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IRRATIONAL PROOFS FOR THREE THEOREMS OF STANLEY

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Abstract. We give new proofs of three theorems of Stanley on generating functions for the integer points in rational cones. The first, Stanley's Reciprocity Theorem, relates the rational generating function $\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{x}) := \sum_{\mathbf{m}\in\mathbf{v}+\mathcal{K}\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$, where $\mathcal{K}$ is a rational cone and $\mathbf{v} \in \mathbb{R}^d$, with $\sigma_{-\mathbf{v}+\mathcal{K}^\circ}(1/\mathbf{x})$. The second, Stanley's Positivity Theorem, asserts that the generating function of the Ehrhart quasipolynomial $L_{\mathcal{P}}(n) := \#\left(n\mathcal{P} \cap \mathbb{Z}^d\right)$ of a rational polytope $\mathcal{P}$ can be written as a rational function with *nonnegative* numerator $\nu_{\mathcal{P}}$. The third, Stanley's Monotonicity Theorem, asserts that if $\mathcal{P} \subset \mathcal{Q}$, then $\nu_{\mathcal{P}} \le \nu_{\mathcal{Q}}$. Our proofs are based on elementary (primary school) counting afforded by irrational decompositions of rational polyhedra.

1. Introduction

For us, a (*convex*) *rational polyhedron* $\mathcal{P}$ is the intersection of (closed) half-spaces in $\mathbb{R}^d$, where each half-space has the form $\left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid\ a_1x_1 + a_2x_2 + \cdots + a_dx_d \le b\right\}$ for some integers $a_1, a_2, \ldots, a_d, b$. A *rational cone* is a rational polyhedron with a unique vertex at the origin. We are interested in the generating function

$$\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{x}) \ := \sum_{\mathbf{m}\in\mathbf{v}+\mathcal{K}\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$$

for the integral points of the shifted ("affine") cone $\mathbf{v} + \mathcal{K}$ and its companion $\sigma_{\mathbf{v}+\mathcal{K}^\circ}(\mathbf{x})$ for the integral points of the (relative) interior $\mathcal{K}^\circ$ of $\mathcal{K}$. Here, $\mathbf{x}^{\mathbf{m}}$ denotes the product $x_1^{m_1}x_2^{m_2}\cdots x_d^{m_d}$. The function $\sigma_{\mathbf{v}+\mathcal{K}}$ (as well as $\sigma_{\mathbf{v}+\mathcal{K}^\circ}$) is a rational function in the variables $\mathbf{x}$. Stanley's *Reciprocity Theorem* [10] relates the functions $\sigma_{\mathbf{v}+\mathcal{K}}$ and $\sigma_{-\mathbf{v}+\mathcal{K}^\circ}$ for any $\mathbf{v} \in \mathbb{R}^d$. We abbreviate the vector $(1/x_1, 1/x_2, \ldots, 1/x_d)$ by $\frac{1}{\mathbf{x}}$.

Theorem 1 (Stanley). *Suppose that $\mathcal{K}$ is a rational cone and $\mathbf{v} \in \mathbb{R}^d$. Then, as rational functions, $\sigma_{\mathbf{v}+\mathcal{K}}(\mathbf{x}) = (-1)^{\dim\mathcal{K}}\, \sigma_{-\mathbf{v}+\mathcal{K}^\circ}\left(\frac{1}{\mathbf{x}}\right)$.*

There are proofs of Theorem 1 which involve local cohomology in commutative algebra [13, Section I.8] and complex analysis [10]. Many proofs, including ours, first prove it for the easy case of simple cones, and then use a decomposition of $\mathcal{K}$ into simple cones to deduce

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Theorem 1. This approach requires some device to handle the subsequent overcounting of integral points that occurs as the cones in the decomposition overlap along (proper) faces. In other proofs, this device is either a shelling argument [15], or a valuation (finitely additive measure) [6], or some other version of inclusion-exclusion. In contrast, our method of 'irrational decomposition' requires no such device as the proper faces of the cones we use contain no integral points.

We use the same construction to prove Stanley's *Positivity Theorem*. A *rational polytope* is a bounded rational polyhedron. A rational polytope is *integral* if its vertices lie in $\mathbb{Z}^d$. For an integral polytope $\mathcal{P} \subset \mathbb{R}^d$, Ehrhart [3] showed that the function

$$L_{\mathcal{P}}(n) \ := \ \#\left(n\mathcal{P} \cap \mathbb{Z}^d\right)$$

is a polynomial in the integer variable n. If the polytope $\mathcal{P}$ is only rational, then the function $L_{\mathcal{P}}(n)$ is a *quasi-polynomial*. More precisely, let p be a positive integer such that $p\mathcal{P}$ is integral. Then there exist polynomials $f_0, f_1, \ldots, f_{p-1}$ so that

$$L_{\mathcal{P}}(n) \ = \ f_{(n \bmod p)}(n)\,.$$

(It is most efficient, but not necessary, to take the minimal such p.)

The generating function for $L_{\mathcal{P}}$ is a rational function with denominator $(1-t^p)^{\dim\mathcal{P}+1}$ (see, for example, [14, Chapter 4] or the proof we give in Section 3). But one can say more [11].

Theorem 2 (Stanley)**.** *Suppose $\mathcal{P}$ is a rational d-polytope with $p\mathcal{P}$ integral and set*

$$1 + \sum_{n\geq 1} L_{\mathcal{P}}(n)\, t^n \ = \ \frac{a_{(d+1)p-1}t^{(d+1)p-1} + a_{(d+1)p-2}t^{(d+1)p-2} + \cdots + a_0}{(1-t^p)^{d+1}}\,. \tag{1}$$

Then $a_0, a_1, \ldots, a_{(d+1)p-1} \geq 0$.

Even more can be said. Suppose that $\mathcal{Q}$ is a rational polytope containing $\mathcal{P}$ and that both $p\mathcal{P}$ and $p\mathcal{Q}$ are integral. Supressing their dependence on p, let $\nu_{\mathcal{P}}$ and $\nu_{\mathcal{Q}}$ be the numerators of the rational generating functions (1) for $\mathcal{P}$ and $\mathcal{Q}$, respectively. We have $d = \dim\mathcal{P} < \dim\mathcal{Q} = e$ and so $\nu_{\mathcal{Q}}$ is the numerator of the rational generating function for $L_{\mathcal{Q}}(n)$, which has denominator $(1-t^p)^e$, while $\nu_{\mathcal{P}}$ is the numerator of the rational generating function for $L_{\mathcal{P}}(n)$, which has denominator $(1-t^p)^e$. Stanley's *Monotonicity Theorem* [12] aserts that every coefficient of $\nu_{\mathcal{Q}}$ dominates the corresponding coefficient of $\nu_{\mathcal{P}}$, that is, $\nu_{\mathcal{P}} \leq \nu_{\mathcal{Q}}$.

Theorem 3 (Stanley)**.** *Suppose $\mathcal{P} \subset \mathcal{Q}$ are rational polytopes with $p\mathcal{P}$ and $p\mathcal{Q}$ integral. Then $\nu_{\mathcal{P}} \leq \nu_{\mathcal{Q}}$.*

If $p\mathcal{P}$ is an integral simplex with Euclidean volume $\frac{1}{d!}$, then $\nu_{\mathcal{P}}(t) = 1$. Since every rational polytope $\mathcal{P}$ with $p\mathcal{P}$ integral contains such a scaled simplex, we see that Theorem 2 follows from Theorem 3.

While Theorem 1 may seem unconnected to Theorems 2 and 3, they are related by a construction which—to the best of our knowledge—is due to Ehrhart. Lift the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$

of a rational polytope $\mathcal{P} \subset \mathbb{R}^d$ into $\mathbb{R}^{1+d}$, by adding 1 as their first coordinate, and let p be a positive integer such that $p\mathcal{P}$ is integral. Then

$$\mathbf{v}_1' = (p, p\mathbf{v}_1), \quad \mathbf{v}_2' = (p, p\mathbf{v}_2), \quad \ldots, \quad \mathbf{v}_m' = (p, p\mathbf{v}_m)$$

are integral. Now we define the *cone over* $\mathcal{P}$ to be

$$\operatorname{cone}(\mathcal{P}) = \{\lambda_1\mathbf{v}_1' + \lambda_2\mathbf{v}_2' + \cdots + \lambda_m\mathbf{v}_m' \mid \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\} \subset \mathbb{R}^{1+d}.$$

We can recover our original polytope $\mathcal{P}$ (strictly speaking, the set $\{(1, \mathbf{x}) \mid \mathbf{x} \in \mathcal{P}\}$) by cutting $\operatorname{cone}(\mathcal{P})$ with the hyperplane $x_0 = 1$. Cutting $\operatorname{cone}(\mathcal{P})$ with the hyperplane $x_0 = 2$, we obtain a copy of $2\mathcal{P}$, cutting with $x_0 = 3$ gives a copy of $3\mathcal{P}$, etc. Hence

$$\sigma_{\operatorname{cone}(\mathcal{P})}(x_0, x_1, \ldots, x_d) = 1 + \sum_{n \geq 1} \sigma_{n\mathcal{P}}(x_1, \ldots, x_d)\, x_0^n .$$

Since $\sigma_{n\mathcal{P}}(1, 1, \ldots, 1) = \#\left(n\mathcal{P} \cap \mathbb{Z}^d\right)$, we obtain

$$\sigma_{\operatorname{cone}(\mathcal{P})}(t, 1, 1, \ldots, 1) = 1 + \sum_{n \geq 1} L_{\mathcal{P}}(n)\, t^n .$$

A nice application of Theorem 1 is the following reciprocity theorem, which was conjectured (and partially proved) by Ehrhart [4] and proved by Macdonald [8].

Corollary 4 (Ehrhart-Macdonald)**.** *The quasi-polynomials $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ satisfy*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t) .$$

As with Theorem 1, most proofs of Theorem 2 use shellings of a polyhedron or finite additive measures (see, e.g., [5, 8, 9]). The only exceptions we are aware of are proofs via complex analysis (see, e.g., [10]) and commutative algebra (see, e.g., [13, Section I.8]). We feel that no existing proof is as elementary as the one we give.

We remark that the same technique gives a similarly elementary and subtraction-free proof of Brion's Theorem [2]. This proof will appear in [1].

2. Stanley's Reciprocity Theorem for cones

Any cone has a triangulation into *simple cones* which are cones with a minimal number of boundary hyperplanes (see, e.g., [7]). This is the starting point for our proof, which differs from other proofs that use such a decomposition. The decomposition that we use is, from the view of integer points, non-overlapping, and thus we need only apply elementary (as in elementary–school) counting arguments, sidestepping any hint of inclusion-exclusion.

Irrational Proof of Theorem 1. Triangulate $\mathcal{K}$ into simple rational cones $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$, all having the same dimension as $\mathcal{K}$. Now there exists a vector $\mathbf{s} \in \mathbb{R}^d$ such that

$$(\mathbf{v} + \mathcal{K}^\circ) \cap \mathbb{Z}^d = (\mathbf{s} + \mathcal{K}) \cap \mathbb{Z}^d \tag{2}$$

and

$$\partial\left(\pm\mathbf{s}+\mathcal{K}_j\right)\cap\mathbb{Z}^d \;=\; \emptyset \qquad \text{for all } j=1,\ldots,m\,. \tag{3}$$

In fact, $\mathbf{s}$ may be any vector in the relative interior of some cone $\mathbf{v}+\mathcal{K}_i$ for which $\mathbf{s}-\mathbf{v}$ is short.

Note that (2) implies $(-\mathbf{v}+\mathcal{K})\cap\mathbb{Z}^d=(-\mathbf{s}+\mathcal{K})\cap\mathbb{Z}^d$. Furthermore, because of (3),

$$\sigma_{-\mathbf{v}+\mathcal{K}}(\mathbf{x}) \;=\; \sigma_{-\mathbf{s}+\mathcal{K}}(\mathbf{x}) \;=\; \sum_{j=1}^{m}\sigma_{-\mathbf{s}+\mathcal{K}_j}(\mathbf{x})$$

and

$$\sigma_{\mathbf{v}+\mathcal{K}^\circ}(\mathbf{x}) \;=\; \sigma_{\mathbf{s}+\mathcal{K}}(\mathbf{x}) \;=\; \sum_{j=1}^{m}\sigma_{\mathbf{s}+\mathcal{K}_j}(\mathbf{x})\,.$$

The result now follows from reciprocity for simple cones, which is Lemma 5 below. □

Despite our title, the vector $\mathbf{s}-\mathbf{v}$ need not be irrational as any short rational vector will do.

Lemma 5. *Fix linearly independent vectors $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_d\in\mathbb{Z}^d$, and let*

$$\mathcal{K} \;=\; \{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d \mid \lambda_1,\ldots,\lambda_d\geq 0\}\,.$$

Then for those $\mathbf{s}\in\mathbb{R}^d$ for which the boundary of the shifted simple cone $\mathbf{s}+\mathcal{K}$ contains no integer point,

$$\sigma_{\mathbf{s}+\mathcal{K}}\left(\tfrac{1}{\mathbf{x}}\right) \;=\; (-1)^d\,\sigma_{-\mathbf{s}+\mathcal{K}}(\mathbf{x})\,.$$

As in Theorem 1, the reciprocity identity is one of rational functions. In the course of the proof, we will show that $\sigma_{\mathbf{s}+\mathcal{K}}$ is indeed a rational function for $\mathbf{s}\in\mathbb{R}^d$.

Proof. If we tile the cone $\mathbf{s}+\mathcal{K}$ with $\mathbb{N}\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_d\}$–translates of the parallelepiped $\mathbf{s}+\mathcal{P}$, where

$$\mathcal{P} \;:=\; \{\lambda_1\mathbf{w}_1+\lambda_2\mathbf{w}_2+\cdots+\lambda_d\mathbf{w}_d \mid 0\leq\lambda_1,\lambda_2,\ldots,\lambda_d<1\}\,,$$

then we can express $\sigma_{\mathbf{s}+\mathcal{K}}$ using geometric series

$$\sigma_{\mathbf{s}+\mathcal{K}}(\mathbf{x}) \;=\; \frac{\sigma_{\mathbf{s}+\mathcal{P}}(\mathbf{x})}{\left(1-\mathbf{x}^{\mathbf{w}_1}\right)\left(1-\mathbf{x}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{x}^{\mathbf{w}_d}\right)}\,. \tag{4}$$

(This proves that $\sigma_{\mathbf{s}+\mathcal{K}}$ is a rational function.) Similarly,

$$\sigma_{-\mathbf{s}+\mathcal{K}}(\mathbf{x}) \;=\; \frac{\sigma_{-\mathbf{s}+\mathcal{P}}(\mathbf{x})}{\left(1-\mathbf{x}^{\mathbf{w}_1}\right)\left(1-\mathbf{x}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{x}^{\mathbf{w}_d}\right)}\,,$$

so we only need to relate the parallelepipeds $\mathbf{s}+\mathcal{P}$ and $-\mathbf{s}+\mathcal{P}$. By assumption, $\mathbf{s}+\mathcal{P}$ contains no integer points on its boundary, and so we may replace $\mathcal{P}$ by its closure. Note that $\mathcal{P}=\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d-\mathcal{P}$, so we have the identity

$$\mathbf{s}+\mathcal{P} \;=\; -(-\mathbf{s}+\mathcal{P})+\mathbf{w}_1+\mathbf{w}_2+\cdots+\mathbf{w}_d\,. \tag{5}$$

In terms of generating functions, (5) implies that

$$\sigma_{\mathbf{s}+\mathcal{P}}\left(\mathbf{x}\right) \ = \ \sigma_{-\mathbf{s}+\mathcal{P}}\left(\tfrac{1}{\mathbf{x}}\right)\mathbf{x}^{\mathbf{w}_1}\mathbf{x}^{\mathbf{w}_2}\cdots\mathbf{x}^{\mathbf{w}_d} \ ,$$

whence

$$\begin{aligned}
\sigma_{\mathbf{s}+\mathcal{K}}\left(\tfrac{1}{\mathbf{x}}\right) &= \frac{\sigma_{\mathbf{s}+\mathcal{P}}\left(\tfrac{1}{\mathbf{x}}\right)}{\left(1-\mathbf{x}^{-\mathbf{w}_1}\right)\left(1-\mathbf{x}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{x}^{-\mathbf{w}_d}\right)} \\
&= \frac{\sigma_{-\mathbf{s}+\mathcal{P}}(\mathbf{x})\,\mathbf{x}^{-\mathbf{w}_1}\mathbf{x}^{-\mathbf{w}_2}\cdots\mathbf{x}^{-\mathbf{w}_d}}{\left(1-\mathbf{x}^{-\mathbf{w}_1}\right)\left(1-\mathbf{x}^{-\mathbf{w}_2}\right)\cdots\left(1-\mathbf{x}^{-\mathbf{w}_d}\right)} \\
&= \frac{\sigma_{-\mathbf{s}+\mathcal{P}}(\mathbf{x})}{\left(\mathbf{x}^{\mathbf{w}_1}-1\right)\left(\mathbf{x}^{\mathbf{w}_2}-1\right)\cdots\left(\mathbf{x}^{\mathbf{w}_d}-1\right)} \\
&= (-1)^d\frac{\sigma_{-\mathbf{s}+\mathcal{P}}(\mathbf{x})}{\left(1-\mathbf{x}^{\mathbf{w}_1}\right)\left(1-\mathbf{x}^{\mathbf{w}_2}\right)\cdots\left(1-\mathbf{x}^{\mathbf{w}_d}\right)} \\
&= (-1)^d\,\sigma_{-\mathbf{s}+\mathcal{K}}(\mathbf{x}).
\end{aligned}$$

□

Lemma 5 is essentially due to Ehrhart. The new idea here is our 'irrational' decomposition.

3. Stanley's Positivity and Monotonicity Theorems for Ehrhart polynomials

Irrational Proof of Theorem 2. As before, triangulate $\operatorname{cone}(\mathcal{P}) \subset \mathbb{R}^{d+1}$ into simple rational cones $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$, each of whose generators are among the generators $(p, p\mathbf{v}_i)$ of $\operatorname{cone}(\mathcal{P})$. Again there exists a vector $\mathbf{s} \in \mathbb{R}^{d+1}$ such that

$$\operatorname{cone}(\mathcal{P}) \cap \mathbb{Z}^d \ = \ (\mathbf{s} + \operatorname{cone}(\mathcal{P})) \cap \mathbb{Z}^d$$

and no facet of any cone $\mathbf{s} + \mathcal{K}_i$ contains any integral points. Thus every integral point in $\mathbf{s} + \operatorname{cone}(\mathcal{P})$ belongs to exactly one simple cone $\mathbf{s} + \mathcal{K}_j$, and we have

$$\operatorname{cone}(\mathcal{P}) \cap \mathbb{Z}^d \ = \ (\mathbf{s} + \operatorname{cone}(\mathcal{P})) \cap \mathbb{Z}^d \ = \ \bigcup_{j=1}^{m} \left((\mathbf{s} + \mathcal{K}_j) \cap \mathbb{Z}^d \right) \ ,$$

and this union is *disjoint*. We obtain the identity of generating functions,

$$\sigma_{\operatorname{cone}(\mathcal{P})}\left(\mathbf{x}\right) \ = \ \sum_{j=1}^{m} \sigma_{\mathbf{s}+\mathcal{K}_j}\left(\mathbf{x}\right).$$

But now we recall from the Introduction that

$$1 + \sum_{n \geq 1} L_{\mathcal{P}}(n)\, t^n \ = \ \sigma_{\operatorname{cone}(\mathcal{P})}\left(t, 1, 1, \ldots, 1\right) \ = \ \sum_{j=1}^{m} \sigma_{\mathbf{s}+\mathcal{K}_j}\left(t, 1, 1, \ldots, 1\right) \ .$$

So it suffices to show that the rational generating functions $\sigma_{\mathbf{s}+\mathcal{K}_j}\left(t, 1, 1, \ldots, 1\right)$ for the *simple* cones $\mathbf{s} + \mathcal{K}_j$ have nonnegative numerators and denominators of the form $(1 - t^p)^{d+1}$.

In this case, the cone $\mathbf{s}+\mathcal{K}_j$ has integral generators of the form $\mathbf{w}_i = (p, p\mathbf{v}_i)$, for some vertices $\mathbf{v}_1, \ldots, \mathbf{v}_{d+1}$ of the polytope $\mathcal{P}$, where p is a positive integer such that $p\mathcal{P}$ is integral. Substituting $(t, 1, 1, \ldots, 1)$ into the concrete form of the rational generating function (4), gives denominator $(1-t^p)^{d+1}$ and numerator the generating function for the integer points in the parallelepiped which is generated by $\mathbf{w}_1, \ldots, \mathbf{w}_{d+1}$ and shifted by $\mathbf{s}$, where the coefficient a_i of t^i counts points with first coordinate i. $\square$

Irrational Proof of Theorem 3. Suppose first that $\dim \mathcal{P} = \dim \mathcal{Q}$. As in the previous proof, suppose that $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$ triangulate cone$(\mathcal{P})$ into simple rational cones, each of whose generators are among the generators $(p, p\mathbf{v}_i)$ of cone$(\mathcal{P})$. We may extend this to a triangulation $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_l$ of cone$(\mathcal{Q})$, where the additional simple cones have generators from the given generators $(p, p\mathbf{v}_i)$ of cone$(\mathcal{P})$ and $(p, p\mathbf{w}_i)$ of cone$(\mathcal{Q})$. The generators of each cone $\mathcal{K}_i$ and the irrational shift vector $\mathbf{s}$ together give a parallelepiped with no lattice points on its boundary, and the coefficient of t^j in $\nu_{\mathcal{P}}$ is the number of integer points with last coordinate j in the union of these parallelepipeds for $\mathcal{K}_1, \ldots, \mathcal{K}_m$. The result follows as the coefficient of t^j in $\nu_{\mathcal{Q}}$ is the number of integer points with last coordinate j in the parallelepipeds for $\mathcal{K}_1, \ldots, \mathcal{K}_l$, and $m < l$.

If however, $\dim \mathcal{P} < \dim_{\mathcal{Q}}$, then the triangulation $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$ of cone$(\mathcal{P})$ extends to a triangulation $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_l$ of cone$(\mathcal{Q})$, where now the simple cones $\mathcal{K}_i$ are d-faces of the simple cones $\mathcal{L}_j$. Note that the irrational decomposition $\mathbf{s}+\mathcal{L}_j$, $j = 1, \ldots, l$ restricts to an irrational decomposition of cone$(\mathcal{P})$ given by some vector $\mathbf{s}' \in \mathbb{R}\cdot \text{cone}(\mathcal{P})$. Moreover, for every $i = 1, \ldots, m$ there is a unique $a(i)$ with $1 \leq a(i) \leq l$ such that $\mathbf{s}'+\mathcal{K}_i \subset \mathbf{s}+\mathcal{L}_{a(i)}$. The same is true for the parallelepipeds generated by the vectors $(p, \mathbf{v})$ along the rays of these cones, and also for their shifts by $\mathbf{s}'$ and $\mathbf{s}$. Then the result follows by the same argument as before once we interpret the coefficients of t^j in $\nu_{\mathcal{P}}$ and $\nu_{\mathcal{Q}}$ as the number of points with second coordinate j in the union of these parallelepipeds. $\square$

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