# ANOTHER REFINEMENT OF THE BENDER-KNUTH (EX-)CONJECTURE 

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#### Abstract

We compute the generating function of column-strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns, $p$ rows of odd length and $k$ parts equal to $n$. This refines both, Krattenthaler's [10] and the author's [5] refinement of the Bender-Knuth (ex-)Conjecture. The result is proved by an extension of the method for proving polynomial enumeration formulas which was introduced by the author in 5 to $q$-quasi-polynomials.


## 1. Introduction

Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}\right)$ be a partition, i.e. $\lambda_{i} \in \mathbb{Z}$ and $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{r} \geq 0$. A strict plane partition of shape $\lambda$ is an array $\Pi=\left(\pi_{i, j}\right)_{1 \leq i \leq r, 1 \leq j \leq \lambda_{i}}$ of non-negative integers such that the rows are weakly decreasing and the columns are strictly decreasing. For instance

| 7 | 6 | 5 | 5 | 2 |
| :--- | :--- | :--- | :--- | :--- |
| 5 | 4 | 2 | 2 |  |
| 4 | 2 |  |  |  |
| 2 | 1 |  |  |  |

is a strict plane partition of shape $(5,4,2,2)$. The norm $n(\Pi)$ of a strict plane partition is defined as the sum of its parts and $\Pi$ is said to be a strict plane partition of the non-negative integer $n(\Pi)$. Thus 47 is the norm of our example. Strict plane partitions and closely related objects have been enumerated subject to a variety of different constraints. In [2, p.50] Bender and Knuth had conjectured that the generating function with respect to the norm of strict plane partitions with at most $c$ columns and parts in $\{1,2, \ldots, n\}$ is equal to

$$
\sum q^{n(\pi)}=\prod_{i=1}^{n} \frac{[c+i ; q]_{i}}{[i ; q]_{i}}
$$

where $[n ; q]=1+q+\cdots+q^{n-1}=\left(1-q^{n}\right) /(1-q)$ and $[a ; q]_{n}=\prod_{i=0}^{n-1}[a+i ; q]$. This conjecture was proved by Andrews [1], Gordon [8], Macdonald [12, Ex. 19, p.53] and Proctor [13, Prop. 7.2]. For related papers, which mostly include generalizations of the Bender-Knuth (ex-)Conjecture, see [3, 4, 5, 9, 10, 14, 17].

In particular, Krattenthaler [10] computed the generating function of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $p$ rows of odd length. On the other hand the author [5] computed the generating function of strict plane partitions with parts in $\{1,2, \ldots, n\}$,
at most $c$ columns and $k$ parts equal to $n$. In this paper we refine these two results. The main result is the following.

Theorem 1. The generating function of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns, $p$ rows of odd length and $k$ parts equal to $n$ is given by

$$
\begin{aligned}
& M_{n, c, p}[k+1 ; q]_{n-1}[k-c-n+1 ; q]_{n-1} q^{k}+L_{n, c, p}\left((-1)^{k} q^{n k}+(-1)^{n} q^{(n-1)(2 c+n) / 2+k}\right. \\
& \times \sum_{i=1}^{n-1}(-1)^{c} q^{\binom{i}{2}} \frac{[k+1 ; q]_{n-1}[k-c-i+1 ; q]_{i-1}[k-c-n+1 ; q]_{n-i-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}} \\
& \left.-q^{\binom{i}{2}} \frac{[k+1 ; q]_{i-1}[k+i+1 ; q]_{n-i-1}[k-c-n+1 ; q]_{n-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}\right)
\end{aligned}
$$

where

$$
L_{n, c, p}= \begin{cases}\left(\frac{q^{\binom{p+1}{2}}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right][c ; q]}{[c+p ; q]_{n}}-\frac{q^{\binom{p}{2}}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right][c+2 n ; q]}{[c+p+1 ; q]_{n}}\right) \frac{[c+1 ; q]_{n-1}[1 ; q]_{n-1}}{2} \prod_{i=1}^{n-1} \frac{[c+2 i+1 ; q]_{n-i}}{[2 i ; q]_{n-i}[2 i ; q]} & 2 \mid c \\
\left(q^{\binom{p+1}{2}}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right]-q^{\binom{p}{2}}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right]\right) \frac{[1 ; q]_{n-1}}{2} \prod_{i=1}^{n-1} \frac{[c+2 i ; q]_{n-i}}{\left.2 i ; q]_{n-i} 22 ; q\right]} & 2 \nmid c\end{cases}
$$

and

$$
\begin{aligned}
& M_{n, c, p}=\frac{(-1)^{n-1} q^{(n-1)(2 c+n) / 2}}{[1 ; q]_{n-2}}
\end{aligned}
$$

In these formulas the notion of the $q$-binomial coefficient is used. It is defined as follows.

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]= \begin{cases}\frac{[n-k+1 ; q]_{k}}{[1 ; q]_{k}} & \text { if } 0 \leq k \leq n \\
0 & \text { otherwise }\end{cases}
$$

At the end of Section 6 we show that Theorem 1 implies Krattenthaler's and the author's refinement of the Bender-Knuth (ex-)Conjecture and with this the Bender-Knuth (ex-)Conjecture itself.

Our method for proving Theorem is an extension of the method for proving polynomial enumeration formulas we have introduced in [5]. It is interesting to note that this elementary method avoids the use of determinants completely, which is quite unusual in the field of plane partition enumeration. The method is divided into the following three steps.
(1) Extension of the combinatorial interpretation. It only makes sense to ask for the number of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $k$ parts equal to $n$ if $k \in\{0,1, \ldots, c\}$. This is because all $n$ 's must be in the first row of the strict plane partition by the columnstrictness. In the first step we find a combinatorial extension of these strict plane partitions to arbitrary integers $k$, i.e. we find new objects
indexed by an arbitrary integer $k$ which are in bijection with strict plane partitions with $k$ parts equal to $n$ if $k \in\{0,1, \ldots, c\}$.
(2) The extending objects are enumerated by a $q$-quasi-polynomial in $k$. With the help of a simple recursion we show that the extending objects are enumerated by a $q$-quasi-polynomial (see Definition (1). Moreover the degree of the $q$-quasi-polynomial is computed.
(3) Exploring properties of the $q$-quasi-polynomial that determine it uniquely. A ( $q$-quasi-) polynomial is determined by a finite number of properties such as zeros or other evaluations. In the last step we derive enough properties of the $q$-quasi-polynomial in order to compute it using the degree estimation from the previous step.
Note that this article contains two types of extensions of the method for proving polynomial enumeration formulas presented in [5]. Firstly, the method is extended to $q$-quasi-polynomials, see Definition 1] More remarkable is, however, the following extension: In [5] we have described a method that is applicable to polynomial enumeration formulas that factorize into distinct linear factors over $\mathbb{Z}$. There the "properties" in the third step are just the integer zeros together with one (easy to compute) non-zero evaluation. (Thus the third step was entitled "Exploring natural linear factors".) In this article we demonstrate that the lack of enough integer zeros can be compensated by other properties of the ( $q$-quasi-) polynomial.

The paper is organized as follows. In Section 2 we give the combinatorial extension of strict plane partitions as proposed in Step 1. In Section 3 we introduce the notion of $q$-quasipolynomials and establish the properties needed in this paper. In Section 4 we show that the generating function of strict plane partitions which is under consideration in this paper is a $q$-quasi-polynomial and we compute its degree (Step 2). In Section 5 we deduce enough properties of the $q$-quasi-polynomial in order to compute it (Step 3). In Section 6 we perform the (complicated) computation and in Section 7 we derive some $q$-summation formulas which are needed in the computation.

Throughout the whole article we use the extended definition of the summation symbol, namely,

$$
\sum_{i=a}^{b} f(i)= \begin{cases}f(a)+f(a+1)+\cdots+f(b) & \text { if } a \leq b  \tag{1.1}\\ 0 & \text { if } b=a-1 \\ -f(b+1)-f(b+2)-\cdots-f(a-1) & \text { if } b+1 \leq a-1\end{cases}
$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain $I$ containing $\mathbb{Q}$ there exists a unique polynomial $q(X)$ over $I$ such that $\sum_{x=0}^{y} p(x)=q(y)$ for all integers $y$. We usually write $\sum_{x=0}^{y} p(x)$ for $q(y)$.

## 2. Extension of the combinatorial interpretation

In this section we establish the combinatorial extension of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $k$ parts equal to $n$ to arbitrary integers $k$. This extension was already introduced in [5, Section 2]. We repeat it here in less detail.

Let $r, n, c$ be integers with $0 \leq r \leq n$. A generalized ( $r, n, c$ ) Gelfand-Tsetlin-pattern (for short: $(r, n, c)$-pattern) is an array $\left(a_{i, j}\right)_{1 \leq i \leq r+1, i-1 \leq j \leq n+1}$ of integers with
(1) $a_{i, i-1}=0$ and $a_{i, n+1}=c$,
(2) if $a_{i, j} \leq a_{i, j+1}$ then $a_{i, j} \leq a_{i-1, j} \leq a_{i, j+1}$
(3) if $a_{i, j}>a_{i, j+1}$ then $a_{i, j}>a_{i-1, j}>a_{i, j+1}$.

The norm of an $(r, n, c)$-pattern is defined as the sum of its parts, where the first and the last part of each row is omitted. A $(3,6, c)$-pattern for example is of the form

$$
\begin{array}{lcccccccccccc} 
& & & 0 & & a_{4,4} & & a_{4,5} & & a_{4,6} & & c & \\
& & 0 & & a_{3,3} & & a_{3,4} & & a_{3,5} & & a_{3,6} & & c \\
& 0 & & a_{2,2} & & a_{2,3} & & a_{2,4} & & a_{2,5} & & a_{2,6} & \\
0 & a_{1,1} & & a_{1,2} & & a_{1,3} & & a_{1,4} & & a_{1,5} & & & \\
0 & a_{1,6} & & c,
\end{array}
$$

such that every entry not in the top row is between its northwest neighbour $w$ and its northeast neighbour $e$, if $w \leq e$ then weakly between, otherwise strictly between. Thus

|  |  |  | 0 |  | 3 |  | -5 |  | 10 |  | 4 |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 0 | 0 | 2 |  | -2 |  | 3 |  | 8 |  | 4 |  |  |
| 0 | 0 | 2 |  | -1 |  | 2 |  | 4 |  | 7 |  | 4 |  |
| 0 | 0 | 0 |  | 1 |  | 2 |  | 5 |  | 6 |  | 4 |  |

is an example of an ( $3,6,4$ )-pattern. Note that a generalized ( $n-1, n, c$ ) Gelfand-Tsetlinpattern $\left(a_{i, j}\right)$ with $0 \leq a_{n, n} \leq c$ is what is said to be a Gelfand-Tsetlin-pattern with $n$ rows and parts in $\{0,1, \ldots, c\}$, see [16, p. 313] or [7, (3)] for the original reference. (Observe that $0 \leq a_{n, n} \leq c$ implies that the third condition in the definition of a generalized Gelfand-Tsetlinpattern never applies.) The following correspondence between Gelfand-Tsetlin-patterns and strict plane partitions is crucial for our paper.

Lemma 1. There exists a norm-preserving bijection between Gelfand-Tsetlin-patterns with $n$ rows, parts in $\{0,1, \ldots, c\}$ and fixed $a_{n, n}=k$, and strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $k$ parts equal to $n$. In this bijection ( $a_{1, n}, a_{1, n-1}, \ldots, a_{1,1}$ ) is the shape of the strict plane partition.

Proof. Given such a Gelfand-Tsetlin-pattern, the corresponding strict plane partition is such that the shape filled by parts greater than $i$ corresponds to the partition given by the $(n-i)$ th row (the top row being the first row) of the Gelfand-Tsetlin-pattern, where the first and the last part of the row in the pattern are omitted. Thus the strict plane partition in the introduction corresponds to the following Gelfand-Tsetlin pattern (first and last parts in the rows are omitted).


Therefore it suffices to compute the generating function with respect to the norm of ( $n-$ $1, n, c)$-patterns with fixed $a_{n, n}=k, 0 \leq k \leq c$, and where exactly $p$ values of $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$
are odd. However, ( $n-1, n, c$ )-patterns are defined for all $a_{n, n} \in \mathbb{Z}$ and thus we have established the combinatorial extension apart from the following technical detail. That is that we actually have to work with a signed enumeration if $a_{n, n} \notin\{0,1, \ldots, c\}$. Therefore we define the sign of a pattern.

A pair $\left(a_{i, j}, a_{i, j+1}\right)$ with $a_{i, j}>a_{i, j+1}$ and $i \neq 1$ is called an inversion of the ( $r, n, c$ )-pattern and $(-1)^{\# \text { of inversions }}$ is said to be the sign of the pattern, denoted by $\operatorname{sgn}(a)$. The $(3,6,4)-$ pattern in the example above has altogether 6 inversions and thus its sign is 1 . We define the following generating function

$$
F_{q}\left(r, n, c, p ; k_{1}, k_{2}, \ldots, k_{n-r}\right)=\left(\sum_{a} \operatorname{sgn}(a) q^{\mathrm{norm}(a)}\right) / q^{k_{1}+k_{2}+\ldots+k_{n-r}}
$$

where the sum is over all $(r, n, c)$-patterns $\left(a_{i, j}\right)$ with top row defined by $k_{i}=a_{r+1, r+i}$ for $i=1, \ldots, n-r$ and such that exactly $p$ of $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$ are odd. It is crucial that for $0 \leq k \leq c F_{q}(n-1, n, c, p ; k) q^{k}$ is the generating function of $(n-1, n, c)$-patterns with $a_{n, n}=k$ and where exactly $p$ of $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$ are odd. This is because an $(n-1, n, c)$-pattern with $0 \leq a_{n, n} \leq c$ has no inversions. Thus $F_{q}(n-1, n, c, p ; k)$ is the quantity we want to compute. It has the advantage that it is well defined for all integers $k$, whereas our original enumeration problem was only defined for $0 \leq k \leq c$.

## 3. $q$-QUASI-POLYNOMIALS AND THEIR PROPERTIES

In the following let $R$ be a ring containing $\mathbb{C}$. A quasi-polynomial (see [15, page 210]) in the variables $X_{1}, X_{2}, \ldots, X_{n}$ over $R$ is an expression of the form

$$
\sum_{\left(m_{1}, m_{2}, \ldots, m_{n}\right) \in \mathbb{Z}^{n}, m_{i} \geq 0} c_{m_{1}, m_{2}, \ldots, m_{n}}\left(X_{1}, X_{2}, \ldots, X_{n}\right) X_{1}^{m_{1}} X_{2}^{m_{2}} \cdots X_{n}^{m_{n}}
$$

where $\left(X_{1}, X_{2}, \ldots, X_{n}\right) \rightarrow c_{m_{1}, m_{2}, \ldots, m_{n}}\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ are periodic functions on $\mathbb{Z}^{n}$ taking values in $R$, that is there exists an integer $t$ with

$$
c_{m_{1}, m_{2}, \ldots, m_{n}}\left(k_{1}, \ldots, k_{i}, \ldots, k_{n}\right)=c_{m_{1}, m_{2}, \ldots, m_{n}}\left(k_{1}, \ldots, k_{i}+t, \ldots, k_{n}\right)
$$

for all $\left(k_{1}, \ldots, k_{n}\right) \in \mathbb{Z}^{n}$ and $i$, and almost all $c_{m_{1}, \ldots, m_{n}}\left(X_{1}, \ldots, X_{n}\right)$ are zero. Let $\left(m_{1}, \ldots, m_{n}\right)$ be with $c_{m_{1}, \ldots, m_{n}}\left(X_{1}, \ldots, X_{n}\right) \neq 0$ such that $m_{1}+\ldots+m_{n}$ is maximal. Then $m_{1}+\ldots+m_{n}$ is said to be the degree of the quasi-polynomial. (The zero-quasi-polynomial is said to be of degree $-\infty$.) The smallest common period of all $c_{m_{1}, \ldots, m_{n}}\left(X_{1}, \ldots, X_{n}\right)$ is said to be the period of the quasi-polynomial. (In this paper we only deal with $q$-quasi-polynomials of period 1 or 2.) In [5], Section 6] we have defined $q$-polynomials. The following definition of $q$-quasi-polynomials is the merge of these two definitions. In this definition let $R_{q}$ denote the ring of quotients with elements from $R[q]$ in the numerator and elements from $\mathbb{C}[q]$ in the denominator.

Definition 1. A q-quasi-polynomial over $R$ in $X_{1}, X_{2}, \ldots, X_{n}$ is a quasi-polynomial over $R_{q}$ in $q^{X_{1}}, q^{X_{2}}, \ldots, q^{X_{n}}$. Let $R_{q q}\left[X_{1}, X_{2}, \ldots, X_{n}\right]$ denote the ring of these $q$-quasi-polynomials.

Observe that $R_{q q}\left[X_{1}, \ldots, X_{n}\right]$ is the ring of $q$-quasi-polynomials in $X_{i}$ over

$$
R_{q q}\left[X_{1}, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n}\right]
$$

(Thus it would have been possible to define $R_{q q}\left[X_{1}, \ldots, X_{n}\right]$ inductively with respect to $n$.) We define $[X ; q]=\left(1-q^{X}\right) /(1-q)$ and $[X ; q]_{n}=\prod_{i=0}^{n-1}[X+i ; q]$. Observe that

$$
\left[X_{1} ; q\right]_{m_{1}}\left[X_{2} ; q\right]_{m_{2}} \cdots\left[X_{n} ; q\right]_{m_{n}},
$$

with $\left(m_{1}, m_{2}, \ldots, m_{n}\right) \in \mathbb{Z}^{n}$ and $m_{i} \geq 0$, is a basis of the $q$-quasi-polynomials over the periodic functions.

The following two properties of polynomials were crucial for our method for proving polynomial enumeration formulas which we have introduced in [5]. Since we want to extend our method to $q$-quasi-polynomials, we have to find $q$-quasi-analogs of these properties.
(1) If $p(X)$ is a polynomial over $R$, then there exists a (unique) polynomial $r(X)$ with $\operatorname{deg} r=\operatorname{deg} p+1$ and

$$
\sum_{x=0}^{y} p(x)=r(y)
$$

for every integer $y$.
(2) If $p(X)$ is a polynomial over $R$ and $a$ is a zero of $p(X)$, then there exists a polynomial $r(X)$ over $R$ with

$$
p(X)=(X-a) r(X)
$$

Regarding the first property we show the following for $q$-quasi-polynomials.
Lemma 2. Let $p(X)$ be a q-quasi-polynomial in $X$ over $R$ with degree $d$ and period $t$. Then $\sum_{x=1}^{y} p(x) q^{x}$ is a q-quasi-polynomial over $R$ in $y$ with degree at most $d+1$ and period at most

In order to prove this lemma we need a definition and another lemma.
Definition 2. Let $\rho \rightarrow f(\rho)$ be a function. Then the $q$-differential-operator $\frac{d}{d_{q} \rho}$ is defined as follows

$$
\frac{d}{d_{q} \rho} f(\rho)=\frac{f(q \rho)-f(\rho)}{\rho(q-1)} .
$$

With $\frac{d}{d_{q} \rho^{n}}$ we denote the $n$-fold application of the operator.
Observe that for a laurent polynomial we have

$$
\begin{equation*}
\frac{d}{d_{q} \rho} \sum_{i=b}^{c} a_{i} \rho^{i}=\sum_{i=b}^{c}[i ; q] a_{i} \rho^{i-1} . \tag{3.1}
\end{equation*}
$$

Note that this is also true if $b>c$.

## Lemma 3.

$$
\sum_{x=0}^{y}[x ; q]_{n} q^{x} \sigma^{x-1}=\frac{d}{d_{q} \sigma^{n}}\left(\frac{\sigma^{n-1}\left((\sigma q)^{y+1}-1\right)}{(\sigma q-1)}\right)
$$

Proof of Lemma 3 , By (3.1) we have the following identity.

$$
\sum_{x=0}^{y}[x ; q]_{n} q^{x} \sigma^{x-1}=\frac{d}{d_{q} \sigma^{n}}\left(\sum_{x=0}^{y} q^{x} \sigma^{x+n-1}\right)
$$

The assertion now follows from

$$
\sum_{x=0}^{y} q^{x} \sigma^{x+n-1}=\frac{\sigma^{n-1}\left((\sigma q)^{y+1}-1\right)}{(\sigma q-1)}
$$

Proof of Lemma 国. Suppose $p(X)$ is a $q$-quasi-polynomial with period $t$. Let $\rho \in \mathbb{C}$ be a primitive $t$-th root of unity. Then $p(X)$ can be expressed as follows

$$
p(X)=p_{0}(X)+\rho^{X} p_{1}(X)+\rho^{2 X} p_{2}(X)+\ldots+\rho^{(t-1) X} p_{t-1}(X)
$$

where $p_{i}(X)$ are $q$-polynomials, i.e. $q$-quasi-polynomials with period 1 . Suppose $d$ is the degree of $p(X)$. Then, for every $i$, we have

$$
p_{i}(X)=\sum_{j=0}^{d} a_{i, j}[X ; q]_{j}
$$

where $a_{i, j}$ are coefficients in $R_{q}$. Thus, by Lemma 3,

$$
\sum_{x=0}^{y} p(x) q^{x}=\sum_{x=0}^{y} \sum_{i=0}^{t-1} \sum_{j=0}^{d} a_{i j}[x ; q]_{j} \rho^{i x} q^{x}=\left.\sum_{i=0}^{t-1} \sum_{j=0}^{d} a_{i, j} \rho^{i} \frac{d}{d_{q} \sigma^{j}}\left(\frac{\sigma^{j-1}\left((\sigma q)^{y+1}-1\right)}{(\sigma q-1)}\right)\right|_{\sigma=\rho^{i}}
$$

The assertion follows after observing that

$$
\left.\frac{d}{d_{q} \sigma^{j}}\left(\frac{\sigma^{j-1}\left((\sigma q)^{y+1}-1\right)}{(\sigma q-1)}\right)\right|_{\sigma=\rho^{i}}
$$

is a $q$-quasi-polynomial in $y$ of degree at most $j+1$.
Next we consider the second important property of polynomials for our method. It suffices to derive an analog for $q$-polynomials. Suppose $p(X)$ is a $q$-polynomial over $R$ and $a$ is an integer zero of $p(X)$. Then there exists a $q$-polynomial $r(X)$ over $R$ with

$$
p(X)=([X ; q]-[a ; q]) r(X)=q^{a}[X-a ; q] r(X) .
$$

The proof follows from the following identity

$$
[X ; q]^{n}-[a ; q]^{n}=([X ; q]-[a ; q]) \sum_{i=0}^{n-1}[X ; q]^{i}[a ; q]^{n-1-i}=q^{a}[X-a ; q] \sum_{i=0}^{n-1}[X ; q]^{i}[a ; q]^{n-1-i}
$$

This property implies that for an integral domain $R$ and distinct zeros $a_{1}, a_{2}, \ldots, a_{r}$ of the $q$-polynomial $p(X)$ there exists a $q$-polynomial $r(X)$ with

$$
p(X)=\left(\prod_{i=1}^{r}\left[X-a_{i} ; q\right]\right) r(X)
$$

This will be fundamental for the " $q$-Lagrange interpolation" we use in Lemma 14 .

## 4. $F_{q}(n-1, n, c, p ; k)$ IS A $q$-QUASI-POLYNOMIAL IN $k$

In this section we show that $F_{q}\left(r, n, c, p ; k_{1}, \ldots, k_{n-r}\right)$ is a $q$-quasi-polynomial in $k_{1}, k_{2}, \ldots, k_{n-r}$ with period 2 . Moreover we show that the degree in $k_{i}$ is at most $2 r$.

The following recursion is fundamental.

$$
\begin{align*}
& F_{q}\left(r, n, c, p ; k_{1}, k_{2}, \ldots, k_{n-r}\right)= \\
& \sum_{l_{1}=0}^{k_{1}} \sum_{l_{2}=k_{1}}^{k_{2}} \sum_{l_{3}=k_{2}}^{k_{3}} \ldots \sum_{l_{n-r}=k_{n-r-1}}^{k_{n-r}} \sum_{l_{n-r+1}=k_{n-r}}^{c} F_{q}\left(r-1, n, c, p ; l_{1}, l_{2}, \ldots, l_{n-r+1}\right) q^{l_{1}+l_{2}+\ldots+l_{n-r+1}} . \tag{4.1}
\end{align*}
$$

Moreover we have

$$
\begin{aligned}
F_{q}\left(0, n, c, p ; k_{1}, \ldots, k_{n}\right)= \begin{cases}1 & \text { if exactly } p \text { of } k_{1}, k_{2}, \ldots, k_{n} \text { are odd } \\
0 & \text { otherwise }\end{cases} \\
\qquad=\sum_{1 \leq i_{1}<i_{2}<\ldots<i_{p} \leq n} \prod_{j=1}^{p} \frac{e_{1,2}\left(k_{i_{j}}\right)}{e_{0,2}\left(k_{i_{j}}\right)} \prod_{j=1}^{n} e_{0,2}\left(k_{j}\right)=: S(n, p)\left(k_{1}, \ldots, k_{n}\right),
\end{aligned}
$$

where $x \rightarrow e_{i, t}(x)$ is the function defined on integers with

$$
e_{i, t}(x)=\left\{\begin{array}{ll}
1 & x \equiv i \bmod t \\
0 & \text { otherwise }
\end{array}=\prod_{0 \leq j \leq t-1, j \neq i} \frac{\rho^{x}-\rho^{j}}{\rho^{i}-\rho^{j}},\right.
$$

where $\rho \in \mathbb{C}$ is a primitive $t$-th root of unity. The identity

$$
F_{q}\left(0, n, c, p ; k_{1}, \ldots, k_{i}, \ldots, k_{n}\right)=F_{q}\left(0, n, c, p ; k_{1}, \ldots, k_{i}+2, \ldots, k_{n}\right)
$$

for all $i, 1 \leq i \leq n$, implies that $F_{q}\left(0, n, c, p ; k_{1}, \ldots, k_{n}\right)$ is a $q$-quasi-polynomial with period 2 . The recursion (4.1) and Lemma 2 implies (inductively with respect to $r$ ) that $F_{q}(r, n, c, p ;$.) is a $q$-quasi-polynomial in $\left(k_{1}, k_{2}, \ldots, k_{n-r}\right)$ with period at most 2 .

For our purpose it is convenient to define the following generalization of $F_{q}(r, n, c, p ;$.).
Definition 3. Let $n, r, r \leq n$, be non-negative integers and $A\left(k_{1}, \ldots, k_{n}\right)$ a function on $\mathbb{Z}^{n}$. We define $G_{q}(r, n, c, A)$ inductively with respect to $r: G_{q}(0, n, c, A)=A$ and

$$
\begin{align*}
& G_{q}(r, n, c, A)\left(k_{1}, \ldots, k_{n-r}\right)= \\
& \quad \sum_{l_{1}=0}^{k_{1}} \sum_{l_{2}=k_{1}}^{k_{2}} \ldots \sum_{l_{n-r+1}=k_{n-r}}^{c} G_{q}(r-1, n, c, A)\left(l_{1}, l_{2}, \ldots, l_{n-r+1}\right) q^{l_{1}+l_{2}+\ldots+l_{n-r+1}} \tag{4.2}
\end{align*}
$$

With this definition we have

$$
F_{q}\left(r, n, c, p ; k_{1}, \ldots, k_{n-r}\right)=G_{q}(r, n, c, S(n, p))\left(k_{1}, \ldots, k_{n-r}\right)
$$

We define

$$
T(n, i)=\sum_{1 \leq j_{1}<j_{2}<\ldots<j_{i} \leq n}(-1)^{k_{j_{1}}+k_{j_{2}}+\ldots+k_{j_{i}}} .
$$

The following lemma shows that $S(n, p)$ is a linear combination of $T(n, 1), T(n, 2), \ldots, T(n, n)$ and $T(n, 0):=1$.

## Lemma 4.

$$
S(n, p)=\frac{1}{2^{n}}\left(\sum_{i=0}^{n} \sum_{l=\max (0, i-n+p)}^{\min (p, i)}(-1)^{l}\binom{i}{l}\binom{n-i}{p-l} T(n, i)\right)
$$

Proof. Set $[n]:=\{1,2, \ldots, n\}$ and fix $P \subseteq[n]$ with $|P|=p$. Then

$$
\begin{aligned}
& \prod_{j \in P} e_{1,2}\left(k_{j}\right) \prod_{j \in[n] \backslash P} e_{0,2}\left(k_{j}\right)=\prod_{j \in P} \frac{1-(-1)^{k_{j}}}{2} \prod_{j \in[n] \backslash P} \frac{1+(-1)^{k_{j}}}{2}= \\
& \quad=\frac{1}{2^{n}} \sum_{i=0}^{n} \sum_{l=\max (0, i-n+p)}^{\min (p, i)}(-1)^{l} \sum_{\substack{1 \leq j_{1}<\ldots<j_{l} \leq n, j_{x} \in P}}(-1)^{k_{j_{1}}+\ldots+k_{j_{l}}} \sum_{\substack{1 \leq m_{1}<\ldots<m_{i-l} \leq n, m_{x} \in[n] \backslash P}}(-1)^{k_{m_{1}}+\ldots+k_{m_{i-l}}},
\end{aligned}
$$

where the second equation follows by expanding the product. In the summation index $i$ counts the number of $\pm(-1)^{k_{x}}$ we choose from the product of the $n$ factors of the form $1 \pm(-1)^{k_{x}}$ and the index $l$ counts the number of $-(-1)^{k_{x}}$ we choose. Observe that

$$
\sum_{\substack{P \subseteq[n],,|\bar{P}|=p}} \sum_{\substack{\leq j_{1}<\ldots<j_{l} \leq n, j_{x} \in P}}(-1)^{k_{j_{1}}+\ldots+k_{j_{l}}} \sum_{\substack{1 \leq m_{1}<\ldots<m_{i-l} \leq n, m_{x} \in[n] \backslash P}}(-1)^{k_{m_{1}}+\ldots+k_{m_{i-l}}}=\binom{i}{l}\binom{n-i}{p-l} T(n, i),
$$

because every $(-1)^{k_{x_{1}}+\ldots+k_{x_{i}}}, 1 \leq x_{1}<\ldots<x_{i} \leq n$, appears with multiplicity $\binom{i}{l}\binom{n-i}{p-l}$ on the left-hand-side, since there are $\binom{i}{l}$ ways to choose the elements from $\left\{x_{1}, \ldots, x_{i}\right\}=: I$ which lie in $P$ and $\binom{n-i}{p-l}$ ways to choose the elements in $[n] \backslash I$ which lie in $P$. The assertion follows.

Lemma 13 from 5 implies that $G_{q}(n-1, n, c, 1)(k)$ is a $q$-polynomial of degree $2 n-2$ at most in $k$. More general we aim to show that the degree of $G_{q}(n-1, n, c, T(n, p))(k)$ in $k$ is at most $2 n-2$ as well. (Thus our result reproves Lemma 13 from [5].) The linearity of $A \rightarrow G_{q}(r, n, c, A)$ and Lemma 4 then implies that the degree of $G_{q}(n-1, n, c, S(n, p))$ is at most $2 n-2$ in $k$.

In fact we show that the degree of $G_{q}(r, n, c, T(n, p))$ in $k_{i}$ is at most $2 r$. This degree estimation is rather complicated. Assume by induction with respect to $r$ that the degree of $G_{q}(r-1, n, c, T(n, p))\left(k_{1}, \ldots, k_{n-r}\right)$ in $k_{i}$ is at most $2 r-2$ as well as the degree in $k_{i+1}$. The degree of $G_{q}(r, n, c, T(n, p))$ in $k_{i}$ is at most the degree of

$$
\sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} G_{q}(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right)
$$

in $k_{i}$ with $k_{0}=0$ and $k_{n-r+1}=c$. By Lemma 2 this allows us to conclude easily that the degree of $G_{q}(r, n, c, T(n, p))$ in $k_{i}$ is at most $4 r-2$, however, we want to establish that the degree is at most $2 r$. The following lemma is fundamental for this purpose. In order to state it we need to define an operator $D_{i}$ which is crucial for the analysis of (4.2).
Definition 4. Let $G\left(k_{1}, \ldots, k_{m}\right)$ be a function in $m$ variables and $1 \leq i \leq m-1$. We set

$$
\begin{aligned}
& D_{i} G\left(k_{1}, \ldots, k_{m}\right)= \\
& \quad G\left(k_{1}, \ldots, k_{i-1}, k_{i}, k_{i+1}, k_{i+2}, \ldots, k_{m}\right)+G\left(k_{1}, \ldots, k_{i-1}, k_{i+1}+1, k_{i}-1, k_{i+2}, \ldots, k_{m}\right)
\end{aligned}
$$

The following lemma shows the importance of this operator for the degree estimation.
Lemma 5. Let $F\left(x_{1}, x_{2}\right)$ be a q-quasi-polynomial in $x_{1}$ and $x_{2}$ which is in $x_{1}$ as well as in $x_{2}$ of degree at most $R$. Moreover assume that $D_{1} F\left(x_{1}, x_{2}\right)$ is of degree at most $R$ as a q-quasipolynomial in $x_{1}$ and $x_{2}$, i.e. the linear combination of "monomials" $\left[x_{1} ; q\right]_{m}\left[x_{2} ; q\right]_{n} \rho_{1}^{x_{1}} \rho_{2}^{x_{2}}$ with $m+n \leq R$ and where $\rho_{1}$ and $\rho_{2}$ are roots of unity. Then $\sum_{x_{1}=a}^{y} \sum_{x_{2}=y}^{b} F\left(x_{1}, x_{2}\right) q^{x_{1}+x_{2}}$ is of degree at most $R+2$ in $y$.

Proof. Set $F_{1}\left(x_{1}, x_{2}\right)=D_{1} F\left(x_{1}, x_{2}\right) / 2$ and $F_{2}\left(x_{1}, x_{2}\right)=\left(F\left(x_{1}, x_{2}\right)-F\left(x_{2}+1, x_{1}-1\right)\right) / 2$. Clearly $F\left(x_{1}, x_{2}\right)=F_{1}\left(x_{1}, x_{2}\right)+F_{2}\left(x_{1}, x_{2}\right)$. Observe that $F_{2}\left(x_{2}+1, x_{1}-1\right)=-F_{2}\left(x_{1}, x_{2}\right)$. Thus $F_{2}\left(x_{1}, x_{2}\right)$ is a linear combination of expressions of the form

$$
\left[x_{1} ; q\right]_{m}\left[x_{2}+1 ; q\right]_{n} \rho_{1}^{x_{1}-1} \rho_{2}^{x_{2}}-\left[x_{1} ; q\right]_{n}\left[x_{2}+1 ; q\right]_{m} \rho_{1}^{x_{2}} \rho_{2}^{x_{1}-1}
$$

with $m, n \leq R$ and where $\rho_{1}$ and $\rho_{2}$ are roots of unity. We set

$$
c(y, n, \rho)=\frac{d}{d_{q} \rho^{n}}\left(\frac{\rho^{n-1}\left((\rho q)^{y+1}-1\right)}{(\rho q-1)}\right)
$$

Lemma 3 implies

$$
\begin{aligned}
& \sum_{x_{1}=a}^{y} \sum_{x_{2}=y}^{b}\left(\left[x_{1} ; q\right]_{m}\left[x_{2}+1 ; q\right]_{n} \rho_{1}^{x_{1}-1} \rho_{2}^{x_{2}}-\left[x_{1} ; q\right]_{n}\left[x_{2}+1 ; q\right]_{m} \rho_{1}^{x_{2}} \rho_{2}^{x_{1}-1}\right) q^{x_{1}+x_{2}+1} \\
& =\left(c\left(y, m, \rho_{1}\right)-c\left(a-1, m, \rho_{1}\right)\right)\left(c\left(b+1, n, \rho_{2}\right)-c\left(y, n, \rho_{2}\right)\right) \\
& \quad-\left(c\left(y, n, \rho_{2}\right)-c\left(a-1, n, \rho_{2}\right)\right)\left(c\left(b+1, m, \rho_{1}\right)-c\left(y, m, \rho_{1}\right)\right) \\
& =c\left(y, m, \rho_{1}\right) c\left(b+1, n, \rho_{2}\right)-c\left(a-1, m, \rho_{1}\right) c\left(b+1, n, \rho_{2}\right)+c\left(a-1, m, \rho_{1}\right) c\left(y, n, \rho_{2}\right)- \\
& c\left(y, n, \rho_{2}\right) c\left(b+1, m, \rho_{1}\right)+c\left(a-1, n, \rho_{2}\right) c\left(b+1, m, \rho_{1}\right)-c\left(a-1, n, \rho_{2}\right) c\left(y, m, \rho_{1}\right)
\end{aligned}
$$

Observe that $c\left(x, n, \rho_{1}\right)$ is a $q$-quasi-polynomial in $x$ of degree at most $n+1$ and thus

$$
\sum_{x_{1}=a}^{y} \sum_{x_{2}=y}^{b} F_{2}(x, y) q^{x+y}
$$

is of degree at most $R+1$ in $y$. By the assumption in the lemma $\sum_{x_{1}=a}^{y} \sum_{x_{2}=y}^{b} F_{1}(x, y) q^{x+y}$ is of degree at most $R+2$ in $y$ and the assertion follows.

Lemma 6. Let $m$ be a positive integer, $1 \leq i \leq m$ and $G(\mathbf{l})$ be a function in $\mathbf{l}=\left(l_{1}, \ldots, l_{m}\right)$. Then

$$
\begin{aligned}
& D_{i} \sum_{l_{1}=k_{1}}^{k_{2}} \sum_{l_{2}=k_{2}}^{k_{3}} \ldots \sum_{l_{m}=k_{m}}^{k_{m+1}} G\left(l_{1}, \ldots, l_{m}\right) \\
&=-\frac{1}{2}\left(\sum_{l_{1}=k_{1}}^{k_{2}} \ldots\right.
\end{aligned} \begin{aligned}
& \sum_{l_{i-2}=k_{i-2}}^{k_{i-1}} \sum_{l_{i-1}=k_{i}+1}^{k_{i+1}+1} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}}^{k_{i+2}} \sum_{k_{i}-1}^{k_{i+3}} \ldots \sum_{l_{i+2}=k_{i+2}}^{k_{m+1}} D_{l_{m}=k_{m}} G(\mathbf{l}) \\
& \\
& \\
& \left.+\sum_{l_{1}=k_{1}}^{k_{2}} \ldots \sum_{l_{i-2}=k_{i-2}}^{k_{i-1}} \sum_{l_{i-1}}^{k_{i}} \sum_{k_{i-1}}^{k_{i+1}} \sum_{l_{i}=k_{i}}^{k_{i+1}-1} \sum_{l_{i+1}=k_{i}-1}^{k_{i+3}} \ldots \sum_{l_{i+2}=k_{i+2}}^{k_{m+1}} D_{l_{m}=k_{m}} G(\mathbf{l})\right),
\end{aligned}
$$

with $D_{0} G(\mathbf{l})=0$ and $D_{m} G(\mathbf{l})=0$.
Proof. We set

$$
g\left(l_{i-1}, l_{i}, l_{i+1}\right)=\sum_{l_{1}=k_{1}}^{k_{2}} \ldots \sum_{l_{i-2}=k_{i-2}}^{k_{i-1}} \sum_{l_{i+2}=k_{i+2}}^{k_{i+3}} \ldots \sum_{l_{m}=k_{m}}^{k_{m+1}} G\left(l_{1}, \ldots, l_{m}\right) .
$$

It suffices to show the following.

$$
\begin{align*}
& \sum_{l_{i-1}=k_{i-1}}^{k_{i}} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i+1}}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right)+ \sum_{l_{i-1}=k_{i-1}}^{k_{i+1}+1} \sum_{l_{i}=k_{i+1}+1}^{k_{i}-1} \sum_{l_{i+1}=k_{i}-1}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right) \\
&=-\frac{1}{2}\left(\sum_{l_{i-1}=k_{i}+1}^{k_{i+1}+1} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right)+g\left(l_{i}+1, l_{i-1}-1, l_{i+1}\right)\right. \\
&\left.+\sum_{l_{i-1}=k_{i-1}}^{k_{i}} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+1}-1} g\left(l_{i-1}, l_{i}, l_{i+1}\right)+g\left(l_{i-1}, l_{i+1}+1, l_{i}-1\right)\right) \tag{4.3}
\end{align*}
$$

By (1.1) the left-hand-side of this equation is equal to

$$
\begin{array}{r}
\sum_{l_{i-1}=k_{i-1}}^{k_{i}} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i+1}}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right)-\sum_{l_{i-1}=k_{i-1}}^{k_{i+1}+1} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right) \\
=\sum_{l_{i-1}=k_{i-1}}^{k_{i}} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i+1}}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right)-\sum_{l_{i-1}=k_{i-1}}^{k_{i}} \sum_{l_{i}=k_{i}}^{k_{i+1}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right) \\
-\sum_{k_{i+1}+1}^{k_{i+1}} \sum_{k_{i+1}}^{k_{i+1}} \sum_{k_{i+2}}^{k_{i}} g\left(l_{i-1}, l_{i}, l_{i+1}\right) \\
=-\sum_{l_{i}=k_{i}}^{k_{i}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+1}} \sum_{k_{i}}^{k_{i+1}-1} g\left(l_{i-1}, l_{i}, l_{i+1}\right)-\sum_{l_{i=k}}^{k_{i}=k_{i}+1} \sum_{l_{i+1}=k_{i}-1}^{k_{i}} \sum_{l_{i+1}=k_{i}-1}^{k_{i+2}} g\left(l_{i-1}, l_{i}, l_{i+1}\right) .
\end{array}
$$

The last expression is obviously equal to the right-hand-side of (4.3) and the assertion of the lemma is proved.

We need another definition before we are able to prove the key-lemma for the degree estimation.

Definition 5. Let $r$ be a non-negative integer and $B(x, y)$ a function in $x$ and $y$. We define $D(r, B)(x, y)$ recursively: $D(0, B)(x, y)=B(x, y)$ and

$$
D(r, B)(x, y)=\sum_{x^{\prime}=x+1}^{y+1} \sum_{y^{\prime}=x}^{y} D(r-1, B)\left(x^{\prime}, y^{\prime}\right) q^{x^{\prime}+y^{\prime}}
$$

The following lemma establishes a recursion which expresses $D_{i} G_{q}(r, n, c, A)$ in terms of $G_{q}\left(r, n-2, c+2, A_{j}^{\prime}\right)$ and $D\left(r, B_{j}\right)$ if $A$ fulfills a certain "decomposition condition".

Lemma 7. Let $n, r$ be integers with $0 \leq r \leq n$ and $A\left(k_{1}, \ldots, k_{n}\right)$ a function on $\mathbb{Z}^{n}$. Assume that there exist two families of functions $\left(B_{j}(x, y)\right)_{1 \leq j \leq m}$ and $\left(A_{j}^{\prime}\left(k_{1}, \ldots, k_{n-2}\right)\right)_{1 \leq j \leq m}$ with the property that

$$
D_{i} A\left(k_{1}, \ldots, k_{n}\right)=\sum_{j=1}^{m} B_{j}\left(k_{i}+i, k_{i+1}+i\right) A_{j}^{\prime}\left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n}+2\right)
$$

for all $i, 1 \leq i \leq n-1$. Then

$$
\begin{aligned}
D_{i} G(r, n, c, A)\left(k_{1}, \ldots, k_{n-r}\right)= & \sum_{j=1}^{m} \frac{(-1)^{r}}{2^{r}} q^{r(-2 n+r+1)} D\left(r, B_{j}\right)\left(k_{i}+i, k_{i+1}+i\right) \\
& \times G\left(r, n-2, c+2, A_{j}^{\prime}\right)\left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n-r}+2\right) .
\end{aligned}
$$

Proof. We show the assertion by induction with respect to $r$. For $r=0$ there is nothing to prove. Thus we assume $r>0$. By the induction hypothesis we may assume that

$$
\begin{array}{r}
D_{i} G(r-1, n, c, A)\left(l_{1}, \ldots, l_{n-r+1}\right)=\sum_{j=1}^{m} \frac{(-1)^{r-1}}{2^{r-1}} q^{(r-1)(-2 n+r)} D\left(r-1, B_{j}\right)\left(l_{i}+i, l_{i+1}+i\right) \\
\times G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i-1}, l_{i+2}+2, \ldots, l_{n-r+1}+2\right) \tag{4.4}
\end{array}
$$

and

$$
\begin{align*}
& D_{i+1} G(r-1, n, c, A)\left(l_{1}, \ldots, l_{n-r+1}\right)= \\
& \qquad \begin{array}{l}
\sum_{j=1}^{m} \frac{(-1)^{r-1}}{2^{r-1}}
\end{array} q^{(r-1)(-2 n+r)} D\left(r-1, B_{j}\right)\left(l_{i+1}+i+1, l_{i+2}+i+1\right) \\
& \quad \times G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i}, l_{i+3}+2, \ldots, l_{n-r+1}+2\right) . \tag{4.5}
\end{align*}
$$

By Lemma 6 we have

$$
\begin{aligned}
& D_{i} G(r, n, c, A)\left(k_{1}, \ldots, k_{n-r}\right)= \\
& -\frac{1}{2}\left(\sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i-1}=k_{i-2}}^{k_{i-1}} \sum_{l_{i}=k_{i}+1}^{k_{i+1}+1} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} \sum_{l_{i+2}=k_{i}-1}^{k_{i+2}} \ldots \sum_{l_{n-r+1}=k_{n-r}}^{c} D_{i} G(r-1, n, c, A)(\mathbf{l}) q^{l_{1}+\ldots+l_{n-r+1}}\right. \\
& \left.\quad+\sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} \sum_{l_{i+2}=k_{i}-1}^{k_{i+1}-1} \sum_{l_{i+3}=k_{i+2}}^{k_{i+3}} \ldots \sum_{l_{n-r+1}=k_{n-r}}^{c} D_{i+1} G(r-1, n, c, A)(\mathbf{l}) q^{l_{1}+\ldots+l_{n-r+1}}\right)
\end{aligned}
$$

In this expression we replace $D_{i} G(r-1, n, c, A)(\mathbf{l})$ by (4.4) and $D_{i+1} G(r-1, n, c, A)(\mathbf{l})$ by (4.5), and obtain

$$
\begin{aligned}
& \frac{(-1)^{r}}{2^{r}} q^{(r-1)(-2 n+r)} \sum_{j=1}^{m}\left(\sum_{l_{i}=k_{i}+1}^{k_{i+1}+1} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} D\left(r-1, B_{j}\right)\left(l_{i}+i, l_{i+1}+i\right) q^{l_{i}+l_{i+1}}\right. \\
& \times \sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i-1}=k_{i-2}}^{k_{i-1}} \sum_{l_{i+2}=k_{i}-1}^{k_{i+2}} \ldots \sum_{l_{n-r+1}=k_{n-r}}^{c} \\
& G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i-1}, l_{i+2}+2, \ldots, l_{n-r+1}+2\right) q^{l_{1}+\ldots+l_{i-1}+l_{i+2}+\ldots+l_{n-r+1}} \\
& +\sum_{l_{i+1}=k_{i}}^{k_{i+1}} \sum_{l_{i+2}=k_{i}-1}^{k_{i+1}-1} D\left(r-1, B_{j}\right)\left(l_{i+1}+i+1, l_{i+2}+i+1\right) q^{l_{i+1}+l_{i+2}} \\
& \times \sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+3}=k_{i+2}}^{k_{i+3}} \ldots \sum_{l_{n-r+1}=k_{n-r}}^{c} \\
& \left.G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i}, l_{i+3}+2, \ldots, l_{n-r+1}+2\right) q^{l_{1}+\ldots+l_{i}+l_{i+3}+\ldots+l_{n-r+1}}\right) \\
& =\frac{(-1)^{r}}{2^{r}} q^{r(-2 n+r+1)} \sum_{j=1}^{m} D\left(r, B_{j}\right)\left(k_{i}+i, k_{i+1}+i\right) \\
& \times\left(\sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i-1}=k_{i-2}}^{k_{i-1}} \sum_{l_{i+2}=k_{i}+1}^{k_{i+2}+2} \sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2} \ldots \sum_{l_{n-r+1}=k_{n-r}+2}^{c+2}\right. \\
& G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right) q^{l_{1}+\ldots+l_{i-1}+l_{i+2}+\ldots+l_{n-r+1}} \\
& +\sum_{l_{1}=0}^{k_{1}} \ldots \sum_{l_{i-1}=k_{i-2}}^{k_{i-1}} \sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+3}=k_{i+2}+2}^{k_{i+3}+2} \ldots \sum_{l_{n-r+1}=k_{n-r}+2}^{c+2} \\
& \left.G\left(r-1, n-2, c+2, A_{j}^{\prime}\right)\left(l_{1}, \ldots, l_{i}, l_{i+3}, \ldots, l_{n-r+1}\right) q^{l_{1}+\ldots+l_{i}+l_{i+3}+\ldots+l_{n-r+1}}\right)
\end{aligned}
$$

The last expression is obviously equal to the right-hand-side of the equation in the statement of the lemma.

In the next lemma we give a bound for the degree of $D(r, B)(x, y)$.
Lemma 8. Suppose $B(x, y)$ is a q-quasi-polynomial in $x$ and $y$ of degree $d$, i.e. the linear combination of terms of the form $[x ; q]_{m}[y ; q]_{n} \rho_{1}^{x} \rho_{2}^{y}$ with $m+n \leq d$ and $\rho_{1}, \rho_{2}$ are roots of unity. Then $D(r, B)(x, y)$ is of degree at most $2 r+d$ in $x$ and $y$.

Proof. By induction with respect to $r$ it suffices to show that

$$
\sum_{x^{\prime}=x+1}^{y+1} \sum_{y^{\prime}=x}^{y}\left[x^{\prime} ; q\right]_{m}\left[y^{\prime} ; q\right]_{n} \rho_{1}^{x^{\prime}-1} \rho_{2}^{y^{\prime}-1} q^{x^{\prime}+y^{\prime}}
$$

is of degree at most $m+n+2$ in $x$ and $y$. Using the notation from Lemma 2 we see that this double sum is equal to

$$
\begin{array}{r}
\left(\sum_{x^{\prime}=x+1}^{y+1}\left[x^{\prime} ; q\right]_{m} \rho_{1}^{x^{\prime}-1} q^{x^{\prime}}\right)\left(\sum_{y^{\prime}=x}^{y}\left[y^{\prime} ; q\right]_{n} \rho_{2}^{y^{\prime}-1} q^{y^{\prime}}\right) \\
=\left(c\left(y+1, m, \rho_{1}\right)-c\left(x, m, \rho_{1}\right)\right)\left(c\left(y, n, \rho_{2}\right)-c\left(x-1, n, \rho_{2}\right)\right)
\end{array}
$$

and the assertion follows.
In order to apply Lemma 7 to our situation we show that $T(n, p)\left(k_{1}, \ldots, k_{n}\right)$ has the "decomposition property" from Lemma 7 Observe that

$$
\begin{aligned}
D_{i} T(n, p)=2 T(n-2, p) & \left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n}+2\right) \\
& +(-1)^{k_{i}+k_{i+1}} 2 T(n-2, p-2)\left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n}+2\right)
\end{aligned}
$$

where $T(n, p)=0$ if $p<0$ or $p>n$. Thus, by Lemma $\mathbf{Z}$

$$
\begin{aligned}
& D_{i} G_{q}(r, n, c, T(n, p))=\frac{(-1)^{r}}{2^{r}} q^{r(-2 n+r+1)}\left(D(r, 1)\left(k_{i}+i, k_{i+1}+i\right)\right. \\
& \quad \times G_{q}(r, n-2, c+2,2 T(n-2, p))\left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n-r}+2\right) \\
& \quad \quad+D\left(r,(-1)^{k_{i}+k_{i+1}}\right)\left(k_{i}+i, k_{i+1}+i\right) \\
& \left.\quad \times G_{q}(r, n-2, c+2,2 T(n-2, p-2))\left(k_{1}, \ldots, k_{i-1}, k_{i+2}+2, \ldots, k_{n-r}+2\right)\right)
\end{aligned}
$$

By Lemma $\quad D(r, 1)\left(k_{i}, k_{i+1}\right)$ as well as $D\left(r,(-1)^{k_{i}+k_{i+1}}\right)\left(k_{i}, k_{i+1}\right)$ are $q$-quasi-polynomials in $k_{i}$ and $k_{i+1}$ of degree at most $2 r$ and thus the same is true for $D_{i} G_{q}(r, n, c, T(n, p))$. Finally we show that this implies that $G_{q}(r, n, c, T(n, p))$ is a $q$-quasi-polynomial in $k_{i}$ of degree at most $2 r$ for all $i$.

Lemma 9. Let $n$, $r$ be positive integers, $r \leq n$ and $0 \leq p \leq n$. Then $G_{q}(r, n, c, T(n, p))$ is a $q$-quasi-polynomial in $k_{i}$ of degree at most $2 r$ for $i=1,2, \ldots, n-r$.

Proof. We show the assertion by induction with respect to $r$. For $r=0$ there is nothing to prove. We assume that $r>0$ and that the assertion is true for $G_{q}(r-1, n, c, T(n, p))$. The degree of $D_{i} G_{q}(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right)$ as a $q$-quasi-polynomial in $l_{i}$ and $l_{i+1}$ is at most $2 r-2$. Therefore, by Lemma 5 the degree of

$$
\sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} G_{q}(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right) q^{l_{i}+l_{i+1}}
$$

in $k_{i}$ is at most $2 r$. By (4.2) the same is true for the degree of $G_{q}(r, n, c, T(n, p))$ in $k_{i}$.
Corollary 1. Let $n$ be a positive integer and $0 \leq p \leq n$. Then $F_{q}(n-1, n, c, p ; k)$ is a q-quasipolynomial over $\mathbb{C}$ of degree at most $2 n-2$ in $k$.
5. Exploring properties of the $q$-QUASI-POLYNOMIAL $F_{q}(n-1, n, c, p ; k)$

First we observe that $F_{q}(n-1, n, c, p ; k)$ is zero for $k=-1,-2, \ldots,-n+1$ and $k=c+1, c+$ $2, \ldots, c+n-1$.

Lemma 10. Let $r, n, p$ be integers, $0 \leq r \leq n$ and $0 \leq p \leq n$. Then $F_{q}(r, n, c, p ;$.) is zero for $k_{1}=-1,-2, \ldots,-r$ and $k_{n-r}=c+1, c+2, \ldots, c+r$.

Proof. It suffices to show that there exists no $(r, n, c)$-pattern with first row

$$
\left(0, k_{1}, \ldots, k_{n-r}, c\right)
$$

if $k_{1}=-1,-2, \ldots,-r$ or $k_{n-r}=c+1, c+2, \ldots, c+r$. Indeed, suppose $\left(a_{i, j}\right)$ is an $(r, n, c)-$ pattern with $a_{r+1, r+1} \in\{-1,-2, \ldots,-r\}$. In particular we have $0>a_{r+1, r+1}$ and thus the definition of $(r, n, c)$-patterns implies that $0>a_{r, r}>a_{r+1, r+1}$. In a similar way we obtain $0>a_{1,1}>a_{2,2}>\ldots>a_{r, r}>a_{r+1, r+1}$. This is, however, a contradiction, since there exist no $r$ distinct integers between 0 and $a_{r+1, r+1}$. The case that $a_{r+1, n} \in\{c+1, c+2, \ldots, c+r\}$ is similar.

The zeros in Lemma 10 do not determine the $q$-quasi-polynomial $F_{q}(n-1, n, c, p ; k)$ uniquely and thus we need additional properties. To this end we have the following two lemmas.

Lemma 11. Let $r$ be a non-negative integer. Then we have

$$
D\left(r,(-1)^{x+y}\right)=(-1)^{x+y} q^{r(x+y)} C+T(x, y)
$$

where $C \in \mathbb{Q}(q)$ and $T(x, y)$ is a $q$-polynomial in $x$ and $y$ over $\mathbb{Q}$.
Proof. The assertion follows from the following identity by induction with respect to $r$.

$$
\sum_{x^{\prime}=x+1}^{y+1} \sum_{y^{\prime}=x}^{y}(-1)^{x^{\prime}+y^{\prime}} Q^{x^{\prime}+y^{\prime}}=\frac{-Q^{2 x+1}-Q^{2 y+3}-2(-1)^{x+y} Q^{x+y+2}}{(1+Q)^{2}}
$$

Suppose $p(X)$ is a $q$-quasi-polynomial in $X$ with period 1 or 2 . Then there exist unique $q$-polynomials $p_{1}(X)$ and $p_{2}(X)$ with the property that

$$
p(X)=(-1)^{X} p_{1}(X)+p_{2}(X)
$$

We say that $p_{1}(X)$ is the signed part of $p(X)$, in symbols $\mathrm{SP}_{X}(p(X))=p_{1}(X)$. The following lemma shows that the signed part of the $q$-quasi-polynomials $F_{q}(r, n, c, p ;$.) have a quite simple structure.

Lemma 12. Let $r, n, i, p$ be integers, $r$ non-negative, $n$ positive, $1 \leq i<n-r$ and $0 \leq p \leq n$. Then

$$
\mathrm{SP}_{k_{i}} F_{q}\left(r, n, c, p ; k_{1}, \ldots, k_{n-r}\right) / q^{r k_{i}}
$$

is independent of $k_{i}$.
Proof. We show the assertion by induction with respect to $r$. For $r=0$ there is nothing to prove. Let $r>0$. It suffices to prove that

$$
\begin{equation*}
\mathrm{SP}_{k_{i}}\left(\sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}} D_{i} G(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right) q^{l_{i}+l_{i+1}}\right) / q^{r k_{i}} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{align*}
\mathrm{SP}_{k_{i}}\left(\sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}}\right. & \left(G(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{i}, l_{i+1}, \ldots, l_{n-r+1}\right)\right. \\
& \left.\left.\quad-G(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{i+1}+1, l_{i}-1, \ldots, l_{n-r+1}\right)\right) q^{l_{i}+l_{i+1}}\right) / q^{r k_{i}} \tag{5.2}
\end{align*}
$$

are independent of $k_{i}$, where $k_{0}=0$ and $k_{n-r+1}=c$. By Lemma 7 and Lemma $11 D_{i} G(r-$ $1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right)$ is of the form

$$
(-1)^{l_{i}+l_{i+1}} q^{(r-1)\left(l_{i}+l_{i+1}\right)} H\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right)+T\left(l_{1}, \ldots, l_{n-r+1}\right),
$$

where $H\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right)$ is a $q$-quasi-polynomial, $T\left(l_{1}, \ldots, l_{n-r+1}\right)$ is a $q$-quasipolynomial in $\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right)$ and a $q$-polynomial in $l_{i}$ and $l_{i+1}$. Therefore $T$ does not contribute to the signed part of (5.1). Moreover we have

$$
\begin{aligned}
& \sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}}(-1)^{l_{i}+l_{i+1}} q^{(r-1)\left(l_{i}+l_{i+1}\right)} H\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right) q^{l_{i}+l_{i+1}} \\
&=\left.\frac{1}{(1+} q^{r}\right)^{2} \\
& \quad\left((-1)^{k_{i}} q^{r k_{i}}(-1)^{k_{i-1}} q^{r k_{i-1}}+(-1)^{k_{i}} q^{r k_{i}}(-1)^{k_{i+1}} q^{r\left(k_{i+1}+2\right)}\right. \\
&\left.\quad \quad+q^{r\left(2 k_{i}+1\right)}+(-1)^{k_{i-1}+k_{i+1}} q^{r\left(k_{i-1}+k_{i+1}+1\right)}\right) H\left(l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r+1}\right)
\end{aligned}
$$

and the first assertion follows. For the second assertion observe that by the induction hypothesis $G(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{n-r+1}\right)$ is a linear combination of expressions of the form

$$
\begin{gathered}
{\left[l_{i} ; q\right]_{m}\left[l_{i+1}+1 ; q\right]_{n},} \\
q^{(r-1) l_{i}}(-1)^{l_{i}-1}\left[l_{i+1}+1 ; q\right]_{n}
\end{gathered}
$$

and

$$
q^{(r-1) l_{i}}(-1)^{l_{i}-1} q^{(r-1)\left(l_{i+1}+1\right)}(-1)^{l_{i+1}}
$$

over $R_{q q}\left[l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r}\right]$ Therefore

$$
\begin{align*}
G(r-1, n, c, T(n, p))\left(l_{1}, \ldots,\right. & \left.l_{i}, l_{i+1}, \ldots, l_{n-r+1}\right) \\
& -G(r-1, n, c, T(n, p))\left(l_{1}, \ldots, l_{i+1}+1, l_{i}-1, \ldots, l_{n-r+1}\right) \tag{5.3}
\end{align*}
$$

is a linear combination of expressions of the form

$$
\begin{gather*}
{\left[l_{i} ; q\right]_{m}\left[l_{i+1}+1 ; q\right]_{n}-\left[l_{i} ; q\right]_{n}\left[l_{i+1}+1 ; q\right]_{m}}  \tag{5.4}\\
q^{(r-1) l_{i}}(-1)^{l_{i}-1}\left[l_{i+1}+1 ; q\right]_{n}-\left[l_{i} ; q\right]_{n} q^{(r-1)\left(l_{i+1}+1\right)}(-1)^{l_{i+1}} \tag{5.5}
\end{gather*}
$$

and

$$
\begin{equation*}
q^{(r-1) l_{i}}(-1)^{l_{i}-1} q^{(r-1)\left(l_{i+1}+1\right)}(-1)^{l_{i+1}}-q^{(r-1) l_{i}}(-1)^{l_{i}-1} q^{(r-1)\left(l_{i+1}+1\right)}(-1)^{l_{i+1}}=0 \tag{5.6}
\end{equation*}
$$

over $R_{q q}\left[l_{1}, \ldots, l_{i-1}, l_{i+2}, \ldots, l_{n-r}\right]$. Expressions of the form (5.4) do not contribute to the signed part of (5.2). For expressions of the form 5.5] observe that

$$
\begin{aligned}
& \sum_{l_{i}=k_{i-1}}^{k_{i}} \sum_{l_{i+1}=k_{i}}^{k_{i+1}}\left(q^{(r-1) l_{i}}(-1)^{l_{i}-1}\left[l_{i+1}+1 ; q\right]_{n}-\left[l_{i} ; q\right]_{n} q^{(r-1)\left(l_{i+1}+1\right)}(-1)^{l_{i+1}}\right) q^{l_{i}+l_{i+1}+1} \\
& =-\frac{(-1)^{k_{i-1}} q^{r} k_{i-1}+q^{r}(-1)^{k_{i}} q^{r k_{i}}}{1+q^{r}} \frac{q}{[n+1 ; q]}\left(\left[k_{i+1}+1 ; q\right]_{n+1}-\left[k_{i} ; q\right]_{n+1}\right) \\
& =\frac{q}{[n+1 ; q]}\left(\left[k_{i} ; q\right]_{n+1}-\left[k_{i-1}-1 ; q\right]_{n+1}\right) \frac{q^{r}(-1)^{k_{i}} q^{r k_{i}}+q^{2 r}(-1)^{k_{i+1}} q^{r k_{i+1}}}{1+q^{r}} \\
& \left(1+q^{r}\right)[n+1 ; q] \\
& \quad\left(-(-1)^{k_{i-1}} q^{r k_{i-1}}\left[k_{i+1}+1 ; q\right]_{n+1}-q^{r}(-1)^{k_{i}} q^{r k_{i}}\left[k_{i+1}+1 ; q\right]_{n+1}\right. \\
& \quad+(-1)^{k_{i-1}} q^{r k_{i-1}}\left[k_{i} ; q\right]_{n+1}+\left[k_{i-1}-1 ; q\right]_{n+1} q^{r}(-1)^{k_{i}} q^{r k_{i}} \\
& \left.\quad-\left[k_{i} ; q\right]_{n+1} q^{2 r}(-1)^{k_{i+1}} q^{r k_{i+1}}+\left[k_{i-1}-1 ; q\right]_{n+1} q^{2 r}(-1)^{k_{i+1}} q^{r k_{i+1}}\right)
\end{aligned}
$$

The assertion follows.
Corollary 2. Let $n$ be a positive integer. Then

$$
F(n-1, n, c, p ; k)=P_{n, c, p}(k)+(-1)^{k} q^{(n-1) k} L_{n, c, p}
$$

where $P_{n, c, p}(k)$ is a $q$-polynomial in $k$ and $L_{n, c, p}$ is independent of $k$.
We define

$$
G_{n, c, p}=\sum_{k=0}^{c} F_{q}(n-1, n, c, p ; k) q^{k} .
$$

This is the generating function of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $p$ odd rows. In the final lemma of this section we prove that some special evaluations of $F_{q}(n-1, n, c, p ; k)$ in $k$ can be expressed in terms of the generating function $G_{n-1, c, p}$. This lemma, together with Lemma 10 and Corollary 2 provides enough properties in order to compute $F_{q}(n-1, n, c, p ; k)$ in the following section.
Lemma 13. Let $n$ be a positive integer. If $p \neq n$ then

$$
F_{q}(n-1, n, c, p ; 0)=G_{n-1, c, p}
$$

and if $p \neq 0$

$$
F_{q}(n-1, n, c, p ;-n)=(-1)^{n-1} q^{-3 n(n-1) / 2} G_{n-1, c+2, p-1}
$$

Moreover we have

$$
F_{q}(n-1, n, c, n ; 1)=q^{(n+2)(n-1) / 2} G_{n-1, c-1,0}
$$

and

$$
F_{q}(n-1, n, c, 0 ;-n-1)=(-1)^{n-1} q^{-(n-1)(2 n+1)} G_{n-1, c+3, n-1} .
$$

Proof. First let $\left(a_{i, j}\right)$ be an ( $n-1, n, c$ )-pattern with $a_{n, n}=0$ and exactly $p$ numbers of $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$ are odd. This implies that $a_{i, i}=0$ for all $i$ and thus $\left(a_{i, j}\right)_{1 \leq i \leq n-1, i \leq j \leq n+1}$ is an ( $n-2, n-1, c$ )-pattern with $p$ of $a_{1,2}, a_{1,3}, \ldots, a_{1, n}$ are odd. In fact this induces a norm-preserving and sign-preserving bijection between these $(n-1, n, c)$-patterns and these $(n-2, n-1, c)$ patterns. The first identity is proved.

Next let $\left(a_{i, j}\right)$ be an $(n-1, n, c)$-pattern with $a_{n, n}=-n$ and exactly $p$ of $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$ are odd. This implies that $a_{i, i}=-i$. Therefore $a_{i, i+1} \notin\{-3,-4, \ldots,-n\}$ for $i=1, \ldots, n-1$. If we set $b_{i, j}:=a_{i, j}+2$ for $i<j$ and $b_{i, i}=0$ then $\left(b_{i, j}\right)_{1 \leq i \leq n-1, i \leq j \leq n+1}$ is an $(n-2, n-1, c+2)$-pattern with $p-1$ of $b_{1,2}, b_{1,3}, \ldots, b_{1, n}$ are odd. Again this induces a bijection. However, the bijection is neither norm-preserving nor sign-preserving and but the factor $(-1)^{n-1} q^{-3 n(n-1) / 2}$ takes into account the changes of norm and sign.

For the third identitiy let $\left(a_{i, j}\right)$ be an $(n-1, n, c)$-pattern with $a_{n, n}=1$ and all $a_{1,1}, a_{1,2}, \ldots, a_{1, n}$ are odd. The first assumption implies that $a_{i, i} \in\{0,1\}$, the second that $a_{1,1}=1$ and therefore $a_{i, i}=1$ for all $i$. If we set $b_{i, j}=a_{i, j}-1$ then $\left(b_{i, j}\right)_{1 \leq i \leq n-1, i \leq j \leq n+1}$ is an $(n-2, n-1, c-1)$-pattern, where all $b_{1,2}, b_{1,3}, \ldots, b_{1, n}$ are even and the identity follows.

The proof of the fourth identity is similar.

## 6. Computation of $F_{q}(n-1, n, c, p ; k)$

In this section we compute $F_{q}(n-1, n, c, p ; k)$ using the properties we have established in the previous section. For these computations we need some $q$-summation formulas which we derive in Section 7 First we see that Corollary [1, Lemma 10 and Corollary 2 imply a first strong assertion on the form of $F_{q}(n-1, n, c, p ; k)$.

Lemma 14. Let $n$ be a positive integer and $0 \leq p \leq n$. Then $F_{q}(n-1, n, c, p ; k)$ is of the form

$$
\begin{aligned}
& M_{n, c, p}[k+1 ; q]_{n-1}[k-c-n+1 ; q]_{n-1}+L_{n, c, p}\left((-1)^{k} q^{(n-1) k}+(-1)^{n} q^{(n-1)(2 c+n) / 2}\right. \\
& \times \sum_{i=1}^{n-1}\left((-1)^{c} q^{\binom{i}{2}} \frac{[k+1 ; q]_{n-1}[k-c-i+1 ; q]_{i-1}[k-c-n+1 ; q]_{n-i-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}\right. \\
& \left.\left.-q^{\binom{i}{2}} \frac{[k+1 ; q]_{i-1}[k+i+1 ; q]_{n-i-1}[k-c-n+1 ; q]_{n-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}\right)\right)
\end{aligned}
$$

Proof. By Lemma 10 and Corollary 2 we know that for $k \in\{-1,-2, \ldots,-n+1\}$ and $k \in\{c+1, c+2, \ldots, c+n-1\}$ we have

$$
P_{n, c, p}(k)=(-1)^{k+1} q^{(n-1) k} L_{n, c, p} .
$$

By Corollary $1 P_{n, c, p}(k)$ is a $q$-polynomial in $k$ of degree at most $2 n-2$. By $q$-Lagrange interpolation the following polynomial is the unique $q$-polynomial of degree at most $2 n-3$ with the same evaluations as $P_{n, c, p}(k)$ at $k \in\{-1,-2, \ldots,-n+1\}$ and $k \in\{c+1, c+2, \ldots, c+n-1\}$.

$$
\begin{aligned}
& \sum_{i=-n+1}^{-1}(-1)^{i+1} q^{(n-1) i} L_{n, c, p} \prod_{-n+1 \leq j \leq-1, j \neq i} \frac{[k-j ; q]}{[i-j ; q]} \prod_{j=1}^{n-1} \frac{[k-c-j ; q]}{[i-c-j ; q]} \\
&+\sum_{i=1}^{n-1}(-1)^{c+i+1} q^{(n-1)(c+i)} L_{n, c, p} \prod_{j=-n+1}^{-1} \frac{[k-j ; q]}{[c+i-j ; q]} \prod_{1 \leq j \leq n-1, j \neq i} \frac{[k-c-j ; q]}{[i-j ; q]} .
\end{aligned}
$$

This is equal to

$$
\begin{aligned}
& \sum_{i=1}^{n-1}\left((-1)^{i+1} q^{-(n-1) i} L_{n, c, p} \prod_{j=1}^{i-1} \frac{[k+j ; q]}{[j-i ; q]} \prod_{j=i+1}^{n-1} \frac{[k+j ; q]}{[j-i ; q]} \prod_{j=1}^{n-1} \frac{[k-c-j ; q]}{[-i-c-j ; q]}\right. \\
& \left.\quad+(-1)^{c+i+1} q^{(n-1)(c+i)} L_{n, c, p} \prod_{j=1}^{n-1} \frac{[k+j ; q]}{[c+i+j ; q]} \prod_{j=1}^{i-1} \frac{[k-c-j ; q]}{[i-j ; q]} \prod_{j=i+1}^{n-1} \frac{[k-c-j ; q]}{[i-j ; q]}\right) \\
& \quad=L_{n, c, p} \sum_{i=1}^{n-1}\left((-1)^{i+1} q^{-(n-1) i} \frac{[k+1 ; q]_{i-1}[k+i+1 ; q]_{n-1-i}[k-c-n+1 ; q]_{n-1}}{[1-i ; q]_{i-1}[1 ; q]_{n-1-i}[-n+1-c-i ; q]_{n-1}}\right. \\
& \left.\quad+(-1)^{c+i+1} q^{(n-1)(c+i)} \frac{[k+1 ; q]_{n-1}[k-c-i+1 ; q]_{i-1}[k-c-n+1 ; q]_{n-1-i}}{[c+i+1 ; q]_{n-1}[1 ; q]_{i-1}[i-n+1 ; q]_{n-1-i}}\right) .
\end{aligned}
$$

The difference of $P_{n, c, p}(k)$ and the $q$-polynomial displayed above is a $q$-polynomial of degree $2 n-2$ at most which vanishes for $k \in\{-1, \ldots,-n+1\}$ and $k \in\{c+1, \ldots, c+n-1\}$. Thus this difference is equal to $M_{n, c, p}[k+1 ; q]_{n-1}[k-c-n+1 ; q]_{n-1}$, where $M_{n, c, p}$ is a factor independent of $k$, which still has to be determined. We use the identity

$$
[z ; q]_{n}=[-z-n+1 ; q]_{n}(-1)^{n} q^{n(z+(n-1) / 2)}
$$

in order to obtain the expression for $P_{n, c, p}(k)$ in the statement of the lemma.
We set

$$
\begin{aligned}
& U_{n, c}(k)= \\
& \begin{aligned}
(-1)^{n} q^{(n-1)(2 c+n) / 2} & \sum_{i=1}^{n-1}\left((-1)^{c} q^{\binom{i}{2}} \frac{[k+1 ; q]_{n-1}[k-c-i+1 ; q]_{i-1}[k-c-n+1 ; q]_{n-i-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}\right. \\
& \left.-q^{\binom{i}{2}} \frac{[k+1 ; q]_{i-1}[k+i+1 ; q]_{n-i-1}[k-c-n+1 ; q]_{n-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}\right)+(-1)^{k} q^{(n-1) k}
\end{aligned}
\end{aligned}
$$

and $W_{n, c}(k)=[k+1 ; q]_{n-1}[k-c-n+1 ; q]_{n-1}$. With these definitions Lemma 14 states that

$$
\begin{equation*}
F_{q}(n-1, n, c, p ; k)=L_{n, c, p} U_{n, c}(k)+M_{n, c, p} W_{n, c}(k) \tag{6.1}
\end{equation*}
$$

It remains to compute $L_{n, c, p}$ and $M_{n, c, p}$. In the following lemma we give recursive formulas for $L_{n, c, p}$ and $M_{n, c, p}$ with respect to $n$. It is an immediate consequence of Lemma 13
Lemma 15. The initial conditions are $L_{1, c, p}=\frac{(-1)^{p}}{2}$ and $M_{1, c, p}=\frac{1}{2}$. If $p \neq 0, n$ we have

$$
L_{n, c, p}=\frac{G_{n-1, c, p} W_{n, c}(-n)+(-1)^{n} q^{-3 n(n-1) / 2} G_{n-1, c+2, p-1} W_{n, c}(0)}{U_{n, c}(0) W_{n, c}(-n)-U_{n, c}(-n) W_{n, c}(0)}
$$

In case that $p=0$ we have the following recursion

$$
L_{n, c, 0}=\frac{(-1)^{n-1} q^{-(n-1)(2 n+1)} G_{n-1, c+3, n-1} W_{n, c}(0)-G_{n-1, c, 0} W_{n, c}(-n-1)}{U_{n, c}(-n-1) W_{n, c}(0)-U_{n, c}(0) W_{n, c}(-n-1)}
$$

and for $p=n$ we have

$$
L_{n, c, n}=\frac{q^{(n+2)(n-1) / 2} G_{n-1, c-1,0} W_{n, c}(-n)+(-1)^{n} q^{-3 n(n-1) / 2} G_{n-1, c+2, n-1} W_{n, c}(1)}{U_{n, c}(1) W_{n, c}(-n)-U_{n, c}(-n) W_{n, c}(1)}
$$

Concerning $M_{n, c, p}$ we have

$$
M_{n, c, p}=\frac{G_{n-1, c, p}-U_{n, c}(0) L_{n, c, p}}{W_{n, c}(0)}
$$

for $p \neq n$ and

$$
M_{n, c, p}=\frac{(-1)^{n-1} q^{-3 n(n-1) / 2} G_{n-1, c+2, p-1}-U_{n, c}(-n) L_{n, c, p}}{W_{n, c}(-n)}
$$

for $p \neq 0$.
Proof. Lemma 13 and (6.1) implies the following equations. If $p \neq 0$ then

$$
L_{n, c, p} U_{n, c}(-n)+M_{n, c, p} W_{n, c}(-n)=(-1)^{n-1} q^{-3 n(n-1) / 2} G_{n-1, c+2, p-1},
$$

and if $p \neq n$ then

$$
L_{n, c, p} U_{n, c}(0)+M_{n, c, p} W_{n, c}(0)=G_{n-1, c, p}
$$

For $p=0$ we have

$$
L_{n, c, 0} U_{n, c}(-n-1)+M_{n, c, 0} W_{n, c}(-n-1)=(-1)^{n-1} q^{-(n-1)(2 n+1)} G_{n-1, c+3, n-1}
$$

and for $p=n$ we have

$$
L_{n, c, n} U_{n, c}(1)+M_{n, c, n} W_{n, c}(1)=q^{(n+2)(n-1) / 2} G_{n-1, c-1,0} .
$$

For every $p \in\{0,1,2, \ldots, n\}$ this gives a system of two linearly independent equations for $L_{n, c, p}$ and $M_{n, c, p}$. With the help of Cramer's rule we obtain the recursions of $L_{n, c, p}$. The recursions for $M_{n, c, p}$ are immediate consequences of the equations.

In the following lemma we see that the denominators in the recursive formulas for $L_{n, c, p}$ in Lemma 15 are products.

Lemma 16. We have

$$
\begin{gathered}
U_{n, c}(0) W_{n, c}(-n)-U_{n, c}(-n) W_{n, c}(0)=\frac{2\left[1 ; q^{2}\right]_{n-1}(1+q)^{2 n-1}}{q^{c(n-1)+2 n(n-1)}} \begin{cases}\frac{\left[(c+2) / 2 ; q^{2}\right]_{n-1}}{1+q} & \text { if } c \text { is even } \\
\frac{\left[(c+1) / 2 ; q^{2}\right]_{n}}{[c+n ; q]} & \text { if } c \text { is odd }\end{cases} \\
U_{n, c}(-n-1) W_{n, c}(0)-U_{n, c}(0) W_{n, c}(-n-1)=-\frac{2\left[1 ; q^{2}\right]_{n-1}[n-1 ; q](1+q)^{2 n-1}}{q^{(n-1) c+2(n-1)(n+1)}} \\
\times \begin{cases}{\left[(c+2) / 2 ; q^{2}\right]_{n}} & \text { if } c \text { is even } \\
\frac{\left[(c+1) / 2 ; q^{2}\right]_{n+1}(1+q)}{[c+n+1 ; q]} & \text { if } c \text { is odd }\end{cases}
\end{gathered}
$$

and

$$
\begin{aligned}
& U_{n, c}(1) W_{n, c}(-n)-U_{n, c}(-n) W_{n, c}(1)=\frac{2\left[1 ; q^{2}\right]_{n-1}[n-1 ; q](1+q)^{2 n-1}}{q^{(n-1) c+n(2 n-3)}[c+n ; q]} \\
& \times \begin{cases}{\left[c / 2 ; q^{2}\right]_{n}} & \text { if } c \text { is even } \\
\frac{\left[(c-1) / 2 ; q^{2}\right]_{n+1}(1+q)}{[c+n-1 ; q]} & \text { if } c \text { is odd }\end{cases}
\end{aligned} .
$$

Proof. The formulas for $U_{n, c}(0), U_{n, c}(-n), U_{n, c}(1)$ and $U_{n, c}(-n-1)$ in Section 7. (7.2) (7.5), imply that the denominators from Lemma 15 are sums of at most 3 products. A lengthy but straightforward calculation shows that theses denominators are actually single products.

We finally give the formulas for $L_{n, c, p}, M_{n, c, p}$ and $G_{n, c, p}$.
Lemma 17. The generating function $G_{n, c, p}$ is equal to

$$
\left.G_{n, c, p}=q^{(p+1} 2\right)\left[\begin{array}{l}
n \\
p
\end{array}\right]\left\{\begin{array}{ll}
\frac{1}{[c+p ; q]_{n+1}} \prod_{i=0}^{n} \frac{[c+2 i ; q]_{n-i+1}}{[2+2 i ; q]_{n-i}} & 2 \mid c \\
\prod_{i=1}^{n} \frac{[c+2 i-1 ; q]_{n-i+1}}{[2 i ; q]_{n-i+1}} & 2 \not \chi_{c}
\end{array} .\right.
$$

For $L_{n, c, p}$ we have

$$
L_{n, c, p}= \begin{cases}\prod_{i=1}^{n-1} \frac{[c+2 i+1 ; q]_{n-i}}{[2 i ; q]_{n-i}[2 i ; q]} \frac{[c+1 ; q]_{n-1}[1 ; q]_{n-1}}{2}\left(\frac{q^{\binom{p+1}{2}}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right][c ; q]}{[c+p ; q]_{n}}-\frac{q^{\binom{p}{2}}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right][c+2 n ; q]}{[c+p+1 ; q]_{n}}\right) & 2 \mid c \\
\prod_{i=1}^{n-1} \frac{[c+2 i ; q]_{n-i}}{[2 i ; q]_{n-i}[2 i ; q]} \frac{[1 ; q]_{n-1}}{2}\left(q^{\binom{p+1}{2}}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right]-q^{\left.\binom{p}{2}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right]\right)}\right. & 2 \nmid c\end{cases}
$$

and for $M_{n, c, p}$ we have

$$
\begin{aligned}
& M_{n, c, p}=\frac{(-1)^{n-1} q^{(n-1)(2 c+n) / 2}}{[1 ; q]_{n-2}} \\
& \quad \times\left\{\begin{array}{ll}
\prod_{i=1}^{n-1} \frac{[c+2 i ; q]_{n-i}}{[2 i ; q]_{n-i}}\left(\frac{q^{\left(\frac{p+1}{2}\right)}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right][c ; q]}{[c+p ; q]_{n}}\left(\frac{1}{[n-1 ; q]}-\frac{[c+2 n-1 ; q]}{[c+n ; q][2 n-2 ; q]}\right)+\frac{q^{\binom{p}{2}}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right]}{[c+p+1 ; q]_{n}}\left[\begin{array}{cc}
{[c+2 n-1 ; q]_{2}} \\
{[c+n][2 n-2 ; q]}
\end{array}\right)\right. & 2 \mid c \\
\prod_{i=1}^{n-1} \frac{[c+2 i+1 ; q]_{n-i-1}}{[2 i ; q]_{n-i}} \frac{1}{[2 n-2 ; q]}\left(q^{\binom{p+1}{2}+n-1}\left[\begin{array}{c}
n-1 \\
p
\end{array}\right]+q^{\binom{p}{2}}\left[\begin{array}{c}
n-1 \\
p-1
\end{array}\right]\right) & 2 \nless c
\end{array} .\right.
\end{aligned}
$$

Proof. We show the assertion by induction with respect to $n$. For $n=1$ observe that $L_{1, c, 0}=1 / 2, L_{1, c, 1}=-1 / 2$ and $M_{1, c, 0}=M_{1, c, 1}=1 / 2$. Moreover

$$
G_{1, c, 0}= \begin{cases}1+q^{2}+q^{4}+\ldots+q^{c}=\frac{1-q^{2+c}}{1-q^{2}} & \text { if } c \text { is even } \\ 1+q^{2}+q^{4}+\ldots+q^{c-1}=\frac{1-q^{1+c}}{1-q^{2}} & \text { if } c \text { is odd }\end{cases}
$$

and

$$
G_{1, c, 1}=\left\{\begin{array}{ll}
q+q^{3}+q^{5}+\ldots+q^{c-1}=\frac{q\left(1-q^{c}\right)}{1-q^{2}} & \text { if } c \text { is even } \\
q+q^{3}+q^{5}+\ldots+q^{c}=\frac{q\left(1-q^{+c}\right)}{1-q^{2}} & \text { if } c \text { is odd }
\end{array} .\right.
$$

Assume that the formulas are proved for $n-1$. Then the formula for $L_{n, c, p}$ and $M_{n, c, p}$ can be checked by using the recursions in Lemma 15, (7.2)-(7.5) and the formula for $G_{n-1, c, p}$ which is
true by the induction hypothesis. By (17.1) and (7.6) we have

$$
\begin{aligned}
& G_{n, c, p}=\sum_{k=0}^{c} L_{n, c, p} U_{n, c}(k) q^{k}+M_{n, c, p} W_{n, c}(k) q^{k} \\
& \quad=L_{n, c, p} \frac{\left(\left(1+(-1)^{c}\right)\left[(c+2) / 2 ; q^{2}\right]_{n-1}\left(1+q^{c+n}\right)+\left(1-(-1)^{c}\right)\left[(c+1) / 2 ; q^{2}\right]_{n}\left(1-q^{2}\right)\right)}{\left[1 / 2 ; q^{2}\right]_{n-1}\left(1+q^{n-1}\right)\left(1+q^{n}\right)} \\
& \quad+M_{n, c, p}(-1)^{n-1} q^{(-n+1)(2 c+n) / 2} \frac{[1 ; q]_{n-1}^{2}[c+1 ; q]_{2 n-1}}{[1 ; q]_{2 n-1}} .
\end{aligned}
$$

A lengthy but straightforward calculation proves the formula for $G_{n, c, p}$.
Now we are able to explain why Theorem 1 implies Krattenthaler's and the author's refinement of the Bender-Knuth (ex-)Conjecture. Krattenthaler's refinement, see [10, Theorem 21], is the generating function $G_{n, c, p}$ we have computed in Lemma 17 and thus we have reproved his result with different methods.

The author's refinement, see [5] Theorem 1], is the generating function of strict plane partitions with parts in $\{1,2, \ldots, n\}$, at most $c$ columns and $k$ parts equal to $n$, i.e. the sum over all $p$ 's, $0 \leq p \leq n$, of the generating function in Theorem In order to deduce Theorem 1 in [5] from Theorem of the present paper one has to show that

$$
\sum_{p=0}^{n} L_{n, c, p}=0
$$

and

$$
\sum_{p=0}^{n}(-1)^{n-1} q^{\left.(n-1)(k-c)-\binom{n}{2}\right)+k} M_{n, c, p}=\frac{q^{k n}}{[1 ; q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1 ; q]_{i-1}}{[i ; q]_{i}}
$$

where

$$
[k-c-n+1 ; q]_{n-1}=(-1)^{n-1} q^{(n-1)(k-c)-\binom{n}{2}}[1+c-k ; q]_{n-1}
$$

explains the factor in front of $M_{n, c, p}$. However, Theorem 1 from [5] was proved with methods similar to the methods we have used to prove Theorem $\square$ and thus we omit to show this implication, because it is surely a detour to prove Theorem 1 from [5] in this way.

## 7. Some basic hypergeometric identities

In this section we derive some basic hypergeometric identities which were needed above. The notation is adopted from [6] page 1-6]. In particular the basic hypergeometric series is defined by

$$
{ }_{r} \phi_{s}\left[\begin{array}{l}
a_{1}, \ldots, a_{r} \\
b_{1}, \ldots, b_{s}
\end{array} q, z\right]=\sum_{n=0}^{\infty} \frac{\left(a_{1} ; q\right)_{n} \cdots\left(a_{r} ; q\right)_{n}}{(q ; q)_{n}\left(b_{1} ; q\right)_{n} \cdots\left(b_{s} ; q\right)_{n}}\left((-1)^{n} q^{\binom{n}{2}}\right)^{s-r+1} z^{n},
$$

where the rising $q$-factorial $(a ; q)_{n}$ is given by $(a ; q)_{n}:=\prod_{i=0}^{n-1}\left(1-a q^{i}\right)$. (Observe that $[x ; q]_{n}=$ $\left(q^{x} ; q\right)_{n} /(1-q)^{n}$.) All identities in this section were handled with Krattenthaler's Mathematica package HYPQ [11. A Mathematica-file containing the computations can be downloaded from my webpage (http://www.uni-klu.ac.at/~ifischer/).

The list of identities is the following. (For the definitions of $U_{n, c}(k)$ and $W_{n, c}(k)$ see Section 6.)

$$
\begin{align*}
& \sum_{k=0}^{c} W_{n, c}(k) q^{k}=(-1)^{n-1} q^{(-n+1)(2 c+n) / 2} \frac{[1 ; q]_{n-1}^{2}[c+1 ; q]_{2 n-1}}{[1 ; q]_{2 n-1}}  \tag{7.1}\\
& U_{n, c}(0)=\frac{2 \prod_{i=1}^{n-2}\left(1+q^{i}\right)}{(1+q)^{n-1}} \begin{cases}\frac{[c+1 ; q]_{n-1}}{[(c+1) / 2 ; q]_{n-1}} & \text { if } c \text { is even } \\
\frac{[c ; q]_{n}}{[c / 2 ; q]_{n}(1+q)} & \text { if } c \text { is odd }\end{cases}  \tag{7.2}\\
& U_{n, c}(-n)=\frac{2(1+q)^{n-1} \prod_{i=1}^{n-2}\left(1+q^{i}\right)}{q^{(n-1)(3 n-2) / 2}} \begin{cases}\frac{[(c+2) / 2 ; q]_{n-1}}{[c+2 ;]_{n}-1} & \text { if } c \text { is even } \\
\frac{[c+1) / 2 ; q q_{n}(1+q)}{[c+1 ; q]_{n}} & \text { if } c \text { is odd }\end{cases}  \tag{7.3}\\
& U_{n, c}(1)=\frac{2 q^{n} \prod_{i=1}^{n-2}\left(1+q^{i}\right)}{(1+q)^{n-1}} \tag{7.4}
\end{align*}
$$

$$
\begin{align*}
& U_{n, c}(-n-1)=\frac{2(-1)^{n}(1+q)^{n-1} \prod_{i=1}^{n-2}\left(1+q^{i}\right)}{q^{(n+1)(3 n-4) / 2}} \\
& \times \begin{cases}\frac{\left[(c+4) / 2 ; q^{2}\right]_{n-1}[n-2 ; q]}{[c+3 q]_{n-1}} & \text { if } c \text { is even } \\
\frac{[(c+3) / 2 ; q]]_{n-1}[n-2 ; q]}{[c+2 ; q]_{n-1}}+q^{c+n} \frac{\left[(c+3) / 2 ; q^{2}\right]_{n-1}[2\lfloor(n-1) / 2\rfloor+1 ; q][2\lfloor n / 2\rfloor ; q]}{[c+2 ; q]_{n}(1+q)^{2}} & \text { if } c \text { is odd }\end{cases}  \tag{7.5}\\
& \sum_{k=0}^{c} U_{n, c}(k) q^{k} \\
& =\frac{2}{\left[1 / 2 ; q^{2}\right]_{n-1}\left(1+q^{n-1}\right)\left(1+q^{n}\right)} \begin{cases}{\left[(c+2) / 2 ; q^{2}\right]_{n-1}\left(1+q^{c+n}\right)} & \text { if } c \text { is even } \\
{\left[(c+1) / 2 ; q^{2}\right]_{n}\left(1-q^{2}\right)} & \text { if } c \text { is odd }\end{cases} \tag{7.6}
\end{align*}
$$

We first consider (7.1). Using the basic hypergeometric notation introduced above, the left-hand-side can be written as

$$
{ }_{2} \phi_{1}\left[\begin{array}{l}
q^{n}, q^{-c} \\
q^{1-c-n} ; q, q
\end{array}\right] \frac{(q ; q)_{-1+n}\left(q^{1-c-n} ; q\right)_{-1+n}}{(1-q)^{2 n-2}} .
$$

If we use the $q$-Chu-Vandermonde summation formula [6, (1.5.3); Appendix (II.6)]

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
a, q^{-n}  \tag{7.7}\\
c
\end{array} ; q, q\right]=\frac{a^{n}(c / a ; q)_{n}}{(c ; q)_{n}},
$$

we obtain

$$
\begin{equation*}
\frac{q^{c n}(q ; q)_{-1+n}\left(q^{1-c-2 n} ; q\right)_{c}\left(q^{1-c-n} ; q\right)_{-1+n}}{(1-q)^{2 n-2}\left(q^{1-c-n} ; q\right)_{c}} \tag{7.8}
\end{equation*}
$$

and this is equivalent to the right-hand-side in (7.1).

Next we consider (7.2), (7.3), (7.4) and (7.5). In all these summations we first use some contiguous relations before we apply the following summation formula

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
a, b  \tag{7.9}\\
a q / b
\end{array} q,-q / b\right]=\frac{(-q ; q)_{\infty}\left(a q ; q^{2}\right)_{\infty}\left(a q^{2} / b^{2} ; q^{2}\right)_{\infty}}{(-q / b ; q)_{\infty}(a q / b ; q)_{\infty}},
$$

see [6, (1.8.1); Appendix (II.9)], where $(a ; q)_{\infty}=\prod_{i=0}^{\infty}\left(1-a q^{i}\right)$. We define

$$
X_{n, c}(k)=\sum_{i=1}^{n-1} q^{\binom{i}{2}} \frac{[k+1 ; q]_{n-1}[k-c-i+1 ; q]_{i-1}[k-c-n+1 ; q]_{n-i-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}
$$

and

$$
Y_{n, c}(k)=\sum_{i=1}^{n-1} q^{\binom{i}{2}} \frac{[k+1 ; q]_{i-1}[k+i+1 ; q]_{n-i-1}[k-c-n+1 ; q]_{n-1}}{[1 ; q]_{i-1}[1 ; q]_{n-1-i}[c+i+1 ; q]_{n-1}}
$$

Observe that

$$
U_{n, c}(k)=(-1)^{n} q^{(n-1)(2 c+n) / 2}\left((-1)^{c} X_{n, c}(k)-Y_{n, c}(k)\right)+(-1)^{k} q^{(n-1) k} .
$$

In order to prove (7.2), it suffices to compute $X_{n, c}(0), Y_{n, c}(0)$. Using basic hypergeometric notation $X_{n, c}(0)$ is equal to

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
q^{c+1}, q^{-n+2} \\
q^{c+n+1}
\end{array} ; q,-q^{n}\right] \frac{(q ; q)_{n-1}\left(q^{1-c-n} ; q\right)_{n-2}}{(q ; q)_{n-2}\left(q^{c+2} ; q\right)_{n-1}} .
$$

If we apply the following contiguous relation

$$
{ }_{r} \phi_{s}\left[\begin{array}{l}
a,(A) \\
b,(B)
\end{array} ; q, z\right]=\frac{1-b / q}{a-b / q} r \phi_{s}\left[\begin{array}{c}
a,(A) \\
b / q,(B)
\end{array} ; q, z / q\right]+\frac{1-a}{b / q-a} r \phi_{s}\left[\begin{array}{c}
a q,(A) \\
b,(B)
\end{array} ; q, z / q\right]
$$

and then (7.9) we obtain the following formula for $X_{n, c}(0)$.

$$
\begin{aligned}
& \frac{q^{-1-c}\left(q^{1-c-n} ; q\right)_{n-2}(-q ; q)_{\infty}\left(q^{c+2} ; q^{2}\right)_{\infty}\left(q^{c+2 n-1} ; q^{2}\right)_{\infty}}{\left(q^{c+2} ; q\right)_{n-2}\left(-q^{n-1} ; q\right)_{\infty}\left(q^{c+n} ; q\right)_{\infty}}+ \\
& \frac{q^{-c-n}\left(q^{c+1} ; q\right)_{1}\left(q^{1-c-n} ; q\right)_{n-2}\left(q^{n-1} ; q\right)_{1}(-q ; q)_{\infty}\left(q^{c+3} ; q^{2}\right)_{\infty}\left(q^{c+2 n} ; q^{2}\right)_{\infty}}{\left(q^{c+2} ; q\right)_{n-1}\left(q^{1-n} ; q\right)_{1}\left(-q^{n-1} ; q\right)_{\infty}\left(q^{c+n+1} ; q\right)_{\infty}}
\end{aligned}
$$

Next observe that $Y_{n, c}(0)$ is equal to

$$
(-1)^{n-1}{ }_{3} \phi_{2}\left[\begin{array}{c}
q^{-n+2}, q^{c+2}, q \\
q^{c+n+1}, q^{2}
\end{array} ; q,-q^{n-1}\right] \frac{\left(q^{n-1} ; q\right)_{1}\left(q^{c+1} ; q\right)_{1}}{(q ; q)_{1}\left(q^{c+n} ; q\right)_{1}} .
$$

Using the contiguous relation

$$
{ }_{r} \phi_{s}\left[\begin{array}{c}
(A), q  \tag{7.10}\\
(B)
\end{array} q, z\right]=\frac{(-1)^{r+s} q^{1-r+s}}{z} \frac{\prod_{i=1}^{s}\left(1-B_{i} / q\right)}{\prod_{i=1}^{r}\left(1-A_{i} / q\right)}\left(1-{ }_{r} \phi_{s}\left[\begin{array}{c}
(A / q), q \\
(B / q)
\end{array} q, q^{-1+r-s} z\right]\right),
$$

we transform the ${ }_{3} \phi_{2}$ series into a ${ }_{2} \phi_{1}$ series. Next we apply the following contiguous relation

$$
{ }_{r} \phi_{s}\left[\begin{array}{c}
a,(A)  \tag{7.11}\\
(B)
\end{array} ; q, z\right]={ }_{r} \phi_{s}\left[\begin{array}{c}
a q,(A) \\
(B)
\end{array} ; q, z\right]+(-1)^{r+s} a z \frac{\prod_{i=1}^{r-1}\left(1-A_{i}\right)}{\prod_{i=1}^{s}\left(1-B_{i}\right)}{ }^{r} \phi_{s}\left[\begin{array}{c}
a q,(q A) \\
(q B)
\end{array} q, q^{1-r+s} z\right]
$$

and finally we are able to apply (7.9) to the remaining two ${ }_{2} \phi_{1}$ series. We obtain a formula for $Y_{n, c}(0)$ in terms of rising $q$-factorials. If we combine this with the formula for $X_{n, c}(0)$ we obtain (7.2).

The situation is similar for (7.3), (7.4), (7.5), however, we do not give the proofs in detail. In order to describe a set of contiguous relations which is needed before (7.9) can be applied in each of these cases, we use Krattenthaler's table of contiguous relations, which can be found in the HYPQ documentation [11]. For details see the Mathematica-file with the computation on my webpage http://www.uni-klu.ac.at/~ifischer/.

| $X_{n, c}(-n)$ | C34 |
| :--- | :--- |
| $Y_{n, c}(-n)$ | C15 |
| $X_{n, c}(1)$ | C14, C42, C34 |
| $Y_{n, c}(1)$ | C16, C15, C02, C41, C11, C37 |
| $X_{n, c}(-n-1)$ | C14, C42, C34 |
| $Y_{n, c}(-n-1)$ | C14, C15, C36, C11, C37 |

Finally we consider (7.6), which is a double sum and thus the most-complicated identity. First we compute $\sum_{k=0}^{c} X_{n, c}(k) q^{k}$. The trick is to consider a more general expression. Observe that $X_{n, c}(k)$ is the unique $q$-polynomial in $k$ of degree at most $2 n-3$ with the property that $X_{n, c}(k)=0$ for $k=-1,-2, \ldots,-n+1$ and $X_{n, c}(c+i)=(-1)^{n-1} q^{-\binom{n}{2}}\left(-q^{n-1}\right)^{i}$ for $i=1,2, \ldots, n-1$. (In Lemma $14 X_{n, c}(k)$ is actually constructed in such a way that these conditions are fulfilled.) Consequently $S_{n, c}(d):=\sum_{k=0}^{d} X_{n, c}(k) q^{k}$ is the unique $q$-polynomial in $d$ of degree at most $2 n-2$ in $d$ with $S_{n, c}(d)=0$ for $d=-1,-2, \ldots,-n$ and

$$
S_{n, c}(c+i)=S_{n, c}(c)+(-1)^{n-1} q^{-\binom{n}{2}+c+n} \frac{-1+\left(-q^{n}\right)^{i}}{1+q^{n}}
$$

for $i=1, \ldots, n-1$. Thus, by $q$-Lagrange interpolation,

$$
\begin{aligned}
S_{n, c}(d)=\sum_{i=1}^{n}\left(S_{n, c}(c)+(-1)^{n-1} q^{-\binom{n}{2}+c+n}\right. & \left.\frac{\left(-q^{n}\right)^{i}-1}{1+q^{n}}\right) q^{\binom{n-i+1}{2}}(-1)^{n+i} \\
& \times \frac{[d+1 ; q]_{n}[d-c-i+2 ; q]_{i-1}[d-c-n+1 ; q]_{n-i}}{[c+i ; q]_{n}[1 ; q]_{i-1}[1 ; q]_{n-i}} .
\end{aligned}
$$

Apriori the degree of this $q$-polynomial is $2 n-1$. Thus the coefficient of $\left(q^{d}\right)^{2 n-1}$

$$
\sum_{i=1}^{n}\left(S_{n, c}(c)+(-1)^{n-1} q^{-\binom{n}{2}+c+n} \frac{\left(-q^{n}\right)^{i}-1}{1+q^{n}}\right) \frac{q^{\binom{n-i+1}{2}}(-1)^{n+i} q^{-1+c+i+n-c n}}{[c+i ; q]_{n}[1 ; q]_{i-1}[1 ; q]_{n-i}}
$$

must be zero. We obtain the following expression for $S_{n, c}(c)=\sum_{k=0}^{c} X_{n, c}(k) q^{k}$.

This formula simplifies since

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{(-1)^{i} q\binom{(n-i+1}{2}+i}{[c+i ; q]_{n}[1 ; q]_{i-1}[1 ; q]_{n-i}}=-q^{(n-1) c+(n+1) n / 2} \frac{[1 ; q]_{2 n-2}}{[1 ; q]_{n-1}^{2}[c+1 ; q]_{2 n-1}} \tag{7.12}
\end{equation*}
$$

In order to see that observe that the left-hand-side in this equation is equal to

$$
-{ }_{2} \phi_{1}\left[\begin{array}{c}
q^{c+1}, q^{-n+1} \\
q^{c+n+1}
\end{array} ; q, q\right] \frac{q^{n^{2} / 2-n / 2+1}(1-q)^{2 n-1}}{(q ; q)_{n-1}\left(q^{c+1} ; q\right)_{n}}
$$

Using (17.7) we obtain (7.12). Thus

$$
\begin{aligned}
& S_{n, c}(c)=\frac{(-1)^{n-1} q^{-\binom{n}{2}+c+n}}{1+q^{n}} \\
& \quad \times\left(-q^{c-n-c n} \sum_{i=0}^{n-1} q^{\binom{i+2}{2}} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}}+1\right) .
\end{aligned}
$$

Similarly one can show that

$$
\begin{aligned}
& T_{n, c}(d)=\sum_{i=0}^{n-1}\left(T_{n, c}(0)+\frac{(-1)^{n} q^{(n-1)(2 c+n) / 2}\left(-1+\left(-q^{-n}\right)^{i}\right)}{1+q^{n}}(-1)^{i+n} q^{\left(i+i^{2}+n+2 c n+2 i n+n^{2}\right) / 2}\right. \\
&\left.\times \frac{[d-c-n ; q]_{n}[d ; q]_{i}[d+i+1 ; q]_{n-i-1}}{[c+i+1 ; q]_{n}[1 ; q]_{i}[1 ; q]_{n-1-i}}\right),
\end{aligned}
$$

where $T_{n, c}(d):=\sum_{k=d}^{c} Y_{n, c}(k) q^{k}$. Again we have that $T_{n, c}(d)$ is a $q$-polynomial of degree at most $2 n-2$, however, the left-hand-side of the equation above is apriori a $q$-polynomial of degree $2 n-1$. Thus we obtain the following formula for $T_{n, c}(0)=\sum_{k=0}^{c} Y_{n, c}(k) q^{k}$.

Again the formula simplifies since

$$
\begin{equation*}
\sum_{i=0}^{n-1} \frac{(-1)^{i} q^{\binom{n+i}{2}}}{[c+i+1 ; q]_{n}[1 ; q]_{i}[1 ; q]_{n-i-1}}=\frac{q^{\binom{n}{2}}[1 ; q]_{2 n-2}}{[c+1 ; q]_{2 n-2}[1 ; q]_{n-1}^{2}} \tag{7.13}
\end{equation*}
$$

In order to see that transform the sum into hypergeometric notation

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
q^{c+1}, q^{1-n} \\
q^{c+n+1}
\end{array} ; q, q^{2 n-1}\right] \frac{q^{n^{2} / 2-n / 2}(1-q)^{2 n-1}}{(q ; q)_{n-1}\left(q^{c+1} ; q\right)_{n}}
$$

and apply the summation formula, see [6, (1.5.2); Appendix (II.7)],

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
a, q^{-n} \\
c
\end{array} ; q, \frac{c q^{n}}{a}\right]=\frac{(c / a ; q)_{n}}{(c ; q)_{n}}
$$

to obtain the result. This implies that

$$
T_{n, c}(0)=\frac{(-1)^{n-1} q^{-(n-1)(2 c+n) / 2}}{\left(1+q^{n}\right)}\left(\sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}}-1\right) .
$$

Therefore

$$
\begin{aligned}
\sum_{k=0}^{c} U_{n, c}(k) q^{k}= & \frac{1+(-1)^{c} q^{(1+c) n}}{1+q^{n}}+\frac{(-1)^{n} q^{-(n-1)(2 c+n) / 2}}{1+q^{n}} \\
& \times\left(\sum_{i=0}^{n-1}(-1)^{c} q^{\binom{i+2}{2}+c} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}}\right. \\
& \left.\quad-\sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}}-\left(-q^{n}\right)^{c+1}+1\right)
\end{aligned}
$$

Consequently it suffices to compute

$$
\begin{align*}
& \sum_{i=0}^{n-1} q^{\binom{i+2}{2}+c} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}} \\
& -\quad-\sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}} \tag{7.14}
\end{align*}
$$

and

$$
\begin{align*}
& \sum_{i=0}^{n-1} q^{\binom{i+2}{2}+c} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}} \\
& \quad+\sum_{i=0}^{n-1} q^{\binom{i}{2}} \frac{[c+1 ; q]_{i}[c+i+n+1 ; q]_{n-i-1}[n-i ; q]_{i}}{[n ; q]_{n-1}[1 ; q]_{i}} . \tag{7.15}
\end{align*}
$$

Using basic hypergeometric notation (7.14) is equal to

$$
{ }_{4} \phi_{3}\left[\begin{array}{c}
q^{1+c}, q^{3 / 2+c / 2},-q^{3 / 2+c / 2}, q^{1-n} \\
q^{1 / 2+c / 2},-q^{1 / 2+c / 2}, q^{1+c+n}
\end{array} ; q,-q^{n-1}\right] \frac{\left(q^{1+c} ; q\right)_{1}\left(q^{1+c+n} ; q\right)_{n-1}}{\left(q^{n} ; q\right)_{n-1}} .
$$

We apply the following transformation

$$
\left.{ }_{4} \phi_{3}\left[\begin{array}{c}
a, a^{1 / 2} q,-a^{1 / 2} q, b  \tag{7.16}\\
a^{1 / 2},-a^{1 / 2}, a q / b
\end{array}\right], t\right]={ }_{2} \phi_{1}\left[\begin{array}{c}
1 / b, t \\
b q t
\end{array} ; q, a q\right] \frac{(a q ; q)_{\infty}(b t ; q)_{\infty}}{(t ; q)_{\infty}(a q / b ; q)_{\infty}},
$$

which can be found in [6, Ex. 2.2] and obtain a ${ }_{2} \phi_{1}$-series. We apply another transformation

$$
{ }_{2} \phi_{1}\left[\begin{array}{c}
a, b  \tag{7.17}\\
c
\end{array} ; q, z\right]={ }_{2} \phi_{1}\left[\begin{array}{c}
c / b, z \\
a z
\end{array} q, b\right] \frac{(b ; q)_{\infty}(a z ; q)_{\infty}}{(c ; q)_{\infty}(z ; q)_{\infty}}
$$

see [6, (1.4.1); Appendix (III.1)], before we are able to apply the summation (7.9). In basic hypergeometric notation (7.15) is equal to

$$
{ }_{4} \phi_{3}\left[\begin{array}{c}
q^{1+c}, i q^{3 / 2+c / 2},-i q^{3 / 2+c / 2}, q^{1-n} \\
i q^{1 / 2+c / 2},-i q^{1 / 2+c / 2}, q^{1+c+n}
\end{array} ; q,-q^{n-1}\right] \frac{\left(q^{2+2 c} ; q\right)_{1}\left(q^{1+c+n} ; q\right)_{n-1}}{\left(q^{c+1} ; q\right)_{1}\left(q^{n} ; q\right)_{n-1}} .
$$

We apply the following transformation rule

$$
\begin{align*}
& { }_{4} \phi_{3}\left[\begin{array}{c}
a, b, c, d \\
a q / b, a q / c, a q / d
\end{array} ; q,-a q^{2} /(b c d)\right]= \\
& \left.\begin{array}{l}
{ }_{8} \phi_{7}\left[\begin{array}{c}
a^{2} q /(b c d), a q^{3 / 2} /(b c d)^{1 / 2},-a q^{3 / 2} /(b c d)^{1 / 2}, a^{1 / 2},-a^{1 / 2}, a q /(c d), a q /(b d), a q /(b c) \\
a q^{1 / 2} /(b c d)^{1 / 2},-a q^{1 / 2} /(b c d)^{1 / 2}, a^{3 / 2} q^{2} /(b c d),-a^{3 / 2} q^{2} /(b c d), a q / b, a q / c, a q / d
\end{array} q,-q\right.
\end{array}\right] \\
& \quad \times \frac{(a q ; q)_{\infty}(-q ; q)_{\infty}\left(a^{3 / 2} q^{2} /(b c d) ; q\right)_{\infty}\left(-a^{3 / 2} q^{2} /(b c d) ; q\right)_{\infty}}{\left(a^{2} q^{2} /(b c d) ; q\right)_{\infty}\left(-a q^{2} /(b c d) ; q\right)_{\infty}\left(a^{1 / 2} q ; q\right)_{\infty}\left(-a^{1 / 2} q ; q\right)_{\infty}} \tag{7.18}
\end{align*}
$$

see [6. Ex. 2.13 (ii)], to obtain a ${ }_{8} \phi_{7}$-series. Next we apply the transformation rule

$$
\begin{align*}
& { }_{8} \phi_{7}\left[\begin{array}{c}
a, a^{1 / 2} q,-a^{1 / 2} q, b, c, d, e, f \\
a^{1 / 2},-a^{1 / 2}, a q / b, a q / c, a q / d, a q / e, a q / f
\end{array} ; a^{2} q^{2} /(b c d e f)\right]= \\
& \begin{array}{l}
{ }_{8} \phi_{7}\left[\begin{array}{c}
a^{2} q /(b c d), a q^{3 / 2} /(b c d)^{1 / 2},-a q^{3 / 2} /(b c d)^{1 / 2}, a q /(c d), a q /(b d), a q /(b c), e, f \\
a q^{1 / 2} /(b c d)^{1 / 2},-a q^{1 / 2} /(b c d)^{1 / 2}, a q / b, a q / c, a q / d, a^{2} q^{2} /(b c d e), a^{2} q^{2} /(b c d f)
\end{array} ; q, a q /(e f)\right. \\
\\
\quad \times \frac{(a q ; q)_{\infty}(a q /(e f) ; q)_{\infty}\left(a^{2} q^{2} /(b c d e) ; q\right)_{\infty}\left(a^{2} q^{2} /(b c d f) ; q\right)_{\infty}}{(a q / e ; q)_{\infty}(a q / f ; q)_{\infty}\left(a^{2} q^{2} /(b c d) ; q\right)_{\infty}\left(a^{2} q^{2} /(b c d e f) ; q\right)_{\infty}}
\end{array}
\end{align*}
$$

see [6, (2.10.1); Appendix (III.23)] and finally the summation formula

$$
\begin{align*}
& { }_{8} \phi_{7}\left[\begin{array}{c}
-(a b / q)^{1 / 2} c, i(a b)^{1 / 4} c^{1 / 2} q^{3 / 4},-i(a b)^{1 / 4} c^{1 / 2} q^{3 / 4}, a, b, c,-c,-(a b q)^{1 / 2} / c \\
i(a b / q)^{1 / 4} c^{1 / 2},-i(a b / q)^{1 / 4} c^{1 / 2},-(b q / a)^{1 / 2} c,-(a q / b)^{1 / 2} c,-(a b q)^{1 / 2},(a b q)^{1 / 2}, c^{2} ; q, \frac{c q^{1 / 2}}{(a b)^{1 / 2}}
\end{array}\right]= \\
& \frac{\left(-(a b q)^{1 / 2} c ; q\right)_{\infty}\left(-c(q / a b)^{1 / 2} ; q\right)_{\infty}}{\left(-(b q / a)^{1 / 2} c ; q\right)_{\infty}\left(-(a q / b)^{1 / 2} c ; q\right)_{\infty}} \frac{\left(a q ; q^{2}\right)_{\infty}\left(b q ; q^{2}\right)_{\infty}\left(c^{2} q / a ; q^{2}\right)_{\infty}\left(c^{2} q / b ; q^{2}\right)_{\infty}}{\left(q q^{2}\right)_{\infty}\left(a b q ; q^{2}\right)_{\infty}\left(c^{2} q ; q^{2}\right)_{\infty}\left(c^{2} q /(a b) ; q^{2}\right)_{\infty}}, \tag{7.20}
\end{align*}
$$

see [6] Ex. 2.17(i); Appendix (II.16)], in order to obtain the closed form for (7.15).

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