# Independence polynomials of well-covered graphs: generic counterexamples for the unimodality conjecture 

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#### Abstract

A graph $G$ is well-covered if all its maximal stable sets have the same size, denoted by $\alpha(G)$ (M. D. Plummer, 1970). If $s_{k}$ denotes the number of stable sets of cardinality $k$ in graph $G$, and $\alpha(G)$ is the size of a maximum stable set, then $I(G ; x)=\sum_{k=0}^{\alpha(G)} s_{k} x^{k}$ is the independence polynomial of $G$ (I. Gutman and F. Harary, 1983). J. I. Brown, K. Dilcher and R. J. Nowakowski (2000) conjectured that $I(G ; x)$ is unimodal (i.e., there is some $j \in\{0,1, \ldots, \alpha(G)\}$ such that $\left.s_{0} \leq \ldots \leq s_{j-1} \leq s_{j} \geq s_{j+1} \geq \ldots \geq s_{\alpha(G)}\right)$ for any well-covered graph $G$. T. S. Michael and W. N. Traves (2002) proved that this assertion is true for $\alpha(G) \leq 3$, while for $\alpha(G) \in\{4,5,6,7\}$ they provided counterexamples.

In this paper we show that for any integer $\alpha \geq 8$, there exists a connected well-covered graph $G$ with $\alpha=\alpha(G)$, whose independence polynomial is not unimodal. In addition, we present a number of sufficient conditions for a graph $G$ with $\alpha(G) \leq 6$ to have the unimodal independence polynomial. key words: stable set, independence polynomial, unimodal sequence, wellcovered graph.


## 1 Introduction

Throughout this paper $G=(V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V=V(G)$ and edge set $E=E(G) . K_{n}, P_{n}, K_{n_{1}, n_{2}, \ldots, n_{p}}$ denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 3$ vertices, and the complete $p$-partite graph on $n_{1}+n_{2}+\ldots+n_{p}$ vertices, where $n_{i} \geq 1,1 \leq i \leq p$.

The disjoint union of the graphs $G_{1}, G_{2}$ is the graph $G=G_{1} \sqcup G_{2}$ having $V(G)=$ $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$. In particular, $\sqcup n G$ denotes the disjoint union of $n>1$ copies of the graph $G$. The Zykov sum (25, 26]) of two disjoint graphs $G_{1}, G_{2}$ is the graph $G_{1}+G_{2}$ that has $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ as a vertex set and $E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\left\{v_{1} v_{2}: v_{1} \in V\left(G_{1}\right), v_{2} \in V\left(G_{2}\right)\right\}$ as an edge set.

A stable set in $G$ is a set of pairwise non-adjacent vertices. The stability number $\alpha(G)$ of $G$ is the maximum size of a stable set in $G$. By $\omega(G)$ we mean $\alpha(\bar{G})$, where $\bar{G}$ is the complement of $G$.

A graph $G$ is called well-covered if all its maximal stable sets are of the same cardinality, (Plummer, [22]). If, in addition, $G$ has no isolated vertices and its order equals $2 \alpha(G)$, then $G$ is very well-covered (Favaron, [6]). For instance, the graph $G^{*}$, obtained from $G$ by appending a single pendant edge to each vertex of $G$ (5], [24]), is well-covered (see, for example, 15]), and $\alpha\left(G^{*}\right)=n$. Moreover, $G^{*}$ is very well-covered, since it is well-covered, it has no isolated vertices, and its order equals $2 \alpha\left(G^{*}\right)$. The following result shows that, under certain conditions, any well-covered graph equals $G^{*}$ for some graph $G$.

Theorem 1.1 [7] Let $H$ be a connected graph of girth $\geq 6$, which is isomorphic to neither $C_{7}$ nor $\bar{K}_{1}$. Then $H$ is well-covered if and only if its pendant edges form a perfect matching.

In other words, Theorem 1.1 shows that apart from $K_{1}$ and $C_{7}$, connected wellcovered graphs of girth $\geq 6$ are very well-covered. For example, a tree $T \neq K_{1}$ could be only very well-covered, and this is the case if and only if $T=G^{*}$ for some tree $G$ (see also Ravindra, [23]).

Let $s_{k}$ be the number of stable sets in $G$ of cardinality $k \in\{0,1, \ldots, \alpha(G)\}$. The polynomial $I(G ; x)=\sum_{k=0}^{\alpha(G)} s_{k} x^{k}$ is called the independence polynomial of $G$ (Gutman and Harary, [10]). It is easy to deduce that

$$
\begin{aligned}
I\left(G_{1} \sqcup G_{2} ; x\right) & =I\left(G_{1} ; x\right) \cdot I\left(G_{2} ; x\right), \\
I\left(G_{1}+G_{2} ; x\right) & =I\left(G_{1} ; x\right)+I\left(G_{2} ; x\right)-1
\end{aligned}
$$

(see also 10, [2], 13]).
A finite sequence of real numbers $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{n}\right)$ is said to be:

- unimodal if there is some $k \in\{0,1, \ldots, n\}$, called the mode of the sequence, such that

$$
a_{0} \leq \ldots \leq a_{k-1} \leq a_{k} \geq a_{k+1} \geq \ldots \geq a_{n}
$$

- log-concave if $a_{i}^{2} \geq a_{i-1} \cdot a_{i+1}$ for $i \in\{1,2, \ldots, n-1\}$.

It is known that any log-concave sequence of positive numbers is also unimodal, while the converse is not generally true.

A polynomial $P=a_{0}+a_{1} x+a_{2} x^{2}+\ldots+a_{n} x^{n}$ is called unimodal (log-concave) if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, the independence polynomial $I\left(K_{1,3} ; x\right)=1+\mathbf{4 x}+3 x^{2}+x^{3}$ is log-concave, while

$$
I\left(K_{25}+\left(K_{3} \sqcup K_{4} \sqcup K_{5} \sqcup K_{5}\right) ; x\right)=1+42 x+\mathbf{1 0 7} x^{2}+295 x^{3}+300 x^{4}
$$

is unimodal, but it is not log-concave, because $107^{2}-42 \cdot 295=-941$.
Hamidoune [12] proved that the independence polynomial of a claw-free graph (i.e., a graph having no induced subgraph isomorphic to $K_{1,3}$ ) is log-concave, and hence, unimodal. However, there are graphs whose independence polynomials are not unimodal, e.g., $I\left(K_{70}+\left(\sqcup 4 K_{3}\right) ; x\right)=1+\mathbf{8 2} x+54 x^{2}+\mathbf{1 0 8} x^{3}+81 x^{4}$ (for other examples, see [1]). Nevertheless, in [1] it is stated the following (still open) unimodality conjecture for trees.

Conjecture 1.2 The independence polynomial of any tree is unimodal.
In [17] and [18, the unimodality of independence polynomials of a number of well-covered trees (e.g., $P_{n}^{*}, K_{1, n}^{*}$ ) is validated, using the result, mentioned above, on claw-free graphs due to Hamidoune, or directly, by identifying the location of the mode. These findings seem promising for proving Conjecture 1.2 in the case of very well-covered trees, since a tree $T$ is well-covered if and only if either $T$ is a well-covered spider (i.e., $T \in\left\{K_{1}, K_{1}^{*}, K_{1, n}^{*}: n \geq 1\right\}$ ), or $T$ is obtained from a well-covered tree $H_{1}$ and a well-covered spider $H_{2}$, by adding an edge joining two non-pendant vertices belonging to $H_{1}, H_{2}$, respectively (see [16). For instance, the trees presented in Figure 11 are well-covered as follows: $T_{2}$ is a well-covered spider, while $T_{1}$ is an edge-join of two well-covered spiders, namely, $K_{1,2}^{*}$ and $K_{1,1}^{*}$.


Figure 1: Two well-covered trees.
In [3] it was conjectured that the independence polynomial of any well-covered graph is unimodal. Michael and Traves 21] proved that this assertion is true for $\alpha(G) \in\{1,2,3\}$, but it is false for $\alpha(G) \in\{4,5,6,7\}$. Nevertheless, the conjecture of Brown et al. is still open for very well-covered graphs.

In [20] it was shown that for any $\alpha \geq 1$, there is a connected very well-covered graph $G$ with $\alpha(G)=\alpha$, whose independence polynomial is unimodal.

In this paper we prove that for any integer number $\alpha \geq 8$, there exists a connected well-covered graph $G$ with $\alpha(G)=\alpha$, whose $I(G ; x)$ is not unimodal. We also give a simple proof for the unimodality of the independence polynomial of a well-covered graph $G$ with $\alpha(G) \leq 3$, while for $\alpha(G) \in\{4,5,6\}$ a number of sufficient conditions ensuring the unimodality of $I(G ; x)$ are presented.

## 2 The small stability number as a reason for wellcovered graphs to have unimodal independence polynomials

Alavi et al. [1] showed that for any permutation $\sigma$ of $\{1,2, \ldots, \alpha\}$ there is a graph $G$ with $\alpha(G)=\alpha$ such that $s_{\sigma(1)}<s_{\sigma(2)}<\ldots<s_{\sigma(\alpha)}$.

Lemma 2.1 If a graph $G$ satisfies $\omega(G) \leq \alpha=\alpha(G)$, then $s_{\alpha} \leq s_{\alpha-1}$.
Proof. Let $H=(\mathcal{A}, \mathcal{B}, \mathcal{W})$ be the bipartite graph defined as follows: $X \in \mathcal{A} \Leftrightarrow X$ is a stable set in $G$ of size $\alpha(G)-1$, then $Y \in \mathcal{B} \Leftrightarrow Y$ is a stable set in $G$ of size $\alpha(G)$, and $X Y \in \mathcal{W} \Leftrightarrow X \subset Y$ in $G$. Since any $Y \in \mathcal{B}$ has exactly $\alpha(G)$ subsets of size $\alpha(G)-1$, it follows that $|\mathcal{W}|=\alpha(G) \cdot s_{\alpha}$. On the other hand, if $X \in \mathcal{A}$ and $X \cup\left\{y_{1}\right\}, X \cup\left\{y_{2}\right\} \in \mathcal{B}$, it implies $y_{1} y_{2} \in E(G)$, because $X$ is stable and $\left|X \cup\left\{y_{1}, y_{2}\right\}\right|>\alpha(G)$. Hence, any $X \in \mathcal{A}$ has at most $\omega(G)$ neighbors. Consequently, $|\mathcal{W}|=\alpha(G) \cdot s_{\alpha} \leq \omega(G) \cdot s_{\alpha-1}$, and this leads to $s_{\alpha} \leq s_{\alpha-1}$, since $\alpha(G) \geq \omega(G)$.

The converse of Lemma 2.1 is not true, e.g., $\alpha\left(K_{4}-e\right)=2<3=\omega\left(K_{4}-e\right)$ and $I\left(K_{4}-e ; x\right)=1+4 x+x^{2}$, where by $K_{4}-e$ we mean the graph obtained from $K_{4}$ by deleting one of its edges.

Proposition 2.2 [21], [19] If $G$ is a well-covered graph having $\alpha(G)=\alpha$,
then $s_{0} \leq s_{1} \leq \ldots \leq s_{\lceil\alpha / 2\rceil}$.
Corollary 2.3 If $G$ is a well-covered graph and $\omega(G) \leq \alpha(G)=3$, then $I(G ; x)$ is log-concave.

Proof. Let $I(G ; x)=s_{0}+s_{1} x+s_{2} x^{2}+s_{3} x^{3}$. By Proposition 2.2 and Lemma 2.1 we get $s_{0} \leq s_{1} \leq s_{2} \geq s_{3}$, which implies that $s_{2}^{2} \geq s_{1} s_{3}$. To complete the proof, let us notice that $s_{1}^{2}=|V(G)|^{2} \geq|E(\bar{G})|=s_{2}=s_{0} s_{2}$.

The roots of the independence polynomials of well-covered graphs are investigated in a number of papers, as [3, [4], [8, (9, 11], 19. Brown et al. showed, by a nice argument, that:

Lemma 2.4 [3] If a graph $G$ has $\alpha(G)=2$, then $I(G ; x)$ has real roots.
The assertion fails for graphs with stability number greater than 2 , e.g., $I\left(K_{1,3} ; x\right)$. Notice that the independence polynomials of the trees from Figure are respectively

$$
\begin{aligned}
& I\left(T_{1} ; x\right)=1+10 x+36 x^{2}+\mathbf{6 0} x^{3}+47 x^{4}+14 x^{5} \\
& I\left(T_{2} ; x\right)=1+8 x+21 x^{2}+\mathbf{2 3} x^{3}+9 x^{4}
\end{aligned}
$$

while only for the first is true that all its roots are real. Let us observe that $T_{1}, T_{2}$ are well-covered and their polynomials are unimodal. Hence, Newton's theorem (stating that if a polynomial with positive coefficients has only real roots, then its coefficients
form a log-concave sequence) is not useful in solving Conjecture 1.2 even for the particular case of very well-covered trees.

Let us mention that there are connected graphs, with stability number equal to 3 , whose independence polynomials are:

- not unimodal, e.g.,

$$
\left.I\left(K_{24}+\left(K_{4} \sqcup K_{3} \sqcup K_{3}\right)\right) ; x\right)=1+\mathbf{3 4} x+33 x^{2}+\mathbf{3 6} x^{3} ;
$$

- unimodal, but not log-concave, e.g.,

$$
\left.I\left(K_{95}+\left(\sqcup 3 K_{7}\right)\right) ; x\right)=1+116 x+\mathbf{1 4 7} x^{2}+343 x^{3}
$$

- unimodal, but not log-concave, while the graphs are also well-covered, e.g.,

$$
I(\left(\sqcup 3 K_{10}\right)+K_{\underbrace{3,3, \ldots, 3}_{120}}^{3,} ; x)=1+390 x+\mathbf{6 6 0} x^{2}+1120 x^{3}
$$

There are also well-covered connected graphs with stability number equal to 4 , whose independence polynomials are:

- not unimodal, e.g.,

$$
I(\left(\sqcup 4 K_{10}\right)+K_{\underbrace{4,4, \ldots, 4}_{1800}}^{4,} ; x)=1+7240 x+\mathbf{1 1 4 0 0} x^{2}+11200 x^{3}+\mathbf{1 1 8 0 0} x^{4}
$$

- unimodal, but not log-concave, e.g.,

$$
I(\left(\sqcup 4 K_{10}\right)+K_{\underbrace{4,4, \ldots, 4}_{25}}^{4,} ; x)=1+140 x+750 x^{2}+4100 x^{3}+10025 x^{4}
$$

- log-concave, e.g.,

$$
I(\left(\sqcup 4 K_{10}\right)+K_{\underbrace{4,4, \ldots, 4}_{10}}^{4, x)=1+80 x+660 x^{2}+4040 x^{3}+10010 x^{4} . . . . ~ . ~}
$$

Let us observe that the product of two unimodal independence polynomials is not always unimodal, e.g., $I\left(K_{100}+\sqcup 3 K_{7} ; x\right)=1+121 x+147 x^{2}+\mathbf{3 4 3} x^{3}$ and $I\left(K_{90}+\right.$ $\left.\sqcup 3 K_{7} ; x\right)=1+111 x+147 x^{2}+\mathbf{3 4 3} x^{3}$, while their product is not unimodal:

$$
1+232 x+13725 x^{2}+34790 x^{3}+\mathbf{1 0 1 1 8 5} x^{4}+100842 x^{5}+\mathbf{1 1 7 6 4 9} x^{6}
$$

Theorem 2.5 14 The product of a log-concave polynomial by a unimodal polynomial is unimodal, while the product of two log-concave polynomials is log-concave.

Theorem 2.5 is best possible for independence polynomials, since the product of a log-concave independence polynomial and a unimodal independence polynomial is not always log-concave. For instance, $I\left(K_{40}+\sqcup 3 K_{7} ; x\right)=1+61 x+147 x^{2}+\mathbf{3 4 3} x^{3}$ is log-concave, $I\left(K_{110}+\sqcup 3 K_{7} ; x\right)=1+131 x+147 x^{2}+\mathbf{3 4 3} x^{3}$ is unimodal, while their product

$$
1+192 x+8285 x^{2}+28910 x^{3}+87465 x^{4}+100842 x^{5}+117649 x^{6}
$$

is not log-concave.
Further we summarize some facts on graphs with small stability numbers.
Proposition 2.6 The following is a list of sufficient conditions ensuring that the independence polynomial of a graph $G$ is unimodal:
(i) any connected component $H$ of $G$ has $\alpha(H) \leq 2$;
(ii) $\alpha(G)=3$ and $G$ is well-covered;
(iii) $\alpha(G)=4, G$ is disconnected and well-covered;
(iv) $\alpha(G)=5, G=H_{1} \sqcup H_{2}, \alpha\left(H_{1}\right)=2$ and $H_{2}$ is well-covered;
(v) $\omega(G) \leq \alpha(G) \leq 5$ and $G$ is well-covered;
(vi) $\alpha(G)=6, G$ is disconnected and any component $H$ of $G$ with $\alpha(H) \in\{3,4,5\}$ is well-covered and satisfies $\omega(H) \leq \alpha(H)$.

Proof. (i) If $H_{1}, H_{2}, \ldots, H_{k}$ are the components of $G$ and $\alpha\left(H_{i}\right) \leq 2,1 \leq i \leq k$, then $I(G ; x)$ is unimodal, by Newton's Theorem, because $I(G ; x)=I\left(H_{1} ; x\right) \cdot \ldots \cdot I\left(H_{k} ; x\right)$ and, consequently, by Lemma [2.4] all its roots are real.
(ii) If $G$ is disconnected, then $I(G ; x)$ is unimodal, by part (i). Assume that $G$ is connected, and let $I(G ; x)=1+n x+s_{2} x^{2}+s_{3} x^{3}$, where $n$ is the order of $G$. Any vertex $v \in V(G)$ is contained in some maximum stable set of $G$, since $G$ is well-covered. Hence, $v$ has at least two neighbors in the complement $\bar{G}$ of $G$, which ensures that $n \leq|E(\bar{G})|=s_{2}$. Consequently, $I(G ; x)$ is unimodal, with the mode 2 or 3, depending on $\max \left\{s_{2}, s_{3}\right\}$, respectively. Let us mention that there are connected well-covered graphs with stability number equal to 3 , whose independence polynomial has non-real roots, e.g., $I\left(K_{3,3,3} ; x\right)=1+9 x+9 x^{2}+3 x^{3}$ has non-real roots.
(iii) If $G$ is disconnected and at least one of its components is a complete graph, then $G=K_{p} \sqcup H$ and $I(G ; x)=I\left(K_{p} ; x\right) \cdot I(H ; x)=(1+p x) \cdot\left(1+s_{1} x+s_{2} x^{2}+s_{3} x^{3}\right)$ is unimodal, by Theorem 2.5. If none of its components is a complete graph, then $G$ has only two components, say $H_{1}$ and $H_{2}$, and $\alpha\left(H_{1}\right)=\alpha\left(H_{2}\right)=2$. Hence, by Lemma [2.4] $I\left(H_{1} ; x\right), I\left(H_{2} ; x\right)$ have only real roots. Therefore, $I(G ; x)=I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)$ is unimodal, by Newton's Theorem.
(iv) According to Lemma 2.4 and Newton's Theorem, $I\left(H_{1} ; x\right)$ is log-concave. Since $G=H_{1} \sqcup H_{2}$, it follows that $I(G ; x)=I\left(H_{1} ; x\right) \cdot I\left(H_{1} ; x\right)$. Hence, using part (ii) and Theorem 2.5 we infer that $I(G ; x)$ is unimodal.
(v) Taking into account the parts (i),(ii), we may assume that $\alpha(G) \in\{4,5\}$.

Suppose that $\alpha(G)=4$. Then, $I(G ; x)=s_{0}+s_{1} x+s_{2} x^{2}+s_{3} x^{3}+s_{4} x^{4}$, and, according to Proposition 2.2, we obtain that $s_{0} \leq s_{1} \leq s_{2}$, since $G$ is well-covered, while by Lemma 2.1] it follows that $s_{3} \geq s_{4}$, because $\omega(G) \leq \alpha(G)$. Therefore, $I(G ; x)$ is unimodal, with the mode 2 or 3 , depending on $\max \left\{s_{2}, s_{3}\right\}$. Now, for $\alpha(G)=5$,
$I(G ; x)=s_{0}+s_{1} x+s_{2} x^{2}+s_{3} x^{3}+s_{4} x^{4}+s_{5} x^{5}$ and Proposition 2.2 implies that $s_{0} \leq s_{1} \leq s_{2} \leq s_{3}$, while Lemma 2.1 assures that $s_{4} \geq s_{5}$, since $\alpha(G) \geq \omega(G)$. Consequently, $I(G ; x)$ is unimodal, with the mode 3 or 4 , depending on $\max \left\{s_{3}, s_{4}\right\}$.
(vi) If $G$ has a component $H$ with $\alpha(H) \in\{4,5\}$, this is unique, and $\alpha(G-H) \leq 2$. Consequently, by parts (i),(v) and Theorem [2.5] $I(G ; x)=I(H ; x) \cdot I(G-H ; x)$ is unimodal. If $G$ has two components $H_{1}, H_{2}$ with $\alpha\left(H_{1}\right)=\alpha\left(H_{2}\right)=3$, then Corollary 2.3 and Theorem 2.5 assure that $I(G ; x)=I\left(H_{1} ; x\right) \cdot I\left(H_{2} ; x\right)$ is unimodal. The other cases follow easily, by applying parts (i),(iii) and Theorem 2.5

## 3 A family of well-covered graphs having non-unimodal independence polynomials

The independence polynomial of $H_{n}=\left(\sqcup 4 K_{10}\right)+K_{\underbrace{4,4, \ldots, 4}_{n}}^{4}, n \geq 1$ is

$$
\begin{aligned}
I\left(H_{n} ; x\right) & =n \cdot(1+x)^{4}+(1+10 x)^{4}-n \\
& =1+(40+4 n) x+(600+6 n) x^{2}+(4000+4 n) x^{3}+(10000+n) x^{4}
\end{aligned}
$$

Let us notice that $\alpha\left(H_{n}\right)=4$ and $H_{n}$ is well-covered. Since $40+4 n<600+6 n$ is true for any $n \geq 1$, it follows that $I\left(H_{n} ; x\right)$ is not unimodal whenever

$$
4000+4 n<\min \{600+6 n, 10000+n\}
$$

which leads to $1700<n<2000$, where the case $n=1701$ is due to Michael and Traves, 21. Moreover, $I\left(H_{n} ; x\right)$ is not log-concave only for $23<n<2453$.

Lemma 3.1 For any integer $k \geq 0$, the following polynomial is not unimodal.

$$
\sum_{i=0}^{k+4} s_{i} x^{i}=\left(1+6844 \cdot x+10806 \cdot x^{2}+10804 \cdot x^{3}+11701 \cdot x^{4}\right) \cdot(1+1000 \cdot k \cdot x)^{k}
$$

Proof. We show that $s_{k+2}>s_{k+3}<s_{k+4}$. Since the result is evident for $k=0$, we may assume that $k \geq 1$.

Let us notice that:

$$
\begin{aligned}
s_{k+4}= & 11701 \cdot 10^{3 k} \cdot k^{k} \\
s_{k+3}= & 10804 \cdot 10^{3 k} \cdot k^{k}+11701 \cdot 10^{3(k-1)} \cdot k^{k}=10^{3(k-1)} \cdot k^{k} \cdot 10815701, \\
s_{k+2}= & 10806 \cdot 10^{3 k} \cdot k^{k}+10804 \cdot 10^{3(k-1)} \cdot k^{k}+ \\
& +11701 \cdot 10^{3(k-2)} \cdot k^{k-1} \cdot(k-1) \cdot 0.5 \\
= & 10^{3(k-2)} \cdot k^{k-1} \cdot(21633619701 \cdot k-11701) \cdot 0.5 .
\end{aligned}
$$

Firstly, we have

$$
\begin{aligned}
s_{k+4}-s_{k+3} & =11701 \cdot 10^{3 k} \cdot k^{k}-10^{3(k-1)} \cdot k^{k} \cdot 10815701 \\
& =10^{3(k-1)} \cdot k^{k} \cdot 885299>0
\end{aligned}
$$

Secondly, we obtain

$$
\begin{aligned}
s_{k+2}-s_{k+3}= & 10^{3(k-2)} \cdot k^{k-1} \cdot(21633619701 \cdot k-11701) \cdot 0.5 \\
& -10^{3(k-1)} \cdot k^{k} \cdot 10815701 \\
= & 10^{3(k-2)} \cdot k^{k-1} \cdot(2217701 \cdot k-11701) \cdot 0.5>0
\end{aligned}
$$

which completes the proof.


Figure 2: Well-covered graphs with non-unimodal independence polynomials.

Theorem 3.2 For any integer $k \geq 4$, there is a well-covered graph $G$ with $\alpha(G)=k$, whose independence polynomial is not unimodal.

Proof. Let $q=k-4$ and $G_{q}$ be the graph depicted in Figure 2 and formally defined as follows:

$$
G_{q}=\left(\sqcup q K_{1000}\right) \sqcup(\sqcup 4 K_{10}+K_{\underbrace{4,4, \ldots, 4}_{1701}}^{4}) .
$$

It is easy to see that $G_{q}$ is a disconnected well-covered graph, $\alpha\left(G_{q}\right)=k$, and its independence polynomial is not unimodal, because $I\left(G_{q} ; x\right)$ is identical to the non-unimodal polynomial from Lemma 3.1

Moreover, the graph $G_{q}+G_{q}$ is well-covered, connected, $\alpha\left(G_{q}+G_{q}\right)=k$, and its independence polynomial is not unimodal, since $I\left(G_{q}+G_{q} ; x\right)=2 \cdot I\left(G_{q} ; x\right)-1$.

## 4 Conclusions

In this paper we demonstrated that for every integer $k \geq 8$ there exists a (dis)connected well-covered graph $G$ with $\alpha(G)=k$, whose independence polynomial is not unimodal. It is worth mentioning that all these graphs are not very well-covered. In other words, the unimodality conjecture remains open for the case of very well-covered graphs.

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