REFINED RESTRICTED INVOLUTIONS

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Abstract

Define $I_n^k(\alpha)$ to be the set of involutions of $\{1,2,\ldots,n\}$ with exactly k fixed points which avoid the pattern $\alpha \in S_i$, for some $i \geq 2$, and define $I_n^k(\emptyset;\alpha)$ to be the set of involutions of $\{1,2,\ldots,n\}$ with exactly k fixed points which contain the pattern $\alpha \in S_i$, for some $i \geq 2$, exactly once. Let $i_n^k(\alpha)$ be the number of elements in $I_n^k(\alpha)$ and let $i_n^k(\emptyset;\alpha)$ be the number of elements in $I_n^k(\emptyset;\alpha)$. We investigate $I_n^k(\alpha)$ and $I_n^k(\emptyset;\alpha)$ for all $\alpha \in S_3$. In particular, we show that $i_n^k(132) = i_n^k(213) = i_n^k(321)$, $i_n^k(231) = i_n^k(312)$, $i_n^k(\emptyset;132) = i_n^k(\emptyset;213)$, and $i_n^k(\emptyset;231) = i_n^k(\emptyset;312)$ for all $0 \leq k \leq n$.

1. Introduction

Recall that $\pi \in S_n$ is called an involution if and only if $\pi^{-1} = \pi$. Equivalently, π is an involution if and only if the cycle structure of π has no cycle of length longer than two. In [RSZ], the study of refined restricted permutations was initiated. In order to describe the objects studied in [RSZ] and below we have need of a few definitions.

Let $\pi \in S_n$ be a permutation of $\{1,2,\ldots,n\}$ written in one-line notation. Let $\alpha \in S_m$, $m \leq n$. We say that π *contains the pattern* α if there exist indices $i_1, i_2, \ldots, i_m$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_m}$ is equivalent to α, where we define equivalence as follows. Define $\overline{\pi}_{i_j} = |\{x : \pi_{i_x} \leq \pi_{i_j}, 1 \leq x \leq m\}|$. If $\alpha = \overline{\pi}_{i_1}\overline{\pi}_{i_2}\ldots\overline{\pi}_{i_m}$ then we say that α and $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_m}$ are

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equivalent. For example, if $\tau = 124635$ then τ contains the pattern 132 by noting that $\tau_2\tau_4\tau_5 = 263$ is equivalent to 132. We say that π is α-*avoiding* if π does not contain the pattern α. In our above example, τ is 321-avoiding.

Let $S = \cup_{i\geq 2} S_i$. Let T be a subset of S and M be a multiset of S. Define $S_n(T;M)$ to be the set of permutations in S_n which avoid all patterns in T and contain each pattern in M exactly once. Let $s_n(T;M)$ be the number of elements in $S_n(T;M)$. If $M = \emptyset$ we write $S_n(T)$ and $s_n(T)$. Further, if T or M contain only one pattern, we omit the set notation.

Consider the following refinement, introduced in [RSZ]. Define $S_n^k(T;M)$ to be the set of permutations in $S_n(T;M)$ with exactly k fixed points. Let $s_n^k(T;M)$ be the number of elements in $S_n^k(T;M)$ where we omit M and the set notation when appropriate.

In this paper, we are concerned with those permutations in $S_n^k(T;M)$ which are involutions. To this end, we define $I_n^k(T;M)$ to be the set of involutions in $S_n^k(T;M)$ and we let $i_n^k(T;M)$ be the number of elements in $I_n^k(T;M)$. As before, we omit M and the set notation when appropriate.

In [RSZ], it was shown that $s_n^k(132) = s_n^k(213) = s_n^k(321)$ and $s_n^k(231) = s_n^k(312)$ for all $0 \leq k \leq n$. In this paper we will show that the same result holds when restricting our permutations to be involutions.

The results $s_n^k(132) = s_n^k(321)$ and $i_n^k(132) = i_n^k(321)$ lend some evidence that there may be a restricted permutation result concerning the cycle structure. However, for a given cycle structure c, in general, the number of 132-avoiding permutations with cycle structure c is *not* equal to the number of 321-avoiding permutations with cycle structure c. As an example, consider $S_6(132)$ and $S_6(321)$. (It should be noted that $n = 6$ is the minimal n such that the number of permutations classified according to their cycle structure differ by restriction.) Below we give the permutations in each according to their cycle structure.

Cycle structure	$S_6(132)$	$S_6(321)$
1^6	1	1
$1^4 2^1$	5	5
$1^3 3^1$	8	8
$1^2 2^2$	9	9
$1^2 4^1$	12	12
$1^1 2^1 3^1$	20	20
$1^1 5^1$	20	20
2^3	5	5
$2^1 4^1$	20	18
3^2	8	10
6^1	24	24
Sum	132	132

Some results concerning restricted involutions along with their fixed point refinement are known. These are stated in the following three theorems. Other results are given in [GM] and [GM2].

Theorem 1.1 *(Simion and Schmidt, [SiS][2]) Let $i_n(\alpha)$ be the number of α-avoiding involutions in S_n. Let $p_1 \in \{123, 132, 213, 321\}$ and $p_2 \in \{231, 312\}$. For $n \geq 1$,*

$$\begin{aligned} i_n(p_1) &= \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \textit{and} \\ i_n(p_2) &= 2^{n-1}. \end{aligned}$$

Theorem 1.2 *(Guibert and Mansour, [GM]) Let $i_n^k(132)$ be the number of 132-avoiding involutions in S_n with k fixed points. For $0 \leq k \leq n$,*

$$i_n^k(132) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \textit{for } k+n \textit{ even} \\ 0 & \textit{for } k+n \textit{ odd.} \end{cases}$$

Theorem 1.3 *(Robertson, Saracino, and Zeilberger, [RSZ]) Let $\gamma \in S_n$ be given by $\gamma_i = n+1-i$ for $1 \leq i \leq n$. For $\pi \in S_n$, let $\pi^\star = \gamma\pi\gamma^{-1}$. Then, for all π, π and $\pi^\star$ have the same number of fixed points. Furthermore, the number of occurrences of the pattern 213 (respectively 312) in π equals the number of occurrences of the pattern 132 (respectively 231) in $\pi^\star$.*

In the next section, we finish the enumeration of $i_n^k(\alpha)$ for all $\alpha \in S_3$ and $0 \leq k \leq n$, as well as provide some bijective results. In the last section, we investigate $I_n^k(\emptyset; \alpha)$ for all $\alpha \in S_3$ and $0 \leq k \leq n$.

Notation We note here that with a transposition (xy) we will always take $x < y$.

2. Involutions Avoiding a Length Three Pattern

In [SiS], Simion and Schmidt completed the study of involutions avoiding a given pattern of length three. Their results are given in Theorem 1.1 above. As done in [RSZ], we refine the enumeration problem by classifying restricted permutations according to the number of fixed points.

We can see from the conjugation given in Theorem 1.3 that $i_n^k(132) = i_n^k(213)$ and $i_n^k(231) = i_n^k(312)$ for all $0 \leq k \leq n$. In this section (Theorem 2.2) we show that $i_n^k(321) = i_n^k(132) = i_n^k(213)$ for all $0 \leq k \leq n$ as well.

We note here that since our permutations are involutions, we clearly require $n+k$ to be even in all theorems below.

[2] Rodica Simion did not like the SS acronym due to its unpleasant connotation (see [Z]). Hence, we use the nonstandard SiS and hope that others will as well.

In the proofs below, we will use the following properties of standard Young tableaux. (For proofs of these properties see [K], [K2], and [S].) Let $\mathcal{Y}_\pi$ be the Young tableaux corresponding (via the Robinson-Schensted algorithm) to $\pi \in S_n$. Let $\mathcal{Y}_\pi$ have shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, where λ_i is the length of the i^{th} row.

1. λ_1 is the maximum length of an increasing subsequence of π.
2. The length of the first column of $\mathcal{Y}_\pi$ is the maximum length of a decreasing subsequence of π.
3. If π is an involution, then the number of fixed points of π equals the number of odd length columns in $\mathcal{Y}_\pi$.
4. For $i \leq \frac{n}{2}$, the number of standard Young tableaux of shape $(n-i, i)$ (or its transpose via changing columns into rows) is $\binom{n}{i} - \binom{n}{i-1}$.

Theorem 2.1 *For $n \geq 1$,*

$$i_n^0(123) = i_n^2(123) = \begin{cases} \binom{n-1}{\frac{n}{2}} & \textit{for } n \textit{ even} \\ 0 & \textit{for } n \textit{ odd} \end{cases}$$

$$i_n^1(123) = \begin{cases} \binom{n}{\frac{n-1}{2}} & \textit{for } n \textit{ odd} \\ 0 & \textit{for } n \textit{ even} \end{cases}$$

$$i_n^k(123) = 0 \textit{ for } k \geq 3.$$

Proof. Clearly for $k \geq 3$ we have an occurrence of 123. Hence, it remains to prove the formulas for $k = 0, 1, 2$.

Consider first $k = 0$ so that n is even. Let $\pi \in I_n^0(123)$. Since π is 123-avoiding, the longest increasing subsequence of π has length at most 2. Keeping in mind that π has no fixed point, we use the above properties of standard Young tableaux to see that

$$i_n^0(123) = \sum_{\substack{j=0 \\ j \text{ even}}}^{\frac{n}{2}} \left(\binom{n}{j} - \binom{n}{j-1} \right) = \binom{n-1}{\frac{n}{2}}.$$

Next, consider $k = 2$, so that again n is even. Since the total number of standard Young tableaux of two columns on $\{1, 2, \ldots, n\}$ with n even is $\binom{n}{\frac{n}{2}}$, we have

$$i_n^2(123) = \binom{n}{\frac{n}{2}} - \binom{n-1}{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1} = \binom{n-1}{\frac{n}{2}}.$$

For $k = 1$ we consider $i^1_{n-1}(123)$, which is equal to the number of standard Young tableaux on $\{1, 2, \ldots, n\}$ with at most 2 columns, with n odd (so that exactly one of the columns is of odd length). Hence,

$$i^1_n(123) = \sum_{j=0}^{\frac{n-1}{2}} \left(\binom{n-1}{j} - \binom{n}{j-1} \right) = \binom{n}{\frac{n-1}{2}}.$$

□

Remark. The case $i^1_n(123)$ also follows from Theorem 1.1 (originally done in [SiS]).

We now provide two bijections between $I^0_n(123)$ and $I^2_n(123)$ since we see that they are enumerated by the same sequence.

The first bijection uses standard Young tableaux. For $\pi \in S_n$, denote by $SYT(\pi)$ the standard Young tableau created by the Robinson-Schensted algorithm. Let $SYT_n(2)$ be the set of all standard Young tableaux on n elements with at most 2 columns with the lengths of the columns having the same parity. Now, let $\pi \in I^0_n(123)$ and consider $SYT(\pi)$. From the properties of standard Young tableaux we see that $SYT(\pi)$ has one or two columns, each of even length. Note that n must be the bottom entry in one of the columns. Let $\gamma : SYT_n(2) \to SYT_n(2)$ be the map which takes n and places it on the bottom of the other column (even if empty). For example,

$$\gamma \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & \\ 6 & \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}.$$

It is easy to check that for $\pi \in I^0_n(123)$, $\gamma(STY_n(\pi)) = STY_n(\tau)$ with $\tau \in I^2_n(123)$ and that γ is a bijection.

The second bijection we present uses Dyck paths. For completeness we make the following definition.

Definition 2.2 *Let $i \geq j \geq 0$ and $i + j \geq 2$ be even. A partial Dyck path is a path in $\mathbb{R}^2$ from $(0,0)$ to (i,j) with $j > 0$ consisting of a sequence of steps of length $\sqrt{2}$ and slope ± 1 which does not fall below the x-axis. We denote these two types of steps by $(1,1)$ and $(1,-1)$, called up-steps and down-steps, respectively. If $j = 0$ we call the path a (standard) Dyck path.*

Notation. We will denote the set of partial/standard Dyck paths from $(0,0)$ to (i,j) by $D(i,j)$.

We now describe, for completeness, a bijection from $S_n(123)$ to $D(2n,0)$ due to Krattenthaler [Kr].

Let $\mathcal{K} : S_n(123) \to D(2n, 0)$ be the bijection defined as follows. Let $\pi_1\pi_2\cdots\pi_n = \pi \in S_n(123)$. Determine the right-to-left maxima of π, i.e. $m = \pi_i$ is a right-to-left maximum if $m > \pi_j$ for all $j > i$. Let π have right-to-left maxima $m_1 < m_2 < \cdots < m_s$, so that we may write

$$\pi = w_s m_s w_{s-1} m_{s-1} \cdots w_1 m_1,$$

where the w_i's are possibly empty. Generate a Dyck path from $(0, 0)$ to $(2n, 0)$ as follows. Read π from right to left. For each m_i do $m_i - m_{i-1}$ up-steps (where we define $m_0 = 0$). For each w_i do $|w_i| + 1$ down-steps.

Using Krattenthaler's bijection, it is easy to check the following.

1. $\mathcal{K}|_{I_n(123)}$ produces a Dyck path that is symmetric about the line $x = n$.
2. $\mathcal{K}|_{I_n^0(123)}$ produces a Dyck path that has an even number of peaks.
3. $\mathcal{K}|_{I_n^2(123)}$ produces a Dyck path that has an odd number of peaks.

For example, to prove 2 and 3, we note that for all i, π_i is right-to-left maximum if and only if $i = 1$, and that if there are two fixed points then the righthand fixed point is a right-to-left maximum but the lefthand fixed point is not.

Using facts 1–3, we define $\Gamma : I_n^0(123) \to I_n^2(123)$ as follows. Let $\pi \in I_n^0(123)$ and generate $\mathcal{K}(\pi)$, which by the above properties must have a valley on the line $x = n$, i.e. it must have a down-step which ends on the line $x = n$ followed by an up-step. To apply Γ, turn the down-step into an up-step and the up-step into a down-step. It is easy to check that Γ is a bijection.

In the next theorem we find the surprising fact that $i_n^k(132) = i_n^k(321)$ for all $0 \le k \le n$.

Theorem 2.3 *Let $\alpha \in \{132, 213, 321\}$. For $0 \le k \le n$,*

$$i_n^k(\alpha) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \text{for } n+k \text{ even} \\ 0 & \text{for } n+k \text{ odd.} \end{cases}$$

Proof. Due to Theorems 1.2 and 1.3, all that remains is to prove the formula for the pattern 321. The proof for 321 uses the properties of standard Young tableaux. Since $\pi \in I_n^k(321)$ may not contain a decreasing subsequence of length greater than 2, we see that the Young tableaux corresponding to π has shape $(n - \frac{n-k}{2}, \frac{n-k}{2})$. Thus,

$$i_n^k(321) = \binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2} - 1},$$

which simplifies to the stated formula. $\square$

We see, in particular, from Theorem 2.3, that the 321-avoiding derangement involutions of $\{1, 2, \ldots, 2n\}$ and the 321-avoiding involutions of $\{1, 2, \ldots, 2n-1\}$ with exactly one fixed point are both enumerated by $C_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. To the best of the authors' knowledge, these are new manifestation of the Catalan numbers. Below, we provide a bijective explanation of this fact, as a special case of the more general bijection δ defined below.

It is well-known that $|D(n,k)| = \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}$ (with $n+k$ even), the formula given in Theorem 2.3. Knowing this, we give a bijection from $I_n^k(321)$ to $D(n,k)$. Note that $D(n,k) = \emptyset$ if $k < 0$ or $k > n$.

Let $\pi \in I_n^k(321)$ with $n+k$ even and define the map $\delta : I_n^k(321) \to D(n,k)$ as follows. Write $\pi = \pi_1\pi_2\cdots\pi_n$. If $\pi_i - i \geq 0$ then the i^{th} step in $\delta(\pi)$ is an up-step. If $\pi_i - i < 0$ then the i^{th} step in $\delta(\pi)$ is a down-step.

We first show that $\delta(\pi) \in D(n,k)$ (i.e., that it does not fall below the x-axis and that it ends at (n,k)). Since π is an involution, if we ignore all fixed points in π, by the definition of δ, each down-step must be coupled with a remaining up-step to its left. Hence, for each $1 \leq i \leq n$, $|\{j : \pi_j \geq j, j \leq i\}| \geq |\{j : \pi_j < j, j \leq i\}|$ thereby showing that $\delta(\pi)$ does not fall below the x-axis. Since k is the number of fixed points in π and we have k more up-steps than down-steps in $\delta(\pi)$, our ending height of $\delta(\pi)$ is clearly k, thereby showing that $\delta(\pi) \in D(n,k)$.

To finish showing that δ is a bijection we provide δ^{-1}. Let $d \in D(n,k)$. Number the steps of d from left to right by $1, 2, \ldots, n$. Proceeding from right to left across d, couple each down-step with the closest uncoupled up-step to its left. Take the two step numbers and create a transposition. For the uncoupled up-steps (if any), take the step number of and create a fixed point. Once we have traversed d we will have an involution with k fixed points.

We now show that the resulting involution is 321-avoiding. We may decompose d as

$$u^{i_1}Pu^{i_2}Pu^{i_3}\cdots u^{i_k}Pu^{i_{k+1}},$$

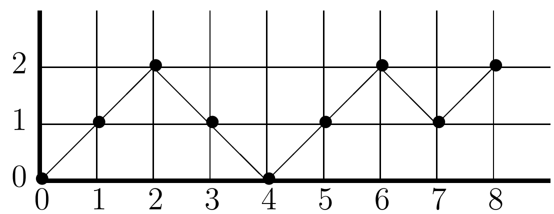
with $k \geq 1$, $i_1, i_2, \ldots, i_{k+1} \geq 0$, and where u^j stands for a sequence of j consecutive up-steps and the P's are nonempty Dyck paths. Hence, each occurrence of u in this decomposition is an uncoupled up-step and yields a fixed point in $\delta^{-1}(d)$. Furthermore, any transposition in π comes from an up-step and down-step that both reside within the same P.

Note that a 321 occurrence, if it exists, may contain at most one fixed point. Hence, a 321 occurrence must come from at least two transposition, say (ab) and (cd). Furthermore, from the description of δ^{-1}, we see that (ab) and (cd) must come from the same P in the decomposition given above since if (ab) comes from a P to the left of the P from which (cd) comes, then necessarily $a, b < c, d$ and neither c nor d can be the smallest element of the 321 pattern.

First, let z be a fixed point (xy) and (uv) be transpositions in $\delta^{-1}(d)$, where (xy) and (uv) come from the same P in the decomposition above. From this decomposition and the description of δ^{-1} we see that either $z < x, y, u, v$ or $z > x, y, u, v$. If $z < x, y, u, v$ then we have either a 123 or a 132 pattern. If $z > x, y, u, v$ then we have either a 123 or a 213 pattern. Hence a fixed point and at most two transpositions cannot create a 321 pattern.

We now let (xy) and (uv) be transpositions in $\delta^{-1}(d)$ which come from the same P in the decomposition given above. Without loss of generality, let $x < u$. From the description of δ^{-1} we must have $x < u < y < v$. This ordering yields a 3412 pattern, and thus no occurrence of 321. Hence, any possible 321 occurrence must consist of one number from each of three transpositions (xy), (uv), and (wz). We may assume that $x < u < w$ and conclude that $x < u < w < y < v < z$. This yields a 456123 pattern, and thus no occurrence of 321.

An example is in order. Consider $\pi = 34125768 \in I_8^2(321)$. Then $\delta(\pi)$ is the partial Dyck path shown below.



$\delta(\mathbf{34125768}) \in \mathbf{D(8,2)}$

For the inverse, we traverse the above partial Dyck path from right to left to get the involution (in cycle notation) $(8)(6\,7)(5)(2\,4)(1\,3) = 34125768$.

We may also use a bijection to $D(n,k)$ to offer an alternative proof for the patterns 132 and 213 (which by Theorem 1.3 are essentially the same). Let $\pi \in I_n^k(213)$ with $n+k$ even and consider the bijection $\zeta : I_n^k(213) \to D(n,k)$ defined as follows.

Create two columns, the left column designated the *up column*, denoted UC, and the right column designated the *down column*, denoted DC. Let $\pi = \pi_1\pi_2\cdots\pi_n$. Read π from left to right while performing the following algorithm.

I. If $\pi_i = i$, move to the next row, place i in UC, and move down another row.

II. If $\pi_i > i$, let x be the largest entry in DC's row. If x does not exist, set $x = 0$.

a) If $\pi_i > x$, place π_i in DC and i in UC.
b) If $\pi_i < x$, move to the next row and place π_i in DC and i in UC. Furthermore, move any element $y \in DC$, $y < \pi_i$, that is in a row above π_i to the row in which π_i was placed.

After placing all elements of π into UC or DC, compute $(u_1, u_2, \ldots, u_t)$ and $(d_1, d_2, \ldots, d_t)$, where u_i is the number of entries in the i^{th} row of UC and d_i is the number of entries in the i^{th} row of DC. (Note that some of the d_i's may be 0.) The partial Dyck path given by $u^{u_1}d^{d_1}u^{u_2}d^{d_2}\cdots u^{u_t}d^{d_t}$, where u^j is j consecutive up-steps and d^j is j consecutive down-steps, is $\zeta(\pi)$

To show that ζ is a bijection, we give ζ^{-1}. Let $d \in D(n,k)$. Traversing d from left to right label the up-steps in order (starting with 1). Once this is done, traversing d from right to left, label the down-steps in order starting with the next number (one more than the number of up-steps in d).

Call an up-step and a down-step to the right of the up-step *matching* if the line segment connecting their midpoints does not intersect the partial Dyck path. Using the labeling of steps given above, create a transposition of the labels for every pair of matching up-steps and down-steps. If an up-step has no matching down-step, create a fixed point with its label.

We now provide a sketch that the resulting permutation is 213-avoiding. We may decompose d as

$$u^{i_1}Pu^{i_2}Pu^{i_3}\cdots u^{i_k}Pu^{i_{k+1}},$$

with $k \geq 1$, $i_1, i_2, \ldots, i_{k+1} \geq 0$, and where u^j stands for a sequence of j consecutive up-steps and the P's are nonempty Dyck paths. Hence, each occurrence of u in this decomposition is an unmatched up-step and yields a fixed point in $\delta^{-1}(d)$. Furthermore, any transposition in $\delta^{-1}(d)$ comes from an up-step and down-step that both reside within the same P.

Let f be a fixed point $\delta^{-1}(d)$. Note that all elements to the left of f in $\delta^{-1}(d)$ are either fixed points or are larger than f. It follows that a 213 occurrence cannot contain two fixed points. Also, if x and y are not fixed points, then only xfy may be a 213 pattern. However, this implies that (xy) is not a transposition, i.e., that (xa) and (yb) are the transpositions (with $a \neq y$). Further, (xa) must come from a P in the decomposition to the left of the up-step corresponding to f and (yb) must come from a P in the decomposition to the right of the up-step corresponding to f. But this implies that $x > y, b$ and so xfy is not an occurrence of 213. Thus, a 213 occurrence cannot contain a fixed point.

Now assume that (xy) and (uv), $x < u$, create a 213 pattern. The only ordering which yields a 213 pattern is $x < y < u < v$. However, this is not possible since we have an up-step (corresponding to u) with a higher label than a down-step (corresponding to y).

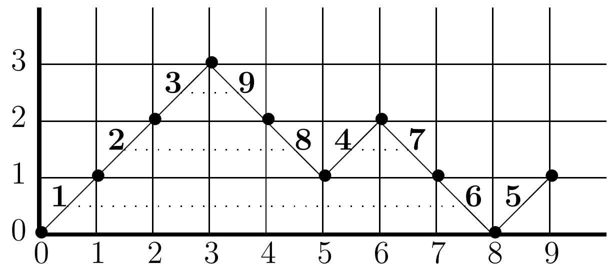
The last remaining case to consider is $(xy), (uv)$, (wz), $x < u < w$, creating a 213 pattern. We must have the ordering $x < u < w < v < y < z$ in order to have a 213 pattern (in fact, two such patterns). However, such an ordering is not possible since the path matching u and v will intersect one of the paths matching x and y or w and z.

To illustrate ζ, consider the following example. Let $\pi = 689751423 \in I_9^1(213)$. We find that our up and down columns are

UC	DC
$1,2,3$	$8,9$
4	$6,7$
5	

.

From here we get $(u_1,u_2,u_3)=(3,1,1)$ and $(d_1,d_2,d_3)=(2,2,0)$. Hence, $\zeta(\pi)$ is the partial Dyck path given below (ignoring the labels and dotted lines).



$\zeta(\mathbf{689751423})\in\mathbf{D}(\mathbf{9},\mathbf{1})$

For the inverse, note that a dotted line connects an up-step with its matching down-step (if it exists). Using this information and the labels on the above partial Dyck path we can immediately construct (in cycle notation) $(1\,6)(2\,8)(3\,9)(4\,7)(5)=689751423$.

The next theorem finishes this section.

Theorem 2.4 *Let $\alpha\in\{231,312\}$. For $n\geq 1$ and $0\leq k\leq n$,*

$$i_n^k(\alpha)=\begin{cases} 2^{\frac{n-k-2}{2}}\left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}}+\binom{\frac{n+k-2}{2}}{\frac{n-k}{2}}\right) & \text{for } n+k \text{ even}\\ 0 & \text{for } n+k \text{ odd.}\end{cases}$$

Proof. In [SiS] it is remarked that $S_n(\{231,312\})=I_n(231)$. Hence, $S_n^k(\{231,312\})=I_n^k(231)$ for all $0\leq k\leq n$. This last equality, coupled with Theorem 2.8 in [MR], gives the stated formula. ◻

3. Involutions Containing a Length Three Pattern Exactly Once

We can see from the conjugation given in Theorem 1.3 that $i_n^k(\emptyset;132)=i_n^k(\emptyset;213)$ and $i_n^k(\emptyset;231)=i_n^k(\emptyset;312)$ for all $0\leq k\leq n$. In this section, we show that these are the only equalities for patterns of length three. We note again that since our permutations are involutions, we clearly require $n+k$ to be even in all theorems below.

Theorem 3.1 *For $n \geq 1$,*

$$i_n^3(\emptyset; 123) = \frac{3}{n}\binom{n}{\frac{n-3}{2}} \quad \textit{for } n \geq 3 \textit{ odd, and}$$

$$i_n^k(\emptyset; 123) = 0 \qquad \textit{otherwise.}$$

Proof. We start with some at-first-sight unrelated results.

Recall that $D(i,j)$ is the set of partial/standard Dyck paths from $(0,0)$ to (i,j). Define $d(n,j)$ to be the size of $D(2n-j-1, j-1)$ for $j \geq 0$. Since a step ending at $(2n-j-1, j-1)$ is either a down-step from $(2n-j-2, j)$ or an up-step from $(2n-j-2, j-2)$ we see that

$$d(n,j) = d(n, j+1) + d(n-1, j-1). \tag{3.1}$$

By definition we have $d(n,1) = C_{n-1}$. From (3.1) we get $d(n,2) = C_{n-1}$ as well. Rearranging (3.1) and making the change of variables $j \mapsto j+1$ and $n \mapsto n+1$ we get

$$d(n,j) = d(n+1, j+1) - d(n+1, j+2). \tag{3.2}$$

From (3.2) we have

$$\sum_{j=2}^{n} d(n,j) = \sum_{j=2}^{n} \left(d(n+1, j+1) - d(n+1, j+2)\right) = d(n+1, 3).$$

Applying (3.1) again we see that

$$\sum_{j=2}^{n} d(n,j) = d(n+1,3) = d(n+1,2) - d(n,1) = C_n - C_{n-1}. \tag{3.3}$$

Now consider $\mathcal{K} : S_n(123) \to D(2n, 0)$, Krattenthaler's bijection as described in section 2.

Let π have right-to-left maxima $m_1 < m_2 < \cdots < m_s$, so that we may write

$$\pi = w_s m_s w_{s-1} m_{s-1} \cdots w_1 m_1,$$

where the w_i's are possibly empty.

We notice that if $\pi_j = n$ then since $m_s = n$ we have $|w_s| = j-1$. We consider the adumbrated permutation (which is technically not a permutation, but obviously corresponds uniquely to a permutation of the same length)

$$\pi^\star = m_s w_{s-1} m_{s-1} \cdots w_1 m_1,$$

(i.e. π with w_s removed). Using the algorithmic steps of $\mathcal{K}$, we may abuse notation and write $\mathcal{K}(\pi^\star)$ to mean $\mathcal{K}(\pi)$ with its last step removed. This partial Dyck path is in $D(2n-j-1, j-1)$. To see this, note that $\mathcal{K}(\pi^\star)$ ends at $(2n-j, j)$ but the last step *must* be an up-step. Hence, the number of permutations in $S_n(123)$ with $\pi(j) = n$ is $d(n, j)$ for any $1 \le j \le n$.

At last we turn our attention to $I_n^k(\emptyset; 123)$. We first argue that if $k \neq 3$ then $i_n^k(\emptyset; 123) = 0$. Clearly, if $k > 3$ we have more than one occurrence of 123. Hence, we assume $k < 3$. Let $\pi \in I_n^k(\emptyset; 123)$ with $k < 3$ and let our 123 pattern be the subsequence abc in π. It is easy to see that if we let $\pi = \pi(1) b \pi(2)$ then $\pi(1)$ is a permutation of $\{a, b+1, b+2, \dots, c-1, c+1, \dots, n\}$ and $\pi(2)$ is a permutation of $\{1, 2, \dots, a-1, a+1, \dots, b-1, c\}$. Since π is an involution, we see that we must have both a and c as fixed points. This in turn implies that b must be fixed, since b is preceded by $b-1$ entries, contradicting our assumption that $k < 3$.

Thus, we restrict our attention to $k = 3$, whereby our 3 fixed points create the single 123 occurrence. Call these fixed points $a < b < c$. From above we see that we must have $b = \frac{n+1}{2}$ and n odd in order for b to be a fixed point.

Since we are restricted to involutions, the placement of $1, 2, \dots, b-1, c$ completely defines $\pi \in I_n^3(\emptyset; 123)$. Thus, $1, 2, \dots, b-1, c$ must be 123-avoiding and can be identified uniquely with some $\tau \in S_b(123)$ with $\tau(j) = b$, where $j = j' - (b-1)$ and j' is defined by $\pi^{-1}(c)$. Since $\tau(1) = a$ we must have $\tau(1) \neq c$ so that $j \neq 1$. Since the number of permutations in $S_n(123)$ with $\pi(j) = n$ is $d(n, j)$ and $b = \frac{n+1}{2}$ we have, using (3.3),

$$i_n^3(\emptyset; 123) = \sum_{j=2}^{\frac{n+1}{2}} d\left(\frac{n+1}{2}, j\right) = C_{\frac{n+1}{2}} - C_{\frac{n-1}{2}}, \tag{3.4}$$

which simplifies to the stated formula. □

As a consequence of Theorem 3.1, we obtain the following obvious corollary.

Corollary 3.2 *For $n \ge 3$, $i_n(\emptyset; 123) = \frac{3}{n}\binom{n}{\frac{n-3}{2}}$.*

The next theorem for the pattern 132 was first proved in [GM]. This combined with Theorem 1.3 yields the following theorem, which we include for completeness.

Theorem 3.3 *For $n \ge 3$, $0 \le k \le n$, and $\alpha \in \{132, 213\}$,*

$$i_n^k(\emptyset; \alpha) = \begin{cases} \frac{k+1}{n-1}\binom{n-1}{\frac{n+k}{2}} & \text{for } n+k \text{ even and } k \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Summing $i_n^k(\emptyset; \alpha)$ for $\alpha \in \{132, 213\}$ over k gives us the following result, first given in [GM].

Corollary 3.4 *For $n \geq 3$ and $\alpha \in \{132, 213\}$, $i_n(\emptyset; \alpha) = \binom{n-2}{\lfloor \frac{n-3}{2} \rfloor}$.*

Theorem 3.5 *For $n \geq 4$, $0 \leq k \leq n$, and $\alpha \in \{231, 312\}$,*

$$i_n^k(\emptyset; \alpha) = \begin{cases} (k-1)2^{\frac{n-k-6}{2}} \left(\binom{\frac{n+k}{2}-2}{\frac{n-k}{2}-1} + 2\binom{\frac{n+k}{2}-3}{\frac{n-k}{2}-1} + \binom{\frac{n+k}{2}-4}{\frac{n-k}{2}-1} \right) & \text{for } n+k \text{ even} \\ \\ 0 & \text{for } n+k \text{ odd.} \end{cases}$$

Proof. Let $a(n,k) = i_n^k(\emptyset; 231)$ and $b(n,k) = i_n^k(231)$ for $0 \leq k \leq n$. Let $\pi \in I_n^k(\emptyset; 231)$. Write $\pi = \pi(1)n\pi(2)j$; if $j = 1$ then $\pi(1) = \emptyset$ and if $j = n$ then $\pi(2) = \emptyset$.

For $j = n$ we clearly have $\pi(1) \in I_{n-1}^{k-1}(\emptyset; 231)$. For $j < n$ we consider two cases: the 231 pattern is to the left of n and the 231 pattern is to the right of n. We argue that the 231 pattern cannot include n. To see this, assume otherwise and let ynx be the 231 pattern. If $x \neq j$ we must have $j > y$ so that $x < j$ and (xy) is not a transposition of π. This implies that $\pi^{-1}(x)nx$ and ynx are distinct 231 patterns (since (xy) is not a transposition of π), a contradiction. If $x = j$ then we have $j < y$. Since (jn) is a transposition of π, we know that $\pi^{-1}(y) \neq j$. Hence, ynj and $yn\pi^{-1}(y)$ are two distinct occurrences of 231, again a contradiction.

First, consider the case where $\pi(1)$ contains the pattern. Note that we must have $\pi(2) = (n-1)(n-2)\cdots(j+2)(j+1)$ and that $\pi(1) \in I_{j-1}^{k-1} \cup I_{j-1}^{k}$, depending upon the parity of $n+j$. Hence, this case contributes

$$\sum_{\substack{j=1 \\ n+j \text{ even}}}^{n-2} a(j-1, k-1) + \sum_{\substack{j=1 \\ n+j \text{ odd}}}^{n-1} a(j-1, k).$$

Next, consider the case where $\pi(2)j$ contains the pattern. In this case it is easy to see that $\pi(2) = (n-2)(n-1)$ and that $j = n-3$. Thus, this case contributes $b(n-4, k-2)$ to the total.

Summing over all j we get

$$a(n,k) = a(n-1, k-1) + b(n-4, k-2) + \sum_{\substack{j=1 \\ n+j \text{ even}}}^{n-2} a(j-1, k-1) + \sum_{\substack{j=1 \\ n+j \text{ odd}}}^{n-1} a(j-1, k)$$

which, using $b(n,k) = 2b(n-2,k) + b(n-1,k-1)$ given in [MR], yields

$$a(n,k) = 2a(n-2,k) + a(n-1,k-1) + b(n-6,k-2) + b(n-5,k-3). \tag{3.5}$$

Define the generating functions $A_k(x) = \sum_{n\geq 0} a(n,k)x^n$ and $B_k(x) = \sum_{n\geq 0} b(n,k)x^n$. Using $b(n,k) = 2b(n-2,k) + b(n-1,k-1)$ it is easy to show that

$$B_k(x) = \frac{x^k(1-x^2)}{(1-2x^2)^{k+1}}. \tag{3.6}$$

Since $b(n,k) = s_n^k(231,312)$ (shown in the proof of Theorem 2.4) we have (from Theorem 2.9 in [MR])

$$B_k(x) = \sum_{\substack{n\geq 1 \\ n+k \text{ even}}} 2^{\frac{n-k-2}{2}} \left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}} + \binom{\frac{n+k-2}{2}}{\frac{n-k}{2}} \right) x^n. \tag{3.7}$$

From (3.5) we have $A_k(x) = 2x^2 A_k(x) + xA_{k-1}(x) + x^6 B_{k-2}(x) + x^5 B_{k-3}(x)$. Using (3.5) and (3.6) we get

$$A_k(x) = \frac{(k-1)x^{k+2}(1-x^2)^2}{(1-2x^2)^k} = (k-1)x^3(1-x^2)B_{k-1}(x).$$

To obtain the stated formula for $a(n,k)$, we extract the coefficient of x^n in $A_k(x)$ using the above equation and (3.7) and simplify. $\square$

Summing $i_n^k(\emptyset;\alpha)$ for $\alpha \in \{231,312\}$ over k gives us the following nice formula.

Corollary 3.6 *For $n \geq 5$ and $\alpha \in \{231,312\}$, $i_n(\emptyset;\alpha) = (n-1)2^{n-6}$.*

Remark. For $n = 4,6,8,\ldots$, $i_n(\emptyset;\alpha) = i_{2n-4}^2(\emptyset;\alpha)$ for $\alpha \in \{231,312\}$.

The last remaining pattern to consider in this section is 321.

Theorem 3.7 *For $n \geq 3$, $0 \leq k \leq n$,*

$$i_n^k(\emptyset;321) = \frac{k(k+3)}{n+1}\binom{n+1}{\frac{n-k}{2}-1}.$$

Proof. Let $i(n,k) = i_n^k(\emptyset;321)$. We first show that

$$i(n,k) = \sum_{\substack{f=1 \\ f \text{ odd}}}^{n-k} i(n-f,k-1)C_{\frac{f-1}{2}} + \sum_{\substack{f=2 \\ f \text{ even}}}^{n-k} i_{n-f}^k(321)C_{\frac{f}{2}},$$

which is equivalent, by Theorem 2.3, to

$$i(n,k) = \sum_{\substack{f=1 \\ f \text{ odd}}}^{n-k} i(n-f,k-1)C_{\frac{f-1}{2}} + \sum_{\substack{f=2 \\ f \text{ even}}}^{n-k} \frac{k+1}{n-f+1}\binom{n-f+1}{\frac{n-k-f}{2}}C_{\frac{f}{2}}, \tag{3.8}$$

where we have initial conditions $i(n,0)=0$ for all $n\geq 3$.

To see that $i(n,0)=0$ for all n let cba be the 321 pattern in π. In order to avoid another 321 pattern, to the left (right) of b we cannot have an element larger (smaller) than b, except c (a) itself. Hence, b is a fixed point. Thus, the restriction of having exactly one 321 pattern implies a fixed point must be present (see Theorem 6.4 in [RSZ] for further details). Hence, we may let f be the smallest fixed point in $\pi\in I_n^k(\emptyset;321)$. We separate the argument into two cases: f odd and f even.

First, let f be odd and write $\pi=\pi(1)f\pi(2)$. In order for π to contain exactly one occurrence of 321 we must have $\pi(1)\in I^0_{f-1}(321)$ and $\pi(2)\in I^{k-1}_{n-f}(\emptyset;321)$. To see that we require $\pi(1)\in I^0_{f-1}(321)$ assume otherwise, that is that $\pi(1)$ is not an involution. Since π is an involution and f is odd, there exist $x\neq y$ both in $\pi(1)$ with $x,y>f$. This produces two occurrences of 321: $xf\pi(x)$ and $yf\pi(y)$, a contradiction. (As an aside, this shows that an odd fixed point cannot be part of a 321 occurrence.) Next, since $\pi(1)\in I^0_{f-1}(321)$, we necessarily must have $\pi(2)\in I^{k-1}_{n-f}(\emptyset;321)$. Summing over valid f and using Theorem 2.3 (for $k=0$) we get

$$\sum_{\substack{f=1\\ f\ odd}}^{n-k} i(n-f,k-1)C_{\frac{f-1}{2}}$$

in this case.

Next, consider f even. Again, write $\pi=\pi(1)f\pi(2)$. Since π is an involution and f is even we must have $x\in\pi(1)$ with $x>f$. This gives the 321 occurrence $xf\pi(x)$. Thus, only one such x may exist. Furthermore, $\pi(2)$ must be 321-avoiding. Now, consider the f leftmost entries in π: $\tau=\tau(1)x\tau(2)f$. Note that τ is a 321-avoiding permutation on f elements. Furthermore, τ does not contain the element $\pi^{-1}(x)$. Thus, τ is a permutation of $\{1,2,\ldots,\pi^{-1}(x)-1,\pi^{-1}(x)+1,\ldots,f-1,f,x\}$. By letting $i\in\tau$ become $i-1$ if $\pi^{-1}(x)+1\leq i\leq f$ and letting x become f we obtain $\tau^\star\in I^0_f(321)$. Next consider $\pi(2)^\star=x\pi(2)$, i.e. $\pi(2)$ with x in the first position. As before, $\pi(2)^\star$ may be identified with $\sigma\in I^{k-1}_{n-f+1}(321)$ with the added condition that $\sigma(1)\neq 1$ since we know that $\pi(2)^\star(1)=x>\pi^{-1}(x)$. Next, we have that the number of 321-avoiding involutions of $\{1,2,\ldots,n-f+1\}$ with $k-1$ fixed points and 1 not a fixed point is $i^{k-1}_{n-f+1}(321)-i^{k-2}_{n-f}(321)$ (where $i^{k-2}_{n-f}(321)$ counts the number of such permutations with 1 being a fixed point). Noting that $i^{k-1}_n(321)-i^{k-2}_{n-1}(321)=i^k_{n-1}(321)$ and summing over valid f we get

$$\sum_{\substack{f=2\\ f\ even}}^{n-k} i^k_{n-f}(321)C_{\frac{f}{2}}.$$

Combining the two cases' results proves (3.8).

We must now show that (3.8) along with the initial conditions yields $i(n,k)=\frac{k(k+3)}{n+1}\binom{n+1}{\frac{n-k}{2}-1}$.

We first show that

$$\frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1} = \sum_{\substack{f=2\\ f\ even}}^{n-k} \frac{k+1}{n-f+1}\binom{n-f+1}{\frac{n-k-f}{2}} C_{\frac{f}{2}},$$

i.e., that

$$\frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1} = \sum_{i=1}^{\frac{n-k}{2}} \frac{k+1}{n-2i+1}\binom{n-2i+1}{\frac{n-k-2i}{2}} C_i. \tag{3.9}$$

For $0 \le k \le n$, denote the lefthand side of (3.9) by $f(n,k)$ and the righthand side of (3.9) by $g(n,k)$. It is straightforward to show that for $k \ge 1$, $f(n,k) = f(n-1,k+1) + f(n-1,k-1)$ and $g(n,k) = g(n-1,k+1) + g(n-1,k-1)$, where we define $f(n,k) = 0$ and $g(n,k) = 0$ if $n < k$. Since $f(2,2) = g(2,2)$, to prove that (3.9) holds it is sufficient to show that $f(n,0) = g(n,0)$ for all $n \ge 2$.

By Theorem 3.1, we see that $f(n,0) = \frac{3}{n+1}\binom{n+1}{\frac{n}{2}-1} = i_n^3(\emptyset;123)$. From (3.4), this gives us $f(n,0) = C_{\frac{n}{2}+1} - C_{\frac{n}{2}}$, where C_n is the Catalan number. Next, since

$$\begin{aligned} g(n,0) &= \textstyle\sum_{i=1}^{\frac{n}{2}} \frac{1}{n-2i+1}\binom{n-2i+1}{\frac{n}{2}-i} C_i \\ &= \textstyle\sum_{i=1}^{\frac{n}{2}} C_{\frac{n}{2}-i} C_i \\ &= \textstyle\sum_{i=0}^{\frac{n}{2}} C_{\frac{n}{2}-i} C_i - C_{\frac{n}{2}} \\ &= C_{\frac{n}{2}+1} - C_{\frac{n}{2}} \\ &= f(n,0) \end{aligned}$$

we have proven (3.9).

We now have

$$i(n,k) = \sum_{\substack{f=1\\ f\ odd}}^{n-k} i(n-f,k-1) C_{\frac{f-1}{2}} + \frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1}, \tag{3.10}$$

with initial conditions $i(n,0) = 0$ for all $n \ge 2$.

We use this and induction on $n+k$ to prove that $i(n,k) = \frac{k(k+3)}{n+1}\binom{n+1}{\frac{n-k}{2}-1}$. Since this holds for $i(1,1)$ and $i(2,0)$, we may assume that $i(n-f,k-1) = \frac{(k-1)(k+2)}{n-f+1}\binom{n-f+1}{\frac{n-f-k-1}{2}}$. Substitution into (3.10) gives

$$i(n,k) = \sum_{i=1}^{\frac{n-k}{2}} \frac{(k-1)(k+2)}{n-2i+2}\binom{n-2i+2}{\frac{n-k}{2}-i} C_{i-1} + \frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1}.$$

Hence, we must show that

$$\sum_{i=1}^{\frac{n-k}{2}} \frac{k+2}{n-2i+2}\binom{n-2i+2}{\frac{n-k}{2}-i}C_{i-1} = \frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1}. \tag{3.11}$$

Denote by $h(n,k)$ the lefthand side of (3.11) and keep $f(n,k)$ as the notation for the right-hand side of (3.11). It is straightforward to show that $h(n,k) = h(n-1,k+1)+h(n-1,k-1)$ and that $h(1,1) = f(1,1)$ and $h(2,2) = f(2,2)$. To prove (3.11), it is sufficient to show that $h(n,0) = f(n,0)$ for all $n \geq 2$. Since

$$\begin{aligned} h(n,0) &= \textstyle\sum_{i=1}^{\frac{n}{2}} \frac{2}{n-2i+2}\binom{n-2i+2}{\frac{n}{2}-i}C_{i-1} \\ &= \textstyle\sum_{i=0}^{\frac{n}{2}-1} \frac{2}{n-2i}\binom{n-2i}{\frac{n}{2}-i-1}C_i \\ &= \textstyle\sum_{i=0}^{\frac{n}{2}-1} C_{\frac{n}{2}-i}C_i \\ &= \textstyle\sum_{i=0}^{\frac{n}{2}} C_{\frac{n}{2}-i}C_i - C_{\frac{n}{2}} \\ &= C_{\frac{n}{2}+1} - C_{\frac{n}{2}} \\ &= f(n,0) \end{aligned}$$

we have proven (3.11), thereby proving the theorem. □

From the proof of Theorem 3.7 we obtain Corollary 3.9 below, for which we have need of the following definition.

Definition 3.8 *Let $dp(n,k) \in D(n,k)$ and let $dp_x(n)$ be a Dyck path with $2n$ steps starting at $(x,0)$. For $1 \leq i \leq \frac{n-k}{2}$, we call a lattice path which results from $dp(n-2i,k) \cup dp_{n-2i}(i)$ a modified Dyck path with a single drop from height k, and denote the set of all such modified Dyck paths by $MDP(n;k)$.*

Using this definition, we can give the following, the proof of which is a direct consequence of (3.9).

Corollary 3.9 *For $n \geq 2$ and $0 \leq k \leq n$ with $n+k$ even, $|MDP(n;k)| = \frac{k+3}{n+1}\binom{n+1}{\frac{n-k}{2}-1}$.*

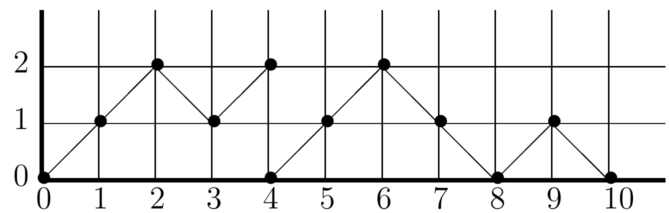
Comparing Corollary 3.9 with the number of partial Dyck paths, we find that $|MDP(n;k)| = |D(n,k+2)|$. We explain this via a bijection.

Let $pdp(n-2i,k) \circ dp(i)$ be the decomposition of an element in $MDP(n;k)$ where *pdp* stands for partial Dyck path and *dp* stands for (standard) Dyck path. To obtain an element in $D(n,k+2)$ we perform the following steps.

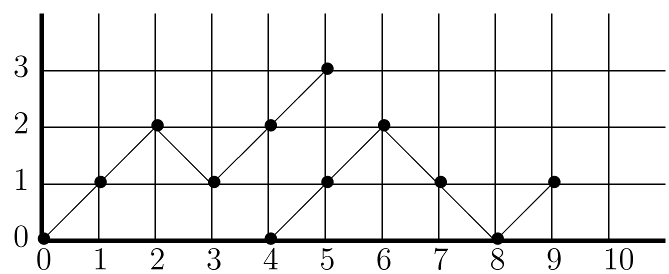
Concatenate one up-step to the end of $pdp(n-2i,k)$. To the end of this new up-step concatenate $d(i)$ and remove the last step of $d(i)$ (necessarily a down-step). The result is an element of $D(n,k+2)$.

For the inverse, perform the following steps to $pdp(n,k+2) \in D(n,k+2)$. Add a down-step to the end of $pdp(n,k+2)$. Next, traverse $pdp(n,k+2)$ from left to right and locate the last occurrence of two consecutive up-steps whose second step has ending point on the line $y=k+2$. From these two up-steps, remove the up-step closest to the origin. We now have a partial Dyck path ending at height k and a Dyck path lying $k+1$ units above the x-axis. Move the Dyck path left 1 unit and down $k+1$ units. The result is a member of $MDP(n;k)$.

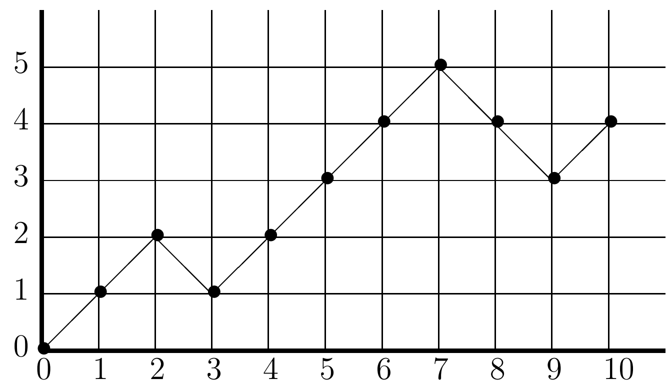
We illustrate this bijection with an example. Consider the following member of $MDP(10;2)$.



We add an up-step to the end of the partial Dyck path and remove the last step of the modified Dyck path to get the following lattice path.



To create an element of $D(n, k+2)$ we concatenate the Dyck path with its last step removed to the end of the partial Dyck path and get the following.



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Appendix

Below we provide values of $i_n^k(\alpha)$ and $i_n^k(\emptyset;\alpha)$ for small n and all $\alpha \in S_3$.

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	0	3	0	0					
4	3	0	3	0	0				
5	0	10	0	0	0	0			
6	10	0	10	0	0	0	0		
7	0	35	0	0	0	0	0	0	
8	35	0	35	0	0	0	0	0	0

$\mathbf{i_n^k(123)}$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	0	2	0	1					
4	2	0	3	0	1				
5	0	5	0	4	0	1			
6	5	0	9	0	5	0	1		
7	0	14	0	14	0	6	0	1	
8	14	0	28	0	20	0	7	0	1

$\mathbf{i_n^k(132) = i_n^k(321) = i_n^k(213)}$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	1	0	1						
3	0	3	0	1					
4	2	0	5	0	1				
5	0	8	0	7	0	1			
6	4	0	18	0	9	0	1		
7	0	20	0	32	0	11	0	1	
8	8	0	56	0	50	0	13	0	1

$\mathbf{i_n^k(231) = i_n^k(312)}$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	0								
1	0	0							
2	0	0	0						
3	0	0	0	1					
4	0	0	0	0	0				
5	0	0	0	3	0	0			
6	0	0	0	0	0	0	0		
7	0	0	0	9	0	0	0	0	
8	0	0	0	0	0	0	0	0	0

$\mathbf{i_n^k}(\emptyset;\mathbf{123})$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	0								
1	0	0							
2	0	0	0						
3	0	1	0	0					
4	0	0	1	0	0				
5	0	2	0	1	0	0			
6	0	0	3	0	1	0	0		
7	0	5	0	4	0	1	0	0	
8	0	0	9	0	5	0	1	0	0

$\mathbf{i_n^k}(\emptyset;\mathbf{132}) = \mathbf{i_n^k}(\emptyset;\mathbf{213})$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	0								
1	0	0							
2	0	0	0						
3	0	0	0	0					
4	0	0	1	0	0				
5	0	0	0	2	0	0			
6	0	0	2	0	3	0	0		
7	0	0	0	8	0	4	0	0	
8	0	0	5	0	18	0	5	0	0

$\mathbf{i_n^k}(\emptyset;\mathbf{231}) = \mathbf{i_n^k}(\emptyset;\mathbf{312})$

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	0								
1	0	0							
2	0	0	0						
3	0	1	0	0					
4	0	0	2	0	0				
5	0	4	0	3	0	0			
6	0	0	10	0	4	0	0		
7	0	14	0	18	0	5	0	0	
8	0	0	40	0	28	0	6	0	0

$\mathbf{i_n^k}(\emptyset;\mathbf{321})$