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ABSTRACT. In this paper we look at polynomials arising from statistics on the classes of involutions,  $I_n$ , and involutions with no fixed points,  $J_n$ , in the symmetric group. Our results are motivated by F. Brenti's conjecture [3] which states that the Eulerian distribution of  $I_n$  is logconcave. Symmetry of the generating functions is shown for the statistics d, maj and the joint distribution (d, maj). We show that exc is logconcave on  $I_n$ , inv is log-concave on  $J_n$  and d is partially unimodal on both  $I_n$  and  $J_n$ . We also give recurrences and explicit forms for the generating functions of the inversions statistic on involutions in Coxeter groups of types  $B_n$  and  $D_n$ . Symmetry and unimodality of inv is shown on the subclass of signed permutations in  $D_n$  with no fixed points. In light of these new results, we present further conjectures at the end of the paper.

#### 1. INTRODUCTION

In this paper we look at polynomials arising from statistics on the classes of involutions and involutions with no fixed points in the symmetric group.

Let  $S_n$  be the symmetric group on [1, n]. Call  $Des(\sigma) := \{i : 1 \leq i < n \text{ and } \sigma_i > \sigma_{i+1}\}$  the descent set of  $\sigma \in S_n$  and the number of descents is denoted  $d(\sigma) := |Des(\sigma)|$ . We further define  $d_i(\sigma) := |\{j \geq i : j \in Des(\sigma)\}|$ , the partial descents of  $\sigma$  for  $1 \leq i < n$ . The major index of  $\sigma$  is  $maj(\sigma) := \sum_{i \in Des(\sigma)} i$  and the number of inversions is  $inv(\sigma) := |\{1 \leq i < j \leq n : \sigma_i > \sigma_j\}|$ . The number of excedances is  $exc(\sigma) := |\{1 \leq i \leq n : \sigma_i > i\}|$  and weak excedances is  $wexc(\sigma) := |\{1 \leq i \leq n : \sigma_i \geq i\}|$ . Let  $fix(\sigma)$  and  $trans(\sigma)$  denote the number of fixed points and transpositions of  $\sigma$ , respectively. We use the notation  $[x^i]P(x)$  for the coefficient of  $x^i$  in the polynomial P(x).

For a statistic  $stat: S_n \to \mathbf{N}_0$ , define the polynomials

$$\mathcal{I}_n^{stat}(x) := \sum_{\sigma \in I_n} x^{stat(\sigma)}, \qquad \mathcal{J}_n^{stat}(x) := \sum_{\sigma \in J_n} x^{stat(\sigma)},$$

where  $I_n := \{ \sigma \in S_n : \sigma^2 = \text{id} \}$  and  $J_n := \{ \sigma \in I_n : fix(\sigma) = 0 \}$ . For an arbitrary collection  $S'_n \subseteq S_n$ , the sequence of coefficients of  $\sum_{\pi \in S'_n} x^{d(\pi)}$ is termed the *Eulerian distribution* of  $S'_n$ . The results in this paper are motivated by

**Conjecture 1.1** (Brenti [3]). The Eulerian distribution of  $I_n$  is log-concave.

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We propose further conjectures concerning statistics on different classes of involutions in the final section.

## 2. Involutions in the Symmetric group

## 2.1. The excedances statistic.

**Theorem 2.1.** The coefficients of the polynomial  $\mathcal{I}_n^{exc}(x)$  are log-concave.

*Proof.* The number  $exc(\sigma)$  is precisely the number of 2-cycles in an involution, so we have

$$\mathcal{I}_{n}^{exc}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!} \left(\frac{x}{2}\right)^{k}.$$
 (2.1)

It is an easy exercise to show log-concavity for  $0 \le j < \lfloor n/2 \rfloor$  since we have a direct expression for the coefficients.

Note that the polynomials  $\mathcal{I}_n^{exc}(x)$  are closely related to the Hermite polynomials  $h_n(x)$ , whereby

$$\sum_{n>0} \frac{h_n(x)t^n}{n!} = \exp(tx - t^2/2),$$

via the equation  $\mathcal{I}_n^{exc}(x) = (-x)^n h_n(-1/2x)$ . The Hermite polynomials are known to be real-rooted (see for example Stanley [10, p. 505]).

The Schützenberger involution on tableaux,  $T \to evac(T)$ , maps involutions to involutions and  $wexc(evac(\sigma)) = n - exc(\sigma)$ , since  $evac(\sigma)_i = n + 1 - \sigma_{n+1-i}$ , so that  $\mathcal{I}_n^{wexc}(x) = x^n \mathcal{I}_n^{exc}(x^{-1})$ , hence

**Corollary 2.2.** The coefficients of the polynomial  $\mathcal{I}_n^{wexc}(x)$  are log-concave.

2.2. The descents and major index statistics. In the spirit of Adin et. al. [1], we define

$$\mathcal{G}_n(x_1,\ldots,x_{n-1}) := \sum_{\sigma \in I_n} x_1^{d_1(\sigma)} x_2^{d_2(\sigma)} \cdots x_{n-1}^{d_{n-1}(\sigma)}.$$

**Theorem 2.3.** The polynomial  $\mathcal{G}_n(x_1, \ldots, x_{n-1})$  satisfies

$$\mathcal{G}_n(x_1,\ldots,x_{n-1}) = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}\mathcal{G}_n(x_1^{-1},\ldots,x_{n-1}^{-1}).$$

*Proof.* If  $\sigma \in I_n$  then the reading and insertion tableau associated with  $\sigma$  under Robinson-Schensted correspondence (Stanley [9, Ch. 7]) are identical. That is, there is a bijection between  $I_n$  and all standard Young tableaux (SYT) on [1, n].

Let  $\sigma \in I_n$  with associated SYT T. The set  $Des(\sigma)$  corresponds to those entries i in the tableau T such that (i + 1) is below and weakly to the left of i. Let  $T^{\perp}$  be the tableau T reflected on its main diagonal. Notice that if (i+1) is below and weakly to the left of i in T, then (i+1) is to the right of and weakly above i in  $T^{\perp}$ . The bijection between the class of SYT on [1, n]and involutions  $I_n$  shows that to  $T^{\perp}$  there corresponds a unique involution  $\sigma^{\perp} \in I_n$ , and has the property that  $\{Des(\sigma), Des(\sigma^{\perp})\}$  is a partition of the set [1, n - 1]. In this manner, the reflection operation is an involution on involutions. It follows that

$$\begin{aligned} d_i(\sigma^{\perp}) &= |\{j \ge i : j \in Des(\sigma^{\perp})\}| \\ &= |\{j \ge i : j \notin Des(\sigma)\}| \\ &= n - i - |\{j \ge i : j \in Des(\sigma)\}| \\ &= n - i - d_i(\sigma). \end{aligned}$$

We have shown that if  $\sigma \in I_n$ , then there is a unique  $\sigma^{\perp} \in I_n$  such that  $(d_1(\sigma^{\perp}), \ldots, d_{n-1}(\sigma^{\perp})) = (n-1-d_1(\sigma), \ldots, 1-d_{n-1}(\sigma)).$ 

Both polynomials  $\mathcal{I}_n^d(q)$  and  $\mathcal{I}_n^{maj}(q)$  are instances of the  $\mathcal{G}$  polynomial since  $\mathcal{I}_n^d(q) = \mathcal{G}_n(q, 1, \ldots, 1)$  and  $\mathcal{I}_n^{maj}(q) = \mathcal{G}_n(q, q, \ldots, q)$ . Comparing coefficients on both sides of the symmetric  $\mathcal{G}$  relation yields

# **Corollary 2.4.** The polynomials $\mathcal{I}_n^d(t)$ and $\mathcal{I}_n^{maj}(t)$ are symmetric.

Symmetry of the polynomials  $\mathcal{I}_n^d(x)$  and  $\mathcal{J}_n^d(t)$  was conjectured by Dumont and first proven by Strehl [11], using a method similar to that of the previous theorem for the coefficients of  $\mathcal{I}_n^d(x)$ . A separate argument was used to prove symmetry of  $\mathcal{J}_n^d(t)$  because for  $\sigma \in J_n$ , it is not necessarily true that  $\sigma^{\perp} \in J_n$ . Theorem 2.3 allows us to show symmetry of the joint distribution of (d, maj) on  $I_n$  since  $\sum_{\sigma \in I_n} t^{d(\sigma)} q^{maj(\sigma)} = \mathcal{G}_n(tq, q, \ldots, q)$ .

Corollary 2.5. The polynomial

$$\mathcal{I}_n^{d,maj}(t,q) = \sum_{\sigma \in I_n} t^{d(\sigma)} q^{maj(\sigma)}$$

is symmetric in the sense that  $[t^i q^j] \mathcal{I}_n^{d,maj}(t,q) = [t^{n-1-i} q^{\binom{n}{2}-j}] \mathcal{I}_n^{d,maj}(t,q).$ 

Hultman [8] recently proved that for any finite Coxeter system (W, S), the associated descent polynomial  $\sum_{w} t^{d_W(w)}$  is symmetric where the sum ranges over all  $w \in W$  with  $w^2 = \mathrm{id}_w$ . Désarménien and Foata [7] use an elegant method involving Schur functions to derive the generating function

$$\sum_{n \ge 0} \frac{H_n(z_1, z_2, t, q)u^n}{(t; q)_n} = \sum_{r \ge 0} t^r \frac{1}{(z_1 u; q)_{r+1}} \prod_{0 \le i < j \le r} \frac{1}{1 - u^2 z_2 q^{i+j}} \quad (2.2)$$

where  $H_n(z_1, z_2, t, q) := \sum_{\sigma \in I_n} z_1^{fix(\sigma)} z_2^{trans(\sigma)} t^{d(\sigma)} q^{maj(\sigma)}$ ,  $(a; q)_0 = 1$  and  $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ . The generating functions for the polynomials  $\mathcal{I}_n^d(t)$ ,  $\mathcal{I}_n^{maj}(q)$  are immediate from this:

$$\sum_{n\geq 0} \frac{\mathcal{I}_n^d(t)u^n}{(1-t)^n} = \sum_{r\geq 0} t^r \left(\frac{1}{(1-u)^{r+1}(1-u^2)^{r(r+1)/2}}\right); \quad (2.3)$$

$$\sum_{n\geq 0} \frac{\mathcal{J}_n^d(t)u^n}{(1-t)^n} = \sum_{r\geq 0} t^r \left(\frac{1}{(1-u^2)^{r(r+1)/2}}\right);$$
(2.4)

$$\sum_{n \ge 0} \frac{\mathcal{I}_n^{maj}(q)u^n}{(q;q)_n} = \sum_{r \ge 0} \frac{1}{(u;q)_{r+1}} \prod_{0 \le i < j \le r} \frac{1}{(1-u^2q^{i+j})}.$$
 (2.5)

By extracting the appropriate coefficients, we now show partial unimodality of  $\mathcal{I}_n^d(q)$  and  $\mathcal{J}_n^d(q)$ . The onerous aspect of proving total unimodality using these direct expressions seems to be the appearance of both r and  $\binom{r+1}{2}$  within binomial terms.

**Theorem 2.6.** For all  $1 \leq i \leq n^{0.925}/10$ ,  $[t^i]\mathcal{J}_n^d(t) < [t^{i+1}]\mathcal{J}_n^d(t)$  and  $[t^{n+1-i}]\mathcal{J}_n^d(t) > [t^{n+2-i}]\mathcal{J}_n^d(t)$ .

*Proof.* Extracting the coefficient of  $u^n$  in Equation (2.4), one finds

$$\mathcal{J}_{n}^{d}(t) = \sum_{p=1}^{n} \alpha_{n,p} t^{p} = \sum_{p=1}^{n} t^{p} \left\{ \sum_{k=0}^{p-1} (-1)^{k} \binom{n+1}{k} \binom{\binom{p-k+1}{2} + n/2 - 1}{n/2} \right\}.$$

Inverting this gives

$$f_n(p) := \binom{\binom{p+1}{2} + n/2 - 1}{n/2} = \sum_{i=0}^{p-1} \binom{n+i}{n} \alpha_{n,p-i}.$$

For  $p \geq 2$ ,

$$\begin{aligned} f_n(p) &- f_n(p-1) \\ &= \alpha_{n,p} - \alpha_{n,p-1} + \sum_{i=1}^{p-1} \binom{n+i}{i} \alpha_{n,p-i} - \sum_{i=1}^{p-2} \binom{n+i}{i} \alpha_{n,p-1-i} \\ &= \alpha_{n,p} - \alpha_{n,p-1} + \binom{n+p-1}{p-1} \alpha_{n,1} + \sum_{i=1}^{p-2} \binom{n+i}{i} (\alpha_{n,p-i} - \alpha_{n,p-1-i}) \\ &\leq \alpha_{n,p} - \alpha_{n,p-1} + \binom{n+p-1}{p-1} \alpha_{n,1} + (n+1) \sum_{i=1}^{p-2} \binom{n+i}{i} (\alpha_{n,p-i} - \alpha_{n,p-1-i}) \\ &\leq \alpha_{n,p} - \alpha_{n,p-1} + (n+1) (f_n(p-1) - f_n(p-2)). \end{aligned}$$

Thus  $\alpha_{n,p} - \alpha_{n,p-1} \ge f_n(p) - f_n(p-1) - (n+1)(f_n(p-1) - f_n(p-2))$ . The right hand side of the previous inequality is positive for p not too large. Notice that

$$\frac{f_n(p)}{f_n(p-1)} \geq \left(1 + \frac{n}{p^2 + p - 2}\right)^p$$
$$\geq \left(1 + \frac{n+2}{2p^2}\right)^p$$

which, in turn, is bounded below by n + 2 when  $p \le n^{0.925}/10$ . The second inequality follows from symmetry as shown in Strehl [11].

**Theorem 2.7.** For all  $1 \le k \le 0.175n^{0.931}$ ,  $[t^{k-1}]\mathcal{I}_n^d(t) < [t^k]\mathcal{I}_n^d(t)$  and  $[t^{n-1-k}]\mathcal{I}_n^d(t) > [t^{n-k}]\mathcal{I}_n^d(t)$ .

*Proof.* Extracting the coefficient of  $u^n$  in Equation (2.3) we find

$$\mathcal{I}_{n}^{d}(t) = \sum_{k=0}^{n-1} \beta_{n,k} t^{k} = \sum_{k=0}^{n-1} t^{k} \left\{ \sum_{j=0}^{k} \binom{n+1}{j} (-1)^{j} \gamma(n,k-j) \right\}$$

where  $\gamma(n,0)$ ; = 1 and  $\gamma(n,r) := \sum_{i=0}^{\lfloor n/2 \rfloor} {\binom{n-2i+r}{r}} {\binom{i+\binom{r+1}{2}-1}{i}}$  for r > 0. Thus for  $0 \le k \le n$ ,

$$\gamma(n,k) = \sum_{i=0}^{k} {\binom{n+i}{i}} \beta_{n,k-i}$$

and so

$$\begin{split} \gamma(n,k) &- \gamma(n,k-1) \\ &= \sum_{i=0}^{k-1} \binom{n+i}{i} (\beta_{n,k-i} - \beta_{n,k-1-i}) + \binom{n+k}{k} \\ &= \beta_{n,k} - \beta_{n,k-1} + \sum_{i=0}^{k-2} \binom{n+i+1}{i+1} (\beta_{n,k-1-i} - \beta_{n,k-2-i}) + \binom{n+k}{k} \\ &< \beta_{n,k} - \beta_{n,k-1} + (n+1) \sum_{i=0}^{k-2} \binom{n+i}{i} (\beta_{n,k-1-i} - \beta_{n,k-2-i}) + \binom{n+k}{k} \\ &= \beta_{n,k} - \beta_{n,k-1} + (n+1) \left( \gamma(n,k-1) - \gamma(n,k-2) - \binom{n+k-1}{k-1} \right) \right) \\ &+ \binom{n+k}{k} \\ &< \beta_{n,k} - \beta_{n,k-1} + (n+1)\gamma(n,k-1). \end{split}$$

It suffices to show that for n and k as stated in the theorem,  $\gamma(n,k) > (n+2)\gamma(n,k-1)$ . One may also write

$$\gamma(n,k) = [u^n] \frac{(1+u)^{k+1}}{(1-u^2)^{\binom{k+2}{2}}}$$

and since n > k + 1 we have

$$\begin{split} \gamma(n,k) &= \sum_{i=0}^{\lfloor k+1/2 \rfloor} \binom{k+1}{n-2\lfloor n/2 \rfloor+2i} [u^{2(\lfloor n/2 \rfloor-i)}] \frac{1}{(1-u^2)^{\binom{k+2}{2}}} \\ &> \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{n-2\lfloor n/2 \rfloor+2i} [u^{2(\lfloor n/2 \rfloor-i)}] \frac{1}{(1-u^2)^{\binom{k+2}{2}}}. \end{split}$$

Now for all  $m \ge (n-k-2)/2$ ,

$$\binom{\binom{k+2}{2}+m}{m+1} > \left(1+\frac{2m}{(k+1)(k+2)}\right)^{k+1} \binom{\binom{k+1}{2}+m}{m+1} \\ > (n+2)\binom{\binom{k+1}{2}+m}{m+1}.$$

for  $k \le 0.175 n^{0.931}$ , hence

$$\begin{split} \gamma(n,k) > & (n+2) \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{n-2\lfloor n/2 \rfloor + 2i} [u^{2(\lfloor n/2 \rfloor - i)}] \frac{1}{(1-u^2)^{\binom{k+1}{2}}} \\ &= & (n+2)\gamma(n,k-1), \end{split}$$

giving the first inequality. Again, symmetry of the  $\mathcal{I}_n^d(t)$  polynomial yields the second inequality.

2.3. The inversions statistic. The generating function for the inversions statistic on involutions is intimately related to the q-Hermite polynomials, as studied by Désarménien [6]. Let  $a_n(k,j)$  be the number of involutions in  $I_n$  with k fixed points and j inversions, and define  $Z_n(x,q) := \sum_{k,j} a_n(k,j)q^jx^k$ . Désarménien [6, Eqns. 3.10,3.11] showed

$$Z_{n+1}(x,q) = xZ_n(x,q) + q\left(\frac{1-q^{2n}}{1-q^2}\right)Z_{n-1}(x,q)$$

for all n > 1 with  $Z_0(x,q) = 1$  and  $Z_1(x,q) = x$ . Setting x = 1, 0, yields the following proposition.

**Proposition 2.8.** For all  $n \ge 0$ ,

$$\mathcal{I}_{n+2}^{inv}(q) = \mathcal{I}_{n+1}^{inv}(q) + q\left(\frac{1-q^{2(n+1)}}{1-q^2}\right) \mathcal{I}_n^{inv}(q)$$

where  $\mathcal{I}_0^{inv}(q), \mathcal{I}_1^{inv}(q) := 1$  and for  $n \ge 0$ ,

$$\mathcal{J}_{n+2}^{inv}(q) = q\left(\frac{1-q^{2(n+1)}}{1-q^2}\right)\mathcal{J}_n^{inv}(q),$$

where  $\mathcal{J}_0^{inv}(q) = 1$ .

The above recurrences can also be derived in a straightforward manner using a special case of Equation (3.1). The coefficients of  $\mathcal{I}_n^{inv}(q)$  are neither log-concave nor unimodal (see Figure 1) but the recursion in the previous proposition admits a solution as a matrix product, which may be of benefit in approaching Conjecture 4.1(ii).

**Proposition 2.9.** Let  $g_i(q) := \sum_{j=0}^{i-2} q^{1+2j}$  and  $\mathbf{A}_n(q) := \prod_{i=2}^n \begin{pmatrix} 1 & 1 \\ g_i(q) & 0 \end{pmatrix}$ , then  $\mathcal{I}_n^{inv}(q) = \mathbf{A}_n(q)_{1,1} + \mathbf{A}_n(q)_{2,1}$ .

*Proof.* For all  $n \geq 2$ , we may write

$$\mathcal{I}_n^{inv}(q) = \prod_{i=1}^n \mathcal{Y}_i(q)$$

where  $\mathcal{Y}_n(q) := \mathcal{I}_n^{inv}(q)/\mathcal{I}_{n-1}^{inv}(q)$  and  $\mathcal{I}_0^{inv}(q), \mathcal{I}_1^{inv}(q) := 1$ . From the first recurrence in Proposition 2.8 the polynomial  $\mathcal{Y}_n(q)$  satisfies the recurrence  $\mathcal{Y}_n(q) = 1 + g_{n-1}(q)/\mathcal{Y}_{n-1}(q)$  for all  $n \geq 2$  where  $g_n(q) := q + q^3 + \ldots + q^{2n-3}$ . Using this, the product  $\mathcal{Y}_i \mathcal{Y}_{i+1} \cdots \mathcal{Y}_n$  may be written in the form  $\alpha_i(q)\mathcal{Y}_i(q) + \beta_i(q)$ . It is easily seen that  $\alpha_n(q) = 1, \beta_n(q) = 0$  and

$$\begin{pmatrix} \alpha_i(q) \\ \beta_i(q) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ g_{i+1}(q) & 0 \end{pmatrix} \begin{pmatrix} \alpha_{i+1}(q) \\ \beta_{i+1}(q) \end{pmatrix}$$

Thus we have  $\mathcal{I}_n^{inv}(q) = \alpha_1(q)\mathcal{Y}_1(q) + \beta_1(q) = \alpha_1(q) + \beta_1(q)$ , since  $\mathcal{Y}_1(q) = 1$ , and

$$\begin{pmatrix} \alpha_1(q) \\ \beta_1(q) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ g_2(q) & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ g_3(q) & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ g_n(q) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

 $\square$ 

**Theorem 2.10.** The coefficients of the polynomial  $\mathcal{J}_n^{inv}(q)$  are log-concave. *Proof.* Solving the second recurrence in Proposition 2.8 we get:

$$\mathcal{J}_{2m}^{inv}(q) = q^m \prod_{i=1}^{m-1} \frac{1 - q^{2(2i+1)}}{1 - q^2}.$$

Set  $u = q^2$  and notice that the sequence of non-zero coefficients in  $\mathcal{J}_n^{inv}(q)$ is the same as  $\prod_{i=1}^{m-1} \frac{1-u^{2i+1}}{1-u}$ . The coefficients of the polynomials  $(1-u^{2i+1})$  $u^{2i+1})/(1-u)$  are non-negative log-concave sequences with no internal zero coefficients. Thus using Stanley [10, Prop. 2], the product of all such polynomials will also be log-concave with no internal zero coefficients.

# 3. Involutions in Coxeter groups of types B and D

In this section we give recursive expressions for the inversion polynomials of involutions for Coxeter groups of types B and D. We use the notation of Björner and Brenti [2].

Coxeter groups of type B, the 'signed permutations', are defined as follows: let  $S_n^B$  be the group of all bijections  $\pi$  on the set  $[\pm n] \setminus \{0\}$  such that  $\pi(-a) = -\pi(a)$  for all  $a \in [\pm n]$ . For  $\pi \in S_n^B$ , define

$$\begin{split} N_1(\pi(1), \dots, \pi(n)) &:= |\{1 \le i \le n : \pi(i) < 0\}| \\ N_2(\pi(1), \dots, \pi(n)) &:= |\{1 \le i < j \le n : \pi(i) + \pi(j) < 0\}|. \end{split}$$

Let  $S_n^D$  be the subgroup of  $S_n^B$  consisting of all signed permutations  $\pi \in S_n^B$ such that there are an even number of negative entries in the window of  $\pi$ , i.e.  $S_n^D := \{\pi \in S_n^B : N_1(\pi) \equiv 0 \pmod{2}\}$ . For completeness let us also define those signed permutations containing an odd number of negative signs in the window of  $\pi$ ,  $S_n^O = S_n^B \setminus S_n^D$ . The inversions statistics on  $S_n^B$  and  $S_n^D$  are defined slightly differently to

inv on  $S_n$ . From [2, Equations (8.1) and (8.18)],

$$inv_B(\pi) := inv(\pi(1), \dots, \pi(n)) + N_1(\pi(1), \dots, \pi(n)) + N_2(\pi(1), \dots, \pi(n)) inv_D(\pi) := inv(\pi(1), \dots, \pi(n)) + N_2(\pi(1), \dots, \pi(n)).$$

Let us mention that in the symmetric group setting,

$$\sum_{\pi \in S_n^B} q^{inv_B(\pi)} = [2]_q [4]_q \dots [2n]_q$$
$$\sum_{\pi \in S_n^D} q^{inv_D(\pi)} = [2]_q [4]_q \dots [2n-2]_q [n]_q.$$

where  $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$  (see [2, Theorem 7.1.5.]) Define  $I_n^B := \{\pi \in S_n^B : \pi^2 = \text{id}\}, I_n^D := \{\pi \in S_n^D : \pi^2 = \text{id}\}$  and  $I_n^O := I_n^B \setminus I_n^D$ . Let

$$\mathcal{IB}_n(q) := \sum_{\pi \in I_n^B} q^{inv_B(\pi)},$$

with  $\mathcal{ID}_n(q)$  and  $\mathcal{IO}_n(q)$  similarly defined. To aid the proof of the following two theorems, we introduce some notation concerning the recursive construction of these signed permutations.

construction of these signed permutations. Let  $\pi \in I_n^B$  and denote by  $\overline{\pi}^{(n+1,n+1)}$  the signed permutation  $\pi' \in I_{n+1}^B$ such that  $\pi'(i) = \pi(i)$ , for  $1 \le i, \le n$  and  $\pi'(n+1) = n+1$ . Similarly let  $\overline{\pi}^{(-(n+1),n+1)}$  be the signed permutation  $\pi' \in I_{n+1}^B$  such that  $\pi'(i) = \pi(i)$  for  $1 \le i \le n$  and  $\pi'(n+1) = -(n+1)$ .

For  $\pi \in I_n^B$  and  $k \in [\pm (n+1)] - \{0\}$ , let  $\overline{\pi}^{(k,n+2)}$  be the signed permutation  $\pi' \in I_{n+2}^B$  such that

- $\pi'(|k|) = (n+2)sgn(k), \ \pi'(n+2) = k,$
- for all  $1 \le i \le n$ ,

$$\pi'(i + \mathbf{1}[i \ge |k|]) = \pi(i) + sgn(\pi(i))\mathbf{1}[|\pi(i)| \ge |k|]$$

where sgn(a) = +1 if a > 0 and -1 otherwise. Consequently  $I_{n+2}^B$ ,  $I_{n+2}^D$  and  $I_{n+2}^O$  may be constructed recursively,

$$I_{n+2}^{B} = \bigcup_{\substack{\pi \in I_{n+1}^{B} \\ \pi \in I_{n}^{B}}} \{ \overline{\pi}^{(n+2,n+2)}, \overline{\pi}^{(-(n+2),n+2)} \} \$$
  
$$\bigcup_{\substack{k=1 \\ \pi \in I_{n}^{B}}}^{n+1} \{ \overline{\pi}^{(k,n+2)}, \overline{\pi}^{(-k,n+2)} \}$$
(3.1)

$$I_{n+2}^{D} = \bigoplus_{\pi \in I_{n+1}^{D}} \{ \overline{\pi}^{(n+2,n+2)} \} \ \uplus \bigoplus_{\pi \in I_{n+1}^{O}} \{ \overline{\pi}^{(-(n+2),n+2)} \} \ \uplus$$
$$\prod_{\substack{n+1\\ \mu \in I_{n}^{D}}} \{ \overline{\pi}^{(k,n+2)}, \overline{\pi}^{(-k,n+2)} \}$$
(3.2)

$$I_{n+2}^{O} = \bigoplus_{\pi \in I_{n+1}^{O}} \{ \overline{\pi}^{(n+2,n+2)} \} \ \uplus \bigoplus_{\pi \in I_{n+1}^{D}} \{ \overline{\pi}^{(-(n+2),n+2)} \} \ \uplus$$
$$\bigoplus_{\substack{n+1\\ \underset{\pi \in I_{n}^{O}}{\overset{k=1}{\longrightarrow}}} \{ \overline{\pi}^{(k,n+2)}, \overline{\pi}^{(-k,n+2)} \}.$$
(3.3)

**Theorem 3.1.** For all  $n \geq 2$ ,

$$\mathcal{IB}_{n+2}(q) = (1+q^{2n+3})\mathcal{IB}_{n+1}(q) + \frac{q(1+q^2)(1-q^{2(n+1)})}{1-q^2}\mathcal{IB}_n(q)$$

with initial polynomials  $\mathcal{IB}_2(q) = 1 + 2q + 2q^3 + q^4$ ,  $\mathcal{IB}_3(q) = 1 + 3q + q^2 + 3q^3 + 2q^4 + 2q^5 + 3q^6 + q^7 + 3q^8 + q^9$ .

*Proof.* Using Equation (3.1),

$$\mathcal{IB}_{n+2}(q) = \sum_{\pi \in I_{n+1}^B} q^{inv_B(\overline{\pi}^{(n+2,n+2)})} + q^{inv_B(\overline{\pi}^{(-(n+2),n+2)})} + \sum_{k=1}^{n+1} \sum_{\pi \in I_n^B} q^{inv_B(\overline{\pi}^{(k,n+2)})} + q^{inv_B(\overline{\pi}^{(-k,n+2)})}.$$

If  $\pi \in I_{n+1}^B$ , then  $inv_B(\overline{\pi}^{(n+2,n+2)}) = inv_B(\pi(1), \dots, \pi(n+1), n+2) = inv_B(\pi)$  and  $inv_B(\overline{\pi}^{(-(n+2),n+2)}) = inv_B(\pi) + 2n + 3$ . Similarly if  $\pi \in I_n^B$  and  $1 \le k \le n+1$ , then  $inv_B(\overline{\pi}^{(k,n+2)}) = inv_B(\pi) + 2n + 3 - 2k$  and  $inv_B(\overline{\pi}^{(-k,n+2)}) = inv_B(\pi) + 2k + 1$ . Hence

$$\mathcal{IB}_{n+2}(q) = \sum_{\pi \in I_{n+1}^B} q^{inv_B(\pi)} + q^{inv_B(\pi)+2n+3} + \sum_{k=1}^{n+1} \sum_{\pi \in I_n^B} q^{inv_B(\pi)+2n-2k+3} + q^{inv_B(\pi)+2k+1} = (1+q^{2n+3})\mathcal{IB}_{n+1}(q) + \mathcal{IB}_n(q) \sum_{k=1}^{n+1} (q^{2n-2k+3}+q^{2k+1}).$$

We may express  $\mathcal{IB}_n(q)$  in a somewhat closed form, as was done in Proposition 2.9; for all  $n \geq 3$ ,  $\mathcal{IB}_n(q) = (\mathbf{V}_n(q)_{1,1} + \mathbf{V}_n(q)_{2,1})(1 + 2q + 2q^3 + q^4)$  where

$$\mathbf{V}_n(q) = \prod_{i=3}^n \left( \begin{array}{cc} u_i(q) & 1\\ v_i(q) & 0 \end{array} \right)$$

and  $u_i(q) := 1 + q^{2i-1}, v_i(q) := (1+q^2)(1-q^{2(i-1)})/(1-q^2).$ 

**Theorem 3.2.** For all  $n \geq 2$ ,

$$\mathcal{ID}_{n+1}(q) = \mathcal{ID}_n(q) + q^{2n} \mathcal{IO}_n(q) + \left(q^{2(n-1)} + \frac{q(1-q^{2n})}{1-q^2}\right) \mathcal{ID}_{n-1}(q)$$
  
$$\mathcal{IO}_{n+1}(q) = \mathcal{IO}_n(q) + q^{2n} \mathcal{ID}_n(q) + \left(q^{2(n-1)} + \frac{q(1-q^{2n})}{1-q^2}\right) \mathcal{IO}_{n-1}(q)$$

with initial polynomials  $\mathcal{ID}_2(q), \mathcal{IO}_2(q) = 1 + q + q^2, \mathcal{ID}_3(q) = (1 + q + q^2 + q^3)(1 + q^3) + 2q$  and  $\mathcal{IO}_3(q) = (1 + q + q^2 + q^3)(1 + q^3) + 2q^5$ .

*Proof.* Using Equation (3.2),

$$\begin{split} \mathcal{ID}_{n+2}(q) &= \sum_{\pi \in I_{n+1}^D} q^{inv_D(\overline{\pi}^{(n+2,n+2)})} + \sum_{\pi \in I_{n+1}^O} q^{inv_D(\overline{\pi}^{(-(n+2),n+2)})} \\ &+ \sum_{k=1}^{n+1} \sum_{\pi \in I_n^D} q^{inv_D(\overline{\pi}^{(k,n+2)})} + q^{inv_D(\overline{\pi}^{(-k,n+2)})}. \end{split}$$

If  $\pi \in I_{n+1}^D$ ,  $I_{n+1}^O$ , then  $inv_D(\overline{\pi}^{(n+2,n+2)}) = inv_D(\pi)$  and  $inv_D(\overline{\pi}^{(-(n+2),n+2)}) = inv_D(\pi) + 2(n+1)$ . Also if  $\pi \in I_n^D$ , then  $inv_D(\overline{\pi}^{(k,n+2)}) = 2n - 2k + 3 + inv_D(\pi)$  and  $inv_D(\overline{\pi}^{(-k,n+2)}) = inv_D(\pi) + 2n$ . Hence,

$$\mathcal{ID}_{n+2}(q) = \sum_{\pi \in I_{n+1}^{D}} q^{inv_{D}(\pi)} + \sum_{\pi \in I_{n+1}^{O}} q^{inv_{D}(\pi)+2(n+1)} + \sum_{k=1}^{n+1} \sum_{\pi \in I_{n}^{D}} q^{inv_{D}(\pi)} (q^{2n-2k+3} + q^{2n}).$$

The second recurrence is derived in the same manner by using Equation (3.3).

Let  $J_n^D \subset I_n^D$  denote the class of all signed permutations such that  $\pi(i) \neq \pm i$  for all  $i \in [1, n]$  and consider the generating function  $\mathcal{JD}_n(q) := \sum_{\pi \in J_n^D} q^{inv_D(\pi)}$ .

**Theorem 3.3.** For all even  $n \geq 2$ ,

$$\mathcal{JD}_n(q) = 2q^{n/2} \prod_{i=1}^{n/2-1} \frac{(1+q^{4i})(1-q^{4i+2})}{1-q^2}$$

*Proof.* Since  $J_n^D$  is a subclass of  $I_n^D$  and from the characterization in Equation (3.2), one has

$$\begin{aligned} \mathcal{JD}_{n+4}(q) &= \sum_{\pi \in J_{n+2}^{D}} q^{inv_{D}(\overline{\pi}^{(n+3,n+4)})} + q^{inv_{D}(\overline{\pi}^{(-(n+3),n+4)})} \\ &+ \sum_{1 \leq i < j \leq n+2} \sum_{\pi \in J_{n}^{D}} \left( q^{inv_{D}(\overline{\pi}^{(i,n+3)(j,n+4)})} + q^{inv_{D}(\overline{\pi}^{(i,n+4)(j,n+3)})} \right. \\ &+ q^{inv_{D}(\overline{\pi}^{(-i,n+3)(j,n+4)})} + q^{inv_{D}(\overline{\pi}^{(-i,n+4)(j,n+3)})} \\ &+ q^{inv_{D}(\overline{\pi}^{(i,n+3)(-j,n+4)})} + q^{inv_{D}(\overline{\pi}^{(-i,n+4)(-j,n+3)})} \\ &+ q^{inv_{D}(\overline{\pi}^{(-i,n+3)(-j,n+4)})} + q^{inv_{D}(\overline{\pi}^{(-i,n+4)(-j,n+4)})} \right). \end{aligned}$$

Now if  $\pi \in J_{n+2}^D$  then  $inv_D(\overline{\pi}^{(n+3,n+4)}) = inv_D(\pi) + 1$  and  $inv_D(\overline{\pi}^{(-(n+3),n+4)}) = inv_D(\pi) + 4n + 9$ . A careful analysis shows that for  $\pi \in J_n^D$  and  $1 \le i, j \le n+2, i \ne j$ ,

$$inv_D(\overline{\pi}^{(i,n+3)(j,n+4)}) = inv_D(\pi) + 4n - 2(i+j) + 10 + 2 \cdot \mathbf{1}[i > j]$$
  

$$inv_D(\overline{\pi}^{(-i,n+3)(-j,n+4)}) = inv_D(\pi) + 4n + 2(i+j) + 2 - 2 \cdot \mathbf{1}[i > j]$$
  

$$inv_D(\overline{\pi}^{(-i,n+3)(j,n+4)}) = inv_D(\pi) + 4n + 2(i-j) + 6 + 2 \cdot \mathbf{1}[i > j]$$
  

$$inv_D(\overline{\pi}^{(i,n+3)(-j,n+4)}) = inv_D(\pi) + 4n + 2(j-i) + 6 - 2 \cdot \mathbf{1}[i > j].$$

Thus we have

$$\begin{aligned} \mathcal{JD}_{n+4}(q) \\ &= \mathcal{JD}_{n+2}(q)(q+q^{4n+9}) + \mathcal{JD}_n(q) \times \\ &\sum_{1 \le i < j \le n+2} \left( (q^{10}+q^{12})q^{4n-2(i+j)} + (q^2+1)q^{4n+2(i+j)} \right. \\ &+ (q^6+q^8)q^{4n+2(i-j)} + (q^6+q^4)q^{4n+2(j-i)} \right) \\ &= \mathcal{JD}_{n+2}(q)(q+q^{4n+9}) + \mathcal{JD}_n(q) \frac{q^4(q^{4(n+1)}-1)(q^{2n+4}-1)(q^{2n}+1)}{(q^2-1)^2}. \end{aligned}$$

The result follows by inserting the expression from the theorem, we omit the details.  $\hfill \Box$ 

Notice that if n/2 is even (resp. odd) then the coefficients of odd (resp. even) powers of q in  $\mathcal{JD}_n(q)$  are zero.

**Theorem 3.4.** The coefficients of the even (resp. odd) powers of q in  $\mathcal{JD}_n(q)$  are symmetric and unimodal when n/2 is even (resp. odd).

*Proof.* From Stanley [10, Proposition 1], we have that if A(q) and B(q) are symmetric and unimodal polynomials, both with non-negative coefficients, then A(q)B(q) is also symmetric and unimodal. From the expression in Theorem 3.3, one may write  $\mathcal{JD}_{n+2}(q) = (q + q^2 + ... + q^{2n} + 2q^{2n+1} + q^{2n})$  $q^{2n+2} + \ldots + q^{4n+1}) \mathcal{JD}_n(q)$ . The result follows inductively. 

The generating function of the descent polynomial over involutions of Coxeter groups of types  $B_n$  and  $D_n$  is also seen to be symmetric, as was mentioned in Section 2, thanks to Hultman's [8] result.

## 4. Comments

Unlike the Eulerian polynomial, whose roots are all real and from which log-concavity of the coefficients follows, the roots of all polynomials with the statistics mentioned above are not real for  $n \leq 14$ . Furthermore, they do not lie in the nice triangular  $\pi/3$  region of the complex plane about the negative real-line from which it would be possible to infer log-concavity (see Stanley [10, Prop. 7].) Log-concavity of the coefficients holds numerically for all  $n \leq 14$ . We extend the original conjecture,

## Conjecture 4.1. For all $n \ge 4$ ,

- (i) the sequence  $\{[q^i]\mathcal{I}_n^{maj}(q)\}_{i=0}^{\binom{n}{2}}$  is log-concave, (ii) for  $2 \le i \le \binom{n}{2} 2$  (see Figure 1)  $([q^i]\mathcal{I}_n^{inv}(q))^2 > ([q^{i-2}]\mathcal{I}_n^{inv}(q))([q^{i+2}]\mathcal{I}_n^{inv}(q)),$
- (iii) the sequences  $\{[q^{2i}]Inv_n^B(q)\}_{i>0}$  and  $\{[q^{2i+1}]Inv_n^B(q)\}_{i>0}$  are unimodal.
- (iv) the sequences  $\{[q^{2i}]Inv_n^D(q)\}_{i\geq 0}$  and  $\{[q^{2i+1}]Inv_n^D(q)\}_{i\geq 0}$  are unimodal,
- (v) the sequences  $\{[q^{2i}]Inv_n^O(q)\}_{i\geq 0}$  and  $\{[q^{2i+1}]Inv_n^O(q)\}_{i\geq 0}$  are unimodal.

We list here those polynomials for n = 10 to exemplify these conjectures,

 $\mathcal{I}^{d}_{10}(x) = 1 + 25x + 289x^{2} + 1397x^{3} + 3036x^{4} + 3036x^{5} + 1397x^{6} + 289x^{7} + 25x^{8} + x^{9}.$ 

$$\mathcal{I}_{10}^{exc}(x) = 1 + 45x + 630x^2 + 3150x^3 + 4725x^4 + 945x^5.$$

 $= 1 + x + 2x^{2} + 4x^{3} + 7x^{4} + 12x^{5} + 19x^{6} + 29x^{7} + 44x^{8} + 64x^{9} + 89x^{10} + 119x^{11}$  $\mathcal{I}_{10}^{maj}(x)$  $+ 158x^{12} + 201x^{13} + 250x^{14} + 304x^{15} + 358x^{16} + 412x^{17} + 464x^{18} + 508x^{19} + 546x^{20} + 546x^{10} + 54$  $+572x^{21} + 584x^{22} + 584x^{23} + 572x^{24} + 546x^{25} + 508x^{26} + 464x^{27} + 412x^{28} + 358x^{29} + 56x^{29} +$  $+304x^{30} + 250x^{31} + 201x^{32} + 158x^{33} + 119x^{34} + 89x^{35} + 64x^{36} + 44x^{37} + 29x^{38}$  $+19x^{39} + 12x^{40} + 7x^{41} + 4x^{42} + 2x^{43} + x^{44} + x^{45}.$ 

$$\begin{split} \mathcal{I}_{10}^{inv}(x) &= & 1+9x+28x^2+43x^3+64x^4+98x^5+114x^6+165x^7+179x^8+234x^9+254x^{10} \\ &+299x^{11}+333x^{12}+353x^{13}+408x^{14}+392x^{15}+471x^{16}+411x^{17}+513x^{18}+409x^{15} \\ &+529x^{20}+380x^{21}+517x^{22}+335x^{23}+478x^{24}+281x^{25}+417x^{26}+225x^{27}+343x^{28} \\ &+171x^{29}+264x^{30}+124x^{31}+189x^{32}+85x^{33}+123x^{34}+56x^{35}+72x^{36}+35x^{37} \\ &+37x^{38}+20x^{39}+16x^{40}+10x^{41}+5x^{42}+4x^{43}+x^{44}+x^{45}. \end{split}$$

$$\mathcal{IB}_{10}(x) = 1 + 10 x + 36 x^{2} + 73 x^{3} + 157 x^{4} + 307 x^{5} + 456 x^{6} + 807 x^{7} + 1121 x^{8} + 1629 x^{9} \\ + 2323 x^{10} + 2835 x^{11} + 4124 x^{12} + 4508 x^{13} + 6468 x^{14} + 6715 x^{15} + 9256 x^{16} \\ + 9469 x^{17} + 12333 x^{18} + 12712 x^{19} + 15500 x^{20} + 16306 x^{21} + 18560 x^{22} + 20048 x^{23}$$

$$\begin{split} +21334\,x^{24}+23730\,x^{25}+23626\,x^{26}+27127\,x^{27}+25285\,x^{28}+29989\,x^{29}+26242\,x^{30} \\ +32053\,x^{31}+26550\,x^{32}+33126\,x^{33}+26310\,x^{34}+33138\,x^{35}+25641\,x^{36}+32124\,x^{37} \\ +24639\,x^{36}+30194\,x^{30}+23393\,x^{40}+27534\,x^{41}+21953\,x^{42}+24364\,x^{43}+23216\,x^{37} \\ +20635\,x^{45}+18657\,x^{46}+17519\,x^{47}+16839\,x^{48}+14343\,x^{49}+14925\,x^{50}+11549\,x^{51} \\ +12956\,x^{52}+9199\,x^{53}+10067\,x^{54}+7288\,x^{55}+0019\,x^{56}+5762\,x^{57}+7178\,x^{58} \\ +4663\,x^{59}+5525\,x^{60}+3633\,x^{61}+4107\,x^{62}+2999\,x^{63}+2962\,x^{64}+2331\,x^{65} \\ +2084\,x^{66}+1858\,x^{67}+1444\,x^{68}+1460\,x^{69}+986\,x^{70}+1123\,x^{71}+671\,x^{72}+834\,x^{73} \\ +454\,x^{74}+589\,x^{75}+312\,x^{76}+394\,x^{77}+217\,x^{78}+255\,x^{79}+156\,x^{80}+156\,x^{81} \\ +111\,x^{82}+91\,x^{83}+79\,x^{84}+52\,x^{85}+56\,x^{86}+30\,x^{87}+40\,x^{88}+17\,x^{89}+26\,x^{90} \\ +10\,x^{91}+15\,x^{92}+5\,x^{33}+5\,x^{94}+3\,x^{95}+2\,x^{96}+3\,x^{97}+x^{98}+3\,x^{99}+x^{100}. \end{split}$$

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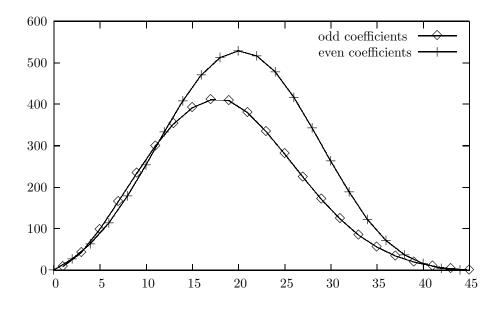


FIGURE 1. The coefficients  $[x^{2i}]\mathcal{I}_{10}^{inv}(x), [x^{2i+1}]\mathcal{I}_{10}^{inv}(x)$ 

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