A COMBINATORIAL PROOF OF GOTZMANN'S PERSISTENCE THEOREM FOR MONOMIAL IDEALS

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ABSTRACT. Gotzmann proved the persistence for minimal growth for ideals. His theorem is called Gotzmann's persistence theorem. In this paper, based on the combinatorics on binomial coefficients, a simple combinatorial proof of Gotzmann's persistence theorem in the special case of monomial ideals is given.

INTRODUCTION

Let K be an arbitrary field, $R = K[x_1, x_2, \ldots, x_n]$ the polynomial ring with $\deg(x_i) = 1$ for i = 1, 2, ..., n. Let M denote the set of variables $\{x_1, x_2, ..., x_n\}$, M^d the set of all monomials of degree d, where $M^0 = \{1\}$, and $\overline{M_i} = M \setminus \{x_i\}$. For a monomial $u \in R$ and for a subset $V \subset M^d$, we define $uV = \{uv | v \in V\}$ and $MV = \{x_i v | v \in V, i = 1, 2, ..., n\}$. For a finite set $V \subset M^d$, we write |V| for the number of the elements of V. Let gcd(V) denote the greatest common divisor of the monomials belonging to V.

Let n and h be positive integers. Then h can be written uniquely in the form, called the *n*th binomial representation of h,

$$h = \binom{h(n)+n}{n} + \binom{h(n-1)+n-1}{n-1} + \dots + \binom{h(i)+i}{i}$$

where $h(n) \ge h(n-1) \ge \cdots \ge h(i) \ge 0$, $i \ge 1$. See [3, Lemma 4.2.6]. Let $\binom{h(1)}{s(1)} + \binom{h(2)}{s(2)} + \cdots + \binom{h(i)}{s(i)}$ be a sum of binomials, where $h(j) \ge s(j)$ for any $j = 1, 2, \ldots, i$. Then we define

$$\left\{ \begin{pmatrix} h(1)\\s(1) \end{pmatrix} + \dots + \begin{pmatrix} h(i)\\s(i) \end{pmatrix} \right\}^{[+1]} = \begin{pmatrix} h(1)+1\\s(1) \end{pmatrix} + \dots + \begin{pmatrix} h(i)+1\\s(i) \end{pmatrix}.$$

Let $h = {\binom{h(n)+n}{n}} + \dots + {\binom{h(i)+i}{i}}$ be the *n*th binomial representation of *h*. We define

$$h^{} = \binom{h(n) + n + 1}{n} + \dots + \binom{h(i) + i + 1}{i},$$

$$h_{} = \binom{h(n) + n}{n - 1} + \dots + \binom{h(i) + i}{i - 1},$$

$$h_{\ll n\gg} = \binom{h(n) + n - 1}{n - 1} + \dots + \binom{h(i) + i - 1}{i - 1},$$

and set $0^{<n>} = 0_{<n>} = 0_{\ll n\gg} = 0$, $1^{<0>} = 1_{\ll 0\gg} = 1$ together with $1_{<0>} = 0$.

The inequality (1) below was proved by F. H. S. Macaulay. See also [3] and [7] for further infomation. Let V be a set of monomials of same degree. Then one has

$$|MV| \ge |V|^{\langle n-1\rangle}.\tag{1}$$

In 1978, Gotzmann [5] proved so-called persistence theorem. In the special case of monomial ideals, the persistence theorem says that

Theorem 0.1 (Persistence Theorem for monomial ideals). Let V be a set of monomials of degree d. If $|MV| = |V|^{\langle n-1 \rangle}$, then $|M^{i+1}V| = |M^iV|^{\langle n-1 \rangle}$ for all $i \geq 0$.

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{Z}_{\geq 0}^n$. The *lexico-graphic order* on \mathbb{Z}^n is defined by A < B if the leftmost nonzero entry of B - A is positive. Moreover, the lexicographic order on monomials of the same degree is defined by $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} < x_1^{b_1}x_2^{b_2}\ldots x_n^{b_n}$ if A < B on $\mathbb{Z}_{>0}^n$.

Let V be a set of monomials of degree d.

- (i) V is called a Gotzmann set if V satisfies $|MV| = |V|^{\langle n-1 \rangle}$.
- (ii) V is called a *lexsegment set* if V is a set of first |V| monomials in lexicographic order. Denote the lexsegment set V of $K[x_1, \ldots, x_n]$ in degree d with |V| = a by Lex(n, d, a).

It is known that lexsegment sets are Gotzmann sets. See [3, §4.2] or [7]. Also, in [8] we determined all integers a > 0 such that every Gotzmann set with |V| = aand with gcd(V) = 1 is lexsegment up to permutation of variables. Related works of Gotzmann's theorem were done by A. Aramova, J. Herzog and T. Hibi [2]. They proved Gotzmann's theorem for exterior algebra. In addition, Z. Furedi and J. R. Griggs [4] determine all integers a > 0 such that every squarefree Gotzmann set with |V| = a is squarefree lexsegment up to permutation.

The inequality (1) and Theorem 0.1 are true for more general case. They need not to be restricted to monomial case. Gotzmann [5] proved the persistence for minimal growth of the Hilbert function of a homogeneous ideal (see [7, Theorem C.17]). M. Green refined Gotzmann's proof (see [3, Theorem 4.3.3]). Green also give a simple proof in [6, Theorem 3.8] using generic initial ideals. On the other hand, in the special case of monomial ideals, in [5] Gotzmann proved the persistence theorem easier than general case using his version of the theory of Castelnuvo– Munford regularity. All of these proofs are completely algebraic. In the present paper we will give a combinatorial proof of persistence theorem for monomial ideals. The advantage of our proof is that we only use the combinatorics on binomials.

In $\S1$, we will prepare some lemmas about binomial representations. In $\S2$, we will give a combinatorial proof of persistence for monomial ideals.

1. BINOMIAL REPRESENTATIONS

In this section we consider some properties about binomial representation and combinatorics which will be used in the main proof. **Definition 1.1.** Let *h* be a positive integer and $h = \sum_{j=i}^{n} \binom{h(j)+j}{j}$ the *n*th binomial representation of *h*. Let $\alpha = \max\{0, \max\{\alpha \in \mathbb{Z} | h - \binom{\alpha+n}{n} > 0\}\}$. We denote $h - {\binom{\alpha+n}{n}}$ by $\bar{h}^{(n)}$, in other words,

- (i) if h = 1, then $\bar{h}^{(n)} = 0$;
- (ii) if h > 1 and i = n, then $\bar{h}^{(n)} = \binom{h(n)+n-1}{n-1};$ (iii) if h > 1 and i < n, then $\bar{h}^{(n)} = \sum_{j=i}^{n-1} \binom{h(j)+j}{j}.$

This construction says $\bar{h}^{(n)} \leq {\binom{\alpha+n}{n-1}}$ and $\bar{h}^{<n>} = {\binom{\alpha+n}{n}}^{<n>} + \bar{h}^{(n)< n-1>}$. Furthermore, if h > 1 then $\bar{h}^{(n)} > 1$.

Firstly, we introduce some easy and fundamental properties.

Lemma 1.2 ([3 Lemma 4.2.7]). Let $a = \sum_{k=i}^{n} \binom{h(k)}{k}$ and $a' = \sum_{k=j}^{n} \binom{h'(k)}{k}$ be the binomial representations. Then one has a < a' if and only if

$$(h(n), h(n-1), \dots, h(i), 0, \dots, 0) < (h'(n), h'(n-1), \dots, h'(j), 0, \dots, 0)$$

in the lexicographic order on \mathbb{Z}^n .

Lemma 1.3. Let h and n be integers with $h \ge 0$ and n > 0. Then, for any integer $1 \leq \alpha \leq h$, one has

$$\binom{h+n}{n} = \binom{\alpha-1+n}{n} + \binom{\alpha+n-1}{n-1} + \binom{\alpha+1+n-1}{n-1} + \dots + \binom{h+n-1}{n-1}$$

and

$$\binom{h+n}{n}^{[+1]} = \left\{ \binom{\alpha-1+n}{n} + \binom{\alpha+n-1}{n-1} + \binom{\alpha+1+n-1}{n-1} + \dots + \binom{h+n-1}{n-1} \right\}^{[+1]}$$

Proof. Use $\binom{n+n}{n} = \binom{n-1+n}{n} + \binom{n-1+n}{n-1}$ to the leftmost binomial coefficient repeatedly, then we have

$$\binom{h+n}{n} = \binom{h-2+n}{n} + \binom{h-1+n-1}{n-1} + \binom{h+n-1}{n-1}$$

$$\vdots$$

$$= \binom{\alpha-1+n}{n} + \binom{\alpha+n-1}{n-1} + \binom{\alpha+1+n-1}{n-1} + \dots + \binom{h+n-1}{n-1},$$

as desired. \Box

as desired.

Lemma 1.4. Let h and n be positive integers. Then,

$$h^{\langle n \rangle} = h + h_{\langle n \rangle}$$

Proof. Let $h = \sum_{j=i}^{n} {\binom{h(j)+j}{j}}$ be the *n*th binomial representation of *h*. Since ${\binom{h+n}{n}} = {\binom{h-1+n}{n}} + {\binom{h-1+n}{n-1}}$, one has

$$h + h_{\langle n \rangle} = \sum_{j=i}^{n} \left\{ \binom{h(j)+j}{j} + \binom{h(j)+j}{j-1} \right\} = \sum_{j=i}^{n} \binom{h(j)+j+1}{j} = h^{\langle n \rangle},$$

as desired.

Next, we introduce some lemmas which will be used in the main proof.

Lemma 1.5. Let a, b and n be positive integers. One has

 $a^{<n>} + b^{<n>} > (a+b)^{<n>}.$

Proof. Assume $n \geq 2$. Then we can take d with $|M^d| > a+b$. Let $V_a = Lex(n, d, a)$, $V_b = Lex(n, d, b)$ and u the minimal element of V_a in the lexicographic order. Let $V = x_1^{d+1}V_a \cup ux_nV_b$. Since $ux_1^{d+1} > ux_1^dx_n$, $x_1^{d+1}V_a \cup ux_nV_b$ is disjoint union if $n \geq 2$. Since $x_1^{d+1}x_nu \in Mx_1^{d+1}V_a \cap Mux_nV_b$, we have $Mx_1^{d+1}V_a \cap Mux_nV_b \neq \emptyset$. By (1) for any positive integer $n \geq 2$, we have

$$(a+b)^{} \le |MV| < |MV_a| + |MV_b| = a^{} + b^{},$$

as desired.

Lemma 1.6. Let a, b, c and α be positive integers. If $\binom{\alpha+n}{n} + a = b + c$ and $a, b, c < \binom{\alpha+n}{n}$, then one has

$$\binom{\alpha+n}{n}^{} + a^{} \le b^{} + c^{}.$$

Especially, if $\binom{\alpha+n}{n} <^{n>} + a^{<n>} = b^{<n>} + c^{<n>}$, then we have $\left\{ \binom{\alpha+n}{n} <^{n>} \right\} <^{n>} + \{a^{<n>}\} <^{n>} = \{b^{<n>}\} <^{n>} + \{c^{<n>}\} <^{n>}.$ (2)

Proof. We use induction on n.

[Case I] Let n = 1.

In general, if h is a positive integer, then $h^{<1>} = \binom{h+1}{1} = h+1$. Thus we have $\binom{\alpha+1}{1}^{<1>} + a^{<1>} = b+1+c+1 = b^{<1>} + c^{<1>}$. Thus we may assume n > 1.

To prove Lemma 1.6, we claim the followings:

(##) Let h and s be positive integers. Assume Lemma 1.6 is true in the case of n = s. If $\binom{h+s}{s} = \sum_{i=1}^{k} h_i + c - d$, $0 < h_i < \binom{h+s}{s}$ for i = 1, 2, ..., k, $k \ge 1$ and $\binom{h+s}{s} > c > d \ge 0$, then one has

$$\binom{h+s}{s}^{~~} \le \sum_{i=1}^{k} h_i^{~~} + c^{~~} - d^{~~}.~~~~~~~~$$

Especially, if $\binom{h+s}{s}^{<s>} = \sum_{i=1}^{k} h_i^{<s>} + c^{<s>} - d^{<s>}$, then we have $\left\{ \binom{h+s}{s}^{<s>} \right\}^{<s>} = \sum_{i=1}^{k} \{h_i^{<s>}\}^{<s>} + \{c^{<s>}\}^{<s>} - \{d^{<s>}\}^{<s>}.$

We will prove the claim. Since Lemmas 1.6 is true for n = s, we have

$$\binom{h+s}{s}^{~~} \le \{\sum_{i=1}^{k} h_i\}^{~~} + c^{~~} - d^{~~}.~~~~~~~~$$
(3)

Moreover, by Lemma 1.5, we have

$$\{\sum_{i=1}^{k} h_i\}^{~~} + c^{~~} \le \sum_{i=1}^{k} h_i^{~~} + c^{~~}.~~~~~~~~$$
(4)

Also, if $k \ge 2$ then (4) is not equal. If k = 1, then (3) is of the form Lemma 1.6. Thus by an assumption we proved the claim (##).

We return to the proof of Lemma 1.6. Let $a = \binom{a(n)+n}{n} + \bar{a}^{(n)}$, $b = \binom{b(n)+n}{n} + \bar{b}^{(n)}$ and $c = \binom{c(n)+n}{n} + \bar{c}^{(n)}$ be the form of Definition 1.1. Let $\bar{a} = \bar{a}^{(n)}$, $\bar{b} = \bar{b}^{(n)}$ and $\bar{c} = \bar{c}^{(n)}$. First, we note fundamental inequalities.

 $\begin{array}{l} (\alpha) \ a < b, a < c, \alpha > b(n), \ \alpha > c(n) \ \text{and} \ a(n) \leq c(n), \\ (\beta) \ \bar{b} \geq 1 \ \text{and} \ \bar{c} \geq 1, \end{array}$

$$(\gamma)$$
 $b < \binom{\alpha+n-1}{n-1}$ and $\bar{c} < \binom{\alpha+n-1}{n-1}$.

The inequality(α) follows from the assumption. We have the inequality(β) since $1 \leq a < b, c$. By Definition 1.1, we have $\bar{b} \leq {\binom{b(n)+n}{n-1}} \leq {\binom{\alpha+n-1}{n-1}}$. But if $\bar{b} = {\binom{b(n)+n}{n-1}}$, then $b(n) < \alpha - 1$ since $b = {\binom{b(n)+1+n}{n}} < {\binom{\alpha+n}{n}}$. Thus we have the inequality(γ). Next, by Lemma 1.3, we can write ${\binom{\alpha+n}{n}}$ and ${\binom{c(n)+n}{n}}$ as follows:

$$\binom{\alpha+n}{n} = \binom{b(n)+n}{n} + \sum_{i=b(n)+1}^{\alpha} \binom{i+n-1}{n-1};$$
$$\binom{c(n)+n}{n} = \binom{a(n)+n}{n} + \sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}.$$

Hence we substitute these equalities for $\binom{\alpha+n}{n} + a = b + c$, then we have

$$\left\{\sum_{i=b(n)+1}^{\alpha} \binom{i+n-1}{n-1}\right\} + \bar{a} = \bar{b} + \bar{c} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}.$$
(5)

Furthermore, $\binom{\alpha+n}{n}^{<n>}+a^{<n>}\leq b^{<n>}+c^{<n>}$ if and only if

$$\left\{\sum_{i=b(n)+1}^{\alpha} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{a}^{}$$

$$\leq \bar{b}^{} + \bar{c}^{} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}^{[+1]}.$$
 (6)

Instead of considering $\binom{\alpha+n}{n} + a = b + c$ and $\binom{\alpha+n}{n}^{<n>} + a^{<n>} \leq b^{<n>} + c^{<n>}$, it is enough to consider (5) and (6). We will consider two cases.

[Case II] Let $\bar{c} \geq \bar{a}$ and n > 1. We will prove that for $i = 0, 1, \ldots, \alpha - (b(n) + 1)$ $\binom{\alpha - i + n - 1}{n - 1}$ can be written

$$\binom{\alpha - i + n - 1}{n - 1} = -d_i + \sum_{j=t_{i+1}}^{t_i - 1} P_j + d_{i+1},\tag{7}$$

where $P_i = \binom{i+n-1}{n-1}$ for $i = a(n)+1, \ldots, c(n), t_{i+1} < t_i \le c(n) - i + 2, 0 \le d_i < P_{t_i-1}$ together with $P_{c(n)+1} = \bar{c}, t_0 = c(n) + 2, d_0 = \bar{a}$ and $d_{\alpha-b(n)} = \bar{b}$.

We use induction on *i*. For i = 0, since $\bar{c} - \bar{a} < {\alpha+n-1 \choose n-1}$, there exists $t_1 \leq c(n) + 1$ such that

$$\sum_{i=t_1}^{c(n)} \binom{i+n-1}{n-1} + \bar{c} - \bar{a} \le \binom{\alpha+n-1}{n-1} < \sum_{i=t_1-1}^{c(n)} \binom{i+n-1}{n-1} + \bar{c} - \bar{a}.$$

Thus we have

$$\binom{\alpha+n-1}{n-1} = \bar{c} - \bar{a} + \sum_{j=t_1}^{c(n)} P_j + d_1 = -d_0 + \sum_{j=t_1}^{c(n)+1} P_j + d_1$$

with $0 \leq d_1 < P_{t_1-1}$. Assume we have the form (7) for $i = 0, \ldots, s-1$. By the assumption of induction and $\alpha > c(n)$ we have $\binom{\alpha - s + n - 1}{n-1} \geq \binom{c(n) - s + 1 + n - 1}{n-1} \geq \binom{t_s - 1 + n - 1}{n-1} = P_{t_s-1}$. Thus $\binom{\alpha - s + n - 1}{n-1} \geq -d_{s+1} + P_{t_s-1}$ and $t_{s+1} < t_s$. By the same way of i = 0, we have (7) for i = s. Especially, if $s = \alpha - (b(n) + 1)$, because of the equality (5), we have

$$\binom{b(n)+n}{n-1} = -d_s + \sum_{j=a(n)+1}^{t_s-1} \binom{j+n-1}{n-1} + \bar{b}.$$
(8)

Thus each $\binom{\alpha - i + n - 1}{n - 1}$ have of the form (7).

Equalities (7) satisfies conditions of (##). By the assumption of induction of n, we have

$$\binom{\alpha-i-1+n-1}{n-1}^{< n-1>} \le -d_{i+1}^{< n-1>} + \sum_{j=t_{i+1}}^{t_i-1} P_j^{< n-1>} + d_{i+2}^{< n-1>}.$$
 (9)

Summating (7) in both sides yields (5), and summating inequalities (9) in both sides yields (6). Furthermore, (6) is equal if and only if (9) are equal for all *i*. Thus if (6) is equal, then (##) says (2) is satisfied.

[Case III] Let $\bar{c} < \bar{a}$ and n > 1. We will prove that for $i = 0, 1, \ldots, \alpha - (b(n) + 1)$

$$\binom{\alpha - i + n - 1}{n - 1} = d_i + \sum_{j = t_{i+1}}^{t_i - 1} \binom{j + n - 1}{n - 1} - d_{i+1}$$
(10)

and

$$\bar{a} = \bar{c} + d_{\alpha-b(n)}, \tag{11}$$

where $0 \leq d_i < {t_i+n-1 \choose n-1}$ and $t_{i+1} < t_i \leq c(n) - i + 1$ together with $d_0 = \bar{b}$, $t_0 = c(n) + 1$ and $t_{\alpha-b(n)} = a(n) + 1$.

For i = 0, since $\binom{\alpha+n-1}{n-1} > \overline{b}$, by the same way of [Case II] we have

$$\binom{\alpha+n-1}{n-1} = \bar{b} + \sum_{j=t_1}^{c(n)} \binom{j+n-1}{n-1} - d_1.$$

Also, if we have equality (10) for $i = 0, 1, \ldots, s - 1$, then we have $\binom{\alpha - s + n - 1}{n - 1} \geq 1$ $\binom{c(n)-(s-1)+n-1}{n-1} \ge \binom{t_s+n-1}{n-1} > d_s$. Thus we have $t_{s+1} < t_s$ and we have equality (10) for i = s by the same way. Finally, since $\bar{a} - \bar{c} < \bar{a} \leq {a(n)+n \choose n-1}$ by definition of \bar{a} , we have $\bar{a} = \bar{c} + d_{\alpha-b(n)}$ and $t_{\alpha-b(n)} = a(n) + 1$. Equalities (10) satisfies the conditions of (##). Thus by the assumption of induction of n, we have

$$\binom{\alpha-i+n-1}{n-1}^{< n-1>} \le d_i^{< n-1>} + \sum_{j=t_{i+1}}^{t_i-1} \binom{j+n-1}{n-1}^{< n-1>} - d_{i+1}^{< n-1>}.$$
 (12)

Furthermore, since $\bar{c} > 0$ and $d_{\alpha-b(n)} > 0$ we have

$$\bar{a}^{\langle n-1 \rangle} < \bar{c}^{\langle n-1 \rangle} + d_{s+1}^{\langle n-1 \rangle}.$$
 (13)

Then, by summating (10) and (11), we have the equality (5). By summating inequalities (12) and (13), we have

$$\left\{\sum_{i=b(n)+1}^{\alpha} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{a}^{} < \bar{b}^{} + \bar{c}^{} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{a}^{} + \bar{c}^{} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{a}^{} + \bar{c}^{} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{a}^{} + \bar{c}^{} + \left\{\sum_{i=a(n)+1}^{c(n)} \binom{i+n-1}{n-1}\right\}^{[+1]} + \bar{c}^{} + \bar{c}^{$$

In this case (6) is not equal. Thus we need not consider the equality (2).

Lemma 1.7. Let h and n be positive integers. Then, one has

$$h^{} < h^{}.$$

Proof. Let $h = \binom{h(n+1)+n+1}{n+1} + \bar{h}^{(n+1)}$. Then $h^{< n+1>} = \binom{h(n+1)+n+1}{n+1}^{< n+1>} + \bar{h}^{(n+1)<n>}$. By Lemma 1.3, we have

$$\binom{h(n+1)+n+1}{n+1} = \binom{n+1}{n+1} + \sum_{i=1}^{h(n+1)} \binom{i+n}{n}$$

Furthermore, we have $\binom{n+1}{n+1}^{[+1]} > \binom{n}{n}^{[+1]}$. By Lemma 1.5, we have

$$\binom{h(n+1)}{n+1}^{< n+1>} + \bar{h}^{(n+1)} > \binom{n}{n}^{< n>} + \sum_{i=1}^{h(n+1)} \binom{i+n}{n}^{< n>} + \bar{h}^{(n+1)}$$

$$\ge \{\binom{n}{n} + \sum_{i=1}^{h(n+1)} \binom{i+n}{n} + \bar{h}^{(n+1)}\}^{< n>} = h^{< n>},$$

 is desired.

as desired.

2. A COMBINATORIAL PROOF OF PERSISTENCE FOR MONOMIAL IDEALS

Let V be a set of monomials of degree d and u = gcd(V). If |V| > 1, we define $K_i(V) = \{v \in M^d | x_i u \text{ divides } v\}$ and $D_i(V) = V \setminus K_i(V)$ for i = 1, 2, ..., n. If |V| = 1, then we define $K_i(V) = V$ and $D_i(V) = \emptyset$. Note that if |V| > 1, then $D_i(V) \neq \emptyset$ and $K_i(V) \neq \emptyset$.

Before giving a combinatorial proof of persistence theorem for monomial ideals, we prepare some lemmas.

Lemma 2.1. Let V be a set of monomials of degree d and u = gcd(V). For any i = 1, 2, ..., n, we have

$$\overline{M_i} \mathcal{D}_i(V) \subset MV \setminus x_i V. \tag{14}$$

$$|MV| \ge |\mathbf{K}_i(V)|^{} + |\mathbf{D}_i(V)|^{}.$$
(15)

Moreover, in (15), the equality holds if and only if $K_i(V)$ is a Gotzmann set of $K[x_1, x_2, \ldots, x_n]$, $\frac{1}{u}D_i(V)$ is a Gotzmann set of $K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ and $x_iD_i(V) \subset \overline{M_i}K_i(V)$.

Proof. Any element of $\overline{M_i}D_i(V)$ can not be divided by ux_i . On the other hand, $\overline{M_i}D_i(V) \subset MV$. Thus we have $\overline{M_i}D_i(V) \subset MV \setminus x_iV$.

Now we have

$$|MV| = |MK_i(V)| + |MD_i(V)| - |\{MK_i(V) \cap MD_i(V)\}|$$

Now we have $MK_i(V) \cap MD_i(V) = MK_i(V) \cap x_iD_i(V) \subset x_iD_i(V)$ and $|MD_i(V)| = |x_iD_i(V)| + |\overline{M_i}D_i(V)|$. On the other hand, the inequality (1) says $|\overline{M_i}D_i(V)| = |D_i(V)|^{\leq n-2}$ since $\frac{1}{u}D_i(V) \subset K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Thus we have

$$|MV| \geq |M\mathbf{K}_i(V)| + |\overline{M_i}\mathbf{D}_i(V)|$$

$$\geq |\mathbf{K}_i(V)|^{} + |\mathbf{D}_i(V)|^{}.$$

Especially, equality holds if and only if $K_i(V)$ and $\frac{1}{u}D_i(V)$ are Gotzmann sets and $MK_i(V) \cap x_iD_i(V) = x_iD_i(V)$.

Next we determine the range of $|D_i(V)|$, when V is a Gotzmann set.

Lemma 2.2. Let V be a Gotzmann set of monomials of degree d. Then, for any i = 1, 2, ..., n, we have

$$\overline{|V|}^{(n-1)} \le |\mathcal{D}_i(V)| \le |V|_{\ll n-1 \gg}.$$
(16)

Proof. If |V| = 0 or |V| = 1, then $\overline{|V|}^{(n-1)} = |D_i(V)| = 0$. Thus we may assume n > 1 and |V| > 1. First, we consider the second inequality of (16). By Lemma 2.1 and by the inequality (1), we have

$$|\mathbf{D}_i(V)|^{} \le |\overline{M_i}\mathbf{D}_i(V)| \le |(MV \setminus x_iV)|.$$

On the other hand, by Lemma 1.4, we have

$$|(MV \setminus x_i V)| = |MV| - |V| = |V|^{} - |V|$$

= $|V|_{}$.

Thus $|D_i(V)|^{<n-2>} \le |V|_{<n-1>}$. Hence we have $|D_i(V)| \le |V|_{\ll n-1\gg}$.

We consider the first inequality of (16). If n = 2, then $\overline{|V|}^{(n-1)} = |D_i(V)| = 1$. Thus we may assume $n \ge 3$. Let $|V| = \binom{a+n-1}{n-1} + \overline{|V|}^{(n-1)}$. If $|D_i(V)| < \overline{|V|}^{(n-1)}$, then $|K_i(V)| = |V| - |D_i(V)| > \binom{a+n-1}{n-1}$. Thus we can write $|K_i(V)| = \binom{a+n-1}{n-1} + b$ with b > 0. By Lemma 2.1, we have

$$|MV| \geq |\mathbf{K}_{i}(V)|^{} + |\mathbf{D}_{i}(V)|^{} \\ = \binom{a+n-1}{n-1}^{} + b^{} + |\mathbf{D}_{i}(V)|^{}.$$

On the other hand, by Lemma 1.5, we have

$$b^{\langle n-2\rangle} + |\mathcal{D}_i(V)|^{\langle n-2\rangle} > \{b + |\mathcal{D}_i(V)|\}^{\langle n-2\rangle} = (\overline{|V|}^{(n-1)})^{\langle n-2\rangle}.$$

Thus we have

$$|MV| > \binom{a+n-1}{n-1}^{< n-1>} + (\overline{|V|}^{(n-1)})^{< n-2>} = |V|^{< n-1>}.$$

This is a contradiction since V is a Gotzmann set.

Now, we finished all preparations for following lemma which proves the Persistence Theorem immediately.

Lemma 2.3. Let V be a Gotzmann set of monomials of degree d with gcd(V) = 1and $V \neq M^d$. Then there exist $i \in \{1, 2, ..., n\}$ which satisfies followings:

- (i) $K_i(V)$ is a Gotzmann set of $K[x_1, \ldots, x_n]$, $D_i(V)$ is a Gotzmann set of $K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ and $|D_i(V)| < |V|_{\ll n-1 \gg}$;
- (ii) $x_i D_i(V) \subset \overline{M_i} K_i(V);$ (iii) $\{ |K_i(V)|^{<n-1>} \}^{<n-1>} + \{ |D_i(V)|^{<n-2>} \}^{<n-2>} = \{ |V|^{<n-1>} \}^{<n-1>}.$

Proof. Now, we set $|V| = a = \sum_{j=p}^{n-1} {a(j)+j \choose j}$, $|\mathbf{D}_i(V)| = b = \sum_{j=q}^{n-2} {b(j)+j \choose j}$ and $|\mathbf{K}_i(V)| = c = \sum_{j=r}^{n-1} {c(j)+j \choose j}$ be the binomial representations. Set $V \neq \emptyset$. [Case(A)] Let |V| = 1 or n = 1.

If |V| = 1, then $V = M^0$ since gcd(V) = 1. If n = 1, then |V| = 1. Thus we may assume |V| > 1 and n > 1.

By Lemma 2.1, if $a^{\langle n-1 \rangle} \leq b^{\langle n-2 \rangle} + c^{\langle n-1 \rangle}$, then $a^{\langle n-1 \rangle} = b^{\langle n-2 \rangle} + c^{\langle n-1 \rangle}$ and $K_i(V)$ and $D_i(V)$ are Gotzmann sets. Thus conditions (i) and (ii) are satisfied. In [Case(B)] and [Case(C)], we will prove that if $b < a_{\ll n-1 \gg}$ then $a^{\langle n-1 \rangle} \leq b^{\langle n-2 \rangle} + c^{\langle n-1 \rangle}$.

If $b < a_{\ll n-1}$, then, by Lemma 1.2, there exists a maximal integer t, such that $n-1 \ge t \ge p$ and

$$0 \le b - \sum_{j=t+1}^{n-1} \binom{a(j)+j-1}{j-1} < \binom{a(t)+t-1}{t-1}.$$

Let

$$a = \sum_{j=t+1}^{n-1} {a(j)+j \choose j} + {a(t)+t \choose t} + a',$$
(17)

$$b = \sum_{j=t+1}^{n-1} {a(j)+j-1 \choose j-1} + b',$$
(18)

and
$$c = a - b = \sum_{j=t+1}^{n-1} {a(j) + j - 1 \choose j} + c'.$$
 (19)

Since $0 \leq b' < \binom{a(t)+t-1}{t-1}$, we have $\binom{a(t)+t-1}{t} < c' < \binom{a(t)+t+1}{t}$. Also, we have

$$a^{\langle n-1 \rangle} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j)+j}{j} \right\}^{[+1]} + \binom{a(t)+t}{t}^{[+1]} + a^{\langle \ell-1 \rangle}$$
(20)

and
$$b^{\langle n-2\rangle} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j)+j-1}{j-1} \right\}^{[+1]} + b^{\langle \ell-1\rangle}.$$
 (21)

 $\begin{bmatrix} \mathbf{Case}(\mathbf{B}) \end{bmatrix} \text{Let } b < a_{\ll n-1 \gg} \text{ and } c' < \binom{a(t)+t}{t}.$ Let $c'' = c' - \binom{a(t)+t-1}{t}$. If b' = 0, then $c' \ge \binom{a(t)+t}{t}$. Thus b' > 0. On the other hand, we have c'' > 0 since $c' > \binom{a(t)+t-1}{t}$. Since $c'' < \binom{a(t)+t-1}{t-1}$, $c = \sum_{j=t}^{n-1} \binom{a(j)+j-1}{j} + \{ (t-1) \text{th binomial representation of } c'' \}$ is (n-1)th binomial representation of c. Thus

$$c^{\langle n-1\rangle} = \left\{ \sum_{j=t}^{n-1} \binom{a(j)+j-1}{j} \right\}^{[+1]} + c''^{\langle t-1\rangle}.$$

Thus, by (21), we have

$$b^{\langle n-2\rangle} + c^{\langle n-1\rangle} = \left\{ \sum_{j=t+1}^{n-1} \binom{a(j)+j}{j} \right\}^{[+1]} + \binom{a(t)+t-1}{t}^{[+1]} + b'^{\langle t-1\rangle} + c''^{\langle t-1\rangle}.$$
(22)

Since $\binom{a(t)+t}{t} = \binom{a(t)+t-1}{t} + \binom{a(t)+t-1}{t-1}$ and a = b + c together with (17), (18) and (19) say $b' + c'' = a' + \binom{a(t)+t-1}{t-1}$. Hence by Lemma 1.5, Lemma 1.6 together with b' > 0and c'' > 0, we have

$$b^{\prime < t-1>} + c^{\prime\prime < t-1>} \ge a^{\prime < t-1>} + \binom{a(t) + t - 1}{t - 1}^{< t-1>}.$$
(23)

Thus by (20) and (22), we have $a^{(n-1)} \leq b^{(n-2)} + c^{(n-1)}$. Furthermore, if (23) is equal, then Lemma 1.6 says

$$\{b^{\prime < t-1>}\}^{< t-1>} + \{c^{\prime\prime < t-1>}\}^{< t-1>} = \{a^{\prime < t-1>}\}^{< t-1>} + \left\{\binom{a(t) + t - 1}{t - 1}^{< t-1>}\right\}^{< t-1>}.$$

Thus we have $\{a^{< n-1>}\}^{< n-1>} = \{b^{< n-2>}\}^{< n-2>} + \{c^{< n-1>}\}^{< n-1>}.$ [Case(C)] Let $b < a_{\ll n-1\gg}$ and $c' \ge \binom{a(t)+t}{t}.$

Let $c'' = c' - \binom{a(t)+t}{t}$ and $\alpha = \max\{i|a(i) = a(t)\}$. Since $\sum_{j=t+1}^{\alpha} \binom{a(j)+j-1}{j} + \binom{a(t)+t}{t} = \binom{a(\alpha)+\alpha}{\alpha}$ and $c'' < \binom{a(t)+t}{t-1} \le \binom{a(\alpha)+\alpha}{\alpha-1}$, we have

$$c^{} = \left\{ \sum_{j=\alpha+1}^{n-1} \binom{a(j)+j-1}{j} \right\}^{[+1]} + \binom{a(\alpha)+\alpha}{\alpha}^{[+1]} + c''^{<\alpha-1>}$$
$$= \left\{ \sum_{j=t+1}^{n-1} \binom{a(j)+j-1}{j} + \binom{a(t)+t}{t} \right\}^{[+1]} + c''^{<\alpha-1>}.$$

Thus, by (21), we have

$$b^{\langle n-2\rangle} + c^{\langle n-1\rangle} = \left\{ \sum_{j=t}^{n-1} \binom{a(j)+j}{j} \right\}^{[+1]} + b^{\langle t-1\rangle} + c^{\prime\prime\langle \alpha-1\rangle}.$$
(24)

Since a = b + c together with (17), (18) and (19), we have a' = b' + c''. By Lemmas 1.5 and 1.7, we have

$$a'^{} \le b'^{} + c''^{} \le b'^{} + c''^{<\alpha-1>}.$$
(25)

Hence by (20) and (24) we have $a^{\langle n-1 \rangle} \leq b^{\langle n-2 \rangle} + c^{\langle n-1 \rangle}$. Furthermore, if the inequality (25) is equal, then c' = 0 or b' = 0 and $\alpha = t$. In each case, we have $\{a'^{\langle t-1 \rangle}\}^{\langle t-1 \rangle} = \{b'^{\langle t-1 \rangle}\}^{\langle t-1 \rangle} + \{c''^{\langle \alpha-1 \rangle}\}^{\langle \alpha-1 \rangle}$. Hence we have $\{a^{\langle n-1 \rangle}\}^{\langle n-1 \rangle} = \{b^{\langle n-2 \rangle}\}^{\langle n-2 \rangle} + \{c^{\langle n-1 \rangle}\}^{\langle n-1 \rangle}$.

[Case(D)] Let $b = a_{\ll n-1 \gg}$.

By Lemma 2.1, we have $\overline{M_i}D_i(V) \subset MV \setminus x_iV$. But, by (1) we have $|\overline{M_i}D_i(V)| \ge b^{<n-2>}$. Now, we have $a_{<n-1>} = a^{<n-1>} - a = |(MV \setminus x_iV)|$ and $b^{<n-2>} = a_{<n-1>}$. Thus we have $\overline{M_i}D_i(V) = MV \setminus x_iV$.

By [Case(B)] and [Case(C)], if $|D_i(V)| < a_{\ll n-1}$ for some *i*, then we have conditions (i), (ii) and (iii). Finally, we will prove that if $|D_i(V)| = a_{\ll n-1}$ for i = 1, 2, ..., n, then $V = M^d$ or $V = \emptyset$. In [Case(D)], we see $\overline{M_i}D_i(V) = MV \setminus x_iV$ if $|D_i(V)| = a_{\ll n-1}$. We claim (#).

(#) Assume $|D_i(V)| = a_{\ll n-1\gg}$ for i = 1, 2, ..., n. If there exist a monomial $v \in M^d$ such that $v \notin V$, then for any x_j and x_i with $x_i | v$, one has $\frac{x_j}{x_i} v \notin V$.

To see why (#) is true, we assume that $v \notin V$ and there exist x_i and x_j such that $\frac{x_j}{x_i}v \in V$. Since $v \notin V$, we have $x_jv \notin x_jV$. Thus we have $x_i\frac{x_j}{x_i}v = x_jv \in MV \setminus x_jV = \overline{M_j}D_j(V)$. But any element in $\overline{M_j}D_j(V)$ does not contain x_j since gcd(V) = 1, this is a contradiction.

By using (#), if there exists a monomial $v \in M^d$ such that $v \notin V$, then all monomials in M^d do not belong to V. Hence we have $V = M^d$ or $V = \emptyset$.

We are now in the position to finish our combinatorial proof of persistence theorem for monomial ideals.

Proof of persistence theorem for monomial ideals. What we have to prove is that if V is a Gotzmann set then MV is also a Gotzmann set.

Let V be a Gotzmann set of degree d. We use induction on |V|. Firstly, for any monomial $u \in R$, V is a Gotzmann set if and only if uV is a Gotzmann set since |V| = |uV| and |MV| = |uMV|. Thus we may assume gcd(V) = 1.

If $V = M^d$, then MV is also a Gotzmann set. If |V| = 1 then $V = M^0$.

If $V \neq M^d$ and |V| > 1. Lemma 2.3 (ii) says there exists $i \in \{1, 2, ..., n\}$ such that $\overline{M_i}K_i(V) \supset x_iD_i(V)$ and $M^2K_i(V) \supset \overline{M_i}^2K_i(V) \supset x_i\overline{M_i}D_i(V)$. Thus $|MV| = |MK_i(V)| + |\overline{M_i}D_i(V)|$ and $|M^2V| = |M^2K_i(V)| + |\overline{M_i}^2D_i(V)|$. By Lemma 2.3 (i) and by assumption of induction, both $MK_i(V)$ and $\overline{M_i}D_i(V)$ are Gotzmann sets. Hence by Lemma 2.3 (iii), we have

$$|M^{2}V| = |M^{2}K_{i}(V)| + |\overline{M_{i}}^{2}D_{i}(V)|$$

= {|K_{i}(V)|^{(n-1)}}^{(n-1)} + {|D_{i}(V)|^{(n-2)}}^{(n-2)}
= {|V|^{(n-1)}}^{(n-1)}
= {|MV|}^{(n-1)}.

This completes the proof.

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