# An umbral setting for cumulants and factorial moments 

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#### Abstract

We provide an algebraic setting for cumulants and factorial moments through the classical umbral calculus. Main tools are the compositional inverse of the unity umbra, connected with the logarithmic power series, and a new umbra here introduced, the singleton umbra. Various formulae are given expressing cumulants, factorial moments and central moments by umbral functions.


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## 1 Introduction

The purpose of this paper is mostly to show how the classical umbral calculus gives a lithe algebraic setting in handling cumulants and factorial moments. The classical umbral calculus consists of a symbolic technique dealing with sequences of numbers $a_{n}$ indexed by nonnegative integers $n=0,1,2,3, \ldots$, where the subscripts are treated as if they were powers. This kind of device was extensively used since the nineteenth century although the mathematical community was sceptic of it, owing to its lack of foundation. To the best of our knowledge, the method was first proposed by Rev. John Blissard in a series of papers as from 1861 (cf. [5] for the full list of papers), nevertheless it is impossible to put the credit of the original idea down to him since the Blissard's calculus has its mathematical source in symbolic differentiation. In the thirties, Bell [1] reviewed the whole subject in several papers, restoring the purport of the Blissard's idea and in [2] he tried to give a rigorous foundation of the mystery at the ground of the umbral calculus but his attempt did not have a hold. Indeed, in the first modern textbook of combinatorics [12] Riordan often employed this symbolic method without giving any formal justification. It was first GianCarlo Rota to disclose the "umbral magic art" of shifting from $a^{n}$ to $a_{n}$ bringing to the light the underlying linear functional (cf. [15]). This idea led Rota and his collaborators to conceive a beautiful theory (cf. [10] and [14]) which has originated a large variety of applications (see [4] for a list of papers updated to 2000). Some years later, Roman and Rota [13] gave rigorous form to the umbral tricks in the setting of Hopf algebra (see also [9]). But in 1994 Rota himself wrote (cf. [18]): "... Although the notation of Hopf algebra satisfied the most ardent advocate of spic-and-span rigor, the translation of "classical" umbral calculus into the newly found rigorous language made the method altogether unwieldy and unmanageable. Not only was the eerie feeling

[^0]of witchcraft lost in the translation, but, after such a translation, the use of calculus to simplify computation and sharpen our intuition was lost by the wayside..." Then, in the paper [18] The Classical Umbral Calculus (1994) Rota, together with Taylor, tries to restore the feeling meant by the founders of the umbral calculus keeping new notation both minimal and indispensable to avoid the misunderstanding of the past. In this new setting, the basic device is to represent an unital sequence of numbers by a symbol $\alpha$, named umbra, i.e. to associate the sequence $1, a_{1}, a_{2}, \ldots$ to the sequence $1, \alpha, \alpha^{2}, \ldots$ of powers of $\alpha$ through an operator $E$ that looks like the expectation of random variables (r.v.'s). This new way of dealing with sequences of numbers has been applied to combinatorial and algebraic subjects (cf. [17], [23] and [8]), wavelet theory (cf. [19]) and difference equations (cf. [24]). Besides it has led to a nimble language for r.v.'s theory, as showed in [16] and [5].

The present work is inspired by this last point of view. As a matter of fact, an umbra looks as the framework of a random variable (r.v.) with no reference to any probability space, someway getting closer to statistical methods. However, the use of symbolic methods in statistics is not a novelty: for instance Stuart and Ord [22] resort to such a technique in handling moments about a point. In addition in the umbral calculus, questions as convergence of series are no matter, as showed hereafter dealing with cumulants.

Among the sequences of numbers related to r.v.'s, cumulants play a central role characterizing all r.v.'s occurring in the classical stochastic processes. For instance, a r.v. having Poisson distribution of parameter $x$ is the unique probability distribution for which all its cumulants are equal to $x$. It seems therefore that a r.v. is better described by its cumulants than by its moments. Moreover, due to their properties of additivity and invariance under translation, the cumulants are not necessarily connected with the moments of any probability distribution. We can define cumulants $\kappa_{j}$ of any sequence $a_{n}, n=1,2,3, \ldots$ by

$$
\sum_{n=0}^{\infty} \frac{a_{n} t^{n}}{n!}=\exp \left\{\sum_{j=1}^{\infty} \frac{\kappa_{j} t^{j}}{j!}\right\}
$$

in disregard of questions of whether any series converges. By this approach, many difficulties connected to the "problem of cumulants" smooth out, where with "problem of cumulants" we refer to characterizations of sequences that are cumulants of some probability distributions. The simplest example is that the second cumulant of a probability distribution must always be nonnegative, and is zero only if all of the higher cumulants are zero. Cumulants are subject to no such constraints when they are analyzed by an algebraic point of view. What is more, in statistics they do not play any dual role compared to factorial moments. Whereas the algebraic setting here proposed comes to the light their close relationship through an umbral analogy with the well known complementary notions of compound and randomized Poisson r.v.'s (cf. [6])

Umbral notations are introduced in Section 2 by means of r.v.'s semantics. Our intention by this way is to make the reader comfortable with the umbral system of calculation without require any prior knowledge. We only skip some technical proofs of formal matters on which the reader is referred to citations. We also resume the theory of Bell umbrae, completely developed in [5], that not only gives the umbral
counterpart of the family of Poisson r.v.'s but it allows an umbral expression of the functional composition of exponential power series. Section 3 is devoted to a new umbra, named singleton umbra, playing a dual role compared to the Bell umbra. Their relationship is encoded by the compositional inverse of the unity umbra. The singleton umbra is the keystone of the umbral presentation of cumulants and factorial moments.

In the last two sections we give various umbral formulae for cumulants and factorial moments that parallel those known in statistics but simplify the proofs as well as the forms. This happens for instance for the equations expressing cumulants in terms of moments (and vice-versa) and also for their recursive formulas. Inversion theorems allowing to obtain an umbra from its cumulants or factorial moments are also stated.

In 1929, Fisher [7] introduced the $k$-statistics as new symmetric functions of the random sample. The aim of Fisher was to estimate the cumulants without using the moment estimators. He used only combinatorial methods. The $k$-statistics are related to the power sum symmetric functions whose variables are the r.v.'s of the sample, but these expressions are very unhandy. We believe that the umbral calculus may seek to simplify the expression of the $k$-statistics (as well as the $h$-statistics for the central moments) taking into account its combinatorial nature.

## 2 Umbrae and random variables

In the following, we resume terminology, notations and some basic definitions of the classical umbral calculus, as it has been introduced by Rota and Taylor in [18] and further developed in [5]. Fundamental is the idea of associating a sequence of numbers $1, a_{2}, a_{3}, \ldots$ to an indeterminate $\alpha$ which is said to represent the sequence. This device is familiar in probability when $a_{i}$ represents the $i-$ th moment of a r.v. $X$. In this case, the sequence $1, a_{1}, a_{2}, \ldots$ results from applying the expectation operator $E$ to the sequence $1, X, X^{2}, \ldots$ consisting of powers of the r.v. $X$.

More formal, an umbral calculus consists of the following data:
a) a set $A=\{\alpha, \beta, \ldots\}$, called the alphabet, whose elements are named umbrae;
b) a commutative integral domain $R$ whose quotient field is of characteristic zero;
c) a linear functional $E$, called evaluation, defined on the polynomial ring $R[A]$ and taking values in $R$ such that
i) $E[1]=1$;
ii) $E\left[\alpha^{i} \beta^{j} \cdots \gamma^{k}\right]=E\left[\alpha^{i}\right] E\left[\beta^{j}\right] \cdots E\left[\gamma^{k}\right]$ for any set of distinct umbrae in $A$ and for $i, j, \ldots, k$ nonnegative integers (uncorrelation property);
d) an element $\epsilon \in A$, called augmentation [13], such that $E\left[\epsilon^{n}\right]=\delta_{0, n}$, for any nonnegative integer $n$, where

$$
\delta_{i, j}=\left\{\begin{array}{ll}
1 & \text { if } i=j \\
0 & \text { if } i \neq j
\end{array} \quad i, j \in N\right.
$$

$e)$ an element $u \in A$, called unity umbra [5], such that $E\left[u^{n}\right]=1$, for any nonnegative integer $n$.

A sequence $a_{0}=1, a_{1}, a_{2}, \ldots$ in $R$ is umbrally represented by an umbra $\alpha$ when

$$
E\left[\alpha^{i}\right]=a_{i}, \quad \text { for } i=0,1,2, \ldots
$$

The elements $a_{i}$ are called moments of the umbra $\alpha$ on the analogy of r.v.'s theory. The umbra $\epsilon$ can be view as the r.v. which takes the value 0 with probability 1 and the umbra $u$ as the r.v. which takes the value 1 with probability 1 . Note that the uncorrelation property among umbrae parallels the analogue one for r.v.'s as well as it is $E\left[\alpha^{n+k}\right] \neq E\left[\alpha^{n}\right] E\left[\alpha^{k}\right]$. Remark as this setting gets out of the well-known "moment problem" for r.v.'s.

Example 2.1 Bell umbra.
The Bell umbra $\beta$ is the umbra such that

$$
E\left[(\beta)_{n}\right]=1 \quad n=0,1,2, \ldots
$$

where $(\beta)_{0}=1$ and $(\beta)_{n}=\beta(\beta-1) \cdots(\beta-n+1)$ is the lower factorial. It results $E\left[\beta^{n}\right]=B_{n}$ where $B_{n}$ is the $n$-th Bell number (cf. [5]), i.e. the number of the partitions of a finite nonempty set with $n$ elements or the $n$-th coefficient in the Taylor series expansion of the function $\exp \left(e^{t}-1\right)$. So $\beta$ is the umbral counterpart of the Poisson r.v. with parameter 1.

We call factorial moments of an umbra $\alpha$ the elements

$$
a_{(n)}= \begin{cases}1, & n=0 \\ E\left[(\alpha)_{n}\right], & n>0\end{cases}
$$

where $(\alpha)_{n}=\alpha(\alpha-1) \cdots(\alpha-n+1)$ is the lower factorial. So the definition of $\beta$ in example 2.1 could be reformulated as follows: the Bell scalar umbra is the umbra whose factorial moments are $b_{(n)}=1$ for any nonnegative integer $n$.

### 2.1 Similar umbrae and dot-product

The notion of similarity among umbrae comes in handy in order to manipulate sequences such

$$
\begin{equation*}
\sum_{i=0}^{n}\binom{n}{i} a_{i} a_{n-i}, \quad n \in N \tag{1}
\end{equation*}
$$

as moments of umbrae. The sequence (1) cannot be represented by using only the umbra $\alpha$ with moments $a_{0}=1, a_{1}, a_{2}, \ldots$. Indeed, being $\alpha$ correlated to itself, the product $a_{i} a_{n-i}$ cannot be written as $E\left[\alpha^{i} \alpha^{n-i}\right]$. So we need two distinct umbrae having the same sequence of moments, as it happens for similar r.v.'s. Therefore, if we choose an umbra $\alpha^{\prime}$ uncorrelated with $\alpha$ but with the same sequence of moments, it is

$$
\begin{equation*}
\sum_{i=0}^{n}\binom{n}{i} a_{i} a_{n-i}=E\left[\sum_{i=0}^{n}\binom{n}{i} \alpha^{i}\left(\alpha^{\prime}\right)^{n-i}\right]=E\left[\left(\alpha+\alpha^{\prime}\right)^{n}\right] . \tag{2}
\end{equation*}
$$

Then the sequence (1) represents the moments of the umbra $\left(\alpha+\alpha^{\prime}\right)$. A way to formalize this matter is to define two equivalence relations among umbrae.

Two umbrae $\alpha$ and $\gamma$ are umbrally equivalent when

$$
E[\alpha]=E[\gamma]
$$

in symbols $\alpha \simeq \beta$. They are similar when

$$
\alpha^{n} \simeq \gamma^{n}, \quad n=0,1,2, \ldots
$$

in symbols $\alpha \equiv \gamma$. We note that equality implies similarity which implies umbral equivalence. The converses are false. Then, we shall denote by the symbol $n . \alpha$ the dot-product of $n$ and $\alpha$, an auxiliary umbra (cf. [18]) similar to the sum $\alpha^{\prime}+\alpha^{\prime \prime}+$ $\ldots+\alpha^{\prime \prime \prime}$ where $\alpha^{\prime}, \alpha^{\prime \prime}, \ldots, \alpha^{\prime \prime \prime}$ are a set of $n$ distinct umbrae each similar to the umbra $\alpha$. So the sequence in (2) is umbrally represented by the umbra $2 . \alpha$. We assume that $0 . \alpha$ is an umbra similar to the augmentation $\epsilon$.

We shall hereafter consider the dot product of $n$ and $\alpha$ as an umbra if we saturate the alphabet $A$ with sufficiently many umbrae similar to any expression whatever. For a formal definition of a saturated umbral calculus see [18]. It can be shown that saturated umbral calculi exist and that every umbral calculus can be embedded in a saturated umbral calculus.

The following statements are easily to be proved:
Proposition 2.2 (i) If $n . \alpha \equiv n . \beta$ for some integer $n \neq 0$ then $\alpha \equiv \beta$;
(ii) if $c \in R$ then $n .(c \alpha) \equiv c(n . \alpha)$ for any nonnegative integer $n$;
(iii) $n .(m . \alpha) \equiv(n m) . \alpha \equiv m .(n . \alpha)$ for any two nonnegative integers $n, m$;
(iv) $(n+m) \cdot \alpha \equiv n \cdot \alpha+m \cdot \alpha^{\prime}$ for any two nonnegative integers $n, m$ and any two distinct umbrae $\alpha \equiv \alpha^{\prime}$;
(v) $(n . \alpha+n . \beta) \equiv n .(\alpha+\beta)$ for any nonnegative integer $n$ and any two distinct umbrae $\alpha$ and $\beta$.

Two umbrae $\alpha$ and $\gamma$ are said to be inverse to each other when $\alpha+\gamma \equiv \varepsilon$. We denote the inverse of the umbra $\alpha$ by $-1 . \alpha^{\prime}$, with $\alpha \equiv \alpha^{\prime}$. Recall that, in dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two inverse umbrae of the same umbra are similar.

Example 2.3 Uniform umbra.
The Bernoulli umbra (cf. [18]) represents the sequence of Bernoulli numbers $B_{n}$ such that

$$
\sum_{k \geq 0}\binom{n}{k} B_{k}=B_{n}
$$

The inverse of the Bernoulli umbra is the umbral counterpart of the uniform r.v. over the interval $[0,1]$ (cf. [23]).

### 2.2 Generating functions

The formal power series in $R[A][[t]]$

$$
\begin{equation*}
u+\sum_{n \geq 1} \alpha^{n} \frac{t^{n}}{n!} \tag{3}
\end{equation*}
$$

is the generating function (g.f.) of the umbra $\alpha$, and it is denoted by $e^{\alpha t}$. The notion of umbrally equivalence and similarity can be extended coefficientwise to formal power series $R[A][[t]]$ (see $[24]$ for a formal construction). So it results

$$
\alpha \equiv \beta \Leftrightarrow e^{\alpha t} \simeq e^{\beta t} .
$$

Moreover, any exponential formal power series ${ }^{1}$ in $R[[t]]$

$$
f(t)=1+\sum_{n \geq 1} a_{n} \frac{t^{n}}{n!}
$$

can be umbrally represented by a formal power series (3) in $R[A][[t]]$. In fact, if the sequence $1, a_{1}, a_{2}, \ldots$ is umbrally represented by $\alpha$ then

$$
f(t)=E\left[e^{\alpha t}\right] \quad \text { i.e. } \quad f(t) \simeq e^{\alpha t}
$$

assuming that we naturally extend $E$ to be linear. We will say that $f(t)$ is umbrally represented by $\alpha$. Note that, from now on, when there is no mistaking, we will just say that $f(t)$ is the g.f. of $\alpha$. For example the g.f. of the augmentation umbra $\epsilon$ is 1 as well as the g.f. of the unity umbra $u$ is $e^{x}$.

Getting back to a r.v. $X$, recall that when $E[\exp (t X)]$ is a convergent function $f(t)$, it admits an exponential expansion in terms of the moments which are completely determined by the related distribution function (and vice-versa). In this case the moment generating function (m.g.f.) encodes all the information of $X$ and the notion of similarity among r.v.'s corresponds to that of umbrae.

The first advantage of the umbral notation introduced for g.f.'s is the representation of operations among g.f.'s with operations among umbrae. For example the multiplication among exponential g.f.'s is umbrally represented by a summation of the corresponding umbrae:

$$
\begin{equation*}
g(t) f(t) \simeq e^{(\alpha+\gamma) t} \quad \text { with } \quad f(t) \simeq e^{\alpha t}, g(t) \simeq e^{\gamma t} . \tag{4}
\end{equation*}
$$

Via (4), the g.f. of $n . \alpha$ is $f(t)^{n}$. If $\alpha$ is an umbra with g.f. $f(t)$, the inverse umbra $-1 . \alpha^{\prime}$ has g.f. $[f(t)]^{-1}$. The summation among exponential g.f.'s is umbrally represented by a disjoint sum of umbrae. The disjoint sum (respectively disjoint difference) of $\alpha$ and $\gamma$ is the umbra $\eta$ (respectively $\iota$ ) with moments

$$
\eta^{n} \simeq\left\{\begin{array} { l l } 
{ u , } & { n = 0 } \\
{ \alpha ^ { n } + \gamma ^ { n } , } & { n > 0 }
\end{array} \quad \left(\text { respectively } \quad \iota^{n} \simeq\left\{\begin{array}{ll}
u, & n=0 \\
\alpha^{n}-\gamma^{n}, & n>0
\end{array}\right)\right.\right.
$$

in symbols $\eta \equiv \alpha \dot{+} \gamma$ (respectively $\iota \equiv \alpha \dot{-} \gamma$ ). By the definition, it follows

$$
f(t) \pm[g(t)-1] \simeq e^{(\alpha \pm \gamma) t}
$$

[^1]Example 2.4 Unbiased estimators.
Suppose to make the disjoint sum of $n$ times the umbra $\alpha$. We will denote this umbra by $\dot{+}_{n} \alpha$. Its g.f. is $1+n[f(t)-1]$. The umbra $\dot{+}_{n} \alpha$ has the following probabilistic counterpart. Let $\left\{X_{i}\right\}_{i=1}^{n}$ be a random sample of independent and identically distributed (i.i.d.) r.v's. As it is well-known the power sum symmetric functions

$$
S_{r}=\sum_{i=1}^{n} X_{i}^{r}
$$

gives the unbiased estimators $S_{r} / n$ of the moments of $X_{i}$. But $E\left[\left(\dot{+}_{n} \alpha\right)^{r}\right]=n a_{r}$, hence the umbral corresponding of the power sum symmetric functions sequence $S_{r}$ is the umbra $\dot{+}_{n} \alpha$.

### 2.3 Auxiliary umbrae

In the following, suppose $\alpha$ an umbra with g.f. $f(t)$ and $\gamma$ an umbra with g.f. $g(t)$. The introduction of the g.f. device leads to the definition of new auxiliary umbrae useful for the development of the system of calculation. For this purpose, we should replace $R$ with whatever polynomial ring having coefficients in $R$ and a number of indeterminates according to necessity. In this paper, we deal with $R[x, y]$. This allows to define the dot-product of $x$ and $\alpha$ via g.f., i.e. $x . \alpha$ is the auxiliary umbra having generating function

$$
e^{(x . \alpha)} \simeq f(t)^{x}
$$

The Proposition 2.2 still holds replacing $n$ with $x$ and $m$ with $y$. Then, an umbra is said to be scalar if the moments are elements of $R$ while it is said to be polynomial if the moments are polynomials.

Example 2.5 Bell polynomial umbra.
The Bell polynomial umbra $\phi$ is the umbra having factorial moments equal to $x^{n}$ (cf. [5]). This umbra has g.f. $\exp \left[x\left(e^{t}-1\right)\right]$ so that $\phi \equiv x . \beta$, where $\beta$ is the Bell umbra. It turns out that the Bell polynomial umbra $x . \beta$ is the umbral counterpart of the Poisson r.v. with parameter $x$.

Example 2.6 Moments about a point.
The moments $E\left[(X-a)^{n}\right]$ about a point $a \in \mathbf{R}$ of a r.v. $X$ are easily represented by umbrae through the following definition: the umbra $\alpha^{a}$ having moments about a point $a \in R$ is defined as

$$
\begin{equation*}
\alpha^{a} \equiv \alpha-a . u . \tag{5}
\end{equation*}
$$

If $a, b \in R$ and $b-a=c$, then

$$
\alpha^{a} \equiv \alpha-(b+c) \cdot u \equiv \alpha^{b}+c \cdot u
$$

is the umbral version of the equations giving the moments about $a$ in terms of the moments about $b$ (cf. [22] for another symbolic expression).

The dot-product $\gamma . \alpha$ of two umbrae is the auxiliary umbra having g.f.

$$
e^{(\gamma \cdot \alpha) t} \simeq[f(t)]^{\gamma} \simeq e^{\gamma \log f(t)} \simeq g[\log f(t)] .
$$

The moments of the dot-product $\gamma . \alpha$ are (cf. [5])

$$
\begin{equation*}
E\left[(\gamma \cdot \alpha)^{n}\right]=\sum_{i=0}^{n} g_{(i)} B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \quad n=0,1,2, \ldots \tag{6}
\end{equation*}
$$

where $g_{(i)}$ are the factorial moments of the umbra $\gamma, B_{n, i}$ are the (partial) Bell exponential polynomials (cf. [12]) and $a_{i}$ are the moments of the umbra $\alpha$. Observe that $E[\gamma . \alpha]=g_{1} a_{1}=E[\gamma] E[\alpha$.$] The following properties hold (cf. [5]):$

Proposition 2.7 a) if $\eta \cdot \alpha \equiv \eta \cdot \gamma$ then $\alpha \equiv \gamma$;
b) if $c \in R$ then $\eta \cdot(c \alpha) \equiv c(\eta \cdot \alpha)$ for any two distinct umbrae $\alpha$ and $\eta$;
c) if $\gamma \equiv \gamma^{\prime}$ then $(\alpha+\eta) \cdot \gamma \equiv \alpha \cdot \gamma+\eta \cdot \gamma^{\prime}$;
d) $\eta \cdot(\gamma \cdot \alpha) \equiv(\eta \cdot \gamma) \cdot \alpha$.

Observe that from property b) it follows

$$
\begin{equation*}
\alpha . x \equiv \alpha .(x u) \equiv x(\alpha . u) \equiv x \alpha . \tag{7}
\end{equation*}
$$

Remark 1 The auxiliary umbra $\gamma . \alpha$ is the umbral version of a random sum. Indeed the m.g.f. $g[\log f(t)]$ corresponds to the r.v. $S_{N}=X_{1}+X_{2}+\cdots+X_{N}$ where $N$ is a discrete r.v. having m.g.f. $g(t)$ and $X_{i}$ are i.i.d. r.v.'s having m.g.f. $f(t)$. The rightdistributive property of the dot-product $\gamma . \alpha$ runs in parallel with the probability theory because the random sum $S_{N+M}$ is similar to $S_{N}+S_{M}$, where $N$ and $M$ are independent discrete r.v.'s. The left-distributive property of the dot-product $\gamma . \alpha$ does not hold as well as it happens in the r.v.'s theory. In fact, let $Z=X+Y$ be a r.v. with $X$ and $Y$ independent r.v.'s. As it is easy to verify, the random sum $S_{N}=Z_{1}+Z_{2}+\cdots+Z_{N}$, with $Z_{i}$ i.i.d. r.v.'s similar to $Z$, is not similar to the r.v. $S_{N}^{X}+S_{N}^{Y}$ with $S_{N}^{X}=X_{1}+X_{2}+\cdots+X_{N}$, and $X_{i}$ i.i.d. r.v.'s similar to $X$ and with $S_{N}^{Y}=Y_{1}+Y_{2}+\cdots+Y_{N}$ and $Y_{i}$ i.i.d. r.v.'s similar to $Y$.

## Example 2.8 Randomized Poisson r.v.

Let us consider the Bell polynomial umbra $x$. $\beta$. If in the place of $x$ we put a generic umbra $\alpha$, we get the auxiliary umbra $\alpha . \beta$ whose factorial moments are

$$
(\alpha \cdot \beta)_{n} \simeq \alpha^{n} \quad n=0,1,2, \ldots
$$

and moments given by the exponential umbral polynomials (cf. [5])

$$
\begin{equation*}
(\alpha \cdot \beta)^{n} \simeq \Phi_{n}(\alpha) \simeq \sum_{i=0}^{n} S(n, i) \alpha^{i} \quad n=0,1,2, \ldots \tag{8}
\end{equation*}
$$

Its g.f. is $f\left[e^{t}-1\right]$. The umbra $\alpha . \beta$ represents a random sum of independent Poisson r.v.'s with parameter 1 indexed by an integer r.v. $Y$, i.e. a randomized Poisson r.v. with parameter $Y$.

As suggested in [14], there is a connection between compound Poisson processes and polynomial sequence of binomial type, i.e. sequence $\left\{p_{n}(x)\right\}$ of polynomials with degree $n$ satisfying the identities

$$
p_{n}(x+y)=\sum_{i=0}^{n}\binom{n}{i} p_{i}(x) p_{n-i}(y)
$$

for any $n$ (cf. for instance [10]). Two different approaches can be found in [3] and in [21]. A natural device to make clear this connection is the $\alpha$-partition umbra $\beta . \alpha$, introduced in [5]. Its g.f. is $\exp [f(t)-1]$ and it suggests to interpret a partition umbra as a compound Poisson r.v. with parameter 1. As well-known, a compound Poisson r.v. with parameter 1 is introduced as a random sum $S_{N}=$ $X_{1}+X_{2}+\cdots+X_{N}$ where $N$ has a Poisson distribution with parameter 1. The umbra $\beta . \alpha$ fits perfectly this probabilistic notion taking into consideration that the Bell scalar umbra $\beta$ plays the role of a Poisson r.v. with parameter 1. What's more, since the Poisson r.v. with parameter $x$ is umbrally represented by the Bell polynomial umbra $x . \beta$, a compound Poisson r.v. with parameter $x$ is represented by the polynomial $\alpha-$ partition umbra $x . \psi \equiv x . \beta . \alpha$ with g.f. $\exp [x(f(t)-1)]$. The name "partition umbra" has a probabilistic ground. Indeed the parameter of a Poisson r.v. is usually denoted by $x=\lambda t$, with $t$ representing a time interval, so that when this interval is partitioned into non-overlapping ones, their contributions are stochastic independent and add to $S_{N}$. This last circumstance is umbrally expressed by the relation

$$
\begin{equation*}
(x+y) \cdot \beta \cdot \alpha \equiv x \cdot \beta \cdot \alpha+y \cdot \beta \cdot \alpha \tag{9}
\end{equation*}
$$

giving the binomial property for the polynomial sequence represented by $x . \beta . \alpha$. In terms of g.f.'s, the formula (9) means that

$$
\begin{equation*}
h_{x+y}(t)=h_{x}(t) h_{y}(t) \tag{10}
\end{equation*}
$$

where $h_{x}(t)$ is the g.f. of $x . \beta . \alpha$. Viceversa every g.f. $h_{x}(t)$ satisfying the equality $(10)$ is the g.f. of a polynomial $\alpha-$ partition umbra. The $\alpha$-partition umbra represents the sequence of partition polynomials $Y_{n}=Y_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ (or complete Bell exponential polynomials [12]), i.e.

$$
\begin{equation*}
E\left[(\beta . \alpha)^{n}\right]=\sum_{i=0}^{n} B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right)=Y_{n}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \tag{11}
\end{equation*}
$$

where $a_{i}$ are the moments of the umbra $\alpha$. Moreover every $\alpha$-partition umbra satisfies the relation

$$
\begin{equation*}
(\beta . \alpha)^{n} \simeq \alpha^{\prime}\left(\beta . \alpha+\alpha^{\prime}\right)^{n-1} \quad \alpha \equiv \alpha^{\prime}, n=0,1,2, \ldots \tag{12}
\end{equation*}
$$

and conversely (see [5] for the proof). The previous property will allow an useful umbral characterization of the cumulant umbra (see corollary 4.12 in section 4.) The umbra $\beta . \alpha$ plays a central role also in the umbral representation of the composition of exponential g.f.'s. Indeed, the composition umbra of $\alpha$ and $\gamma$ is the umbra $\tau \equiv \gamma . \beta . \alpha$. The umbra $\tau$ has g.f. $g[f(t)-1]$ and moments

$$
\begin{equation*}
E\left[\tau^{n}\right]=\sum_{i=0}^{n} g_{i} B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \tag{13}
\end{equation*}
$$

with $g_{i}$ and $a_{i}$ moments of the umbra $\gamma$ and $\alpha$ respectively. We denote by $\alpha^{<-1>}$ the compositional inverse of $\alpha$, i.e. the umbra having g.f. $f^{-1}(t)$ such that $f^{-1}[f(t)-1]=$ $f\left[f^{-1}(t)-1\right]=1+t$. For an intrinsic umbral expression of the compositional inverse umbra see [5], where it is also stated an umbral version of the Lagrange inversion formula.

Example 2.9 Randomized compound Poisson r.v.
As already underlined in example 2.8 , the umbra $\gamma . \beta$ represents a randomized Poisson r.v. Hence it is natural to look at the composition umbra as a compound randomized Poisson r.v., i.e. a random sum indexed by a randomized Poisson r.v. Moreover, being ( $\gamma . \beta$ ). $\alpha \equiv \gamma .(\beta . \alpha)$ (cf. statement $d$ ) of Proposition 2.7), the previous relation allows to see this r.v. from another side: the umbra $\gamma .(\beta . \alpha)$ generalizes the concept of a random sum of i.i.d. compound Poisson r.v. with parameter 1 indexed by an integer r.v. $X$, i.e. a randomized compound Poisson r.v. with random parameter $X$.

At the end, the symbol $\alpha^{n}$ denotes an auxiliary umbra similar to the product $\alpha^{\prime} \alpha^{\prime \prime} \cdots \alpha^{\prime \prime \prime}$ where $\alpha^{\prime}, \alpha^{\prime \prime}, \ldots, \alpha^{\prime \prime \prime}$ are a set of $n$ distinct umbrae each similar to the umbra $\alpha$. We assume that $\alpha^{0}$ is an umbra similar to the unity umbra $u$. The moments of $\alpha^{n}$ are:

$$
\begin{equation*}
E\left[\left(\alpha^{n}\right)^{k}\right]=E\left[\left(\alpha^{k}\right)^{\cdot n}\right]=a_{k}^{n}, \quad k=0,1,2, \ldots, \tag{14}
\end{equation*}
$$

i.e. the $n$-th power of the moments of the umbra $\alpha$. Thanks to this notation in [5], the umbral expression of the Bell exponential polynomials was given as follows:

$$
\begin{equation*}
B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \simeq\binom{n}{i} \alpha^{i}(i . \bar{\alpha})^{n-i} \tag{15}
\end{equation*}
$$

whenever $a_{1} \neq 0$ and where $\bar{\alpha}$ is the umbra with moments

$$
\begin{equation*}
E\left[\bar{\alpha}^{n}\right]=\frac{a_{n+1}}{a_{1}(n+1)}, \quad n=1,2, \ldots \tag{16}
\end{equation*}
$$

Example 2.10 The central umbra.
We call central umbra the umbra $\alpha^{a_{1}}$ having moments about $a_{1}=E[\alpha]$. From (5), the classical relation between moments and central moments of a r.v. has the following umbral expression:

$$
\left(\alpha^{a_{1}}\right)^{n} \simeq \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\alpha^{\prime}\right)^{k} \alpha^{(n-k)}, \quad \alpha \equiv \alpha^{\prime} n=1,2, \ldots
$$

being $E\left[\left(a_{1} \cdot u\right)^{k}\right]=a_{1}^{k}=E\left[\alpha^{k}\right]$ from (14).

## 3 The singleton umbra

The singleton umbra plays a dual role compared to the Bell umbra, even if it has not a probabilistic counterpart. Besides, the singleton umbra turns out to be an effective symbolic tool in order to umbrally represent some well-known r.v.'s as well as cumulants and factorial moments.

Definition 3.1 (The singleton umbra) An umbra $\chi$ is said to be a singleton umbra if

$$
\chi^{n} \simeq \delta_{1, n} \quad n=1,2, \cdots
$$

The g.f. of the singleton umbra $\chi$ is $1+t$.
Example 3.2 Gamma r.v.
The m.g.f. of a Gamma r.v. with parameters $a$ and $c$ is

$$
M(t)=\frac{1}{(1-c t)^{a}}
$$

This g.f. is umbrally represented by the inverse of $-c(a . \chi)$ (see (ii) of Proposition 2.2 replacing $n$ by $a \in R$ ).

Table 1 lights up the duality between the singleton umbra $\chi$ and the Bell umbra $\beta$.

| Umbra | Generating function |
| :---: | :---: |
| $\chi$ | $(1+t)=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} s(n, k)\right] \frac{t^{n}}{n!}$ |
| $x \cdot \chi$ | $(1+t)^{x}=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} s(n, k) x^{k}\right] \frac{t^{n}}{n!}$ |
| $\alpha \cdot \chi$ | $f[\log (1+t)]$ |
| $\chi \cdot \alpha$ | $1+\log [f(t)]$ |
| $\beta$ | $e^{e^{t}-1}=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} S(n, k)\right] \frac{t^{n}}{n!}$ |
| $x \cdot \beta$ | $e^{x\left(e^{t}-1\right)}=1+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} S(n, k) x^{k}\right] \frac{t^{n}}{n!}$ |
| $\alpha \cdot \beta$ | $f\left(e^{t}-1\right)$ |
| $\beta . \alpha$ | $e^{f(t)-1}$ |

Table 1.
The connection between the singleton umbra $\chi$ and the Bell umbra $\beta$ is made clear in the following proposition.

Proposition 3.3 Let $\chi$ be the singleton umbra, $\beta$ the Bell umbra and $u^{<-1>}$ the compositional inverse of the unity umbra u. It results

$$
\begin{align*}
& \chi \equiv u^{<-1>} \cdot \beta \equiv \beta \cdot u^{<-1>}  \tag{17}\\
& \beta \cdot \chi \equiv u \equiv \chi \cdot \beta \tag{18}
\end{align*}
$$

Proof. The g.f. of $u^{<-1>} . \beta . u$ is $1+t$, being $u^{<-1>}$ and $u$ compositional inverses. So equivalence (17) follows by property $a$ ) of proposition 2.7 being

$$
u^{<-1>} \cdot \beta \cdot u \equiv \chi \equiv \chi \cdot u
$$

Equivalence (18) follows via g.f.'s in Table 1.
Distributive properties of the singleton umbra respect to the sum and the disjoint sum of umbrae are given in the following.

Proposition 3.4 It results

$$
\begin{align*}
\chi \cdot(\alpha+\gamma) & \equiv \chi \cdot \alpha \dot{+} \chi \cdot \gamma  \tag{19}\\
(\alpha \dot{+} \gamma) \cdot \chi & \equiv \alpha \cdot \chi \dot{+} \gamma \cdot \chi \tag{20}
\end{align*}
$$

Proof. Let $f(t)$ be the g.f. of $\alpha$ and $g(t)$ the g.f. of $\gamma$. Equivalence (19) follows observing that the g.f. of $\chi \cdot(\alpha+\beta)$ is $1+\log [f(t) g(t)]=1+\log [f(t)]+\log [g(t)]$, i.e. the g.f. of $\chi . \alpha \dot{+} \chi . \beta$. Equivalence (20) follows observing that the g.f. of $(\alpha \dot{+} \beta) \cdot \chi$ is $f[\log (1+t)]+g[\log (1+t)]-1$, i.e. the g.f. of $\alpha . \chi \dot{+} \beta . \chi$.

The notion of mixture of r.v.'s has an umbral counterpart in the disjoint sum $\dot{+}$. Indeed let $\left\{\alpha_{i}\right\}_{i=1}^{n}$ be $n$ umbrae and $\left\{p_{i}\right\}_{i=1}^{n} \in R$ be $n$ weights such that

$$
\sum_{i=1}^{n} p_{i}=1 .
$$

The mixture umbra $\gamma$ of $\left\{\alpha_{i}\right\}_{i=1}^{n}$ is the following weighted disjoint sum of $\left\{\alpha_{i}\right\}_{i=1}^{n}$

$$
\begin{equation*}
\gamma \equiv \chi \cdot p_{1} \cdot \beta \cdot \alpha_{1} \dot{+} \chi \cdot p_{2} \cdot \beta \cdot \alpha_{2} \dot{+} \ldots \dot{+} \chi \cdot p_{n} \cdot \beta \cdot \alpha_{n} \tag{21}
\end{equation*}
$$

where $\beta$ is the Bell umbra and $\chi$ is the singleton umbra. From (19) equivalence (21) can be rewritten as

$$
\gamma \equiv \chi \cdot\left(p_{1} \cdot \beta \cdot \alpha_{1}+p_{2} \cdot \beta \cdot \alpha_{2}+\ldots+p_{n} \cdot \beta \cdot \alpha_{n}\right) .
$$

Since the g.f. of $\sum_{i=1}^{n} p_{i} . \beta . \alpha_{i}$ is $\exp \left(\sum_{i=1}^{n} p_{i}\left[f_{i}(t)-1\right]\right)$, where $f_{i}(t)$ is the g.f. of $\alpha_{i}$, from Table 1 it follows that the g.f. of $\gamma$ is $\sum_{i=1}^{n} p_{i} f_{i}(t)$.

Example 3.5 Bernoulli umbral r.v.
Let us consider the Bernoulli r.v. $X$ of parameter $p$. Its m.g.f. is $g(t)=q+p e^{t}$ with $q=1-p$. The Bernoulli umbral r.v. is the mixture of the umbra $\varepsilon$ and the unity umbra $u$ :

$$
\xi \equiv \chi \cdot q \cdot \beta \cdot \varepsilon \dot{+} \chi \cdot p \cdot \beta \cdot u
$$

Recalling that $\chi \cdot q \cdot \beta \cdot \varepsilon \equiv \varepsilon$ it is

$$
\xi \equiv \chi \cdot p . \beta .
$$

Indeed it is

$$
E\left[e^{\xi t}\right]=1+\log \left[e^{p\left(e^{t}-1\right)}\right]=q+p e^{t} .
$$

Example 3.6 Binomial umbral r.v.
As it is well-known a binomial r.v. $Y$ with parameters $n \in \mathbf{N}, p \in[0,1]$, is the sum of $n$ i.i.d. Bernoulli r.v.'s having parameter $p$. Then the binomial umbral r.v. is

$$
n . \xi \equiv n \cdot \chi \cdot p . \beta .
$$

The parallelism is evident if we recall that the m.g.f. of the binomial r.v. $Y$ is $f(t)=\left(q+p e^{t}\right)^{n}$.

## 4 The cumulant umbra

For a r.v. having moments $a_{1}, a_{2}, \ldots, a_{n}$ and cumulants $\kappa_{1}, \kappa_{2}, \ldots, \kappa_{n}$ it is

$$
\begin{equation*}
a_{n}=\sum_{\pi} c_{\pi} \kappa_{\pi} \quad \text { and } \quad \kappa_{n}=\sum_{\pi} d_{\pi} a_{\pi} \tag{22}
\end{equation*}
$$

the sums here are taken over the partitions $\pi=\left[j_{1}^{m_{1}}, j_{2}^{m_{2}}, \ldots, j_{k}^{m_{k}}\right]$ of the integer $n$, and

$$
\begin{aligned}
c_{\pi} & =\frac{n!}{\left(j_{1}!\right)^{m_{1}}\left(j_{2}!\right)^{m_{2}} \cdots\left(j_{k}!\right)^{m_{k}}} \frac{1}{m_{1}!m_{2}!\cdots m_{k}!} \\
d_{\pi} & =c_{\pi}(-1)^{\nu_{\pi}-1}\left(\nu_{\pi}-1\right)!\quad \text { and } \quad \nu_{\pi}=m_{1}+m_{2}+\cdots+m_{k} \\
a_{\pi} & =\prod_{j \in \pi} a_{j} \text { and } \kappa_{\pi}=\prod_{j \in \pi} \kappa_{j} .
\end{aligned}
$$

In this section we show how the umbral calculus simplifies the above expressions, as well as the recursive formulae which give moments in terms of cumulants.

Let $\alpha$ be an umbra with g.f. $f(t)$.
Definition 4.1 The cumulant of an umbra $\alpha$ is the umbra $\kappa_{\alpha}$ defined by

$$
\kappa_{\alpha} \equiv \chi . \alpha
$$

where $\chi$ is the singleton umbra.
Definition 4.1 gives the umbral version of the second equality in (22). Moreover the first moment of the cumulant umbra $\kappa_{\alpha}$ is $a_{1}$, i.e. the first moment of the umbra $\alpha$, being $E\left[\kappa_{\alpha}\right]=E[\chi] E[\alpha]=E[\alpha]=a_{1}$.

Example 4.2 Cumulant of the umbra $\varepsilon$.
Since $\varepsilon \equiv \chi . \varepsilon$, the umbra $\varepsilon$ is the cumulant umbra of itself, i.e. $\kappa_{\varepsilon} \equiv \varepsilon$.
Example 4.3 Cumulant of the umbra u.
Since $\chi \equiv \chi . u$, the umbra $\chi$ is the cumulant umbra of the umbra $u$, i.e. $\kappa_{u} \equiv \chi$.
Example 4.4 Cumulant of the Bell umbra.
Since $u \equiv \chi \cdot \beta$ (see (18)), the umbra $u$ is the cumulant umbra of the Bell umbra $\beta$, i.e. $\kappa_{\beta} \equiv u$. From example 2.1, the Poisson r.v. of parameter 1 has cumulants equal to 1 .

Proposition 4.5 The cumulant umbra $\kappa_{\alpha}$ has g.f.

$$
\begin{equation*}
k(t)=1+\log [f(t)] . \tag{23}
\end{equation*}
$$

Proof. See Table 1.
Example 4.6 Cumulant of the singleton umbra.
Since $1+\log (1+t)$ is the g.f. of the umbra $u^{<-1>}$, this umbra is the cumulant umbra of the umbra $\chi$, i.e. $\kappa_{\chi} \equiv u^{<-1>}$.

Example 4.7 Cumulant of the Bernoulli umbral r.v.
From example 3.5, the cumulant umbra of the Bernoulli umbral r.v. is $\chi \cdot(\chi \cdot p . \beta) \equiv$ $u^{<-1>}$.p. $\beta$.

Example 4.8 Cumulant of the Binomial umbral r.v.
From example 3.6, the cumulant umbra of the Binomial umbral r.v. is $\chi .(n . \chi . p . \beta)$, i.e.

$$
\chi \cdot\left(\chi^{\prime} \cdot p \cdot \beta^{\prime}+\chi^{\prime \prime} \cdot p \cdot \beta^{\prime \prime}+\cdots+\chi^{\prime \prime \prime} \cdot p \cdot \beta^{\prime \prime \prime}\right),
$$

where $\chi^{\prime}, \chi^{\prime \prime}, \ldots, \chi^{\prime \prime \prime}$ are a set of $n$ distinct umbrae each similar to the singleton umbra $\chi$ as well as $\beta^{\prime}, \beta^{\prime \prime}, \ldots, \beta^{\prime \prime \prime}$ are a set of $n$ distinct umbrae each similar to the Bell umbra $\beta$. From (19) and recalling examples 2.4 and 4.6 , it results

$$
\chi \cdot n \cdot \chi \cdot p \cdot \beta \equiv \chi \cdot \chi^{\prime} \cdot p \cdot \beta^{\prime} \dot{+} \chi \cdot \chi^{\prime \prime} \cdot p \cdot \beta^{\prime \prime} \dot{+} \cdot \cdot \dot{+} \chi \cdot \chi^{\prime \prime \prime} \cdot p \cdot \beta^{\prime \prime \prime} \equiv \dot{+}_{n} u^{<-1>} \cdot p \cdot \beta .
$$

This parallels the analogous result in probability theory.
From (23), the moments of the cumulant umbra $\kappa_{\alpha}$ are

$$
\left(k_{a}\right)_{n}=E\left[\kappa_{\alpha}^{n}\right]=\left[\frac{d^{n}}{d t^{n}} \log \{f(t)\}\right]_{t=0}
$$

that is equivalent to the definition of the $n$-th cumulant of a r.v. $X$ having m.g.f. $f(t)$.

To state the explicit version of the second equality in (22)

$$
\begin{equation*}
k_{n}=\sum_{i=1}^{n}(-1)^{i-1}(i-1)!B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \tag{24}
\end{equation*}
$$

giving cumulants in terms of moments, usually requires laborious computations (cf. for example [11]). The umbral definition of cumulants allows a simple proof of (24). Indeed, being $\chi \equiv u^{<-1>} . \beta$, the cumulant umbra of $\alpha$ is the umbral composition of $u^{<-1>}$ and $\alpha$ :

$$
\kappa_{\alpha} \equiv u^{<-1>} . \beta . \alpha
$$

and then its moments are given by (13). Equality (24) follows recalling that the moments of $u^{<-1>}$ are the coefficient of the exponential expansion

$$
1+\log (1+t)=1+\sum_{i=1}^{\infty}(-1)^{i-1}(i-1)!\frac{t^{i}}{i!} .
$$

Similarly, the three main algebraic properties of cumulants can be easily recovered from next theorem.

Theorem 4.9 It is
a) (the additivity property)

$$
\begin{equation*}
\chi \cdot(\alpha+\gamma) \equiv \chi \cdot \alpha \dot{+} \chi \cdot \gamma, \tag{25}
\end{equation*}
$$

i.e. the cumulant umbra of a sum of two umbrae is equal to the disjoint sum of the two corresponding cumulant umbrae;
b) (the semi-invariance under traslation property) for any $c \in R$

$$
\chi \cdot(\alpha+c \cdot u) \equiv \chi \cdot \alpha \dot{+} \chi \cdot c ;
$$

c) (the homogeneity property) for any $c \in R$

$$
\chi \cdot(c \alpha) \equiv c(\chi \cdot \alpha)
$$

Proof. Property $a$ ) follows from (19). Property $b$ ) follows from (25), setting $\gamma \equiv$ $c . u$ for any $c \in R$. At the end, property $c$ ) follows from $b$ ) of proposition 2.7.

Example 4.10 Cumulant of the central umbra.
The sequence of cumulants related to the central umbra $\alpha^{a_{1}}$ is the same of $\alpha$ excepting the first equal to 0 . Indeed, by the additivity property of the cumulant umbra it is

$$
\chi \cdot\left(\alpha-a_{1} \cdot u\right) \equiv \chi \cdot \alpha \dot{-} \chi \cdot a_{1}
$$

The results follows from (7).
The umbral version of the first equality in (22) is given in the following theorem.
Theorem 4.11 (Inversion theorem) Let $\kappa_{\alpha}$ be the cumulant umbra of $\alpha$, then

$$
\alpha \equiv \beta . \kappa_{\alpha}
$$

where $\beta$ is the Bell umbra.
Proof. It is

$$
\beta \cdot \kappa_{\alpha} \equiv \beta \cdot u^{<-1>} \cdot \beta \cdot \alpha \equiv \chi \cdot \beta \cdot \alpha \equiv u . \alpha \equiv \alpha
$$

The inversion theorem allows to calculate the moments of the umbra $\alpha$ according to its cumulants. Recalling (11) it is

$$
\begin{equation*}
a_{n}=Y_{n}\left[\left(k_{a}\right)_{1},\left(k_{a}\right)_{2}, \ldots,\left(k_{a}\right)_{n}\right] \tag{26}
\end{equation*}
$$

with $a_{n}$ the $n$-th moment of the umbra $\alpha$ and $\left(k_{a}\right)_{n}$ the $n$-th moment of the umbra $\kappa_{\alpha}$. Equation (26) is the explicit version of the first equality in (22).

Remark 2 The complete Bell polynomials in (11) are a polynomial sequence of binomial type. Since from the inversion theorem any umbra $\alpha$ could be seen as the partition umbra of its cumulant $\kappa_{\alpha}$, it is possible to prove a more general result: every polynomial sequence of binomial type is completely determined by its sequence of formal cumulants. Indeed, in [5] it is proved that any polynomial sequence of binomial type represents the moments of a polynomial umbra $x . \alpha$ and viceversa. So from the inversion theorem any polynomial sequence of binomial type represents the moments of a polynomial umbra $x . \beta . \kappa_{\alpha}$.

The next corollary follows from (12) and from the inversion theorem.
Corollary 4.12 If $\kappa_{\alpha}$ is the cumulant umbra of $\alpha$, then

$$
\begin{equation*}
\alpha^{n} \simeq \kappa_{\alpha}\left(\kappa_{\alpha}+\alpha\right)^{n-1} \tag{27}
\end{equation*}
$$

for any nonnegative integer $n$.

Equivalences (27) were assumed by Shen and Rota in [16] as definition of the cumulant umbra. In terms of moments, equivalences (27) give

$$
a_{n}=\sum_{j=0}^{n-1}\binom{n-1}{j} a_{j}\left(k_{a}\right)_{n-j}
$$

that is largely used in statistic framework [20].
Example 4.13 Lévy process.
Let $\left(X_{t}, t \geq 0\right)$ be a real-value Lévy process, i.e. a process starting from 0 and with stationary and independent increments. According to the Lévy-Khintchine formula (cf. [6]), if we assume that $X_{t}$ has a convergent m.g.f. in some neighbourhood of 0 , it is

$$
\begin{equation*}
E\left[e^{\theta X_{t}}\right]=e^{t k(\theta)} \tag{28}
\end{equation*}
$$

where $k(\theta)$ is the cumulant g.f. of $X_{1}$. The inversion theorem gives the umbral version of equation (28):

$$
t . \alpha \equiv t . \beta . \kappa_{\alpha} .
$$

### 4.1 Cumulants of the Poisson r.v.'s

From example 2.9, the umbra $\gamma . \beta . \alpha$ corresponds to a compound randomized Poisson r.v., i.e. a random sum $S_{N}=X_{1}+\cdots+X_{N}$ with $N$ a randomized Poisson r.v. of parameter the r.v. $Y$. In particular $\alpha$ corresponds to $X$ and $\gamma$ corresponds to $Y$. Since $\chi \cdot(\gamma . \beta . \alpha) \equiv \kappa_{\gamma} . \beta . \alpha$ the cumulant umbra of the composition of $\alpha$ and $\gamma$ is the composition of $\alpha$ and $\kappa_{\gamma}$. Then from (13), the cumulants of a compound randomized Poisson r.v. are given by

$$
\begin{equation*}
\sum_{i=1}^{n} k_{i} B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \tag{29}
\end{equation*}
$$

where $a_{i}$ are the moments of the r.v. $X$ and $k_{i}$ are the cumulants of the r.v. $Y$. Now set $\gamma \equiv x . u$ in $\gamma . \beta . \alpha$. This means to consider a r.v. $Y$ such that $P(Y=x)=1$. Then, the random sum $S_{N}$ becomes a compound Poisson r.v. of parameter $x$ corresponding to the polynomial $\alpha$-partition umbra $x . \beta . \alpha$, with $\alpha$ the umbral counterpart of $X$. and cumulants

$$
\begin{equation*}
\sum_{i=1}^{n} k_{i} B_{n, i}\left(a_{1}, a_{2}, \ldots, a_{n-i+1}\right) \simeq x a_{n} . \tag{30}
\end{equation*}
$$

Indeed (30) follows from (29) since the moments $k_{i}$ of $\chi \cdot x$ are equal to 0 , except the first equal to $x$. If $x=1$ the cumulant of $\alpha$-partition umbra is $\alpha$ so that the moments of $X$ are the cumulants of the corresponding compound Poisson r.v. Now, in $x . \beta$. $\alpha$ take $\alpha \equiv u$. From (30), the cumulants of the Bell polynomial umbra $x . \beta$ are equals to $x$ as well as for the Poisson r.v. of parameter $x$.

At the end, in $\gamma . \beta . \alpha$ set $\alpha \equiv u$. The cumulant umbra of $\gamma . \beta$ is $\kappa_{\gamma} . \beta$ with $\kappa_{\gamma}$ the cumulant umbra of $\gamma$. Its probabilistic counterpart is a randomized Poisson r.v. of parameter the r.v. $Y$, corresponding to the umbra $\gamma$. From (8) the cumulants of a randomized Poisson r.v. of parameter the r.v. $Y$ are the moments of $\kappa_{\gamma} . \beta$, i.e.

$$
\sum_{i=0}^{n} S(n, i) k_{i}
$$

with $k_{i}$ the cumulants of the r.v. $Y$.

## 5 The factorial umbra

The factorial moments of a r.v. do not play a very prominent role in statistics, but they provide very concise formulae for the moments of some discrete distributions, like the binomial one.

Let $\alpha$ be an umbra with g.f. $f(t)$.
Definition 5.1 An umbra $\varphi_{\alpha}$ is said to be an $\alpha$-factorial umbra if

$$
\varphi_{\alpha} \equiv \alpha \cdot \chi
$$

where $\chi$ is the singleton umbra.

Example $5.2 \varepsilon$-factorial umbra.
Since $\varepsilon \equiv \varepsilon \cdot \chi$, the $\varepsilon$-factorial umbra is similar to the umbra $\varepsilon$, i.e. $\varphi_{\varepsilon} \equiv \varepsilon$.
Example 5.3 -factorial umbra.
Since $\chi \equiv u . \chi$, the $u$-factorial umbra is similar to the umbra $\chi$, i.e. $\varphi_{u} \equiv \chi$.
Example $5.4 \beta$-factorial umbra.
Since $u \equiv \beta \cdot \chi$ from (18), the $\beta$-factorial umbra is similar to the unity umbra $u$, i.e. $\varphi_{\beta} \equiv u$. From example 2.1, the Poisson r.v. of parameter 1 has factorial moments equal to 1.

Example $5.5 \chi$-factorial umbra.
From example 4.6 , it is $\chi \cdot \chi \equiv u^{<-1>}$. The $\chi$-factorial umbra turns out to be $u^{<-1>}$, i.e. $\varphi_{\chi} \equiv u^{<-1>}$.

Proposition 5.6 The $\alpha$-factorial umbra has g.f.

$$
\begin{equation*}
g(t)=f[\log (1+t)] \tag{31}
\end{equation*}
$$

Proof. See Table 1.

The $\alpha$-factorial umbra has moments equal to the factorial moments of the umbra $\alpha$, as the following proposition shows.

Proposition 5.7 Let $\varphi_{\alpha}$ be an $\alpha$-factorial umbra. Then

$$
\varphi_{\alpha}^{n} \simeq(\alpha)_{n}, n=0,1,2, \ldots
$$

Proof. By equation (6) and definition 5.1 it is

$$
\begin{equation*}
E\left[\left(\varphi_{\alpha}\right)^{n}\right]=E\left[(\alpha \cdot \chi)^{n}\right]=\sum_{k=0}^{n}(a)_{k} B_{n, k}\left(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1, n-k+1}\right) \tag{32}
\end{equation*}
$$

where $(a)_{k}$ are the factorial moments of the umbra $\alpha$ and $\delta_{1, i}$ are the moments of the umbra $\chi$. By (15) it results

$$
B_{n, k}\left(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1, n-k+1}\right) \simeq\binom{n}{k} \chi^{k}(k \cdot \bar{\chi})^{n-k}
$$

Since the umbra $\bar{\chi}$ has moments equal to 0 and $\chi^{k} \simeq 1$, then

$$
B_{n, k}\left(\delta_{1,1}, \delta_{1,2}, \ldots, \delta_{1, n-k+1}\right)= \begin{cases}0, & \text { if } n>k  \tag{33}\\ 1, & \text { if } n=k\end{cases}
$$

Hence the equation (32) becomes $E\left[\left(\varphi_{\alpha}\right)^{n}\right]=(a)_{n}$.
Example 5.8 Factorial umbra of the central umbra.
From property $c$ ) of Proposition 2.7, it is

$$
\alpha^{a_{1}} \cdot \chi \equiv\left(\alpha-a_{1} \cdot u\right) \cdot \chi \equiv \alpha \cdot \chi-a_{1} \cdot \chi^{\prime} \equiv \varphi_{\alpha}-\varphi_{a_{1} \cdot u},
$$

with $\chi^{\prime} \equiv \chi$. Then the factorial umbra of the central umbra $\alpha^{a_{1}}$ is the difference between the factorial umbra of $\alpha$ and the factorial umbra of the umbra having moments equal to $a_{1}$. By (31) its g.f. results $f[\log (1+t)](1-t)^{a_{1}}$.

Example 5.9 Factorial moments of the binomial r.v.
Since the factorial moments characterize the binomial r.v., we show how to evaluate them by umbral methods. As showed in example 3.6, the umbral counterpart of the binomial r.v. is $n \cdot \chi \cdot p . \beta$. Due to (18) and (7) the corresponding factorial umbra is $n \cdot(\chi \cdot p \cdot \beta) \cdot \chi \equiv n \cdot \chi \cdot p \equiv p(n \cdot \chi)$. Its g.f. is

$$
g(t)=(1+t p)^{n}=\sum_{j=0}^{n}(n)_{j} p^{j} \frac{t^{j}}{j!}
$$

and so the factorial moments are $(n)_{j} p^{j}$. If $n=1$, the factorial umbra is $p \chi$ and from example 3.5 the first factorial moment of the Bernoulli r.v. is equal to $p$ while the others are equal to 0 .

Example 5.10 Factorial umbra of the cumulant umbra.
If $\kappa_{\alpha}$ is the cumulant umbra of $\alpha$, then $\kappa_{\alpha} \cdot \chi$ is the factorial cumulant umbra of $\alpha$, with g.f. $1+\log [f(1+t)]$ by (31).

The following theorem allows to obtain the umbra $\alpha$ from its factorial umbra $\varphi_{\alpha}$.
Theorem 5.11 (Inversion theorem) Let $\varphi_{\alpha}$ be the factorial umbra of $\alpha$. It is

$$
\alpha \equiv \varphi_{\alpha} \cdot \beta
$$

with $\beta$ the Bell umbra.
Proof. By the Proposition 3.3 it results

$$
\varphi_{\alpha} \cdot \chi \equiv \alpha \cdot \chi \cdot \beta \equiv \alpha
$$

### 5.1 Factorial moments of the Poisson r.v.'s

Being $(\gamma \cdot \beta \cdot \alpha) \cdot \chi \equiv \gamma \cdot \beta \cdot\left(\varphi_{\alpha}\right)$ the factorial umbra of the umbral composition $\gamma \cdot \beta . \alpha$ is the umbral composition of $\gamma$ and the factorial umbra of $\alpha$. From (13) the compound randomized Poisson r.v. $S_{N}=X_{1}+\cdots+X_{N}$ with $N$ a Poisson r.v. with parameter the r.v. $Y$ has factorial moments

$$
\begin{equation*}
\sum_{k=1}^{n} g_{k} B_{n, k}\left[(\mu)_{1},(\mu)_{2}, \ldots,(\mu)_{n-k+1}\right] \tag{34}
\end{equation*}
$$

where $(\mu)_{i}$ are the factorial moments of the r.v. $X$ and $g_{k}$ are the moments of the r.v. $Y$. Now setting $\gamma \equiv x . u$ in $\gamma . \beta . \alpha$, we have $g_{k}=x^{k}$. Then from (34)

$$
\begin{equation*}
\sum_{k=1}^{n} x^{k} B_{n, k}\left[(\mu)_{1},(\mu)_{2}, \ldots,(\mu)_{n-k+1}\right] \tag{35}
\end{equation*}
$$

are the factorial moments of a compound Poisson r.v. with parameter $x$. Set $\alpha \equiv u$ in $x . \beta . \alpha$. We have $(x . \beta . u) \cdot \chi \equiv x . \beta \cdot \chi \equiv x . u$ so that the factorial moments of $x . \beta . \alpha$ are equals to $x^{n}$ as well as for its probabilistic counterpart, the Poisson r.v. with parameter $x$.

At the end set $\alpha \equiv u$ in $\gamma . \beta . \alpha$. We have $(\gamma \cdot \beta . u) \cdot \chi \equiv \gamma \cdot \beta \cdot \chi \equiv \gamma$ so that the factorial moments of $\gamma . \beta$ are equals to the moments of $\gamma$. Then a randomized Poisson r.v. with parameter a r.v. $Y$ has factorial moments equal to the moments of the r.v. $Y$.

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[^1]:    ${ }^{1}$ Observe that with this approach we disregard of questions of whether any series converges.

