# PERMUTATION STATISTICS ON INVOLUTIONS 

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#### Abstract

In this paper we look at polynomials arising from statistics on the classes of involutions, $I_{n}$, and involutions with no fixed points, $J_{n}$, in the symmetric group. Our results are motivated by F. Brenti's conjecture [3] which states that the Eulerian distribution of $I_{n}$ is logconcave. Symmetry of the generating functions is shown for the statistics $d, m a j$ and the joint distribution (d,maj). We show that exc is logconcave on $I_{n}$, inv is log-concave on $J_{n}$ and $d$ is partially unimodal on both $I_{n}$ and $J_{n}$. We also give recurrences and explicit forms for the generating functions of the inversions statistic on involutions in Coxeter groups of types $B_{n}$ and $D_{n}$. Symmetry and unimodality of inv is shown on the subclass of signed permutations in $D_{n}$ with no fixed points. In light of these new results, we present further conjectures at the end of the paper.


## 1. Introduction

In this paper we look at polynomials arising from statistics on the classes of involutions and involutions with no fixed points in the symmetric group.

Let $S_{n}$ be the symmetric group on $[1, n]$. Call $\operatorname{Des}(\sigma):=\{i: 1 \leq i<$ $n$ and $\left.\sigma_{i}>\sigma_{i+1}\right\}$ the descent set of $\sigma \in S_{n}$ and the number of descents is denoted $d(\sigma):=|\operatorname{Des}(\sigma)|$. We further define $d_{i}(\sigma):=|\{j \geq i: j \in \operatorname{Des}(\sigma)\}|$, the partial descents of $\sigma$ for $1 \leq i<n$. The major index of $\sigma$ is $\operatorname{maj}(\sigma):=$ $\sum_{i \in \operatorname{Des}(\sigma)} i$ and the number of inversions is $\operatorname{inv}(\sigma):=\mid\left\{1 \leq i<j \leq n: \sigma_{i}>\right.$ $\left.\sigma_{j}\right\} \mid$. The number of excedances is $\operatorname{exc}(\sigma):=\left|\left\{1 \leq i \leq n: \sigma_{i}>i\right\}\right|$ and weak excedances is $\operatorname{wexc}(\sigma):=\left|\left\{1 \leq i \leq n: \sigma_{i} \geq i\right\}\right|$. Let fix $(\sigma)$ and $\operatorname{trans}(\sigma)$ denote the number of fixed points and transpositions of $\sigma$, respectively. We use the notation $\left[x^{i}\right] P(x)$ for the coefficient of $x^{i}$ in the polynomial $P(x)$.

For a statistic stat : $S_{n} \rightarrow \mathbf{N}_{0}$, define the polynomials

$$
\mathcal{I}_{n}^{\text {stat }}(x):=\sum_{\sigma \in I_{n}} x^{\text {stat }(\sigma)}, \quad \mathcal{J}_{n}^{\text {stat }}(x):=\sum_{\sigma \in J_{n}} x^{\text {stat }(\sigma)},
$$

where $I_{n}:=\left\{\sigma \in S_{n}: \sigma^{2}=\mathrm{id}\right\}$ and $J_{n}:=\left\{\sigma \in I_{n}:\right.$ fix $\left.(\sigma)=0\right\}$. For an arbitrary collection $S_{n}^{\prime} \subseteq S_{n}$, the sequence of coefficients of $\sum_{\pi \in S_{n}^{\prime}} x^{d(\pi)}$ is termed the Eulerian distribution of $S_{n}^{\prime}$. The results in this paper are motivated by

Conjecture 1.1 (Brenti [3]). The Eulerian distribution of $I_{n}$ is log-concave.

[^0]We propose further conjectures concerning statistics on different classes of involutions in the final section.

## 2. Involutions in the Symmetric Group

### 2.1. The excedances statistic.

Theorem 2.1. The coefficients of the polynomial $\mathcal{I}_{n}^{e x c}(x)$ are log-concave.
Proof. The number $\operatorname{exc}(\sigma)$ is precisely the number of 2-cycles in an involution, so we have

$$
\begin{equation*}
\mathcal{I}_{n}^{e x c}(x)=\sum_{k=0}^{\lfloor n / 2\rfloor} \frac{n!}{k!(n-2 k)!}\left(\frac{x}{2}\right)^{k} . \tag{2.1}
\end{equation*}
$$

It is an easy exercise to show log-concavity for $0 \leq j<\lfloor n / 2\rfloor$ since we have a direct expression for the coefficients.

Note that the polynomials $\mathcal{I}_{n}^{e x c}(x)$ are closely related to the Hermite polynomials $h_{n}(x)$, whereby

$$
\sum_{n \geq 0} \frac{h_{n}(x) t^{n}}{n!}=\exp \left(t x-t^{2} / 2\right)
$$

via the equation $\mathcal{I}_{n}^{e x c}(x)=(-x)^{n} h_{n}(-1 / 2 x)$. The Hermite polynomials are known to be real-rooted (see for example Stanley [10, p. 505]).

The Schützenberger involution on tableaux, $T \rightarrow \operatorname{evac}(T)$, maps involutions to involutions and $\operatorname{wexc}(\operatorname{evac}(\sigma))=n-\operatorname{exc}(\sigma)$, since $\operatorname{evac}(\sigma)_{i}=$ $n+1-\sigma_{n+1-i}$, so that $\mathcal{I}_{n}^{\text {wexc }}(x)=x^{n} \mathcal{I}_{n}^{\text {exc }}\left(x^{-1}\right)$, hence

Corollary 2.2. The coefficients of the polynomial $\mathcal{I}_{n}^{\text {wexc }}(x)$ are log-concave.
2.2. The descents and major index statistics. In the spirit of Adin et. al. [1], we define

$$
\mathcal{G}_{n}\left(x_{1}, \ldots, x_{n-1}\right):=\sum_{\sigma \in I_{n}} x_{1}^{d_{1}(\sigma)} x_{2}^{d_{2}(\sigma)} \cdots x_{n-1}^{d_{n-1}(\sigma)}
$$

Theorem 2.3. The polynomial $\mathcal{G}_{n}\left(x_{1}, \ldots, x_{n-1}\right)$ satisfies

$$
\mathcal{G}_{n}\left(x_{1}, \ldots, x_{n-1}\right)=x_{1}^{n-1} x_{2}^{n-2} \cdots x_{n-1} \mathcal{G}_{n}\left(x_{1}^{-1}, \ldots, x_{n-1}^{-1}\right) .
$$

Proof. If $\sigma \in I_{n}$ then the reading and insertion tableau associated with $\sigma$ under Robinson-Schensted correspondence (Stanley [9, Ch. 7]) are identical. That is, there is a bijection between $I_{n}$ and all standard Young tableaux (SYT) on $[1, n]$.

Let $\sigma \in I_{n}$ with associated SYT $T$. The set $\operatorname{Des}(\sigma)$ corresponds to those entries $i$ in the tableau $T$ such that $(i+1)$ is below and weakly to the left of $i$. Let $T^{\perp}$ be the tableau $T$ reflected on its main diagonal. Notice that if $(i+1)$ is below and weakly to the left of $i$ in $T$, then $(i+1)$ is to the right of and weakly above $i$ in $T^{\perp}$. The bijection between the class of SYT on $[1, n]$ and involutions $I_{n}$ shows that to $T^{\perp}$ there corresponds a unique involution $\sigma^{\perp} \in I_{n}$, and has the property that $\left\{\operatorname{Des}(\sigma), \operatorname{Des}\left(\sigma^{\perp}\right)\right\}$ is a partition of the set $[1, n-1]$. In this manner, the reflection operation is an involution on involutions.

It follows that

$$
\begin{aligned}
d_{i}\left(\sigma^{\perp}\right) & =\left|\left\{j \geq i: j \in \operatorname{Des}\left(\sigma^{\perp}\right)\right\}\right| \\
& =|\{j \geq i: j \notin \operatorname{Des}(\sigma)\}| \\
& =n-i-|\{j \geq i: j \in \operatorname{Des}(\sigma)\}| \\
& =n-i-d_{i}(\sigma) .
\end{aligned}
$$

We have shown that if $\sigma \in I_{n}$, then there is a unique $\sigma^{\perp} \in I_{n}$ such that $\left(d_{1}\left(\sigma^{\perp}\right), \ldots, d_{n-1}\left(\sigma^{\perp}\right)\right)=\left(n-1-d_{1}(\sigma), \ldots, 1-d_{n-1}(\sigma)\right)$.

Both polynomials $\mathcal{I}_{n}^{d}(q)$ and $\mathcal{I}_{n}^{\text {maj }}(q)$ are instances of the $\mathcal{G}$ polynomial since $\mathcal{I}_{n}^{d}(q)=\mathcal{G}_{n}(q, 1, \ldots, 1)$ and $\mathcal{I}_{n}^{\text {maj }}(q)=\mathcal{G}_{n}(q, q, \ldots, q)$. Comparing coefficients on both sides of the symmetric $\mathcal{G}$ relation yields
Corollary 2.4. The polynomials $\mathcal{I}_{n}^{d}(t)$ and $\mathcal{I}_{n}^{\text {maj }}(t)$ are symmetric.
Symmetry of the polynomials $\mathcal{I}_{n}^{d}(x)$ and $\mathcal{J}_{n}^{d}(t)$ was conjectured by Dumont and first proven by Strehl 11, using a method similar to that of the previous theorem for the coefficients of $\mathcal{I}_{n}^{d}(x)$. A separate argument was used to prove symmetry of $\mathcal{J}_{n}^{d}(t)$ because for $\sigma \in J_{n}$, it is not necessarily true that $\sigma^{\perp} \in J_{n}$. Theorem [2.3] allows us to show symmetry of the joint distribution of ( $d, m a j$ ) on $I_{n}$ since $\sum_{\sigma \in I_{n}} t^{d(\sigma)} q^{\operatorname{maj}(\sigma)}=\mathcal{G}_{n}(t q, q, \ldots, q)$.
Corollary 2.5. The polynomial

$$
\mathcal{I}_{n}^{d, m a j}(t, q)=\sum_{\sigma \in I_{n}} t^{d(\sigma)} q^{m a j(\sigma)}
$$

is symmetric in the sense that $\left[t^{i} q^{j}\right] \mathcal{I}_{n}^{d, m a j}(t, q)=\left[t^{n-1-i} q^{\binom{n}{2}-j}\right] \mathcal{I}_{n}^{d, m a j}(t, q)$.
Hultman [8] recently proved that for any finite Coxeter system ( $W, S$ ), the associated descent polynomial $\sum_{w} t^{d_{W}(w)}$ is symmetric where the sum ranges over all $w \in W$ with $w^{2}=\operatorname{id}_{w}$. Désarménien and Foata 7 use an elegant method involving Schur functions to derive the generating function

$$
\begin{equation*}
\sum_{n \geq 0} \frac{H_{n}\left(z_{1}, z_{2}, t, q\right) u^{n}}{(t ; q)_{n}}=\sum_{r \geq 0} t^{r} \frac{1}{\left(z_{1} u ; q\right)_{r+1}} \prod_{0 \leq i<j \leq r} \frac{1}{1-u^{2} z_{2} q^{i+j}} \tag{2.2}
\end{equation*}
$$

where $H_{n}\left(z_{1}, z_{2}, t, q\right):=\sum_{\sigma \in I_{n}} z_{1}^{f i x(\sigma)} z_{2}^{\operatorname{trans}(\sigma)} t^{d(\sigma)} q^{\operatorname{maj}(\sigma)},(a ; q)_{0}=1$ and $(a ; q)_{n}=(1-a)(1-a q) \cdots\left(1-a q^{n-1}\right)$. The generating functions for the polynomials $\mathcal{I}_{n}^{d}(t), \mathcal{I}_{n}^{\text {maj }}(q)$ are immediate from this:

$$
\begin{align*}
\sum_{n \geq 0} \frac{\mathcal{I}_{n}^{d}(t) u^{n}}{(1-t)^{n}} & =\sum_{r \geq 0} t^{r}\left(\frac{1}{(1-u)^{r+1}\left(1-u^{2}\right)^{r(r+1) / 2}}\right) ;  \tag{2.3}\\
\sum_{n \geq 0} \frac{\mathcal{J}_{n}^{d}(t) u^{n}}{(1-t)^{n}} & =\sum_{r \geq 0} t^{r}\left(\frac{1}{\left(1-u^{2}\right)^{r(r+1) / 2}}\right) ;  \tag{2.4}\\
\sum_{n \geq 0} \frac{\mathcal{I}_{n}^{m a j}(q) u^{n}}{(q ; q)_{n}} & =\sum_{r \geq 0} \frac{1}{(u ; q)_{r+1}} \prod_{0 \leq i<j \leq r} \frac{1}{\left(1-u^{2} q^{i+j}\right)} . \tag{2.5}
\end{align*}
$$

By extracting the appropriate coefficients, we now show partial unimodality of $\mathcal{I}_{n}^{d}(q)$ and $\mathcal{J}_{n}^{d}(q)$. The onerous aspect of proving total unimodality using
these direct expressions seems to be the appearance of both $r$ and $\binom{r+1}{2}$ within binomial terms.

Theorem 2.6. For all $1 \leq i \leq n^{0.925} / 10$, $\left[t^{i}\right] \mathcal{J}_{n}^{d}(t)<\left[t^{i+1}\right] \mathcal{J}_{n}^{d}(t)$ and $\left[t^{n+1-i}\right] \mathcal{J}_{n}^{d}(t)>\left[t^{n+2-i}\right] \mathcal{J}_{n}^{d}(t)$.

Proof. Extracting the coefficient of $u^{n}$ in Equation (2.4), one finds
$\mathcal{J}_{n}^{d}(t)=\sum_{p=1}^{n} \alpha_{n, p} t^{p}=\sum_{p=1}^{n} t^{p}\left\{\sum_{k=0}^{p-1}(-1)^{k}\binom{n+1}{k}\binom{\binom{p-k+1}{2}+n / 2-1}{n / 2}\right\}$.
Inverting this gives

$$
f_{n}(p):=\binom{\binom{p+1}{2}+n / 2-1}{n / 2}=\sum_{i=0}^{p-1}\binom{n+i}{n} \alpha_{n, p-i} .
$$

For $p \geq 2$,

$$
\begin{aligned}
& f_{n}(p)-f_{n}(p-1) \\
& =\alpha_{n, p}-\alpha_{n, p-1}+\sum_{i=1}^{p-1}\binom{n+i}{i} \alpha_{n, p-i}-\sum_{i=1}^{p-2}\binom{n+i}{i} \alpha_{n, p-1-i} \\
& =\alpha_{n, p}-\alpha_{n, p-1}+\binom{n+p-1}{p-1} \alpha_{n, 1}+\sum_{i=1}^{p-2}\binom{n+i}{i}\left(\alpha_{n, p-i}-\alpha_{n, p-1-i}\right) \\
& \leq \alpha_{n, p}-\alpha_{n, p-1}+\binom{n+p-1}{p-1} \alpha_{n, 1}+(n+1) \sum_{i=1}^{p-2}\binom{n+i}{i}\left(\alpha_{n, p-i}-\alpha_{n, p-1-i}\right) \\
& \leq \alpha_{n, p}-\alpha_{n, p-1}+(n+1)\left(f_{n}(p-1)-f_{n}(p-2)\right) .
\end{aligned}
$$

Thus $\alpha_{n, p}-\alpha_{n, p-1} \geq f_{n}(p)-f_{n}(p-1)-(n+1)\left(f_{n}(p-1)-f_{n}(p-2)\right)$. The right hand side of the previous inequality is positive for $p$ not too large.
Notice that

$$
\begin{aligned}
\frac{f_{n}(p)}{f_{n}(p-1)} & \geq\left(1+\frac{n}{p^{2}+p-2}\right)^{p} \\
& \geq\left(1+\frac{n+2}{2 p^{2}}\right)^{p}
\end{aligned}
$$

which, in turn, is bounded below by $n+2$ when $p \leq n^{0.925} / 10$. The second inequality follows from symmetry as shown in Strehl [11.

Theorem 2.7. For all $1 \leq k \leq 0.175 n^{0.931},\left[t^{k-1}\right] \mathcal{I}_{n}^{d}(t)<\left[t^{k}\right] \mathcal{I}_{n}^{d}(t)$ and $\left[t^{n-1-k}\right] \mathcal{I}_{n}^{d}(t)>\left[t^{n-k}\right] \mathcal{I}_{n}^{d}(t)$.

Proof. Extracting the coefficient of $u^{n}$ in Equation (2.3) we find

$$
\mathcal{I}_{n}^{d}(t)=\sum_{k=0}^{n-1} \beta_{n, k} t^{k}=\sum_{k=0}^{n-1} t^{k}\left\{\sum_{j=0}^{k}\binom{n+1}{j}(-1)^{j} \gamma(n, k-j)\right\}
$$

where $\gamma(n, 0) ;=1$ and $\gamma(n, r):=\sum_{i=0}^{\lfloor n / 2\rfloor}\binom{n-2 i+r}{r}\left(\begin{array}{c}i+\binom{r+1}{i}-1\end{array}\right)$ for $r>0$. Thus for $0 \leq k \leq n$,

$$
\gamma(n, k)=\sum_{i=0}^{k}\binom{n+i}{i} \beta_{n, k-i}
$$

and so

$$
\begin{aligned}
& \gamma(n, k)-\gamma(n, k-1) \\
&= \sum_{i=0}^{k-1}\binom{n+i}{i}\left(\beta_{n, k-i}-\beta_{n, k-1-i}\right)+\binom{n+k}{k} \\
&= \beta_{n, k}-\beta_{n, k-1}+\sum_{i=0}^{k-2}\binom{n+i+1}{i+1}\left(\beta_{n, k-1-i}-\beta_{n, k-2-i}\right)+\binom{n+k}{k} \\
&< \beta_{n, k}-\beta_{n, k-1}+(n+1) \sum_{i=0}^{k-2}\binom{n+i}{i}\left(\beta_{n, k-1-i}-\beta_{n, k-2-i}\right)+\binom{n+k}{k} \\
&= \beta_{n, k}-\beta_{n, k-1}+(n+1)\left(\gamma(n, k-1)-\gamma(n, k-2)-\binom{n+k-1}{k-1}\right) \\
& \quad+\binom{n+k}{k} \\
&< \beta_{n, k}-\beta_{n, k-1}+(n+1) \gamma(n, k-1) .
\end{aligned}
$$

It suffices to show that for $n$ and $k$ as stated in the theorem, $\gamma(n, k)>$ $(n+2) \gamma(n, k-1)$. One may also write

$$
\gamma(n, k)=\left[u^{n}\right] \frac{(1+u)^{k+1}}{\left.\left(1-u^{2}\right)^{(k+2}\right)}
$$

and since $n>k+1$ we have

$$
\begin{aligned}
\gamma(n, k) & =\sum_{i=0}^{\lfloor k+1 / 2\rfloor}\binom{k+1}{n-2\lfloor n / 2\rfloor+2 i}\left[u^{2(\lfloor n / 2\rfloor-i)}\right] \frac{1}{\left(1-u^{2}\right)^{\binom{k+2}{2}}} \\
& >\sum_{i=0}^{\lfloor k / 2\rfloor}\binom{k}{n-2\lfloor n / 2\rfloor+2 i}\left[u^{2(\lfloor n / 2\rfloor-i)}\right] \frac{1}{\left(1-u^{2}\right)^{\binom{k+2}{2}} .}
\end{aligned}
$$

Now for all $m \geq(n-k-2) / 2$,

$$
\begin{aligned}
\binom{\binom{k+2}{2}+m}{m+1} & >\left(1+\frac{2 m}{(k+1)(k+2)}\right)^{k+1}\binom{\binom{k+1}{2}+m}{m+1} \\
& >(n+2)\binom{\binom{k+1}{2}+m}{m+1} .
\end{aligned}
$$

for $k \leq 0.175 n^{0.931}$, hence

$$
\begin{aligned}
\gamma(n, k) & >(n+2) \sum_{i=0}^{\lfloor k / 2\rfloor}\binom{k}{n-2\lfloor n / 2\rfloor+2 i}\left[u^{2(\lfloor n / 2\rfloor-i)}\right] \frac{1}{\left(1-u^{2}\right)^{(k+1)}} \\
& =(n+2) \gamma(n, k-1),
\end{aligned}
$$

giving the first inequality. Again, symmetry of the $\mathcal{I}_{n}^{d}(t)$ polynomial yields the second inequality.
2.3. The inversions statistic. The generating function for the inversions statistic on involutions is intimately related to the $q$-Hermite polynomials, as studied by Désarménien [6]. Let $a_{n}(k, j)$ be the number of involutions in $I_{n}$ with $k$ fixed points and $j$ inversions, and define $Z_{n}(x, q):=$ $\sum_{k, j} a_{n}(k, j) q^{j} x^{k}$. Désarménien [6, Eqns. 3.10,3.11] showed

$$
Z_{n+1}(x, q)=x Z_{n}(x, q)+q\left(\frac{1-q^{2 n}}{1-q^{2}}\right) Z_{n-1}(x, q)
$$

for all $n>1$ with $Z_{0}(x, q)=1$ and $Z_{1}(x, q)=x$. Setting $x=1,0$, yields the following proposition.
Proposition 2.8. For all $n \geq 0$,

$$
\mathcal{I}_{n+2}^{i n v}(q)=\mathcal{I}_{n+1}^{i n v}(q)+q\left(\frac{1-q^{2(n+1)}}{1-q^{2}}\right) \mathcal{I}_{n}^{i n v}(q),
$$

where $\mathcal{I}_{0}^{\text {inv }}(q), \mathcal{I}_{1}^{\text {inv }}(q):=1$ and for $n \geq 0$,

$$
\mathcal{J}_{n+2}^{i n v}(q)=q\left(\frac{1-q^{2(n+1)}}{1-q^{2}}\right) \mathcal{J}_{n}^{i n v}(q),
$$

where $\mathcal{J}_{0}^{\text {inv }}(q)=1$.
The above recurrences can also be derived in a straightforward manner using a special case of Equation (3.1). The coefficients of $\mathcal{I}_{n}^{i n v}(q)$ are neither log-concave nor unimodal (see Figure 1) but the recursion in the previous proposition admits a solution as a matrix product, which may be of benefit in approaching Conjecture 4.1(ii).
Proposition 2.9. Let $g_{i}(q):=\sum_{j=0}^{i-2} q^{1+2 j}$ and $\mathbf{A}_{n}(q):=\prod_{i=2}^{n}\left(\begin{array}{cc}1 & 1 \\ g_{i}(q) & 0\end{array}\right)$,
then $\mathcal{I}_{n}^{\text {inv }}(q)=\mathbf{A}_{n}(q)_{1,1}+\mathbf{A}_{n}(q)_{2,1}$.
Proof. For all $n \geq 2$, we may write

$$
\mathcal{I}_{n}^{i n v}(q)=\prod_{i=1}^{n} \mathcal{Y}_{i}(q)
$$

where $\mathcal{Y}_{n}(q):=\mathcal{I}_{n}^{i n v}(q) / \mathcal{I}_{n-1}^{i n v}(q)$ and $\mathcal{I}_{0}^{i n v}(q), \mathcal{I}_{1}^{i n v}(q):=1$. From the first recurrence in Proposition 2.8 the polynomial $\mathcal{Y}_{n}(q)$ satisfies the recurrence $\mathcal{Y}_{n}(q)=1+g_{n-1}(q) / \mathcal{Y}_{n-1}(q)$ for all $n \geq 2$ where $g_{n}(q):=q+q^{3}+\ldots+$ $q^{2 n-3}$. Using this, the product $\mathcal{Y}_{i} \mathcal{Y}_{i+1} \cdots \mathcal{Y}_{n}$ may be written in the form $\alpha_{i}(q) \mathcal{Y}_{i}(q)+\beta_{i}(q)$. It is easily seen that $\alpha_{n}(q)=1, \beta_{n}(q)=0$ and

$$
\binom{\alpha_{i}(q)}{\beta_{i}(q)}=\left(\begin{array}{cc}
1 & 1 \\
g_{i+1}(q) & 0
\end{array}\right)\binom{\alpha_{i+1}(q)}{\beta_{i+1}(q)} .
$$

Thus we have $\mathcal{I}_{n}^{\text {inv }}(q)=\alpha_{1}(q) \mathcal{Y}_{1}(q)+\beta_{1}(q)=\alpha_{1}(q)+\beta_{1}(q)$, since $\mathcal{Y}_{1}(q)=1$, and

$$
\binom{\alpha_{1}(q)}{\beta_{1}(q)}=\left(\begin{array}{cc}
1 & 1 \\
g_{2}(q) & 0
\end{array}\right)\left(\begin{array}{cc}
1 & 1 \\
g_{3}(q) & 0
\end{array}\right) \cdots\left(\begin{array}{cc}
1 & 1 \\
g_{n}(q) & 0
\end{array}\right)\binom{1}{0} .
$$

Theorem 2.10. The coefficients of the polynomial $\mathcal{J}_{n}^{\text {inv }}(q)$ are log-concave.
Proof. Solving the second recurrence in Proposition [2.8 we get:

$$
\mathcal{J}_{2 m}^{i n v}(q)=q^{m} \prod_{i=1}^{m-1} \frac{1-q^{2(2 i+1)}}{1-q^{2}}
$$

Set $u=q^{2}$ and notice that the sequence of non-zero coefficients in $\mathcal{J}_{n}^{\text {inv }}(q)$ is the same as $\prod_{i=1}^{m-1} \frac{1-u^{2 i+1}}{1-u}$. The coefficients of the polynomials ( $1-$ $\left.u^{2 i+1}\right) /(1-u)$ are non-negative log-concave sequences with no internal zero coefficients. Thus using Stanley [10, Prop. 2], the product of all such polynomials will also be log-concave with no internal zero coefficients.

## 3. Involutions in Coxeter groups of types $B$ and $D$

In this section we give recursive expressions for the inversion polynomials of involutions for Coxeter groups of types $B$ and $D$. We use the notation of Björner and Brenti [2].

Coxeter groups of type $B$, the 'signed permutations', are defined as follows: let $S_{n}^{B}$ be the group of all bijections $\pi$ on the set $[ \pm n] \backslash\{0\}$ such that $\pi(-a)=-\pi(a)$ for all $a \in[ \pm n]$. For $\pi \in S_{n}^{B}$, define

$$
\begin{aligned}
& N_{1}(\pi(1), \ldots, \pi(n)):=|\{1 \leq i \leq n: \pi(i)<0\}| \\
& N_{2}(\pi(1), \ldots, \pi(n)):=|\{1 \leq i<j \leq n: \pi(i)+\pi(j)<0\}| .
\end{aligned}
$$

Let $S_{n}^{D}$ be the subgroup of $S_{n}^{B}$ consisting of all signed permutations $\pi \in S_{n}^{B}$ such that there are an even number of negative entries in the window of $\pi$, i.e. $S_{n}^{D}:=\left\{\pi \in S_{n}^{B}: N_{1}(\pi) \equiv 0(\bmod 2)\right\}$. For completeness let us also define those signed permutations containing an odd number of negative signs in the window of $\pi, S_{n}^{O}=S_{n}^{B} \backslash S_{n}^{D}$.

The inversions statistics on $S_{n}^{B}$ and $S_{n}^{D}$ are defined slightly differently to inv on $S_{n}$. From [2] Equations (8.1) and (8.18)],

$$
\begin{aligned}
\operatorname{inv}_{B}(\pi):= & \operatorname{inv}(\pi(1), \ldots, \pi(n))+N_{1}(\pi(1), \ldots, \pi(n)) \\
& +N_{2}(\pi(1), \ldots, \pi(n)) \\
\operatorname{inv}_{D}(\pi):= & \operatorname{inv}(\pi(1), \ldots, \pi(n))+N_{2}(\pi(1), \ldots, \pi(n)) .
\end{aligned}
$$

Let us mention that in the symmetric group setting,

$$
\begin{aligned}
\sum_{\pi \in S_{n}^{B}} q^{i n v_{B}(\pi)} & =[2]_{q}[4]_{q} \cdots[2 n]_{q} \\
\sum_{\pi \in S_{n}^{D}} q^{i n v_{D}(\pi)} & =[2]_{q}[4]_{q} \cdots[2 n-2]_{q}[n]_{q}
\end{aligned}
$$

where $[i]_{q}:=1+q+q^{2}+\ldots+q^{i-1}$ (see [2, Theorem 7.1.5.])
Define $I_{n}^{B}:=\left\{\pi \in S_{n}^{B}: \pi^{2}=\mathrm{id}\right\}, I_{n}^{D}:=\left\{\pi \in S_{n}^{D}: \pi^{2}=\mathrm{id}\right\}$ and $I_{n}^{O}:=I_{n}^{B} \backslash I_{n}^{D}$. Let

$$
\mathcal{I B}_{n}(q):=\sum_{\pi \in I_{n}^{B}} q^{i n v_{B}(\pi)},
$$

with $\mathcal{I D}_{n}(q)$ and $\mathcal{I} \mathcal{O}_{n}(q)$ similarly defined. To aid the proof of the following two theorems, we introduce some notation concerning the recursive construction of these signed permutations.

Let $\pi \in I_{n}^{B}$ and denote by $\bar{\pi}^{(n+1, n+1)}$ the signed permutation $\pi^{\prime} \in I_{n+1}^{B}$ such that $\pi^{\prime}(i)=\pi(i)$, for $1 \leq i, \leq n$ and $\pi^{\prime}(n+1)=n+1$. Similarly let $\bar{\pi}^{(-(n+1), n+1)}$ be the signed permutation $\pi^{\prime} \in I_{n+1}^{B}$ such that $\pi^{\prime}(i)=\pi(i)$ for $1 \leq i \leq n$ and $\pi^{\prime}(n+1)=-(n+1)$.

For $\pi \in I_{n}^{B}$ and $k \in[ \pm(n+1)]-\{0\}$, let $\bar{\pi}^{(k, n+2)}$ be the signed permutation $\pi^{\prime} \in I_{n+2}^{B}$ such that

- $\pi^{\prime}(|k|)=(n+2) \operatorname{sgn}(k), \pi^{\prime}(n+2)=k$,
- for all $1 \leq i \leq n$,

$$
\pi^{\prime}(i+\mathbf{1}[i \geq|k|])=\pi(i)+\operatorname{sgn}(\pi(i)) \mathbf{1}[|\pi(i)| \geq|k|]
$$

where $\operatorname{sgn}(a)=+1$ if $a>0$ and -1 otherwise. Consequently $I_{n+2}^{B}, I_{n+2}^{D}$ and $I_{n+2}^{O}$ may be constructed recursively,

$$
\left.\begin{array}{rl}
I_{n+2}^{B}= & \biguplus_{\substack{\pi \in I_{n+1}^{B}}}\left\{\bar{\pi}^{(n+2, n+2)}, \bar{\pi}^{(-(n+2), n+2)}\right\} \uplus \\
I_{n+2}^{D}= & \biguplus_{\substack{k=1 \\
\pi \in I_{n}^{B}}}^{\substack{n+1}}\left\{\bar{\pi}^{(k, n+2)}, \bar{\pi}^{(-k, n+2)}\right\} \\
& \left.\biguplus_{\substack{k=1 \\
\pi \in I_{n+1}^{D}}}^{\substack{n+1}}\left\{\bar{\pi}^{(n+2, n+2)}\right\} \uplus \bigcup_{\pi}^{(k, n+2)}, \bar{\pi}^{(-k, n+2)}\right\} \\
I_{n+2}^{O}= & \biguplus_{\substack{\pi \\
\pi \in I_{n+1}^{O}}}\left\{\bar{\pi}^{(-(n+2), n+2)}\right\} \uplus \\
& \biguplus_{\substack{k=1 \\
n+1}}\left\{\bar{\pi}^{(k, n+2, n+2)}\right\} \uplus I_{n}^{O}
\end{array} \biguplus_{\pi \in I_{n+1}^{D}}\left\{\bar{\pi}^{(-(n+2), n+2)}\right\} \uplus\right)
$$

Theorem 3.1. For all $n \geq 2$,

$$
\mathcal{I B}_{n+2}(q)=\left(1+q^{2 n+3}\right) \mathcal{I B}_{n+1}(q)+\frac{q\left(1+q^{2}\right)\left(1-q^{2(n+1)}\right)}{1-q^{2}} \mathcal{I B}_{n}(q)
$$

with initial polynomials $\mathcal{I B}_{2}(q)=1+2 q+2 q^{3}+q^{4}, \mathcal{I B}_{3}(q)=1+3 q+q^{2}+$ $3 q^{3}+2 q^{4}+2 q^{5}+3 q^{6}+q^{7}+3 q^{8}+q^{9}$.
Proof. Using Equation (3.1),

$$
\begin{aligned}
\mathcal{I B}_{n+2}(q)= & \sum_{\pi \in I_{n+1}^{B}} q^{i n v_{B}\left(\bar{\pi}^{(n+2, n+2)}\right)}+q^{i n v_{B}\left(\bar{\pi}^{(-(n+2), n+2)}\right)} \\
& +\sum_{k=1}^{n+1} \sum_{\pi \in I_{n}^{B}} q^{i n v_{B}\left(\bar{\pi}^{(k, n+2)}\right)}+q^{i n v_{B}\left(\bar{\pi}^{(-k, n+2)}\right)}
\end{aligned}
$$

If $\pi \in I_{n+1}^{B}$, then $\operatorname{inv}_{B}\left(\bar{\pi}^{(n+2, n+2)}\right)=\operatorname{inv}_{B}(\pi(1), \ldots, \pi(n+1), n+2)=$ $i n v_{B}(\pi)$ and $i n v_{B}\left(\bar{\pi}^{(-(n+2), n+2)}\right)=i n v_{B}(\pi)+2 n+3$. Similarly if $\pi \in I_{n}^{B}$ and $1 \leq k \leq n+1$, then $i n v_{B}\left(\bar{\pi}^{(k, n+2)}\right)=i n v_{B}(\pi)+2 n+3-2 k$ and $i n v_{B}\left(\bar{\pi}^{(-k, n+2)}\right)=i n v_{B}(\pi)+2 k+1$. Hence

$$
\begin{aligned}
\mathcal{I B}_{n+2}(q)= & \sum_{\pi \in I_{n+1}^{B}} q^{i n v_{B}(\pi)}+q^{i n v_{B}(\pi)+2 n+3} \\
& +\sum_{k=1}^{n+1} \sum_{\pi \in I_{n}^{B}} q^{i n v_{B}(\pi)+2 n-2 k+3}+q^{i n v_{B}(\pi)+2 k+1} \\
= & \left(1+q^{2 n+3}\right) \mathcal{I B}_{n+1}(q)+\mathcal{I B}_{n}(q) \sum_{k=1}^{n+1}\left(q^{2 n-2 k+3}+q^{2 k+1}\right)
\end{aligned}
$$

We may express $\mathcal{I B}_{n}(q)$ in a somewhat closed form, as was done in Proposition 2.9, for all $n \geq 3, \mathcal{I B}_{n}(q)=\left(\mathbf{V}_{n}(q)_{1,1}+\mathbf{V}_{n}(q)_{2,1}\right)(1+2 q+$ $2 q^{3}+q^{4}$ ) where

$$
\mathbf{V}_{n}(q)=\prod_{i=3}^{n}\left(\begin{array}{cc}
u_{i}(q) & 1 \\
v_{i}(q) & 0
\end{array}\right)
$$

and $u_{i}(q):=1+q^{2 i-1}, v_{i}(q):=\left(1+q^{2}\right)\left(1-q^{2(i-1)}\right) /\left(1-q^{2}\right)$.
Theorem 3.2. For all $n \geq 2$,

$$
\begin{aligned}
& \mathcal{I} \mathcal{D}_{n+1}(q)=\mathcal{I D}_{n}(q)+q^{2 n} \mathcal{I} \mathcal{O}_{n}(q)+\left(q^{2(n-1)}+\frac{q\left(1-q^{2 n}\right)}{1-q^{2}}\right) \mathcal{I} \mathcal{D}_{n-1}(q) \\
& \mathcal{I} \mathcal{O}_{n+1}(q)=\mathcal{I} \mathcal{O}_{n}(q)+q^{2 n} \mathcal{I} \mathcal{D}_{n}(q)+\left(q^{2(n-1)}+\frac{q\left(1-q^{2 n}\right)}{1-q^{2}}\right) \mathcal{I} \mathcal{O}_{n-1}(q)
\end{aligned}
$$

with initial polynomials $\mathcal{I D}_{2}(q), \mathcal{I} \mathcal{O}_{2}(q)=1+q+q^{2}, \mathcal{I D}_{3}(q)=\left(1+q+q^{2}+\right.$ $\left.q^{3}\right)\left(1+q^{3}\right)+2 q$ and $\mathcal{I} \mathcal{O}_{3}(q)=\left(1+q+q^{2}+q^{3}\right)\left(1+q^{3}\right)+2 q^{5}$.
Proof. Using Equation (3.2),

$$
\begin{aligned}
\mathcal{I} \mathcal{D}_{n+2}(q)= & \sum_{\pi \in I_{n+1}^{D}} q^{i n v_{D}\left(\bar{\pi}^{(n+2, n+2)}\right)}+\sum_{\pi \in I_{n+1}^{O}} q^{i n v_{D}\left(\bar{\pi}^{(-(n+2), n+2)}\right)} \\
& +\sum_{k=1}^{n+1} \sum_{\pi \in I_{n}^{D}} q^{i n v_{D}\left(\bar{\pi}^{(k, n+2)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(-k, n+2)}\right)}
\end{aligned}
$$

If $\pi \in I_{n+1}^{D}, I_{n+1}^{O}$, then $i n v_{D}\left(\bar{\pi}^{(n+2, n+2)}\right)=i n v_{D}(\pi)$ and $i n v_{D}\left(\bar{\pi}^{(-(n+2), n+2)}\right)=$ $i n v_{D}(\pi)+2(n+1)$. Also if $\pi \in I_{n}^{D}$, then $i n v_{D}\left(\bar{\pi}^{(k, n+2)}\right)=2 n-2 k+3+$ $i n v_{D}(\pi)$ and $i n v_{D}\left(\bar{\pi}^{(-k, n+2)}\right)=i n v_{D}(\pi)+2 n$. Hence,

$$
\begin{aligned}
\mathcal{I} \mathcal{D}_{n+2}(q)= & \sum_{\pi \in I_{n+1}^{D}} q^{i n v_{D}(\pi)}+\sum_{\pi \in I_{n+1}^{O}} q^{i n v_{D}(\pi)+2(n+1)} \\
& +\sum_{k=1}^{n+1} \sum_{\pi \in I_{n}^{D}} q^{i n v_{D}(\pi)}\left(q^{2 n-2 k+3}+q^{2 n}\right)
\end{aligned}
$$

The second recurrence is derived in the same manner by using Equation (3.3).

Let $J_{n}^{D} \subset I_{n}^{D}$ denote the class of all signed permutations such that $\pi(i) \neq \pm i$ for all $i \in[1, n]$ and consider the generating function $\mathcal{J D}_{n}(q):=$ $\sum_{\pi \in J_{n}^{D}} q^{i n v_{D}(\pi)}$.

Theorem 3.3. For all even $n \geq 2$,

$$
\mathcal{J}_{n}(q)=2 q^{n / 2} \prod_{i=1}^{n / 2-1} \frac{\left(1+q^{4 i}\right)\left(1-q^{4 i+2}\right)}{1-q^{2}}
$$

Proof. Since $J_{n}^{D}$ is a subclass of $I_{n}^{D}$ and from the characterization in Equation (3.2), one has

$$
\begin{aligned}
& \mathcal{J D}_{n+4}(q)= \sum_{\pi \in J_{n+2}^{D}} q^{i n v_{D}\left(\bar{\pi}^{(n+3, n+4)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(-(n+3), n+4)}\right)} \\
&+\sum_{1 \leq i<j \leq n+2} \sum_{\pi \in J_{n}^{D}}\left(q^{i n v_{D}\left(\bar{\pi}^{(i, n+3)(j, n+4)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(i, n+4)(j, n+3)}\right)}\right. \\
& \quad+q^{i n v_{D}\left(\bar{\pi}^{(-i, n+3)(j, n+4)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(-i, n+4)(j, n+3)}\right)} \\
& \quad \quad+q^{i n v_{D}\left(\bar{\pi}^{(i, n+3)(-j, n+4)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(i, n+4)(-j, n+3)}\right)} \\
&\left.\quad \quad+q^{i n v_{D}\left(\bar{\pi}^{(-i, n+3)(-j, n+4)}\right)}+q^{i n v_{D}\left(\bar{\pi}^{(-i, n+4)(-j, n+4)}\right)}\right) .
\end{aligned}
$$

Now if $\pi \in J_{n+2}^{D}$ then $i n v_{D}\left(\bar{\pi}^{(n+3, n+4)}\right)=i n v_{D}(\pi)+1$ and $i n v_{D}\left(\bar{\pi}^{(-(n+3), n+4)}\right)=$ $\operatorname{inv}_{D}(\pi)+4 n+9$. A careful analysis shows that for $\pi \in J_{n}^{D}$ and $1 \leq i, j \leq$ $n+2, i \neq j$,

$$
\begin{aligned}
\operatorname{inv}_{D}\left(\bar{\pi}^{(i, n+3)(j, n+4)}\right) & =\operatorname{inv}_{D}(\pi)+4 n-2(i+j)+10+2 \cdot \mathbf{1}[i>j] \\
i n v_{D}\left(\bar{\pi}^{(-i, n+3)(-j, n+4)}\right) & =\operatorname{inv}_{D}(\pi)+4 n+2(i+j)+2-2 \cdot \mathbf{1}[i>j] \\
i n v_{D}\left(\bar{\pi}^{(-i, n+3)(j, n+4)}\right) & =\operatorname{inv}_{D}(\pi)+4 n+2(i-j)+6+2 \cdot \mathbf{1}[i>j] \\
i n v_{D}\left(\bar{\pi}^{(i, n+3)(-j, n+4)}\right) & =\operatorname{inv}_{D}(\pi)+4 n+2(j-i)+6-2 \cdot \mathbf{1}[i>j] .
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \mathcal{J} \mathcal{D}_{n+4}(q) \\
& \begin{aligned}
&= \mathcal{J} \mathcal{D}_{n+2}(q)\left(q+q^{4 n+9}\right)+\mathcal{J}_{n}(q) \times \\
& \sum_{1 \leq i<j \leq n+2}\left(\left(q^{10}+q^{12}\right) q^{4 n-2(i+j)}+\left(q^{2}+1\right) q^{4 n+2(i+j)}\right. \\
&\left.\quad+\left(q^{6}+q^{8}\right) q^{4 n+2(i-j)}+\left(q^{6}+q^{4}\right) q^{4 n+2(j-i)}\right) \\
&= \mathcal{J}_{n+2}(q)\left(q+q^{4 n+9}\right)+\mathcal{J D}_{n}(q) \frac{q^{4}\left(q^{4(n+1)}-1\right)\left(q^{2 n+4}-1\right)\left(q^{2 n}+1\right)}{\left(q^{2}-1\right)^{2}} .
\end{aligned} .
\end{aligned}
$$

The result follows by inserting the expression from the theorem, we omit the details.

Notice that if $n / 2$ is even (resp. odd) then the coefficients of odd (resp. even) powers of $q$ in $\mathcal{J}_{n}(q)$ are zero.

Theorem 3.4. The coefficients of the even (resp. odd) powers of $q$ in $\mathcal{J}_{n}(q)$ are symmetric and unimodal when $n / 2$ is even (resp. odd).

Proof. From Stanley [10, Proposition 1], we have that if $A(q)$ and $B(q)$ are symmetric and unimodal polynomials, both with non-negative coefficients, then $A(q) B(q)$ is also symmetric and unimodal. From the expression in Theorem [3.3, one may write $\mathcal{J}_{n+2}(q)=\left(q+q^{2}+\ldots+q^{2 n}+2 q^{2 n+1}+\right.$ $\left.q^{2 n+2}+\ldots+q^{4 n+1}\right) \mathcal{J} \mathcal{D}_{n}(q)$. The result follows inductively.

The generating function of the descent polynomial over involutions of Coxeter groups of types $B_{n}$ and $D_{n}$ is also seen to be symmetric, as was mentioned in Section 2, thanks to Hultman's [8] result.

## 4. Comments

Unlike the Eulerian polynomial, whose roots are all real and from which log-concavity of the coefficients follows, the roots of all polynomials with the statistics mentioned above are not real for $n \leq 14$. Furthermore, they do not lie in the nice triangular $\pi / 3$ region of the complex plane about the negative real-line from which it would be possible to infer log-concavity (see Stanley [10] Prop. 7].) Log-concavity of the coefficients holds numerically for all $n \leq 14$. We extend the original conjecture,

Conjecture 4.1. For all $n \geq 4$,
(i) the sequence $\left\{\left[q^{i}\right] \mathcal{I}_{n}^{m a j}(q)\right\}_{i=0}^{\binom{n}{2}}$ is log-concave,
(ii) for $2 \leq i \leq\binom{ n}{2}-2$ (see Figure 1)

$$
\left(\left[q^{i}\right] \mathcal{I}_{n}^{i n v}(q)\right)^{2} \geq\left(\left[q^{i-2}\right] \mathcal{I}_{n}^{i n v}(q)\right)\left(\left[q^{i+2}\right] \mathcal{I}_{n}^{i n v}(q)\right),
$$

(iii) the sequences $\left\{\left[q^{2 i}\right] \operatorname{Inv} v_{n}^{B}(q)\right\}_{i \geq 0}$ and $\left\{\left[q^{2 i+1}\right] \operatorname{Inv} v_{n}^{B}(q)\right\}_{i \geq 0}$ are unimodal,
(iv) the sequences $\left\{\left[q^{2 i}\right] \operatorname{Inv}_{n}^{D}(q)\right\}_{i \geq 0}$ and $\left\{\left[q^{2 i+1}\right] \operatorname{Inv}_{n}^{D}(q)\right\}_{i \geq 0}$ are unimodal,
(v) the sequences $\left\{\left[q^{2 i}\right] \operatorname{Inv} v_{n}^{O}(q)\right\}_{i \geq 0}$ and $\left\{\left[q^{2 i+1}\right] \operatorname{Inv}_{n}^{O}(q)\right\}_{i \geq 0}$ are unimodal.
We list here those polynomials for $n=10$ to exemplify these conjectures,

$$
\begin{aligned}
\mathcal{I}_{10}^{d}(x)= & 1+25 x+289 x^{2}+1397 x^{3}+3036 x^{4}+3036 x^{5}+1397 x^{6}+289 x^{7}+25 x^{8}+x^{9} . \\
\mathcal{I}_{10}^{\text {exc }}(x)= & 1+45 x+630 x^{2}+3150 x^{3}+4725 x^{4}+945 x^{5} . \\
\mathcal{I}_{10}^{\text {maj}}(x)= & 1+x+2 x^{2}+4 x^{3}+7 x^{4}+12 x^{5}+19 x^{6}+29 x^{7}+44 x^{8}+64 x^{9}+89 x^{10}+119 x^{11} \\
& +158 x^{12}+201 x^{13}+250 x^{14}+304 x^{15}+358 x^{16}+412 x^{17}+464 x^{18}+508 x^{19}+546 x^{20} \\
& +572 x^{21}+584 x^{22}+584 x^{23}+572 x^{24}+546 x^{25}+508 x^{26}+464 x^{27}+412 x^{28}+358 x^{29} \\
& +304 x^{30}+250 x^{31}+201 x^{32}+158 x^{33}+119 x^{34}+89 x^{35}+64 x^{36}+44 x^{37}+29 x^{38} \\
& +19 x^{39}+12 x^{40}+7 x^{41}+4 x^{42}+2 x^{43}+x^{44}+x^{45} . \\
& \\
\mathcal{I}_{10}^{\text {inv }}(x)= & 1+9 x+28 x^{2}+43 x^{3}+64 x^{4}+98 x^{5}+114 x^{6}+165 x^{7}+179 x^{8}+234 x^{9}+254 x^{10} \\
& +299 x^{11}+333 x^{12}+353 x^{13}+408 x^{14}+392 x^{15}+471 x^{16}+411 x^{17}+513 x^{18}+409 x^{19} \\
& +529 x^{20}+380 x^{21}+517 x^{22}+335 x^{23}+478 x^{24}+281 x^{25}+417 x^{26}+225 x^{27}+343 x^{28} \\
& +171 x^{29}+264 x^{30}+124 x^{31}+189 x^{32}+85 x^{33}+123 x^{34}+56 x^{35}+72 x^{36}+35 x^{37} \\
& +37 x^{38}+20 x^{39}+16 x^{40}+10 x^{41}+5 x^{42}+4 x^{43}+x^{44}+x^{45} . \\
\mathcal{I B}_{10}(x)= & 1+10 x+36 x^{2}+73 x^{3}+157 x^{4}+307 x^{5}+456 x^{6}+807 x^{7}+1121 x^{8}+1629 x^{9} \\
& +2323 x^{10}+2835 x^{11}+4124 x^{12}+4508 x^{13}+6468 x^{14}+6715 x^{15}+9256 x^{16} \\
& +9469 x^{17}+12712 x^{19}+15500 x^{20}+16306 x^{21}+18560 x^{22}+20048 x^{23}
\end{aligned}
$$

$$
\begin{aligned}
& +21334 x^{24}+23730 x^{25}+23626 x^{26}+27127 x^{27}+25285 x^{28}+29989 x^{29}+26242 x^{30} \\
& +32053 x^{31}+26550 x^{32}+33126 x^{33}+26310 x^{34}+33138 x^{35}+25641 x^{36}+32124 x^{37} \\
& +24639 x^{38}+30194 x^{39}+23393 x^{40}+27534 x^{41}+21953 x^{42}+24364 x^{43}+20369 x^{44} \\
& +20935 x^{45}+18657 x^{46}+17519 x^{47}+16839 x^{48}+14343 x^{49}+14925 x^{50}+11549 x^{51} \\
& +12956 x^{52}+9199 x^{53}+10967 x^{54}+7288 x^{55}+9019 x^{56}+5762 x^{57}+7178 x^{58} \\
& +4563 x^{59}+5525 x^{60}+3633 x^{61}+4107 x^{62}+2909 x^{63}+2962 x^{64}+2331 x^{65} \\
& +2084 x^{66}+1858 x^{67}+1444 x^{68}+1460 x^{69}+986 x^{70}+1123 x^{71}+671 x^{72}+834 x^{73} \\
& +454 x^{74}+589 x^{75}+312 x^{76}+394 x^{77}+217 x^{78}+255 x^{79}+156 x^{80}+156 x^{81} \\
& +111 x^{82}+91 x^{83}+79 x^{84}+52 x^{85}+56 x^{86}+30 x^{87}+40 x^{88}+17 x^{89}+26 x^{90} \\
& +10 x^{91}+15 x^{92}+5 x^{93}+5 x^{94}+3 x^{95}+2 x^{96}+3 x^{97}+x^{98}+3 x^{99}+x^{100} \text {. } \\
& \mathcal{I D}_{10}(x)=1+10 x+35 x^{2}+61 x^{3}+97 x^{4}+158 x^{5}+204 x^{6}+308 x^{7}+370 x^{8}+495 x^{9}+595 x^{10} \\
& +734 x^{11}+887 x^{12}+1034 x^{13}+1229 x^{14}+1381 x^{15}+1607 x^{16}+1764 x^{17}+2014 x^{18} \\
& +2182 x^{19}+2432 x^{20}+2606 x^{21}+2827 x^{22}+3012 x^{23}+3175 x^{24}+3377 x^{25}+3451 x^{26} \\
& +3663 x^{27}+3654 x^{28}+3863 x^{29}+3781 x^{30}+3970 x^{31}+3819 x^{32}+3964 x^{33}+3766 x^{34} \\
& +3859 x^{35}+3642 x^{36}+3670 x^{37}+3432 x^{38}+3402 x^{39}+3156 x^{40}+3085 x^{41}+2844 x^{42} \\
& +2736 x^{43}+2511 x^{44}+2378 x^{45}+2188 x^{46}+2036 x^{47}+1877 x^{48}+1707 x^{49}+1568 x^{50} \\
& +1396 x^{51}+1284 x^{52}+1128 x^{53}+1035 x^{54}+899 x^{55}+818 x^{56}+708 x^{57}+642 x^{58} \\
& +553 x^{59}+497 x^{60}+428 x^{61}+380 x^{62}+322 x^{63}+284 x^{64}+236 x^{65}+206 x^{66}+168 x^{67} \\
& +142 x^{68}+116 x^{69}+98 x^{70}+81 x^{71}+68 x^{72}+54 x^{73}+46 x^{74}+36 x^{75}+32 x^{76}+23 x^{77} \\
& +21 x^{78}+18 x^{79}+14 x^{80}+11 x^{81}+8 x^{82}+5 x^{83}+4 x^{84}+2 x^{85}+x^{86}+2 x^{87}+x^{88} \\
& +3 x^{89}+x^{90} \text {. } \\
& \mathcal{I} \mathcal{O}_{10}(x)=1+8 x+23 x^{2}+41 x^{3}+77 x^{4}+120 x^{5}+180 x^{6}+268 x^{7}+332 x^{8}+461 x^{9}+547 x^{10} \\
& +718 x^{11}+835 x^{12}+1040 x^{13}+1181 x^{14}+1407 x^{15}+1569 x^{16}+1808 x^{17}+1994 x^{18} \\
& +2236 x^{19}+2448 x^{20}+2672 x^{21}+2875 x^{22}+3078 x^{23}+3245 x^{24}+3421 x^{25}+3545 x^{26} \\
& +3679 x^{27}+3758 x^{28}+3847 x^{29}+3877 x^{30}+3926 x^{31}+3899 x^{32}+3906 x^{33}+3826 x^{34} \\
& +3797 x^{35}+3664 x^{36}+3610 x^{37}+3422 x^{38}+3358 x^{39}+3128 x^{40}+3067 x^{41}+2800 x^{42} \\
& +2744 x^{43}+2461 x^{44}+2408 x^{45}+2138 x^{46}+2080 x^{47}+1835 x^{48}+1759 x^{49}+1542 x^{50} \\
& +1446 x^{51}+1268 x^{52}+1164 x^{53}+1025 x^{54}+919 x^{55}+812 x^{56}+708 x^{57}+634 x^{58} \\
& +535 x^{59}+491 x^{60}+400 x^{61}+374 x^{62}+296 x^{63}+284 x^{64}+218 x^{65}+214 x^{66}+156 x^{67} \\
& +150 x^{68}+110 x^{69}+104 x^{70}+81 x^{71}+72 x^{72}+54 x^{73}+46 x^{74}+34 x^{75}+28 x^{76}+23 x^{77} \\
& +17 x^{78}+16 x^{79}+10 x^{80}+11 x^{81}+6 x^{82}+7 x^{83}+4 x^{84}+4 x^{85}+x^{86}+2 x^{87}+x^{88} \\
& +x^{89}+x^{90} \text {. }
\end{aligned}
$$

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Figure 1. The coefficients $\left[x^{2 i}\right] \mathcal{I}_{10}^{i n v}(x),\left[x^{2 i+1}\right] \mathcal{I}_{10}^{i n v}(x)$
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