# ACTIONS ON PERMUTATIONS AND UNIMODALITY OF DESCENT POLYNOMIALS 

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#### Abstract

We study a group action on permutations due to Foata and Strehl and use it to prove that the descent generating polynomial of certain sets of permutations has a nonnegative expansion in the basis $\left\{t^{i}(1+t)^{n-1-2 i}\right\}_{i=0}^{m}$, $m=\lfloor(n-1) / 2\rfloor$. This property implies symmetry and unimodality. We prove that the action is invariant under stack-sorting which strengthens recent unimodality results of Bóna. We prove that the generalized permutation patterns $(13-2)$ and $(2-31)$ are invariant under the action and use this to prove unimodality properties for a $q$-analog of the Eulerian numbers recently studied by Corteel, Postnikov, Steingrímsson and Williams.

We also extend the action to linear extensions of sign-graded posets to give a new proof of the unimodality of the $(P, \omega)$-Eulerian polynomials of sign-graded posets and a combinatorial interpretations (in terms of Stembridge's peak polynomials) of the corresponding coefficients when expanded in the above basis.

Finally, we prove that the statistic defined as the number of vertices of even height in the unordered decreasing tree of a permutation has the same distribution as the number of descents on any set of permutations invariant under the action. When restricted to the set of stack-sortable permutations we recover a result of Kreweras.


## 1. Introduction

The $n$-th Eulerian polynomial, $A_{n}(t)=A_{n 1}+A_{n 2} t+\cdots+A_{n(n-1)} t^{n-1}$, may be defined as the generating polynomial for the number of descents over the symmetric group $\mathfrak{S}_{n}$, i.e.,

$$
A_{n}(t)=\sum_{\pi \in \mathfrak{S}_{n}} t^{\operatorname{des}(\pi)}
$$

where $\operatorname{des}(\pi)=\left|\left\{i: a_{i}>a_{i+1}\right\}\right|$ and where $\pi: i \rightarrow a_{i}(1 \leq i \leq n)$ is identified with the word $a_{1} a_{2} \cdots a_{n}$ in the distinct $n$ letters $a_{1}, \ldots, a_{n}$ taken out of $[n]:=$ $\{1,2, \ldots, n\}$.

In a series of papers [20, 23, 24] Foata and Strehl studied a group action on the symmetric group, $\mathfrak{S}_{n}$, with the following properties. The number of orbits is the $n$-th tangent number or secant number, according as $n$ is odd or even, and if an orbit, $\operatorname{Orb}(\pi)$, of a permutation $\pi \in \mathfrak{S}_{n}$ is enumerated according to the number of descents then

$$
\begin{equation*}
\sum_{\sigma \in \operatorname{Orb}(\pi)} t^{\operatorname{des}(\sigma)}=(2 t)^{\mathrm{v}(\pi)}(1+t)^{n-1-2 \mathrm{v}(\pi)} \tag{1.1}
\end{equation*}
$$

[^0]where $\mathrm{v}(\pi)=\left|\left\{i: a_{i-1}>a_{i}<a_{i+1}\right\}\right|$. From (1.1) it follows that $A_{n}(t)$ has nonnegative coefficients when expanded in the basis $\left\{t^{k}(1+t)^{n-1-2 k}\right\}_{k=0}^{\lfloor(n-1) / 2\rfloor}$, a result which can also be proven analytically [14, 22]. This implies that the sequence $\left\{A_{n i}\right\}_{i=0}^{n-1}$ is symmetric and unimodal, i.e., that $A_{n i}=A_{n(n-1-i)}, 1 \leq i \leq n-1$ and
$$
A_{n 0} \leq A_{n 1} \leq \cdots \leq A_{n c} \geq A_{n(c+1)} \geq \cdots \geq A_{n(n-1)}
$$
where $c=\lfloor(n-1) / 2\rfloor$. Indeed, $\left\{A_{n i}\right\}_{i=0}^{n-1}$ is a nonnegative sum of unimodal and symmetric sequences with the same center of symmetry.

We will in this paper study a slightly modified version of the Foata-Strehl action and show that interesting subsets of $\mathfrak{S}_{n}$ are invariant under the action. In particular we show that the set of $r$-stack sortable permutations is invariant under the action which strengthens the recent result of Bóna [4, 5] claiming that the corresponding descent generating polynomial is symmetric and unimodal.

In Section 5 we prove that the generalized permutation patterns (13-2) and $(2-31)$ are invariant under the modified Foata-Strehl action. This is used to prove unimodality properties for a $q$-analog of the Eulerian numbers recently studied by Corteel, Postnikov, Steingrímsson and Williams [17, 18, 36, 42, 51 and which appears as a translation of the polynomial enumerating the cells in the totally nonnegative part of a Grassmannian [36, 51, and also in the stationary distribution of the ASEP model in statistical mechanics [17, 18].

We will in Section 6 define an action on the set of linear extensions of a signgraded poset, see Section 6 for relevant definitions. This enables us to give a combinatorial interpretation in terms of Stembridge's peak polynomials of the coefficients of the $(P, \omega)$-Eulerian polynomials when expanded in the basis $\left\{t^{i}(1+t)^{d-2 i}\right\}_{i=0}^{\lfloor d / 2\rfloor}$, $d=|P|-r-1$.

In Section 7 we study the statistic $\pi \rightarrow \operatorname{veh}(\pi)$ on permutations which is defined as the number of vertices of even height in the unordered increasing tree of $\pi$. We prove that veh has the same distribution as des on every subset of $\mathfrak{S}_{n}$ invariant under the action. This can be seen as a generalization of a result of Kreweras [32]. In Section 8.2 we also find a Mahonian partner for veh.

Finally, in Section 10, we discuss further directions and open problems.

## 2. The Action of Foata and Strehl

Let $\pi=a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$ and let $x \in[n]$. We may write $\pi$ as the concatenation $\pi=w_{1} w_{2} x w_{4} w_{5}$ where $w_{2}$ is the maximal contiguous subword immediately to the left $x$ whose letters are all smaller than $x$, and $w_{4}$ is the maximal contiguous subword immediately to the right of $x$ whose letters are all smaller than $x$. This is the $x$-factorization of $\pi$. Define $\varphi_{x}(\pi)=w_{1} w_{4} x w_{2} w_{5}$. Then $\varphi_{x}$ is an involution acting on $\mathfrak{S}_{n}$ and it is not hard to see that $\varphi_{x}$ and $\varphi_{y}$ commute for all $x, y \in[n]$. Hence for any subset $S \subseteq[n]$ we may define the function $\varphi_{S}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ by

$$
\varphi_{S}(\pi)=\prod_{x \in S} \varphi_{x}(\pi)
$$

The group $\mathbb{Z}_{2}^{n}$ acts on $\mathfrak{S}_{n}$ via the functions $\varphi_{S}, S \subseteq[n]$. This action was studied by Foata and Strehl in [20, 23, 24]. To be precise, Foata and Strehl defined the action as $C \circ \varphi_{S} \circ C$, where $C: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ is the involution described by $a_{1} a_{2} \cdots a_{n} \mapsto$ $b_{1} b_{2} \cdots b_{n}$, where $b_{i}=n+1-a_{i}, 1 \leq i \leq n$. Sometimes it is preferable to define the action on the decreasing binary tree of the permutation. The decreasing binary tree
of a permutation of a finite subset of $\{1,2,3, \ldots\}$ is defined recursively as follows. The empty tree corresponds to the empty word. If $\pi$ is non-empty then we may write $\pi$ as the concatenation $\pi=L m R$ where $m$ and $L$ and $R$ are the subwords to the left and right of $m$ respectively. The tree corresponding to $\pi$ has a root labeled $m$ and as left subtree the tree corresponding to $L$ and as right subtree the tree corresponding to $R$. This describes a bijective correspondence between the set of decreasing binary trees with labels $[n]$ and $\mathfrak{S}_{n}$. It is not hard to see that the tree of $\varphi_{x}(\pi)$ is obtained by exchanging the subtrees rooted at $x$, if any. Another action on permutations with similar properties was studied by Hetyei and Reiner [30] and subsequently by Foata and Han [21].

Let $\pi=a_{1} a_{2} \cdots a_{n}$ be a permutation in $\mathfrak{S}_{n}$ and let $a_{0}=a_{n+1}=n+1$. If $k \in[n]$ then $a_{k}$ is a
valley if $a_{k-1}>a_{k}<a_{k+1}$,
peak if $a_{k-1}<a_{k}>a_{k+1}$,
double ascent if $a_{k-1}<a_{k}<a_{k+1}$, and
double descent if $a_{k-1}>a_{k}>a_{k+1}$.
Let $x \in[n]$ and let $\pi=a_{1} a_{2} \ldots a_{n} \in \mathfrak{S}_{n}$. We make the following observation.

- If $x$ is a double descent then $\varphi_{x}(\pi)$ is obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the right of $x$ such that $a_{i}<x<a_{i+1}$.
- If $x$ is a double ascent then $\varphi_{x}(\pi)$ is obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the left of $x$ such that $a_{i}>x>a_{i+1}$.
We modify the Foata-Strehl action in the following way. If $x \in[n]$ then

$$
\varphi_{x}^{\prime}(\pi)= \begin{cases}\varphi_{x}(\pi) & \text { if } x \text { is a double ascent or double descent } \\ \pi & \text { if } x \text { is a valley or a peak }\end{cases}
$$

The functions are easily visualized when a permutation is represented graphically. Let $\pi=a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$ and imagine marbles at coordinates $\left(i, a_{i}\right)$, $i=0,1, \ldots, n+1$ in the $\operatorname{grid} \mathbb{N} \times \mathbb{N}$. For $i=0,1, \ldots, n$ connect $\left(i, a_{i}\right)$ and $\left(i+1, a_{i+1}\right)$ with a wire. Suppose that gravity acts on the marbles from above and suppose that $x$ is not at an equilibrium. If we release $x$ from the wire it will slide and stop when it has reached the same height again. The resulting permutation will be $\varphi_{x}^{\prime}(\pi)$, see Fig. 1] The functions $\varphi_{x}^{\prime}$ were studied by Shapiro, Woan and Getu unaware ${ }^{11}$ that they are essentially the same as the functions defining the Foata-Strehl action.

Again it is clear that the $\varphi_{x}^{\prime}$ 's are involutions and that they commute. Hence, for any subset $S \subseteq[n]$ we may define the function $\varphi_{S}^{\prime}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ by

$$
\varphi_{S}^{\prime}(\pi)=\prod_{x \in S} \varphi_{x}^{\prime}(\pi)
$$

Hence the group $\mathbb{Z}_{2}^{n}$ acts on $\mathfrak{S}_{n}$ via the functions $\varphi_{S}^{\prime}$, $S \subseteq[n]$. Subsequently we will refer to this action as the modified Foata-Strehl action, or the MFS-action for short.

## 3. Properties of the Modified Foata-Strehl Action

For $\pi \in \mathfrak{S}_{n}$ let $\operatorname{Orb}(\pi)=\left\{g(\pi): g \in \mathbb{Z}_{2}^{n}\right\}$ be the orbit of $\pi$ under the MFSaction. There is a unique element in $\operatorname{Orb}(\pi)$ which has no double descents and

[^1]Figure 1. Graphical representation of $\pi=573148926$. The dotted lines indicates where the double ascents/descents move to.

which we denote by $\hat{\pi}$. The next theorem follows from the work in [24, 46], but we prove it here for completeness.

Theorem 3.1. Let $\pi \in \mathfrak{S}_{n}$. Then

$$
\sum_{\sigma \in \operatorname{Orb}(\pi)} t^{\operatorname{des}(\sigma)}=t^{\operatorname{des}(\hat{\pi})}(1+t)^{n-1-2 \operatorname{des}(\hat{\pi})}=t^{\operatorname{peak}(\pi)}(1+t)^{n-1-2 \operatorname{peak}(\pi)}
$$

where $\operatorname{peak}(\pi)=\left|\left\{i: a_{i-1}<a_{i}>a_{i+1}\right\}\right|$.
Proof. If $x$ is a double ascent in $\pi$ then $\operatorname{des}\left(\varphi_{x}^{\prime}(\pi)\right)=\operatorname{des}(\pi)+1$. It follows that

$$
\sum_{\sigma \in \operatorname{Orb}(\pi)} t^{\operatorname{des}(\sigma)}=t^{\operatorname{des}(\hat{\pi})}(1+t)^{a}
$$

where $a$ is the number of double ascents in $\hat{\pi}$. If we delete all double descents from $\hat{\pi}$ we get an alternating permutation

$$
n+1>b_{1}<b_{2}>b_{3}<\cdots>b_{n-a}<n+1
$$

with the same number of descents. Hence $n-a=2 \operatorname{des}(\hat{\pi})+1$. Clearly $\operatorname{des}(\hat{\pi})=$ $\operatorname{peak}(\pi)$ and the theorem follows.

For a subset $T$ of $\mathfrak{S}_{n}$ let

$$
W(T ; t)=\sum_{\pi \in T} t^{\operatorname{des}(\pi)} \quad \text { and } \quad \bar{W}(T ; t)=\sum_{\pi \in T} t^{\operatorname{peak}(\pi)}
$$

Corollary 3.2. Suppose that $T \subseteq \mathfrak{S}_{n}$ is invariant under the MFS-action. Then

$$
W(T ; t)=2^{-n+1}(1+t)^{n-1} \bar{W}\left(T ; 4 t(1+t)^{-2}\right) .
$$

Equivalently

$$
W(T ; t)=\sum_{i=0}^{\lfloor n / 2\rfloor} b_{i}(T) t^{i}(1+t)^{n-1-2 i}
$$

where

$$
b_{i}(T)=2^{-n+1+2 i}|\{\pi \in T: \operatorname{peak}(\pi)=i\}|
$$

Proof. It is enough to prove the theorem for an orbit of a permutation $\pi \in \mathfrak{S}_{n}$. Since the number of peaks is constant on $\operatorname{Orb}(\pi)$ the equality follows from Theorem 3.1

Remark 3.3. If we want to prove " combinatorially" that the coefficients of $W(T ; t)$ form a symmetric and unimodal sequence then we can construct an involution proving symmetry and an injection proving unimodality easily as follows.

Define $f: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ by $f=\varphi_{[n]}^{\prime}$. Clearly $f$ is an involution and restricts to any subset of $\mathfrak{S}_{n}$ invariant under the MFS-action. Moreover,

$$
\begin{equation*}
\operatorname{des}(f(\pi))+\operatorname{des}(\pi)=n-1 \tag{3.1}
\end{equation*}
$$

so $f$ has the desired properties. The involution $f$ was defined differently in [4]. To find an injection

$$
g_{j}:\{\pi \in T: \operatorname{des}(\pi)=j\} \rightarrow\{\pi \in T: \operatorname{des}(\pi)=j+1\}
$$

for $j=1,2, \ldots,\lfloor(n-1) / 2\rfloor$ it suffices to find an injection from the set of subsets of cardinality $k$ of $[m$ ] to the set of subsets of cardinality $k+1$ of $[m$ ], for $1 \leq k \leq$ $\lfloor m / 2\rfloor$. This can done as in e.g. [38].

## 4. Invariance Under Stack Sorting

Much has been written on the combinatorics of the stack-sorting problem (cf. [6]) since it was introduced by Knuth 31. The stack-sorting operator $S$ can be defined recursively on permutations of finite subsets of $\{1,2, \ldots\}$ as follows. If $w$ is empty then $S(w)=w$ and if $w$ is non-empty write $w$ as the concatenation $w=L m R$, where $m$ is the greatest element of $w$ and $L$ and $R$ are the subwords to the left and right of $m$ respectively. Then $S(w)=S(L) S(R) m$.

Let $\pi=a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$. Recall that $i \in[n-1]$ is a descent in $\pi$ if $a_{i}>a_{i+1}$. If $i$ is a descent in $\pi$ we let $r_{i}(\pi)$ be the permutation obtained by inserting $a_{i}$ between the first pair of letters $a_{j}, a_{j+1}$ to the right of $x$ such that $a_{j}<x<a_{j+1}$ $\left(a_{n+1}=n+1\right)$. The following theorem describes a new way of computing $S(\pi)$.

Theorem 4.1. Let $i_{1}<i_{2}<\cdots<i_{d}$ be the descents in the permutation $\pi=$ $a_{1} a_{2} \cdots a_{n}$. Then

$$
S(\pi)=r_{i_{d}} r_{i_{d-1}} \cdots r_{i_{1}}(\pi)
$$

Proof. Let $S^{\prime}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ be defined by $S^{\prime}(\pi)=r_{i_{d}} r_{i_{d-1}} \cdots r_{i_{1}}(\pi)$. It is straightforward to check that $S^{\prime}$ satisfies the same recursion as $S$.

From the above description of $S$ we see that $S\left(\varphi_{x}(\pi)\right)=S(\pi)$, hence the following corollary.

Corollary 4.2. If $\sigma, \tau \in \mathfrak{S}_{n}$ are in the same orbit under the MFS-action then $S(\sigma)=S(\tau)$.

Corollary 4.2 can also be deduced from [7, Proposition 2.1].
Let $r \in \mathbb{N}$. A permutation $\pi \in \mathfrak{S}_{n}$ is said to be $r$-stack sortable if $S^{r}(\pi)=$ $12 \cdots n$. Denote by $\mathfrak{S}_{n}^{r}$ the set of $r$-stack sortable permutations in $\mathfrak{S}_{n}$. By Corollary 4.2 we have that $\mathfrak{S}_{n}^{r}$ is invariant under the MFS-action for all $n, r \in \mathbb{N}$ so Corollary 3.2 applies.

Figure 2. Computing $S(573148926)=r_{7} r_{3} r_{2}(573148926)=513478269$.


Corollary 4.3. For all $n, r \in \mathbb{N}$ we have

$$
W\left(\mathfrak{S}_{n}^{r} ; t\right)=\sum_{i=0}^{\lfloor n / 2\rfloor} b_{i}\left(\mathfrak{S}_{n}^{r}\right) t^{i}(1+t)^{n-1-2 i}
$$

where

$$
b_{i}\left(\mathfrak{S}_{n}^{r}\right)=2^{-n+1+2 i}\left|\left\{\pi \in \mathfrak{S}_{n}^{r}: \operatorname{peak}(\pi)=i\right\}\right|
$$

An immediate consequence of Corollary 4.3 is the following theorem due to Bóna.
Theorem 4.4 (Bóna [4, 5]). For all $n, r \in \mathbb{N}$, the coefficients of $W\left(\mathfrak{S}_{n}^{r} ; t\right)$ form $a$ symmetric and unimodal sequence.

An open problem posed by Bóna [4] is to determine whether the polynomial $W\left(\mathfrak{S}_{n}^{r} ; t\right)$ has the stronger property of having all zeros real for $n, r \in \mathbb{N}$. This is known for $r \geq n-1$ because then $W\left(\mathfrak{S}_{n}^{r} ; t\right)=A_{n}(t)$ and the Eulerian polynomials are known to have all zeros real (cf. [29]), and for $r=1$ as we then get the Narayana polynomials (4.1) which are known to have all zeros real by e.g. Malo's theorem (cf. [33]). In [11] we prove real-rootedness whenever $r=2$ or $r=n-2$. It is easy to see (cf. [9]) that if all zeros of $p(t)=\sum_{i=0}^{n} a_{i} t^{i}$ are real and $\left\{a_{i}\right\}_{i=0}^{n}$ is nonnegative and symmetric with center of symmetry $d / 2$, then

$$
p(t)=\sum_{i=0}^{\lfloor d / 2\rfloor} b_{i} t^{i}(1+t)^{d-2 i}
$$

where $b_{i}, i=0, \ldots,\lfloor d / 2\rfloor$ are nonnegative. Hence Corollary 4.3 can be seen as further evidence for a positive answer to Bóna's question.

Knuth 31 proved that the 1-stack sortable permutations are exactly the permutations that avoid the pattern 231, i.e., permutations $\pi=a_{1} a_{2} \cdots a_{n}$ such that $a_{k}<a_{j}<a_{i}$ for no $1 \leq i<j<k \leq n$. The set of 231-avoiding permutations in $\mathfrak{S}_{n}$ is denoted by $\mathfrak{S}_{n}(231)$. Simion [39] proved that the $n$-th Narayana polynomial is the descent generating polynomial of $\mathfrak{S}_{n}(231)$, i.e.,

$$
\begin{align*}
W\left(\mathfrak{S}_{n}(231) ; t\right) & =\sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1} t^{k} \\
& =\sum_{k=0}^{\lfloor n / 2\rfloor} \frac{1}{k+1}\binom{2 k}{k}\binom{n-1}{2 k} t^{k}(1+t)^{n-1-2 k} \tag{4.1}
\end{align*}
$$

where the second equality can be derived using hypergeometric formulas, see also [41. Hence we have the following corollary.

Corollary 4.5. Let $n, k \in \mathbb{N}$. Then

$$
\left|\left\{\pi \in \mathfrak{S}_{n}(231): \operatorname{peak}(\pi)=k\right\}\right|=2^{n-1-2 k} \frac{1}{k+1}\binom{2 k}{k}\binom{n-1}{2 k}
$$

## 5. A Refinement of the Eulerian Polynomials

The statistic $(2-31): \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ is an instance of a generalized permutation pattern as introduced by Babson and Steingrímsson [2]. Let $\pi=a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$. Then $(2-31)(\pi)$ is the number of pairs $1 \leq i<j \leq n-1$ such that $a_{j+1}<a_{i}<a_{j}$. Similarly, let $(13-2)(\pi)$ be the number of pairs $2 \leq i<j \leq n$ such that $a_{i-1}<$ $a_{j}<a_{i}$.

Theorem 5.1. The statistics $(2-31)$ and (13-2) are constant on any orbit under the MFS-action.

Proof. An alternative description of $(2-31)(\pi), \pi=a_{1} a_{2} \cdots a_{n}$ is the number triples $\left(a_{i}, a_{j}, a_{k}\right)$ such that $1 \leq i<j<k \leq n$ and $a_{k}<a_{i}<a_{j}$, where ( $a_{j}, a_{k}$ ) is a pair of consecutive peak and valley. By consecutive we mean that there are no other peaks or valleys in between $a_{j}$ and $a_{k}$. The number of such triples is invariant under the action since $a_{j}$ and $a_{k}$ cannot move and $a_{i}$ cannot move over the peak $a_{j}$. A similar reasoning applies to (13-2).

Define a $(p, q)$-refinement of the Eulerian polynomial by

$$
A_{n}(p, q, t)=\sum_{\pi \in \mathfrak{S}_{n}} p^{(13-2)(\pi)} q^{(2-31)(\pi)} t^{\operatorname{des}(\pi)}
$$

These polynomials (or at least $A_{n}(p, 1, t)$ and $\left.A_{n}(1, q, t)\right)$ have been in focus in several recent papers [17, 18, 36, 42, 51]. A fascinating property of the polynomial $A_{n}(p, 1, t)$ is that it appears as a translation of the polynomial enumerating the cells in the totally nonnegative part of a Grassmannian [36, 51], and also in the stationary distribution of the ASEP model in statistical mechanics [17, 18.

From Theorem 5.1 and Theorem 3.2 we get that

$$
A_{n}(p, q, t)=\sum_{i=0}^{\lfloor(n-1) / 2\rfloor} b_{n, i}(p, q) t^{i}(1+t)^{n-1-2 i}
$$

where

$$
\begin{equation*}
b_{n, i}(p, q)=2^{-n+1+2 i} \sum_{\substack{\pi \in \mathfrak{S}_{n} \\ \operatorname{peak}(\pi)=i}} p^{(13-2)(\pi)} q^{(2-31)(\pi)} \tag{5.1}
\end{equation*}
$$

Proposition 5.2. Let $n \in \mathbb{N}$. Then

$$
A_{n}(p, q, t)=A_{n}(q, p, t)
$$

Proof. Let $f$ be as in Remark 3.3 and let $R: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ be defined by

$$
R(\pi)=a_{n} \cdots a_{2} a_{1}, \quad \text { if } \pi=a_{1} a_{2} \cdots a_{n}
$$

Let $\pi^{\prime}=R(f(\pi))$. Then

$$
\left(\operatorname{des}\left(\pi^{\prime}\right),(13-2)\left(\pi^{\prime}\right),(2-31)\left(\pi^{\prime}\right)\right)=(\operatorname{des}(\pi),(2-31)(\pi),(13-2)(\pi))
$$

and the proposition follows.
A further striking property of $A_{n}(p, q, t)$ is that

$$
A_{n}\left(q, q^{2}, q\right)=A_{n}\left(q^{2}, q, q\right)=[n]_{q}[n-1]_{q} \cdots[1]_{q}
$$

where $[k]_{q}=1+q+q^{2}+\cdots+q^{k-1}$. This is because the statistics
$S_{1}=(13-2)+(13-2)+(2-31)+$ des $\quad$ and $\quad S_{2}=(13-2)+(2-31)+(2-31)+\mathrm{des}$ are Mahonian (see Section 8.2), a fact due to Simion and Stanton [40], see also [2].

## 6. An Action on the Linear Extensions of a Sign-Graded Poset

Recall that a labeled poset is a pair $(P, \omega)$ where $P$ is a finite poset and $\omega$ : $P \rightarrow \mathbb{Z}$ is an injection. The Jordan-Hölder set, $\mathcal{L}(P, \omega)$, is the set of permutations $\pi=a_{1} a_{2} \cdots a_{p}(p=|P|)$ of $\omega(P)$ such that if $x$ is smaller than $y$ in $P\left(x<_{P} y\right)$, then $\omega(x)$ precedes $\omega(y)$ in $\pi$. The $(P, \omega)$-Eulerian polynomial is defined by

$$
W(P, \omega ; t)=\sum_{\pi \in \mathcal{L}(P, \omega)} t^{\operatorname{des}(\pi)}
$$

Hence the $n$-th Eulerian polynomial is the $(P, \omega)$-Eulerian polynomial of an antichain of size $n$. The $(P, \omega)$-Eulerian polynomials have been intensively studied since they were introduced by Stanley 47] in 1972. For example, the Neggers-Stanley conjecture which asserts that these polynomials always have real zeros has attracted widespread attention [1, 3, 8, 9, 10, 13, 26, 34, 35, 45, 49, 50. A labeled poset is naturally labeled if $x<_{P} y$ implies $\omega(x)<\omega(y)$. Neggers 34 made the conjecture for naturally labeled posets in 1978 and Stanley formulated the conjecture in its general form in 1986. However, in [10], we found a family of counterexamples to the Neggers-Stanley conjecture and subsequently Stembridge 45 found counterexamples that are naturally labeled thus disproving Neggers original conjecture.

Although the Neggers-Stanley conjecture is refuted many questions regarding the $(P, \omega)$-Eulerian polynomials remain open. A question which is still open is whether the coefficients of $W(P, \omega ; t)$ always form a unimodal sequence. It is easy to see that real-rootedness implies unimodality. This weaker property was recently established by Reiner and Welker [35] for a large and important class of posets, namely the class of naturally labeled and graded posets. A poset $P$ is graded if every saturated chain in $P$ has the same length. Prior to [35], Gasharov [26] proved unimodality for graded naturally labeled posets of rank at most 2 . In 9 we proved unimodality for
$(P, \omega)$-Eulerian polynomials of labeled posets which we call sign-graded posets. The class of sign-graded posets contains the class of naturally labeled graded posets.

If $(P, \omega)$ is a labeled poset we may associate signs to the edges of the Hassediagram, $E(P)$, of $P$ as follows. Let $\epsilon: E(P) \rightarrow\{-1,1\}$ be defined by

$$
\epsilon(x, y)=\left\{\begin{array}{rc}
1 \text { if } & \omega(x)<\omega(y) \\
-1 \text { if } & \omega(x)>\omega(y)
\end{array}\right.
$$

It is not hard to prove that the $(P, \omega)$-Eulerian polynomial only depends on $\epsilon$, see [9]. A labeled poset $(P, \omega)$ is sign-graded if for every maximal chain $x_{1}<x_{2}<\cdots<x_{k}$ in $P$, the sum of signs

$$
\sum_{i=1}^{k} \epsilon\left(x_{i-1}, x_{i}\right)
$$

is the same. Note that this definition extends the notion of graded posets since if $(P, \omega)$ is naturally labeled then all signs are equal to one and the above sum is just the length of the chain. The common value, $r$, of the above sum is called the rank of $(P, \omega)$. One may associate a (generalized) rank function $\rho: P \rightarrow \mathbb{Z}$ to a sign-graded poset by

$$
\rho(x)=\sum_{i=1}^{k} \epsilon\left(x_{i-1}, x_{i}\right)
$$

where $x_{1}<x_{2}<\cdots<x_{k}=x$ is any saturated chain from a minimal element to $x$. In [9] we prove the following theorem.

Theorem 6.1 (Brändén (9). Let $(P, \omega)$ be a sign-graded poset of rank $r$ and let $d=p-r-1$. Then

$$
W(P, \omega ; t)=\sum_{i=0}^{\lfloor d / 2\rfloor} a_{i}(P, \omega) t^{i}(1+t)^{d-2 i}
$$

where $a_{i}(P, \omega), i=0,1, \ldots,\lfloor d / 2\rfloor$ are nonnegative integers.
From the proof of Theorem 6.1] [9] it is not evident what the numbers $a_{i}(P, \omega)$ count. We will now give an alternative proof of Theorem 6.1 by extending the MFS-action to $\mathcal{L}(P, \omega)$. This will also give us an interpretation of the numbers $a_{i}(P, \omega), i=0, \ldots,\lfloor d / 2\rfloor$. If both $(P, \omega)$ and $(P, \lambda)$ are sign-graded one can prove [9, Corollary 2.4] that up to a multiple of $t$ the corresponding Eulerian polynomials are the same. Moreover, in [9] we prove that if $(P, \omega)$ is sign-graded then there exists a labeling $\mu$ of $P$ such that
(1) $(P, \mu)$ is sign-graded,
(2) the rank function of $(P, \mu)$ has values in $\{0,1\}$,
(3) all elements of rank 0 have negative labels and
(4) all elements of rank 1 have positive labels

Such a labeling will be called canonical. Hence it is no restriction in assuming that the sign-graded poset is labeled canonically.

Definition 6.2. Let $(P, \omega)$ be sign-graded with $\omega$ canonical. For $x \in \omega(P)$ define a map $\psi_{x}: \mathcal{L}(P, \omega) \rightarrow \mathcal{L}(P, \omega)$ as follows. Let $\pi=a_{1} a_{2} \cdots a_{p} \in \mathcal{L}(P, \omega)$ and let $a_{0}=a_{p+1}=0$.

Figure 3. The dotted lines indicates where the double ascents/descents are mapped.


- If $x<0$ is a double descent let $\psi_{x}(\pi)$ be the permutation obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the right of $x$ such that $a_{i}<x<a_{i+1}$.
- If $x<0$ is a double ascent let $\psi_{x}(\pi)$ be the permutation obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the left of $x$ such that $a_{i}>x>a_{i+1}$.
- If $x>0$ is a double descent let $\psi_{x}(\pi)$ be the permutation obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the left of $x$ such that $a_{i}<x<a_{i+1}$
- If $x>0$ is a double ascent $\psi_{x}(\pi)$ be the permutation obtained by inserting $x$ between the first pair of letters $a_{i}, a_{i+1}$ to the right of $x$ such that $a_{i}>$ $x>a_{i+1}$.
- If $x$ is a peak or a valley let $\psi_{x}(\pi)=\pi$.

See Fig. 3.
It is not immediate that this definition makes sense, i.e., that the resulting permutation represents a linear extension of $P$. Suppose that $x<0$ is a letter of $\pi \in \mathcal{L}(P, \omega)$. Then $x$ is a letter of a maximal contigous subword $w$ of $\pi$ whose letters are all negative. By construction $\psi_{x}$ will not move $x$ outside of the word $w$. We claim that

$$
\omega^{-1}(w)=\{y \in P: \omega(y) \text { is a letter of } w\}
$$

is an anti-chain. Suppose that $y_{1}<_{P} y_{2}$ are elements in $\omega^{-1}(w)$. Then, since $\rho\left(y_{1}\right)=\rho\left(y_{2}\right)=0$, there must be an element $z \in P$ such that $y_{1}<_{P} z<_{P} y_{2}$, $\rho(z)=1$ and $\omega(z)>0$. This means that $\omega(z)$ is between $\omega\left(y_{1}\right)$ and $\omega\left(y_{2}\right)$ in $\pi$, so $\omega(z)$ is a letter of $w$ contrary to the assumption that all letters of $w$ are negative. Since $\omega^{-1}(w)$ is an anti-chain and since $\psi_{x}$ does not move $x$ outside $\omega^{-1}(w)$ we have that $\psi_{x}(\pi) \in \mathcal{L}(P, \omega)$. The case $x>0$ is analogous.

We may now define a $\mathbb{Z}_{2}^{P}$-action on $\mathcal{L}(P, \omega)$ by

$$
\psi_{S}(\pi)=\prod_{x \in S} \psi_{\omega(x)}(\pi), \quad S \subseteq P
$$

Let $\hat{\pi}$ be the unique permutation in $\operatorname{Orb}(\pi)$ such that $0 \hat{\pi} 0$ has no double descents.
Theorem 6.3. Let $(P, \omega)$ be a sign-graded poset of rank $r$ where $\omega$ is canonical and let $\pi \in \mathcal{L}(P, \omega)$. Then

$$
\sum_{\sigma \in \operatorname{Orb}(\pi)} t^{\operatorname{des}(\sigma)}=t^{\operatorname{des}(\hat{\pi})}(1+t)^{p-r-1-2 \operatorname{des}(\hat{\pi})}
$$

Moreover, if $r=0$ then $\operatorname{peak}(\cdot)$ is invariant under the $\mathbb{Z}_{2}^{P}$-action and $\operatorname{peak}(\pi)=$ $\operatorname{des}(\hat{\pi})$ for all $\pi \in \mathcal{L}(P, \omega)$.
Proof. If $x$ is a double ascent in $0 \pi 0$ then $\operatorname{des}\left(\psi_{x}(\pi)\right)=\operatorname{des}(\pi)+1$. It follows that

$$
\sum_{\sigma \in \operatorname{Orb}(\pi)} t^{\operatorname{des}(\sigma)}=t^{\operatorname{des}(\hat{\pi})}(1+t)^{a}
$$

where $a$ is the number of double ascents in $\pi$. Suppose $r=0$. Deleting all double ascents in $\hat{\pi}$ results in an alternating permutation

$$
0>a_{1}<a_{2}>a_{3}<\cdots>a_{p-a}<0
$$

with the same number of peaks/descents as $\pi$. Hence $p-a=2 \operatorname{peak}(\pi)+1$.
If $r=1$, deleting all double ascents in $\hat{\pi}$ results in an alternating permutation

$$
0>a_{1}<a_{2}>a_{3}<\cdots<a_{p-a}>0
$$

with the same number of descents. Hence $p-a-2=2 \operatorname{des}(\hat{\pi})$.
Stembridge 44 developed a theory of "enriched $P$-partitions" in which the distribution of peaks in $\mathcal{L}(P, \omega)$ and the polynomial, viz.,

$$
\bar{W}(P, \omega ; t)=\sum_{\pi \in \mathcal{L}(P, \omega)} t^{\operatorname{peak}(\pi)}
$$

play a significant role. For a canonically labeled poset $(P, \omega)$ let $(\hat{P}, \hat{\omega})$ be any canonically labeled poset such that $\hat{P}$ is obtained from $P$ by adjoining a greatest element.

Theorem 6.4. Let $(P, \omega)$ be a canonically labeled sign-graded poset of rank $r$. If $r=0$ then

$$
W(P, \omega ; t)=2^{-p+1}(1+t)^{p-1} \bar{W}\left(P, \omega ; 4 t(1+t)^{-2}\right) .
$$

Equivalently,

$$
a_{i}(P, \omega)=2^{-p+1+2 i}|\{\pi \in \mathcal{L}(P, \omega): \operatorname{peak}(\pi)=i\}|
$$

If $r=1$ then

$$
W(P, \omega ; t)=2^{-p} t^{-1}(1+t)^{p} \bar{W}\left(\hat{P}, \hat{\omega} ; 4 t(1+t)^{-2}\right)
$$

Equivalently,

$$
a_{i}(P, \omega)=2^{-p+2+2 i}|\{\pi \in \mathcal{L}(\hat{P}, \hat{\omega}): \operatorname{peak}(\pi)=i+1\}|
$$

Proof. Note that $W(\hat{P}, \hat{\omega} ; t)=t^{-r} W(P, \omega ; t)$, so we may assume that $r=0$. By Theorem 6.3 the proof follows just as the proof of Corollary 3.2 ,

Figure 4. The decreasing unordered tree corresponding to 652419738


## 7. Vertices Of Even Height

To any permutation $w$ of a finite subset of $\{1,2, \ldots\}$ we may associate a decreasing unordered tree as follows. Let $\infty$ be a symbol which is greater than every letter in $w$. If $w$ is empty then $T(w ; \infty)$ is the tree with a single vertex labeled $\infty$. Otherwise write $w$ as $w=m_{1} w_{1} m_{2} w_{2} \cdots m_{k} w_{k}$ where $m_{i}$ are the left-to-right maxima of $w$. Then $T(w ; \infty)$ is the labeled tree with $T\left(w_{i} ; m_{i}\right)$ as subtrees of the root, see Fig. 4. Let $\operatorname{veh}(\pi)$ be the number of (non-root) vertices of even height in $T(\pi ; \infty)$. As Fig. 4 suggests

$$
\operatorname{veh}(652419738)=|\{1,5,7,8\}|=4
$$

We will here show that veh and des have the same distribution on any subset of $\mathfrak{S}_{n}$ invariant under the MFS-action. For $\pi \in \mathfrak{S}_{n}$ and $x \in[n]$ let $r_{\pi}(x)$ be the number of right edges in the path from the root to $x$ in the decreasing binary tree associated with $\pi$. It is plain to see that $r_{\pi}(x)+1$ is equal to the height of $x$ as a vertex $T(\pi ; \infty)$. Let $\operatorname{Odd}(\pi)$ the set of all $x \in[n]$ for which $r_{\pi}(x)$ is odd. Hence $\operatorname{Odd}(\pi)$ is the set of vertices of even height in $T(\pi ; \infty)$. Also, let Redge $(\pi)$ be the set of vertices in the decreasing binary tree which are ends of right edges. Clearly, $\operatorname{des}(\pi)=|\operatorname{Redge}(\pi)|$. Define $\Psi, \Phi: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ by

$$
\begin{aligned}
& \Psi(\pi)=\prod_{x} \varphi_{x}(\pi) \quad(x \in \operatorname{Odd}(\pi)) \\
& \Phi(\pi)=\prod_{x} \varphi_{x}(\pi) \quad(x \in \operatorname{Redge}(\pi))
\end{aligned}
$$

Theorem 7.1. The transformations $\Psi$ and $\Phi$ are inverses of each other. Moreover, if $\pi \in \mathfrak{S}_{n}$ then

$$
\begin{aligned}
\operatorname{Odd}(\pi) & =\operatorname{Redge}\left(\pi^{\prime}\right) \quad \text { and } \\
\operatorname{Redge}(\pi) & =\operatorname{Odd}\left(\pi^{\prime \prime}\right),
\end{aligned}
$$

where $\pi^{\prime}=\Psi(\pi)$ and $\pi^{\prime \prime}=\Phi(\pi)$.

Proof. Note that it is enough to prove the first equality since then

$$
\begin{aligned}
\Phi(\Psi(\pi)) & =\prod_{x} \varphi_{x} \prod_{y} \varphi_{y}(\pi) \quad\left(x \in \operatorname{Redge}\left(\pi^{\prime}\right), y \in \operatorname{Odd}(\pi)\right) \\
& =\prod_{x} \varphi_{x} \prod_{y} \varphi_{y}(\pi) \quad(x, y \in \operatorname{Odd}(\pi)) \\
& =\pi
\end{aligned}
$$

because the involutions $\varphi_{x}$ commute.
Let $T$ and $T^{\prime}$ be the decreasing binary trees corresponding to $\pi$ and $\pi^{\prime}$, respectively. Suppose that $x \in \operatorname{Odd}(\pi)$ and let $y$ be its father in $T$. If there is a right-edge between $x$ and $y$ in $T$ then $y \notin \operatorname{Odd}(\pi)$, which means that there will also be a rightedge between $x$ and $y$ in $T^{\prime}$, so that $x \in \operatorname{Redge}\left(\pi^{\prime}\right)$. If there is a left-edge between $x$ and $y$ in $T$ then also $y \in \operatorname{Odd}(\pi)$ so that $\Psi$ will turn this edge to a right-edge and hence $x \in \operatorname{Redge}\left(\pi^{\prime}\right)$.

The fact that $x \notin \operatorname{Odd}(\pi)$ implies $x \notin \operatorname{Redge}(\pi)$ follows similarly.
Note that $\Psi$ and $\Phi$ restricts to bijections on all subsets of $\mathfrak{S}_{n}$ invariant under the "proper" Foata-Strehl action, but not on subsets invariant under the modified Foata-Strehl action. Define a transformation $\Psi^{\prime}: \mathfrak{S}_{n} \rightarrow \mathfrak{S}_{n}$ by

$$
\Psi^{\prime}(\pi)=\prod_{x} \varphi_{x}^{\prime}(\pi) \quad(x \in \operatorname{Odd}(\pi))
$$

Theorem 7.2. Let $T \subseteq \mathfrak{S}_{n}$ be invariant under the modified Foata-Strehl action. Then $\Psi^{\prime}: T \rightarrow T$ is a bijection and

$$
\operatorname{veh}(\pi)=\operatorname{des}\left(\Psi^{\prime}(\pi)\right), \quad \pi \in T
$$

Proof. Since the involutions $\varphi_{x}$ commute we may write $\Psi^{\prime}$ as $\Psi^{\prime}=F \circ \Psi$ where $F$ is defined by

$$
F(\pi)=\prod_{x} \varphi_{x}(\pi) \quad(x \in \operatorname{Redge}(\pi), c(x)=2)
$$

and where $c(x)$ is the number children of $x$ in the decreasing binary tree of $\pi$. Clearly, $\operatorname{des}(\pi)=\operatorname{des}(F(\pi))$ so it remains to prove that $\Psi^{\prime}$ is a bijection.

Let $f$ be defined as in Remark 3.3 and let $\pi \in \mathfrak{S}_{n}$. Then since the involutions $\varphi_{x}$ commute we have

$$
\begin{aligned}
f\left(\Psi^{\prime}(\pi)\right) & =\prod_{y} \varphi_{y}^{\prime} \prod_{x} \varphi_{x}^{\prime}(\pi) \quad(y \in[n], x \in \operatorname{Odd}(\pi)) \\
& =\prod_{x} \varphi_{x}^{\prime}(\pi) \quad(x \notin \operatorname{Odd}(\pi)) .
\end{aligned}
$$

It follows that $\Psi^{\prime}$ can be defined recursively on the set of permutations of any finite subset of $\{1,2, \ldots\}$ as follows. The empty word is mapped by $\Psi^{\prime}$ to itself, and if $w=L n R$ where $n$ is the greatest element of $w$ and $L$ and $R$ are the words to the left and right of $n$ respectively then

$$
\Psi^{\prime}(w)=\Psi^{\prime}(L) n f\left(\Psi^{\prime}(R)\right)
$$

where $f$ is as in Remark 3.3. From this recursive definition it is plain to see that $\Psi^{\prime}$ is bijective.

Corollary 7.3. Let $n, r \in \mathbb{N}$. Then veh and des have the same distribution over $\mathfrak{S}_{n}^{r}$.

To every unordered decreasing tree $T(\pi ; \infty)$ corresponding to a permutation $\pi \in \mathfrak{S}_{n}(231)$ there is a unique ordered unlabeled tree obtained by ordering the children of a vertex decreasingly from left to right and dropping the labels. Recall that a Dyck-path of length $2 n$ is a lattice path in $\mathbb{N}^{2}$ starting at the origon and ending at $(2 n, 0)$, using steps $u=(1,1)$ and $d=(1,-1)$, and never going below the $x$-axis. If we traverse the ordered tree in pre-order and write a $u$ every time we go down an edge and write a $d$ every time we go up an edge we obtain a Dyck path. This describes a bijection between the set of Dyck path of length $2 n$ and the set of ordered trees with $n+1$ vertices (and by the above also between the set of Dyck path of length $2 n$ and $\left.\mathfrak{S}_{n}(231)\right)$. Note that a vertex of even height translates into an up-step of even height in the Dyck path, and a descent translates into a double up-step $u u$ in the path. We have thus recovered the following classical result of Kreweras 32].

Corollary 7.4. The statistics "up-steps at even height" and "double up-steps" have the same distribution over the set of Dyck paths of a given length.

When restricted to $\mathfrak{S}_{n}(231)$ one may express veh as the following alternating sum of permutation patterns 12

$$
\operatorname{veh}(\pi)=d_{1}(\pi)-2 d_{2}(\pi)+4 d_{3}(\pi)-\cdots+(-2)^{n-2} d_{n-1}(\pi)
$$

where $d_{i}(\pi)$ is the number of decreasing subsequences of length $i+1$ in $\pi$.

## 8. A Mahonian Partner for Vertices of Even Height

Recall that the descent set of a permutation $\pi=a_{1} a_{2} \cdots a_{n}$ is defined by $\operatorname{Des}(\pi)=\left\{i \in[n-1]: a_{i}>a_{i+1}\right\}$ and that the major index of $\pi$ as

$$
\operatorname{MAJ}(\pi)=\sum_{i \in \operatorname{Des}(\pi)} i
$$

A statistic $B: \mathfrak{S}_{n} \rightarrow \mathbb{N}$ is said to be Mahonian if it has the same distribution as MAJ on $\mathfrak{S}_{n}$, i.e.,

$$
\sum_{\pi \in \mathfrak{S}_{n}} q^{B(\pi)}=[n]_{q}[n-1]_{q} \cdots[1]_{q},
$$

where $[k]_{q}=1+q+\cdots+q^{k-1}$. A bi-statistic $(A, B)$ is Euler-Mahonian if it has the same distribution as (des, MAJ) on $\mathfrak{S}_{n}$. We will now redefine the statistic veh so that we can define a Mahonian partner for it. To every permutation $\pi=$ $a_{1} a_{2} \cdots a_{n} \in \mathfrak{S}_{n}$ we associate an increasing unordered tree, $T^{\prime}(\pi)$, as follows. If $b$ is a right-to-left minimum of $\pi$ then $b$ is a successor of the root, which is labeled 0 . Otherwise $b$ is the successor of the leftmost element $a$ to the right of $b$ which is smaller than $b$, see Fig. 5 .

Let $n \in \mathbb{N}$. We (re-)define the statistic vertices of even height, veh ${ }^{\prime}: \mathfrak{S}_{n} \rightarrow \mathbb{N}$, by letting $\operatorname{veh}^{\prime}(\pi)$ be the number of (non-root) vertices in $T^{\prime}(\pi)$ of even height. Thus $\operatorname{veh}^{\prime}(586317492)=|\{3,4,8,9\}|=4$. We define the even vertex set, $\operatorname{EV}(\pi)$, as the set of indices $1 \leq i \leq n$ such that the vertex $a_{i}$ is of even height.

The complement, $\pi^{c}$, of a permutation $\pi=a_{1} a_{2} \cdots a_{n}$ is the permutation $b_{1} b_{2} \cdots b_{n}$ on the same letters as $\pi$ such that $a_{i}<a_{j}$ if and only if $b_{i}>b_{j}$ for all $1 \leq i<j \leq n$. We define a transformation $\Theta$ on permutations of any finite subset of $\{1,2, \ldots\}$ recursively as follows. The empty permutation is mapped onto itself and if $\pi$ is the

Figure 5. The increasing unordered tree corresponding to 586317492

concatenation $\sigma m \tau$ where $m$ is the smallest letter in $\pi$, then $\Theta(\pi)=\Theta\left(\sigma^{c}\right) m \Theta(\tau)$. It is clear that $\Theta$ restricted to the symmetric group is a bijection.

$$
\begin{aligned}
\Theta(586317492) & =\Theta(6358) 1 \Theta(7492) \\
& =63 \Theta(58) 1 \Theta(794) 2 \\
& =635819742 .
\end{aligned}
$$

If $S \subset \mathbb{Z}$ and $k \in \mathbb{Z}$ let $S+k:=\{s+k: s \in S\}$.
Theorem 8.1. Let $n \in \mathbb{N}$. For all permutations $\pi \in \mathfrak{S}_{n}$ we have

$$
\operatorname{EV}(\Theta(\pi))=\operatorname{Des}(\pi)
$$

Proof. The proof is by induction over the length $n=|\pi|$ of $\pi$. The case $n=0$ is clear. Suppose that $n>0$. Then we can write $\pi \in \mathfrak{S}_{n}$ as the concatenation $\sigma 1 \tau$. Let $k=|\sigma|$. If $1 \leq i \leq k$ then clearly $i \in \operatorname{EV}(\sigma)$ if and only if $i \notin \mathrm{EV}(\pi)$. Hence

$$
\begin{aligned}
\operatorname{EV}(\pi) & =([k] \backslash \operatorname{EV}(\sigma)) \cup(\operatorname{EV}(\tau)+k+1) \quad \text { and } \\
\operatorname{Des}(\pi) & =\left([k] \backslash \operatorname{Des}\left(\sigma^{c}\right)\right) \cup(\operatorname{Des}(\tau)+k+1)
\end{aligned}
$$

since $[n] \backslash \operatorname{Des}(\pi)=\{n\} \cup \operatorname{Des}\left(\pi^{c}\right)$ for all $\pi$ of length $n$. Using induction we get

$$
\begin{aligned}
\operatorname{EV}(\Theta(\pi)) & =\operatorname{EV}\left(\Theta\left(\sigma^{c}\right) 1 \Theta(\tau)\right) \\
& =\left([k] \backslash \operatorname{EV}\left(\Theta\left(\sigma^{c}\right)\right)\right) \cup(\operatorname{EV}(\Theta(\tau))+k+1) \\
& =\left([k] \backslash \operatorname{Des}\left(\sigma^{c}\right)\right) \cup(\operatorname{Des}(\tau)+k+1) \\
& =\operatorname{Des}(\pi)
\end{aligned}
$$

It is desirable to find a bijection which is not defined recursively and which proves Theorem 8.1.

We may now define a Mahonian partner for veh ${ }^{\prime}$. The statistic sum of indices of vertices even height, SIVEH : $\mathfrak{S}_{n} \rightarrow \mathbb{N}$, is defined by

$$
\operatorname{SIVEH}(\pi)=\sum_{i \in \operatorname{EV}(\pi)} i
$$

Corollary 8.2. For all $n \in \mathbb{N}$ the bistatistic (veh', SIVEH) is Euler-Mahonian on $\mathfrak{S}_{n}$.

## 9. Gal's Conjecture on $\gamma$-Polynomials

Recall that the $h$-polynomial of a simplicial complex $\Delta$ of dimension $d-1$ is the polynomial $h_{\Delta}(t)=h_{0}(\Delta)+h_{1}(\Delta) t+\cdots+h_{d}(\Delta) t^{d}$ defined by the polynomial identity

$$
\sum_{i=0}^{d} h_{i}(\Delta) t^{i}(1+t)^{d-i}=\sum_{i=0}^{d} f_{i-1}(\Delta) t^{i}
$$

where $f_{i}(\Delta),-1 \leq i \leq d-1$ is the number of faces of $\Delta$ of dimension $i$. If $\Delta$ is a simplicial homology sphere then the Cohen-Macaulay property and the DehnSommerville equations imply that $\left\{h_{i}(\Delta)\right\}_{i=0}^{d}$ is nonnegative and symmetric. Hence one may define the $\gamma$-polynomial of $\Delta, \gamma_{\Delta}(t)=\sum_{i=0}^{\lfloor d / 2\rfloor} \gamma_{i}(\Delta) t^{i}$, by

$$
h_{\Delta}(t)=\sum_{i=0}^{\lfloor d / 2\rfloor} \gamma_{i}(\Delta) t^{i}(1+t)^{d-2 i}
$$

A simplicial complex $\Delta$ is flag if the minimal non-faces of $\Delta$ have cardinality two. The following conjecture generalizes the Charney-Davis conjecture [15.

Conjecture 9.1 (Gal [25). If $\Delta$ is a flag simplicial homology sphere of dimension $d-1$, then

$$
\gamma_{i}(\Delta) \geq 0, \quad 0 \leq i \leq\lfloor d / 2\rfloor .
$$

It is desirable to find a combinatorial, geometrical or ring-theoretical description of the numbers $\gamma_{i}(\Delta)$. In 35] Reiner and Welker associated to any graded naturally labeled poset $(P, \omega)$ a simplicial polytopal sphere, $\Delta_{e q}(P)$, whose $h$-polynomial is the $(P, \omega)$-Eulerian polynomial. Hence, Theorem 6.3 gives a combinatorial description of the $\gamma$-polynomial of $\Delta_{e q}(P)$ and verifies Conjecture 9.1 for $\Delta_{e q}(P)$.

In 37 Postnikov, Reiner and Williams extended the MFS-action to give a combinatorial interpretation of the $\gamma$-polynomials of tree-associahedra which confirms Conjecture 9.1 in this case. Also, Chow [16] has given a combinatorial interpretation of the $\gamma$-polynomials of the Coxeter complexes of type $B$ and $D$ and confirming Conjecture 9.1 for these complexes.

## 10. Further Directions and Open Problems

Let $\mathcal{I}_{n}$ be the set of involutions in $\mathfrak{S}_{n}$ and let

$$
I_{n}(t)=\sum_{\pi \in \mathcal{I}_{n}} t^{\operatorname{des}(\pi)}=\sum_{k=0}^{n-1} I_{n, k} t^{k}
$$

Brenti has conjectured that the sequence $\left\{I_{n, k}\right\}_{k=0}^{n-1}$ has no internal zeros and is log-concave, i.e.,

$$
I_{n, k}^{2} \geq I_{n, k+1} I_{n, k-1}, \quad 1 \leq k \leq n-2
$$

see 19 where progress on this conjecture was made. Motivated by Brenti's conjecture Guo and Zeng [28] proved the weaker statement that $\left\{I_{n, k}\right\}_{k=0}^{n-1}$ is unimodal. Also, Strehl 48 proved symmetry for $\left\{I_{n, k}\right\}_{k=0}^{n-1}$ and the following conjecture was made in 28].

Conjecture 10.1 (Guo-Zeng [28]). Let $n \in \mathbb{N}$. Then

$$
I_{n}(t)=\sum_{i=0}^{\lfloor(n-1) / 2\rfloor} a_{n, i} t^{i}(1+t)^{n-1-2 i}
$$

where $a_{n, i} \in \mathbb{N}$ for $0 \leq i \leq\lfloor(n-1) / 2\rfloor$.
Gessel [27] has conjectured a fascinating property of the joint distribution of descents and inverse descents.

Conjecture 10.2 (Gessel [27]). Let $\tau \in \mathfrak{S}_{n}$. Then

$$
\begin{equation*}
\sum_{\pi \in \mathfrak{S}_{n}} s^{\operatorname{des}(\pi)} t^{\operatorname{des}\left(\pi^{-1} \tau\right)}=\sum_{k, j} c_{n}(\tau ; k, j)(s+t)^{k}(s t)^{j}(1+s t)^{n-k-1-2 j} \tag{10.1}
\end{equation*}
$$

where $c_{n}(\tau ; k, j) \in \mathbb{N}$ for all $k, j \in \mathbb{N}$.
Symmetry properties imply that an expansion such as (10.1) with $c_{n}(\tau ; k, j) \in \mathbb{Z}$, $k, j \in \mathbb{N}$ exists. Moreover, $c_{n}(\tau ; k, j)$ only depends on the number of descents of $\tau$. In light of Conjectures 10.1 and 10.2 there might be another $\mathbb{Z}_{2}^{n}$-action on permutations which also behaves well with respect to the inverse permutation.

Recall the definition of $A_{n}(p, q, t)$ of Section 5. The first nontrivial examples are

$$
\begin{aligned}
A_{3}(p, q, t)= & (1+t)^{2}+(p+q) t \\
A_{4}(p, q, t)= & (1+t)^{3}+(p+q)(p+q+2) t(1+t) \\
A_{5}(p, q, t)= & \left.(1+t)^{4}+(p+q)\left((p+q)^{2}+2(p+q)+3\right)\right) t(1+t)^{2}+ \\
& (p+q)^{2}\left(p^{2}+p q+q^{2}+1\right) t^{2}
\end{aligned}
$$

Conjecture 10.3. Let $b_{n, i}(q)$ be defined by (5.1). Then $(p+q)^{i} \mid b_{n, i}(p, q)$ for all $0 \leq i \leq\lfloor(n-1) / 2\rfloor$.

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[^1]:    ${ }^{1}$ The present author was also unaware of this until it was pointed out by the referee.

