$ and a unique $z \in \ell$ such that $\partial(x, z)=i$ and $\partial(x, y)=i+1$ for all $y \in \ell-\{z\}$.

We give an alternate description of a near polygon. Let $K_{1,2,1}$ denote the graph with 4 vertices $s, x, y, s^{\prime}$ such that $\partial(s, x)=\partial(s, y)=\partial(x, y)=$ $\partial\left(x, s^{\prime}\right)=\partial\left(y, s^{\prime}\right)=1$ and $\partial\left(s, s^{\prime}\right)=2$. Then by [2, Theorem 6.4.1] $\Gamma$ is a near polygon if and only if both the following (i')-(ii') hold.
(i') $\Gamma$ does not contain an induced $K_{1,2,1}$ subgraph;
(ii')

$$
\begin{equation*}
a_{i}=a_{1} c_{i} \quad(0 \leq i \leq d-1) . \tag{2.12}
\end{equation*}
$$

Assume $\Gamma$ is a near polygon. Then

$$
\begin{equation*}
a_{d} \geq a_{1} c_{d} . \tag{2.13}
\end{equation*}
$$

Moreover $a_{d}=a_{1} c_{d}$ if and only if no maximal clique satisfies (iia) above [2, Theorem 6.4.1]. In this case we call $\Gamma$ a near $2 d$-gon. Otherwise we call $\Gamma$ a near $(2 d+1)$-gon. Assume $\Gamma$ is a near polygon. The Hoffman bound states that

$$
\begin{equation*}
\theta_{d} \geq-\frac{k}{a_{1}+1} \tag{2.14}
\end{equation*}
$$

with equality if and only if $\Gamma$ is a near $2 d$-gon [2, Proposition 4.4.6(i)].

Definition 2.1. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq$ 3 . We say $\Gamma$ has classical parameters $(d, q, \alpha, \beta)$ whenever the intersection numbers are given by

$$
\begin{gather*}
c_{i}=\left[\begin{array}{l}
i \\
1
\end{array}\right]\left(1+\alpha\left[\begin{array}{c}
i-1 \\
1
\end{array}\right]\right) \quad(0 \leq i \leq d),  \tag{2.15}\\
b_{i}=\left(\left[\begin{array}{c}
D \\
1
\end{array}\right]-\left[\begin{array}{l}
i \\
1
\end{array}\right]\right)\left(\beta-\alpha\left[\begin{array}{l}
i \\
1
\end{array}\right]\right) \quad(0 \leq i \leq d), \tag{2.16}
\end{gather*}
$$

where

$$
\left[\begin{array}{l}
j  \tag{2.17}\\
1
\end{array}\right]:=1+q+q^{2}+\cdots+q^{j-1} .
$$

We give two examples of near polygon distance-regular graphs with classical parameters $(d, q, \alpha, \beta)$.

Example 2.2. The Hamming graph $H(d, n)(n \geq 2)$ [4], [5], [6], [8].

$$
\begin{aligned}
& X=\text { all } d \text {-tuples of elements from the set }\{1,2, \cdots, n\}, \\
& x y \in R \text { iff } x, y \text { differ in exactly } 1 \text { coordinate } \quad(x, y \in X), \\
& q=1, \quad \alpha=0, \quad \beta=n-1, \\
& c_{i}=i, \quad b_{i}=(d-i)(n-1), \quad a_{i}=(n-2) i \quad(0 \leq i \leq d), \\
& \theta_{i}=(d-i)(n-1)-i \quad(0 \leq i \leq d) .
\end{aligned}
$$

Example 2.3. The Dual polar graphs [3], [7].
Let $U$ denote a finite vector space with one of the following non-degenerate forms:

| name | $\operatorname{dim}(U)$ | field | form | $\epsilon$ |
| :---: | :---: | :---: | :---: | :---: |
| $B_{d}\left(p^{n}\right)$ | $2 d+1$ | $G F\left(p^{n}\right)$ | quadratic | 1 |
| $C_{d}\left(p^{n}\right)$ | $2 d$ | $G F\left(p^{n}\right)$ | symplectic | 1 |
| $D_{d}\left(p^{n}\right)$ | $2 d$ | $G F\left(p^{n}\right)$ | $\frac{\text { quadratic }}{\text { (Witt index d) }}$ | 0 |
| ${ }^{2} D_{d+1}\left(p^{n}\right)$ | $2 d+2$ | $G F\left(p^{n}\right)$ | $\frac{\text { quadratic }}{\text { (Witt index d) }}$ | 2 |
| ${ }^{2} A_{2 d}\left(p^{n}\right)$ | $2 d+1$ | $G F\left(p^{2 n}\right)$ | Hermitean | $\frac{3}{2}$ |
| ${ }^{2} A_{2 d-1}\left(p^{n}\right)$ | $2 d$ | $G F\left(p^{2 n}\right)$ | Hermitean | $\frac{1}{2}$ |

A subspace of $U$ is called isotropic whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is $d$.
$X=$ set all maximal isotropic subspaces of $U$,
$x y \in R$ iff $\operatorname{dim}(x \cap y)=d-1 \quad(x, y \in X)$,
$\alpha=0, \quad \beta=q^{\epsilon}$,
$c_{i}=\frac{q^{i}-1}{q-1}, \quad a_{i}=\frac{q^{i+\epsilon}-q^{i}-q^{\epsilon}+1}{q-1} \quad(0 \leq i \leq d)$,
$b_{i}=\frac{q^{i+\epsilon}\left(q^{d-i}-1\right)}{q-1} \quad(0 \leq i \leq d-1)$,
$\theta_{i}=\frac{q^{d+\epsilon-i}-q^{\epsilon}-q^{i}+1}{q-1} \quad(0 \leq i \leq d)$,
where
$q=p^{n}, p^{n}, p^{n}, p^{n}, p^{2 n}, p^{2 n}$ respectively.

The following three theorems will be used in the proof of our results.

Theorem 2.4. ([9, Theorem 4.1]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and let $q$ denote a real number at least 1 . Then the following conditions (i), (ii) are equivalent.
(i) $\Gamma$ has a nontrivial cosine sequence $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ such that $\sigma_{i-1}-q \sigma_{i}$ is independent of $i \quad(1 \leq i \leq d)$.
(ii) The intersection numbers of $\Gamma$ are such that $q c_{i}-b_{i}-q\left(q c_{i-1}-b_{i-1}\right)$ is independent of $i \quad(1 \leq i \leq d)$.

Furthermore, if (i), (ii) hold, then

$$
\begin{equation*}
c_{3} \geq\left(c_{2}-q\right)\left(1+q+q^{2}\right) . \tag{2.18}
\end{equation*}
$$

Theorem 2.5. ([9, Theorem 4.2]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$, and let $q$ denote a real number at least 1 . Then the following conditions (i), (ii) are equivalent.
(i) Statements (i), (ii) hold in Theorem 2.4, and $c_{3}=\left(c_{2}-q\right)\left(1+q+q^{2}\right)$.
(ii) There exists $\alpha, \beta \in \mathbb{R}$ such that $\Gamma$ has classical parameters $(d, q, \alpha, \beta)$.

Theorem 2.6. ([2, Theorem 9.4.4]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ with classical parameters $(d, q, 0, \beta)$. Assume the intersection numbers $a_{1}>0$ and $c_{2}>1$. Suppose $\Gamma$ is a near polygon. Then $\Gamma$ is a dual polar graph or a Hamming graph.

## 3 The inequality

In this section we obtain the inequality in Theorem 1.1.
Lemma 3.1. Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency $k$, and intersection numbers $a_{1}>0, c_{2}>1$. Let $\theta_{1}$ denote the second largest eigenvalue of $\Gamma$. Then

$$
\begin{equation*}
\theta_{1} \leq \frac{k-a_{1}-c_{2}}{c_{2}-1} \tag{3.1}
\end{equation*}
$$

Proof. Abbreviate $E=E_{1}$. Let $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ denote the cosine sequence associated with $\theta_{1}$. Fix any two vertices $x, y \in X$ with $\partial(x, y)=2$. We consider the vectors

$$
\begin{align*}
u & =\sum_{z \in A(x, y)} E \hat{z}-\sum_{w \in A(y, x)} E \hat{w},  \tag{3.2}\\
v & =E \hat{x}-E \hat{y} . \tag{3.3}
\end{align*}
$$

By the Cauchy-Schwartz inequality,

$$
\begin{equation*}
\|u\|^{2}\|v\|^{2} \geq\langle u, v\rangle^{2} \tag{3.4}
\end{equation*}
$$

We compute the terms in (3.4). Using (2.11), (3.2), (3.3) we find

$$
\begin{align*}
\|v\|^{2} & =2 m|X|^{-1}\left(\sigma_{0}-\sigma_{2}\right)  \tag{3.5}\\
\langle u, v\rangle & =2 m a_{2}|X|^{-1}\left(\sigma_{1}-\sigma_{2}\right) \tag{3.6}
\end{align*}
$$

We now compute $\|u\|^{2}$. To do this we first discuss the distances between vertices in $A(x, y)$ and vertices in $A(y, x)$. We claim that for all $z \in A(x, y)$, $z$ is adjacent to $c_{2}-1$ vertices in $A(y, x)$ and is at distance 2 from the remaining $a_{2}-c_{2}+1$ vertices in $A(y, x)$. To see this fix $z \in A(x, y)$. Then $\ell:=$ $A(x, z) \cup\{x, z\}$ is a maximal clique; hence there exists a unique vertex $s \in \ell$
with $\partial(s, y)=1$. That is $s \in C(x, y) \cap C(z, y)$. Observe $|C(x, y) \cap C(z, y)|=1$, since any other $s^{\prime} \in C(x, y) \cap C(z, y)$ will cause either $x s s^{\prime} y$ or $s x z s^{\prime}$ to be a $K_{1,2,1}$ subgraph. Hence there are $c_{2}-1$ vertices in $C(z, y) \cap A(y, x)$. Observe for $w \in A(y, x)$ we have $\partial(w, x)=2$ and $\partial(w, s) \leq 2$ so $\partial(w, z) \leq 2$. We have now proved the claim. Using the claim and applying (2.11) we find

$$
\begin{align*}
\|u\|^{2} & =\left\|\sum_{z \in A(x, y)} E \hat{z}\right\|^{2}+\left\|\sum_{w \in A(y, x)} E \hat{w}\right\|^{2}-2\left\langle\sum_{z \in A(x, y)} E \hat{z}, \sum_{w \in A(y, x)} E \hat{w}\right\rangle \\
& =2 m a_{2}|X|^{-1}\left(\sigma_{0}+\left(a_{1}-c_{2}\right) \sigma_{1}+\left(c_{2}-a_{1}-1\right) \sigma_{2}\right) . \tag{3.7}
\end{align*}
$$

Evaluating (3.4) using (3.5)-(3.7) we routinely find

$$
\begin{equation*}
\left(\sigma_{0}+\left(a_{1}-c_{2}\right) \sigma_{1}+\left(c_{2}-a_{1}-1\right) \sigma_{2}\right)\left(\sigma_{0}-\sigma_{2}\right) \geq a_{2}\left(\sigma_{1}-\sigma_{2}\right)^{2} \tag{3.8}
\end{equation*}
$$

Evaluating (3.8) using (2.8), (2.9), (2.12) we obtain

$$
\begin{equation*}
\left(\theta_{1}-k\right)^{2}\left(\theta_{1}\left(a_{1}+1\right)+k\right)\left(k-\theta_{1}\left(c_{2}-1\right)-a_{1}-c_{2}\right) \geq 0 . \tag{3.9}
\end{equation*}
$$

Clearly $\left(\theta_{1}-k\right)^{2}>0$. By (2.14) and since $\theta_{1}>\theta_{d}$ we find $\theta_{1}\left(a_{1}+1\right)+k>0$. Evaluating (3.9) using these comments we find

$$
k-\theta_{1}\left(c_{2}-1\right)-a_{1}-c_{2} \geq 0
$$

and (3.1) follows.

Remark 3.2. Referring to Example 2.2 and Example 2.3, the eigenvalue $\theta_{1}$ satisfies (3.1) with equality.

We comment on the proof of Lemma 3.1.
Lemma 3.3. With the notation of Lemma 3.1, the following (i)-(iii) are equivalent.
(i) Equality is attained in (3.1).
(ii) For all $x, y \in X$ such that $\partial(x, y)=2$,

$$
\begin{equation*}
\sum_{z \in A(x, y)} E \hat{z}-\sum_{w \in A(y, x)} E \hat{w} \in \operatorname{Span}(E \hat{x}-E \hat{y}) \tag{3.10}
\end{equation*}
$$

(iii) There exist $x, y \in X$ such that $\partial(x, y)=2$ and

$$
\begin{equation*}
\sum_{z \in A(x, y)} E \hat{z}-\sum_{w \in A(y, x)} E \hat{w} \in \operatorname{Span}(E \hat{x}-E \hat{y}) . \tag{3.11}
\end{equation*}
$$

Here $E=E_{1}$.
Proof. Observe from the proof of Lemma 3.1that equality is attained in (3.1) if and only if equality is attained in (3.4). We claim $v \neq 0$. This will follow from (3.5) provided we can show $\sigma_{0} \neq \sigma_{2}$. Suppose $\sigma_{0}=\sigma_{2}$. Setting $\theta=\theta_{1}$ and $\sigma_{2}=\sigma_{0}$ in (2.10) and simplifying the result we find $\theta_{1}=-b_{1}-1$. This is inconsistent with (2.14) and $\theta_{1}>\theta_{d}$. We have now shown $\sigma_{0} \neq \sigma_{2}$ and it follows $v \neq 0$. We now see that equality is attained in (3.4) if and only if $u \in \operatorname{Span}(v)$. The result follows.

## 4 The case of equality

In this section we consider the case of equality in (3.1).
Lemma 4.1. Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_{1}>0, c_{2}>1$. Let $\theta_{1}$ denote the second largest eigenvalue of $\Gamma$ and let $\sigma_{0}, \sigma_{1}, \cdots, \sigma_{d}$ denote the corresponding cosine sequence. Suppose equality holds in (3.1). Then $\sigma_{i-1}-q \sigma_{i}$ is independent of $i(1 \leq i \leq d)$, where $q=c_{2}-1$.

Proof. Setting $c_{2}=q+1$ in (3.1) and using $k-a_{1}-1=b_{1}$ we find $\theta_{1}+1=$ $b_{1} q^{-1}$. In particular $\theta_{1} \neq-1$. Observe $\sigma_{1} \neq \sigma_{2}$; otherwise $\sigma_{0}=\sigma_{1}$ by (2.10) forcing $\theta_{1}=k$ by (2.8), a contradiction. Evaluating (2.10) using $\theta_{1}+1=b_{1} q^{-1}$ we find

$$
\begin{equation*}
\frac{\sigma_{0}-\sigma_{1}}{\sigma_{1}-\sigma_{2}}=q . \tag{4.1}
\end{equation*}
$$

Fix two vertices $x, y \in X$ with $\partial(x, y)=2$. Abbreviate $E=E_{1}$. By Lemma3.3 there exists $\lambda \in \mathbb{R}$ such that

$$
\begin{equation*}
\sum_{z \in A(x, y)} E \hat{z}-\sum_{w \in A(y, x)} E \hat{w}=\lambda(E \hat{x}-E \hat{y}) . \tag{4.2}
\end{equation*}
$$

Fix an integer $i \quad(1 \leq i \leq d-1)$ and pick $u \in X$ with $\partial(u, x)=i-1$ and $\partial(u, y)=i+1$. Taking the inner product of $E \hat{u}$ with both sides of (4.2),

$$
\begin{equation*}
a_{2}\left(\sigma_{i}-\sigma_{i+1}\right)=\lambda\left(\sigma_{i-1}-\sigma_{i+1}\right) \tag{4.3}
\end{equation*}
$$

Setting $i=1$ in (4.3) we find $a_{2}\left(\sigma_{1}-\sigma_{2}\right)=\lambda\left(\sigma_{0}-\sigma_{2}\right)$. From (4.1) we find $\sigma_{0}-\sigma_{2}=\left(\sigma_{1}-\sigma_{2}\right)(1+q)$. By these comments $\lambda=a_{2} /(q+1)$. Evaluating (4.3) using this we find

$$
\sigma_{i-1}-q \sigma_{i}=\sigma_{i}-q \sigma_{i+1} \quad(1 \leq i \leq d-1)
$$

From this we find $\sigma_{i-1}-q \sigma_{i}$ is independent of $i$ for $1 \leq i \leq d$.

Lemma 4.2. Let $\Gamma$ denote a near polygon distance-regular graph with $d \geq 3$ and intersection numbers $a_{1}>0, c_{2}>1$. Let $\theta_{1}$ denote the second largest eigenvalue of $\Gamma$ and assume equality holds in (3.1). Then $\Gamma$ has classical parameters ( $d, q, 0, \beta$ ).

Proof. Let the scalar $q$ be as in Lemma 4.1 By Lemma 4.1] we have Theorem[2.4(i) and hence Theorem[2.4(ii). Applying Theorem[2.4(ii) with $i=2,3$ we find

$$
\begin{equation*}
q c_{2}-b_{2}-q\left(q c_{1}-b_{1}\right)=q c_{3}-b_{3}-q\left(q c_{2}-b_{2}\right) \tag{4.4}
\end{equation*}
$$

Simplifying (4.4) using (2.5) and $c_{2}=q+1, a_{2}=a_{1} c_{2}$ we obtain

$$
\begin{equation*}
\left(a_{1}+1+q\right)\left(1+q+q^{2}-c_{3}\right)=a_{3}-a_{1} c_{3} . \tag{4.5}
\end{equation*}
$$

By (2.12) we have $a_{3}=a_{1} c_{3}$ if $d>3$, and by (2.13) we have $a_{3} \geq a_{1} c_{3}$ if $d=3$. In any case $a_{3} \geq a_{1} c_{3}$ so the right-hand side of (4.5) is nonnegative. Also $a_{1}+1+q>0$ since $q=c_{2}-1$. Evaluating (4.5) using these comments we find

$$
\begin{equation*}
c_{3} \leq 1+q+q^{2} . \tag{4.6}
\end{equation*}
$$

By (2.18) and using $c_{2}=1+q$ we find $c_{3} \geq 1+q+q^{2}$. Now apparently $c_{3}=1+q+q^{2}$. We can now check that the assumption $c_{3}=\left(c_{2}-q\right)\left(1+q+q^{2}\right)$ in Theorem [2.5(i) holds. Applying Theorem [2.5] we find there exist real numbers $\alpha, \beta$ such that $\Gamma$ has classical parameters $(d, q, \alpha, \beta)$. By (2.15) we find $c_{2}=(1+q)(1+\alpha)$. By the construction $c_{2}=q+1$. Comparing these equations we find $\alpha=0$.

Proof of Theorem 1.1, The inequality (1.1) is from (3.1).
(i) $\Longrightarrow$ (iii). By Lemma 4.2, $\Gamma$ has classical parameters $(d, q, 0, \beta)$. By this and Theorem [2.6] we find $\Gamma$ is a dual polar graph or a Hamming graph.
(iii) $\Longrightarrow$ (ii) This is immediate from [2, Corollary 8.5.3].
(ii) $\Longrightarrow$ (i) Lemma 3.3(ii) holds by [9, Theorem 3.3], so Lemma 3.3(i) holds and the result follows.

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Paul Terwilliger
Department of Mathematics
University of Wisconsin-Madison
Van Vleck Hall
480 Lincoln Drive

Madison, WI 53706-1388
USA
email: terwilli@math.wisc.edu

Chih-wen Weng
Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsueh Road,
Hsinchu
Taiwan 30050
email: weng@math.nctu.edu.tw


[^0]:    *Department of Mathematics, University of Wisconsin-Madison, USA
    ${ }^{\dagger}$ Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C.

