

# The rank of connection matrices and the dimension of graph algebras

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## Abstract

Connection matrices were introduced in [1], where they were used to characterize graph homomorphism functions. The goal of this note is to determine the exact rank of these matrices. The result can be rephrased in terms of graph algebras, also introduced in [1]. Yet another version proves that if two  $k$ -tuples of nodes behave the same way from the point of view of graph homomorphisms, then they are equivalent under the automorphism group.

# 1 Introduction

For two finite graphs  $F$  and  $G$ , let  $\text{hom}(F, G)$  denote the number of homomorphisms (adjacency-preserving mappings) from  $F$  to  $G$ .

For every fixed  $G$ , let us construct the following (infinite) matrix  $M(k, G)$ . The rows and columns are indexed by finite graphs  $F$  in which  $k$  nodes are labeled  $1, \dots, k$  (there can be any number of unlabeled nodes). The entry in the intersection of the row corresponding to  $F_1$  and the column corresponding to  $F_2$  is  $\text{hom}(F_1 F_2, G)$ , where  $F_1 F_2$  is obtained by considering the disjoint union of  $F_1$  and  $F_2$ , and identifying the nodes labeled the same way. This matrix is called the *k-th connection matrix for homomorphisms into G*.

One can extend this definition to the case when  $G$  has edgeweights and nodeweights (see Section 2 for the exact definitions). Connection matrices were introduced by Freedman, Lovász and Schrijver [1], where they were used to characterize graph homomorphism functions  $\text{hom}(\cdot, G)$ . In particular, it was shown that *connection matrices are positive semidefinite and  $M(k, G)$  has rank at most  $|V(G)|^k$* . (We'll reproduce the simple proof of this assertion in section 2. The main result in [1] is a converse to this statement, which we don't quote here.)

This assertion raises the question: what is the exact rank of  $M(k, G)$ ? The aim of this paper is to determine this rank.

The operation of gluing together two graphs along their labeled nodes gives rise to a commutative algebra defined on formal linear combinations of graphs. This is a tool that was introduced in [1], and will be very useful for us too. The results of this paper can also be viewed as describing the dimension of these algebras.

A third version of these results is motivated by the following. One often classifies nodes of a graph by their degrees. We can also consider the following stronger classification: for every simple graph  $F$  with a specified node, consider the number  $\text{hom}_v(F, G)$  of those homomorphisms of  $F$  into  $G$  that map the specified node onto  $v$ . This way each node  $v \in V(G)$  is assigned an infinite vector  $h_v = (\text{hom}_v(F_1, G), \text{hom}_v(F_2, G), \dots)$ , where  $(F_1, F_2, \dots)$  is any enumeration of all simple graphs with a specified node. Are there any linear relations between these vectors? Clearly two vectors  $h_u, h_v$  are the same if there is an automorphism of  $G$  that moves  $u$  to  $v$ . For unweighted graphs, this turns out to be all; for weighted graphs, the situation is a bit more complicated, but we'll determine all relations; they are still trivial in some sense. These results extend to graphs  $F$  with  $k$  specified nodes instead of 1.

These results have various applications; for example, Lovász and Sós [3] use it to characterize generalized quasirandom graphs.

## 2 Homomorphisms and connection matrices

We start with extending the notions introduced above to weighted graphs. A *weighted graph*  $G$  is a graph with a positive real weight  $\alpha_G(i)$  associated with each node and a real weight  $\beta_G(i, j)$  associated with each edge  $ij$ . An edge with weight 0 will play the same role as no edge between those nodes, so we can assume that all the edge weights are nonzero, or that  $G$  is a complete graph with loops at each node, whichever is more convenient.

Let  $F$  be an unweighted graph (possibly with multiple edges, but no loops) and  $G$ , a weighted graph. To every  $\phi : V(F) \rightarrow V(G)$ , we assign two weights:

$$\alpha_\phi = \prod_{u \in V(F)} \alpha_G(\phi(u))$$

and

$$\text{hom}_\phi(F, G) = \prod_{u, v \in V(F)} \beta_G(\phi(u), \phi(v)).$$

Define

$$\text{hom}(F, G) = \sum_{\phi: V(F) \rightarrow V(G)} \alpha_\phi \text{hom}_\phi(F, G).$$

If all the node-weights and edge-weights in  $G$  are 1, then this is the number of homomorphisms from  $F$  into  $G$  (with no weights).

For the purpose of this paper, it will be convenient to assume that  $G$  is a complete graph with a loop at all nodes (missing edges can be added with weight 0). Then the weighted graph  $G$  is completely described by a positive real vector  $a = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}^m$  and a real symmetric matrix  $B = (\beta_{ij}) \in \mathbf{R}^{m \times m}$ . It will be convenient to assume that

$$\sum_{i=1}^m \alpha_i = 1;$$

this only means scaling of the hom function by an appropriate power of  $\sum_i \alpha_i$ , and will not influence the results.

A *k-labeled graph* ( $k \geq 0$ ) is a finite graph in which  $k$  nodes are labeled by  $1, 2, \dots, k$ . Two  $k$ -labeled graphs are *isomorphic*, if there is a label-preserving isomorphism between them. We denote by  $K_k$  the  $k$ -labeled complete graph on  $k$ -nodes, and by  $E_k$ , the  $k$ -labeled graph on  $k$  nodes with no edges.

Let  $F_1$  and  $F_2$  be two  $k$ -labeled graphs. Their *product*  $F_1F_2$  is defined as follows: we take their disjoint union, and then identify nodes with the same label. For two 0-labeled graphs,  $F_1F_2$  is just their disjoint union.

The definition of connection matrices can be extended to the case when  $G$  is weighted in a trivial way: The rows and columns of  $M(k, G)$  are indexed by isomorphism types of  $k$ -labeled graphs. The entry in the intersection of the row corresponding to  $F_1$  and the column corresponding to  $F_2$  is  $\text{hom}(F_1F_2, G)$ . Let us also recall their main properties:

**Lemma 2.1** *The connection matrices  $M(k, G)$  are positive semidefinite and  $M(k, G)$  has rank at most  $|V(G)|^k$ .*

This lemma will follow very easily if we introduce two further matrices. Let us extend our notation by defining, for any  $k$ -labeled graph  $F$  and mapping  $\phi : [1, k] \rightarrow V(G)$ ,

$$\text{hom}_\phi(F, G) = \sum_{\substack{\psi: V(F) \rightarrow V(G) \\ \psi \text{ extends } \phi}} \frac{\alpha_\psi}{\alpha_\phi} \text{hom}_\psi(F, G) \quad (1)$$

So

$$\text{hom}(F, G) = \sum_{\phi: [1, k] \rightarrow V(G)} \alpha_\phi \text{hom}_\phi(F, G).$$

Furthermore, for any two  $k$ -labeled graph  $F_1$  and  $F_2$ , we have the important equation

$$\text{hom}_\phi(F_1F_2, G) = \text{hom}_\phi(F_1, G) \text{hom}_\phi(F_2, G). \quad (2)$$

This expresses that once we mapped the common part, the mapping can be extended to the rest of  $F_1$  and  $F_2$  independently.

Let  $N(k, G)$  denote the matrix in which rows are indexed by maps  $\phi : [1, k] \rightarrow V(G)$ , columns are indexed by  $k$ -labeled graphs  $F$ , and the entry in the intersection of the row  $\phi$  and column  $F$  is  $\text{hom}_\phi(F, G)$ . Let  $A(k, G)$  denote the diagonal matrix whose rows and columns are indexed by maps  $\phi : [1, k] \rightarrow V(G)$ , and the diagonal entry in row  $\phi$  is  $\alpha_\phi$ . Then (1) and (2) imply

$$M(k, G) = N(k, G)^\top A(k, G) N(k, G). \quad (3)$$

This equation immediately implies that  $M(k, G)$  is positive semidefinite and

$$\text{rk}(M(k, G)) = \text{rk}(N(k, G)) \leq |V(G)|^k.$$

When does the rank of a connection matrix attain this upper bound? There are two (related, but different) types of degeneracy that causes lower rank.

**Twins.** The first of these causes is easy to handle. We call two nodes  $i, j \in V(G)$  *twins*, if for every node  $l \in V(G)$ ,  $\beta_{il} = \beta_{jl}$  (note: the condition includes  $l = i$  and  $l = j$ ; the node weights  $\alpha_i$  play no role in this definition). We say that  $G$  is *twin-free*, if no two different nodes are twins.

Suppose that  $G$  is not twin-free, so that it has two twin nodes  $i$  and  $j$ . Let us identify the equivalence classes of twin nodes, define the node-weight  $\alpha$  of a new node as the sum of the node-weights of its pre-images, and define the weight of an edge as the weight of any of its pre-images (which all have the same weight). This way we get a twin-free graph  $\bar{G}$  such that  $\text{hom}(F, G) = \text{hom}(F, \bar{G})$  for every graph  $F$ . It follows that the rank of the connection matrices  $M(k, G)$  and  $M(k, \bar{G})$  are the same, and this rank is at most  $|V(\bar{G})|^k < |V(G)|^k$ .

From now on, we assume that  $G$  is twin-free.

**Automorphisms.** The second reason for rank loss in the connection matrices will take more work to handle. Suppose that  $G$  has a proper automorphism (a permutation of the nodes that preserves both the node-weights and edge-weights). Then any two rows of  $N(k, G)$  defined by mappings  $\phi : [1, k] \rightarrow V(G)$  and  $\phi\sigma$  ( $\sigma \in \text{Aut}(G)$ ) are equal. So the rank of  $N(k, G)$  (and  $M(k, G)$ ) is at most the number of orbits of the automorphism group of  $G$  on ordered  $k$ -tuples of its nodes. The main result of this paper is that equality holds here.

**Theorem 2.2** *Let  $G$  be twin-free weighted graph. Let  $\text{orb}_k(G)$  denote the number of orbits of the automorphism group of  $G$  on ordered  $k$ -tuples of its nodes. Then  $\text{rk}(M(k, G)) = \text{orb}_k(G)$  for every  $k$ .*

**Corollary 2.3** *Let  $G$  be a weighted graph that has no twins and no automorphisms. Then  $\text{rk}(M(k, G)) = |V(G)|^k$  for every  $k$ .*

Note that swapping twins  $i$  and  $j$  is almost an automorphism: the only additional condition needed is that  $\alpha_i = \alpha_j$ . So in particular, for unweighted graphs the condition that there are no automorphisms implies that there are no twins.

Along the lines, we'll prove two lemmas, which are of independent interest:

**Lemma 2.4** *Let  $G$  be a twin-free weighted graph, let  $\phi, \psi \in V(G)^k$ , and suppose that for every  $k$ -labeled graph  $F$ ,  $\text{hom}_\phi(F, G) = \text{hom}_\psi(F, G)$ . Then there exists an automorphism  $\sigma$  of  $H$  such that  $\psi = \phi\sigma$ .*

Fix an integer  $k \geq 1$  and a weighted graph  $G$ . We say that a vector  $f : V(G)^k \rightarrow \mathbf{R}$  is *invariant under automorphisms of  $G$* , if  $f(\phi\sigma) = f(\phi)$  for every  $\sigma \in \text{Aut}(G)$ . Trivially, every column of  $N(h, G)$  is invariant under automorphisms.

**Lemma 2.5** *The column space of  $N(k, G)$  consists of precisely those vectors  $f : V(G)^k \rightarrow \mathbf{R}$  that are invariant under automorphisms of  $G$ .*

As a final application, we prove an extension of an old result from [2] to weighted graphs:

**Corollary 2.6** *Let  $G_1$  and  $G_2$  be twin-free weighted graphs, and assume that for every simple graph  $F$ ,  $\text{hom}(F, G_1) = \text{hom}(F, G_2)$ . Then  $G_1$  and  $G_2$  are isomorphic.*

### 3 The algebra of graphs

A  $k$ -labeled quantum graph is a formal linear combination (with real coefficients) of  $k$ -labeled graphs. Let  $\mathcal{G}_k$  denote the (infinite dimensional) vector space of all  $k$ -labeled quantum graphs. We can turn  $\mathcal{G}_k$  into an algebra by using  $F_1 F_2$  introduced above as the product of two generators, and then extending this multiplication to the other elements linearly. Clearly  $\mathcal{G}_k$  is associative and commutative, and the empty graph  $E_k$  is a unit element in  $\mathcal{G}_k$ .

We need to introduce some further (rather trivial) algebras. Let  $\mathcal{A}_k$  be the algebra of formal linear combinations of maps  $\phi : [1, k] \rightarrow V(G)$ , where multiplication is defined in a trivial way: for two maps  $\phi$  and  $\psi$ , let  $\phi * \psi = \phi$  if  $\phi = \psi$  and 0 otherwise. The sum

$$u_k = \sum_{\phi : [1, k] \rightarrow V(G)} \phi$$

is the unit element of this algebra.

Next define  $f(\cdot) = \text{hom}(\cdot, G)$ . Extend  $f$  linearly to quantum graphs. This function  $f$  gives rise to additional structure. We introduce an inner product on  $\mathcal{G}$  by

$$\langle x, y \rangle = f(xy). \tag{4}$$

We'll see that this inner product is semidefinite:  $\langle x, x \rangle \geq 0$  for all  $x$ . We also introduce an inner product on  $\mathcal{A}_k$  as follows: for two basis elements  $\phi, \psi : [1, k] \rightarrow V(G)$ , let

$$\langle \phi, \psi \rangle = \begin{cases} \alpha_\phi & \text{if } \phi = \psi, \\ 0 & \text{otherwise,} \end{cases}$$

and then extend this bilinearly. Trivially, this inner product is positive definite.

The function  $f$  is multiplicative over connected components: if  $F_1, F_2 \in \mathcal{G}_0$ , then

$$f(F_1 F_2) = f(F_1) f(F_2).$$

This means that as a map  $\mathcal{G}_0 \rightarrow \mathcal{A}_0$  it is an algebra homomorphism.

The graph  $G$  also gives rise to a map  $f_k : \mathcal{G}_k \rightarrow \mathcal{A}_k$  by

$$f_k(F) = \sum_{\phi: [1, k] \rightarrow [1, m]} \text{hom}_\phi(F, G) \phi.$$

We extend this map linearly to quantum graphs. For two  $k$ -labeled quantum graphs  $x, y$  we say that

$$x \equiv y \pmod{G}$$

if  $f_k(x) = f_k(y)$ .

It is easy to check that the mapping  $f_k : \mathcal{G}_k \rightarrow \mathcal{A}_k$  is an algebra homomorphism, and preserves inner product. This in particular implies that the inner product defined by (4) is positive semidefinite. Furthermore, since the inner product in  $\mathcal{A}_k$  is positive definite, the kernel of  $f_k$  is exactly the nullspace of the inner product (4). If we factor out this nullspace, we get an algebra  $\mathcal{G}'_k$ . It is easy to check that

$$\dim(\mathcal{G}'_k) = \text{rk}(M(k, G)). \quad (5)$$

So we know that this dimension is at most  $|V(G)|^k$ ; in particular it is finite.

For every  $k > 0$ , we define the *trace*  $\text{tr} : \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$  simply by erasing the label  $k$ . We also have a linear map  $\text{tr} : \mathcal{A}_k \rightarrow \mathcal{A}_{k-1}$  defined by

$$\text{tr}(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \alpha_{i_k}(e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}).$$

These operators correspond to each other in the sense that for every  $k$ -labeled graph  $F$ ,

$$\text{tr}(f_k(F)) = f_{k-1}(\text{tr}(F)).$$

This implies that

$$\text{tr}(\mathcal{A}''_k) \subseteq \mathcal{A}''_{k-1}. \quad (6)$$

## 4 Proof of Theorem 2.2

### 4.1 A lemma about twin-free graphs

We start with a simple lemma about twin-free weighted graphs.

**Lemma 4.1** *Let  $G$  be a twin-free weighted graph. Then every map  $\phi : V(G) \rightarrow V(G)$  such that  $\beta_{\phi(i)\phi(j)} = \beta_{ij}$  for every  $i, j \in V(G)$  is bijective.*

**Proof.** The mapping  $\beta$  has some power  $\gamma = \beta^s$  that is idempotent. We claim that  $i$  and  $\gamma(i)$  are twins. Indeed,

$$\beta_{il} = \beta_{\gamma(i)\gamma(l)} = \beta\gamma^2(i)\gamma(l) = \beta_{\gamma(i)l}$$

for every  $l \in V(G)$ . Since  $G$  is twin-free, this implies that  $\gamma$  is the identity, and so  $\beta$  must be bijective.  $\square$

### 4.2 From Lemma 2.4 to Lemma 2.5 to Theorem 2.2

Now we turn to the proof of Theorem 2.2. Let  $\mathcal{A}'_k$  be the subalgebra of elements of  $\mathcal{A}_k$  invariant under the automorphisms of  $G$ , and let  $\mathcal{A}''_k = f_k(\mathcal{G}_k)$ . It is trivial that  $\mathcal{A}''_k \subseteq \mathcal{A}'_k$ . Furthermore, we have

$$\dim(\mathcal{A}'_k) = \frac{|V(G)|^k}{\text{orb}_k(G)}$$

and

$$\dim(\mathcal{A}''_k) = \dim(\mathcal{G}'_k) = \text{rk}(M(k, G)).$$

Thus it follows that  $\text{rk}(M(k, G)) \leq |V(G)|^k / \text{orb}_k(G)$ ; to prove Theorem 2.2, it suffices to prove that algebras  $\mathcal{A}''_k$  and  $\mathcal{A}'_k$  are the same. This is just the content of lemma 2.5. Thus it suffices to prove this lemma.

The algebra  $\mathcal{A}''_k$  is a finite dimensional commutative algebra with a unit element, and so it has a basis  $w_1, \dots, w_r$  consisting of idempotents. Expressing these idempotents in the basis of the whole algebra  $\mathcal{A}_k$ , we get that for each  $i$  there is a set  $\Psi_i \subseteq V(G)^k$  such that

$$w_i = \sum_{\psi \in \Psi_i} \psi.$$

Since  $\sum_k w_k$  is the unit element, it follows that the sets  $\Psi_i$  ( $i = 1, \dots, r$ ) form a partition of  $V(G)^k$ . We say that  $\phi, \psi \in [1, m]^k$  are *equivalent*, if they belong to the same set  $\Psi_i$ . Clearly  $\phi$  and  $\psi$  are equivalent if and only if

$$\text{hom}_\phi(F, G) = \text{hom}_\psi(F, G)$$



for every  $k$ -labeled graph  $F$ . The subalgebra  $\mathcal{A}_k''$  consists of those elements in which any two maps  $\phi$  and  $\psi$  that are equivalent occur with the same coefficient. Analogously, the subalgebra  $\mathcal{A}_k''$  consists of those elements in which any two maps  $\phi$  and  $\psi$  such that  $\psi = \phi\sigma$  for some automorphism  $\sigma$  occur with the same coefficient. The fact that these two are the same is just the content of lemma 2.4. So it suffices to prove this Lemma.

### 4.3 Proof of Lemma 2.4

For any map  $\phi : [1, k] \rightarrow [1, m]$ , let  $\phi'$  denote its restriction to  $[1, k-1]$ .

**Claim 4.1** *If the maps  $\phi, \psi \in [1, m]^k$  are equivalent, then so are  $\phi'$  and  $\psi'$ .*

Indeed, assume that  $\phi'$  and  $\psi'$  are not equivalent, then there is a  $(k-1)$ -labeled graph  $F$  such that

$$\text{hom}_{\phi'}(F, G) \neq \text{hom}_{\psi'}(F, G).$$

Then for  $F' = F \otimes E_1$  we have

$$\text{hom}_{\phi}(F', G) = \text{hom}_{\phi'}(F, G) \neq \text{hom}_{\psi'}(F, G) = \text{hom}_{\psi}(F', G),$$

which contradicts the assumption that  $\phi$  and  $\psi$  are equivalent.

**Claim 4.2** *Suppose that  $\phi, \psi \in [1, m]^k$  are equivalent. Then for every  $\mu \in [1, m]^{k+1}$  such that  $\phi = \mu'$  there exists a  $\nu \in [1, m]^{k+1}$  such that  $\psi = \nu'$  and  $\mu$  and  $\nu$  are equivalent.*

Let  $\Psi$  be the set of maps equivalent to  $\mu$ . By definition, we have

$$\sum_{\eta \in \Psi} \eta \in \mathcal{A}_{k+1}''.$$

Applying the trace operator, we see by (6) that

$$\sum_{\eta \in \Psi} \alpha(\eta(k+1))\eta' \in \mathcal{A}_k''.$$

Here  $\phi$  occurs with non-zero coefficient; since  $\phi$  and  $\psi$  are equivalent,  $\psi$  must occur with non-zero coefficient, which shows that there must be a  $\nu \in \Psi$  such that  $\nu' = \psi$ . This proves Claim 4.2.

To prove the Lemma, we want to show that if  $\phi$  and  $\psi$  are equivalent, then there exists an automorphism  $\sigma$  of  $G$  such that  $\psi = \phi\sigma$ . We prove this assertion for an increasing class of mappings.

**Claim 4.3** *If  $k = m$  the maps  $\phi, \psi : [1, m] \rightarrow [1, m]^m$  are equivalent, and  $\phi$  is bijective, then there is an automorphism  $\sigma$  of  $G$  such that  $\psi = \phi\sigma$ .*

We may assume that the nodes of  $G$  are labeled so that  $\phi$  is the identity. Let  $\Psi$  be the set of maps equivalent to  $\phi$ . We claim that every  $\psi \in \Psi$ , viewed as a map of  $V(G)$  into itself, satisfies

$$\beta_{ij} = \beta_{\psi(i)\psi(j)} \quad (7)$$

for every  $j$ . Indeed, let  $k_{ij}$  be the  $k$ -labeled graph consisting of  $k$  nodes and a single edge connecting nodes  $i$  and  $j$ . Then

$$\beta_{ij} = \text{hom}_\phi(k_{ij}, G) = \text{hom}_\psi(k_{ij}, G) = \beta_{\psi(i)\psi(j)}.$$

Since  $G$  is twin-free, it follows by Lemma 4.1 that  $\psi$  is one-to-one.

To complete the proof of the Claim, it suffices to show that for every  $\psi \in \Psi$ ,

$$\alpha(j) = \alpha(\psi(j)) \quad (j = 1, \dots, m). \quad (8)$$

It suffices to prove this for the case  $j = m$ . By the definition of equivalence, we have

$$\sum_{\psi \in \Psi} \psi \in \mathcal{A}_m''.$$

Applying the trace operator, we see by (6) that

$$\sum_{\psi \in \Psi} \alpha(\psi(m))\psi' \in \mathcal{A}_{m-1}''.$$

As we have seen, all maps  $\psi \in \Psi$  are bijective, which implies that the maps  $\psi'$  are all different. Since these maps are equivalent by Claim 4.1, it follows that all coefficients  $\alpha(\psi(m))$  are the same. This completes the proof of Claim 4.3.

**Claim 4.4** *If the maps  $\phi, \psi : [1, k] \rightarrow [1, m]$  are equivalent, and  $\phi$  is surjective, then there is an automorphism  $\sigma$  of  $G$  such that  $\psi = \phi\sigma$ .*

By permuting the labels  $1, \dots, k$  if necessary, we may assume that  $\phi(1) = 1, \dots, \phi(m) = m$ . Claim 4.1 implies that the restriction of  $\psi$  to  $[1, m]$  is equivalent to the restriction of  $\phi$  to  $[1, m]$ , and so by Claim 4.3, there is an automorphism  $\sigma$  of  $G$  such that  $\psi(i) = \sigma(i)$  for  $i = 1, \dots, m$ .

Consider any  $m+1 \leq j \leq k$ , and let  $\phi(j) = r$ . We claim that  $\psi(j) = \psi(r)$ . Indeed, the restriction of  $\phi$  to  $\{1, \dots, r-1, r+1, \dots, m, j\}$  is bijective,

and equivalent to the restriction of  $\psi$  to this set; hence the restriction of  $\psi$  to this set must be bijective, which implies that  $\psi(j) = \psi(r)$ . This implies that for every  $1 \leq i \leq k$ ,  $\psi(i) = \sigma(\phi(i))$  as claimed.

Now we are ready to prove the Lemma for arbitrary equivalent maps  $\phi, \psi : [1, k] \rightarrow [1, m]$ . We can extend  $\phi$  to a mapping  $\mu : [1, \ell] \rightarrow [1, m]$  ( $\ell \geq k$ ) which is surjective. By Claim 4.2, there is a mapping  $\nu : [1, \ell] \rightarrow [1, m]$  extending  $\psi$  such that  $\mu$  and  $\nu$  are equivalent. Then by Claim 4.4, there is an automorphism  $\sigma$  of  $G$  such that  $\nu = \mu\sigma$ . Restricting this relation to  $[1, k]$ , the assertion follows.

#### 4.4 Proof of Corollary 2.6

Let  $G$  be the graph obtained by taking the disjoint union of  $G_1$  and  $G_2$ , creating two new nodes  $v_1$  and  $v_2$ , and connecting  $v_i$  to all nodes of  $G_i$ . Also add loops at  $v_i$ . The new nodes and new edges have weight 1.

We claim that for every 1-labeled graph  $F$ ,

$$\text{hom}_{v_1}(F, G) = \text{hom}_{v_2}(F, G) \quad (9)$$

Indeed, if  $F$  is not connected, then those components not containing the labeled node contribute the same factors to both sides. So it suffices to consider the case when  $F$  is connected. Then we have

$$\text{hom}_{v_1}(F, G) = \sum_{\substack{S \subseteq V(F) \\ S \ni v_1}} \text{hom}(F \setminus S, G_1).$$

Indeed, every map  $\phi : [1, k] \rightarrow V(G)$  such that  $\phi(1) = v_1$  maps some subset  $S \subseteq V(F)$ ,  $S \ni v_1$  to the new node  $v_1$ ; if we fix this set, then the restriction  $\phi'$  of  $\phi$  to  $V(F) \setminus S$  is a map into  $V(G_1)$  (else, the contribution of the map to  $\text{hom}(F, G)$  is 0), and the contribution of  $\phi$  to  $\text{hom}_{v_1}(F, G)$  is the same as the contribution of  $\phi'$  to  $\text{hom}(F, G)$ .

Since  $\text{hom}_{v_2}(F, G)$  can be expressed by a similar formula, and the sums on the right hand sides are equal by hypothesis, this proves (9).

Now Lemma 2.4 implies that there is an automorphism of  $G$  mapping  $v_1$  to  $v_2$ . This automorphism gives an isomorphism between  $G_1$  and  $G_2$ .

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