# NEW RESULTS RELATED TO A CONJECTURE OF MANICKAM AND SINGHI 

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#### Abstract

In 1998 Manickam and Singhi conjectured that for every positive integer $d$ and every $n \geq 4 d$, every set of $n$ real numbers whose sum is nonnegative contains at least $\binom{n-1}{d-1}$ subsets of size $d$ whose sums are nonnegative. In this paper we establish new results related to this conjecture. We also prove that the conjecture of Manickam and Singhi does not hold for $n=2 d+2$.


## 1. Introduction

In this paper we establish new results related to a conjecture of Manickam and Singhi (from now on, (MS)-conjecture). In order to illustrate the (MS)-conjecture and our results we need to introduce the following notation. Let $n \in \mathbb{N}$ and let $I_{n}$ be the set $\{1,2, \ldots, n\}$. A function $f: I_{n} \rightarrow \mathbb{R}$ is called a $n$-weight function if

$$
\sum_{x \in I_{n}} f(x) \geq 0
$$

Let $W_{n}(\mathbb{R})$ denote the set of all $n$-weight functions. If $f \in W_{n}(\mathbb{R})$ we set

$$
f^{+}:=\left|\left\{x \in I_{n}: f(x) \geq 0\right\}\right| .
$$

If $d$ is an integer with $1 \leq d \leq n$ and $Y$ is a subset of $I_{n}$ having $d$ elements such that

$$
\sum_{y \in Y} f(y) \geq 0
$$

we call $Y$ a $\left(d^{+}, n\right)$-subset of $f$. If $f \in W_{n}(\mathbb{R})$, we denote by $\phi(f, d)$ the number of distinct $\left(d^{+}, n\right)$-subsets of $f$.

Furthermore, we set

$$
\psi(n, d):=\min \left\{\phi(f, d): f \in W_{n}(\mathbb{R})\right\} .
$$

In 1988, Manickam and Singhi [14] stated the following conjecture:
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(MS)-Conjecture : If $d$ is a positive integer and $f$ is a $n$-weight function with $n \geq 4 d$, then

$$
\psi(n, d) \geq\binom{ n-1}{d-1}
$$

We remark that, as previously observed in [8, the conjecture is equivalent to require that $\psi(n, d)=\binom{n-1}{d-1}$.

This conjecture is interesting for several reasons. It is deeply related with the first distribution invariant of the Johnson-scheme [6, 12, 13, 14]. The distribution invariants were introduced by Bier [5] and later investigated in [7, 11, 12, 14]. Manickam and Singhi [14] claim that this conjecture is, in some sense, dual to the theorem of Erdös-Ko-Rado [9]. Also, as pointed out by Srinivasan [17], this conjecture settles some cases of another conjecture on multiplicative functions by Alladi, Erdös and Vaaler [1].

In general the conjecture of Manickam and Singhi still remains open. So far the following partial results have been achieved:
(1) The (MS)-conjecture is true if $n=u d, u \geq 4$ (Corollary 1 of [6]).
(2) The (MS)-conjecture is true if $d=2$ (Corollary 3 of [6]).
(3) If we set $n$ in the form $n=u d+v$, where $u \geq 4$ and $v=1, \ldots, d-1$, the (MS)-conjecture is true if $r \leq \min \left\{\frac{n-v-1}{(d-1)(d-2)}, \frac{n-v}{d}\right\}$ (Lemma 2 of [6]).
(4) The (MS)-conjecture is true if $d=3$ and $n \geq 93$ (Theorem of [6]).
(5) The (MS)-conjecture is true if $r \leq d \leq n / 2$ (Proposition 2 of [8]).
(6) The (MS)-conjecture is true if it is true when $d<r \leq \frac{d-1}{d} n$ (Proposition 5 of [8]).
(7) The (MS)-conjecture is true if $d=3$ (Section 3 of [16]).
(8) The (MS)-conjecture is true if $n \geq 2^{d} d^{d+1}+2 d^{3}$ (Theorem 3 of [15]).
(9) The (MS)-conjecture is true if $d>3$ and $n \geq d(d-1)^{d}(d-2)^{d}+d(d-1)^{2}(d-$ 2) $+d[n]_{k}$, where $[n]_{k}$ denotes the smallest positive integer congruent to $n(\bmod$ $k$ ) (Main Theorem in [6]).
(10) The (MS)-conjecture is true if $n \geq 2^{d+1} e^{d} d^{d+1}$ (Theorem 1 of [3]).

We point out that for $d>4$ the best estimate between (8), (9) and (10) is (8).
Different techniques have been used to attempt to tackle the (MS)-conjecture. In [6, 8, 15] the approach is combinatorial. In particular Bier and Manickam [6] use a result of Baranyai (see for example [2, 18]). Manickam and Miklos [15] use a circle permutation method, previously utilized by Katona [10] for a simpler proof of the theorem of Erdös-Ko-Rado. The approach in [3, 4, [16] is somewhat different. In fact the techniques in (16] are analytical-combinatorial and in [3, 4] are probabilistic.

A natural question arises when one studies the (MS)-conjecture:

What is the value of $\psi(n, d)$ for each $d \leq n$ ?
In order to provide an answer to this question, in [8] the following numbers were introduced:

$$
\gamma(n, d, r)=\min \left\{\phi(f, d): f \in W_{n}(\mathbb{R}), f^{+}=r\right\}
$$

where $r, d \in \mathbb{N}$, with $r, d \leq n$.
It is clear that a complete computation of these numbers would also provide a complete determination of the numbers $\psi(n, d)$, since

$$
\begin{equation*}
\psi(n, d)=\min _{1 \leq r \leq n} \gamma(n, d, r) . \tag{1.1}
\end{equation*}
$$

In particular, the knowledge of $\gamma(n, d, r)$ when $n \geq 4 d$ and $r$ is an arbitrary integer such that $r \leq n$, would supply an answer to the (MS)-conjecture.

Remark 1.1. In general the computation of $\gamma(n, d, r)$, started in [8], is not an easy task. For some values of $n, d, r$, this has been done in [8]. Nevertheless we stress that there is a gap in the proof of Proposition 2 of [8]. In particular, this means that it is not clear whether the identity

$$
\gamma(n, d, r)=\binom{n-1}{d-1}
$$

holds or not for $r \leq d \leq n / 2$.
In this paper we continue the study of the numbers $\gamma(n, d, r)$. Here we establish some lower and upper bounds for $\gamma(n, d, r)$ when $d \leq r \leq \frac{d-1}{d} n$. From these inequalities we obtain that

$$
\begin{equation*}
\gamma(n, d, r)=\binom{r}{d}+\binom{r}{d-1} \tag{1.2}
\end{equation*}
$$

when $n=2 d+2$ with $r=2 d-1$ and when $r=\frac{d-1}{d} n$. Combining our results with the ones in [8] we obtain the following values of $\gamma(n, d, r)$ :

$$
\gamma(n, d, r)= \begin{cases}\binom{n-1}{d-1} & \text { if } r \leq d \leq \frac{n}{2} \quad(\star) \\ \binom{n-r}{d-r} & \text { if } r \leq d<n \quad \text { and } \quad r<\frac{n}{n-d} \\ \binom{r}{d} & \text { if } d<r<n \quad \text { and } \quad r>\frac{d-1}{d} n \\ \binom{n-1}{d-1} & \text { if } r=1 \\ \binom{r}{d}+\binom{r-1}{d-1} & \text { if } n=2 d+2 \quad \text { and } \quad r=2 d-1 \\ \binom{r}{d}+\binom{r-1}{d-1} & \text { if } r \geq d \quad \text { and } \quad r=\frac{d-1}{d} n,\end{cases}
$$

where $(\star)$ in the first row means that the equality in that case is uncertain (see Remark 1.1). We stress that the determination of the numbers $\gamma(n, d, r)$ in general is an open problem.

A straightforward consequence of (1.2) is that the (MS)-conjecture does not hold if $n=2 d+2$. This provides another range of values of $n$, when $n<4 d$, for which the (MS)conjecture does not hold. Note that Bier and Manickam [6] already proved that (MS)conjecture does not hold in general. In particular they proved that the (MS)-conjecture does not hold for $n=2 d+1$, with $d \geq 2$, and for $n=3 d+1$, with $d \geq 3$.

We also prove that, for $n=2 d+2$ and $r=2 d-1$, (1.2) improves the results of Lemma 1 of [6].

A key tool in our paper is Hall's Theorem, as far as we know, used here for the first time in this context. We use this Theorem to determine, in a non constructive way, certain biunivocal functions between complementary $q$-subsets of a set with $2 q+1$ elements. Such functions are important to compute the numbers $\gamma(n, d, r)$ in the case $n=2 d+2$ and $r=2 d-1$.

We also suggest a new algorithm to determine the previous functions also in a constructive way.

## 2. Preliminaries

In this Section we introduce some notation and prove some elementary arithmetical preliminaries useful in the sequel of this paper.

We shall assume that a generic weight function $f \in W_{n}(\mathbb{R})$, with $f^{+}=r$, has the form

$$
\begin{array}{cccccc}
1 & \cdots & r & r+1 & \cdots & n  \tag{2.1}\\
\downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow \\
x_{1} & \cdots & x_{r} & y_{1} & \cdots & y_{n-r}
\end{array},
$$

with

$$
x_{1} \geq x_{2} \geq \ldots \geq x_{r} \geq 0>y_{1} \geq y_{2} \geq \ldots \geq y_{n-r} .
$$

Let us call the indexes $1, \ldots, r$ the non-negative elements of $f$ and the indexes $r+1, \ldots, n$ the negative elements of $f$. The real numbers $x_{1}, \ldots x_{r}$ are said to be the non-negative values of $f$ and the numbers $y_{1}, \ldots y_{n-r}$ are said to be the negative values of $f$.

If $i_{1}, \ldots, i_{\alpha}$ are non-negative elements of $f$ and $j_{1}, \ldots, j_{\beta}$ are negative elements of $f$, with $i_{1}<\ldots<i_{\alpha}$ and $j_{1}<\ldots<j_{\beta}$, a subset $A$ of $\{1, \ldots, n\}$ is said to be of type

$$
\begin{equation*}
\left[i_{1}, \ldots, i_{\alpha}\right]_{a}^{+}\left[j_{1}, \ldots, j_{\beta}\right]_{b}^{-} \tag{2.2}
\end{equation*}
$$

if $A$ is made of $a$ elements chosen in $\left\{i_{1}, \ldots, i_{\alpha}\right\}$ and $b$ elements chosen in $\left\{j_{1} \ldots, j_{\beta}\right\}$.
Let $X$ be a finite set of integers. If $q$ is an integer less or equal than $|X|$, we call $q-$ string on $X$ a sequence $a_{1} \ldots a_{q}$, where $a_{1}, \ldots, a_{q}$ are distinct elements of $X$ such that $a_{1}<\ldots<a_{q}$. The family of all the $q$-strings on X will be denoted by $X^{(q)}$. In this paper, each subset $Y$ of $X$ with $q$ elements will be identified with the $q$-string of his elements ordered in an increasing way. When $i_{1}, \ldots, i_{k}$ are non-negative elements of $f$ and $j_{1}, \ldots, j_{l}$ are negative elements of $f$, with $i_{1}<\ldots<i_{k}<j_{1}<\ldots<j_{l}$, the $(k+l)$-string $i_{1} \ldots i_{k} j_{1} \ldots j_{l}$ will be written in the form

$$
i_{1} \ldots i_{k} \mid\left(j_{1}-r\right) \ldots\left(j_{l}-r\right)
$$

(thus $j_{1}-r, \ldots, j_{l}-r \in\{1, \ldots, n-r\}$ ).

For example, if $n=10$ and $r=7$, the 4 -string 1269 will be written in the form $126 \mid 2$.

Using the string-terminology instead of the set-terminology, in the sequel we call a $\left(d^{+}, n\right)$-subset of $f$ a $\left(d^{+}, n\right)-$ string of $f$.

Let us consider now the partition $\mathcal{P}$ of the real interval ( $\left.0, \frac{d-1}{d} n\right]$ :

$$
\begin{equation*}
\mathcal{P}=\left\{0, \frac{d-1}{d}, 2 \frac{d-1}{d}, \ldots,(n-1) \frac{d-1}{d}, n \frac{d-1}{d}\right\} . \tag{2.3}
\end{equation*}
$$

The following Proposition establishes when an interval determined by $\mathcal{P}$ contains an integer.

Proposition 2.1. If $k=1, \ldots, n$ and if $n-k \not \equiv_{d} 0$, there exists a unique integer $r$ such that

$$
\begin{equation*}
\frac{d-1}{d}(n-k)<r \leq \frac{d-1}{d}(n-k+1), \tag{2.4}
\end{equation*}
$$

and $r$ coincides with $\left\lfloor\frac{d-1}{d}(n-k+1)\right\rfloor$. Furthermore if $n-k \equiv_{d} 0$ no integer $r$ satisfies (2.4).

Proof. Let $k \in\{1, \ldots, n\}$ and set $m=n-k+1$. Since the interval $\left(\frac{d-1}{d}(m-1), \frac{d-1}{d} m\right]$ has length $\frac{d-1}{d}<1$, there is at most one integer $r$ that satisfies (2.4). Let us now write $m$ in the form

$$
\begin{equation*}
m=\tilde{q} d+s \tag{2.5}
\end{equation*}
$$

where $\tilde{q}, s$ are integers such that $\tilde{q} \geq 0,1 \leq s \leq d$. Let us suppose now that $n-k \not \equiv_{d} 0$, that is $m \not \equiv_{d} 1$; then we have $2 \leq s \leq d$.

Let $r=\left\lfloor\frac{d-1}{d} m\right\rfloor$. We show that $r$ satisfies (2.4).
Firstly, the second inequality of (2.4) is straightforward; secondly, for the first inequality we observe

$$
\frac{d-1}{d}(m-1)=\frac{d-1}{d}(\tilde{q} d+s-1)=\tilde{q}(d-1)+(s-1) \frac{d-1}{d} .
$$

Furthermore

$$
\begin{equation*}
r=\left\lfloor\frac{d-1}{d}(\tilde{q} d+s)\right\rfloor=\left\lfloor\tilde{q}(d-1)+s-\frac{s}{d}\right\rfloor=\tilde{q}(d-1)+(s-1) . \tag{2.6}
\end{equation*}
$$

Therefore $r>\frac{d-1}{d}(m-1)$, since $s \geq 2$.
If $n-k \equiv_{d} 0$, that is $m \equiv_{d} 1$, in (2.5) we have $s=1$ and (2.4) becomes

$$
\begin{equation*}
\tilde{q}(d-1)<r \leq \tilde{q}(d-1)+\frac{d-1}{d} . \tag{2.7}
\end{equation*}
$$

Note that (2.7) has no integer solutions.
Lemma 2.2. Let $r$ be a positive integer such that

$$
\begin{equation*}
d \leq r \leq \frac{d-1}{d} n . \tag{2.8}
\end{equation*}
$$

Then there exists a unique positive integer $b(r) \in\{1, \ldots, n-r-1\}$ that satisfies

$$
\begin{equation*}
\frac{d-1}{d}(n-b(r))<r \leq \frac{d-1}{d}(n-b(r)+1), \tag{2.9}
\end{equation*}
$$

Proof. By construction of partition $\mathcal{P}$, as in (2.3), there exists a unique $b(r) \in\{1, \ldots, n\}$ such that (2.9) holds.

We now show that $b(r)$ cannot exceed $n-r-1$.
Firstly, we suppose that $b(r)>n-r$. Then, we write $b(r)$ in the form $b(r)=n-r+\zeta$, with $\zeta$ integer such that $1 \leq \zeta \leq r$. Since $r$ satisfies (2.9), we have

$$
\frac{d-1}{d}(r-\zeta)<r \leq \frac{d-1}{d}(r-\zeta+1)
$$

that is

$$
\begin{equation*}
\zeta(1-d)<r \leq(d-1)(1-\zeta) \tag{2.10}
\end{equation*}
$$

Since $\zeta(1-d)<0$ and $(d-1)(1-\zeta) \leq 0$, there is no positive integer $r$ that satisfies (2.10).

Secondly, if $b(r)=n-r$, (2.9) becomes

$$
\frac{d-1}{d} r<r \leq \frac{d-1}{d}(r+1),
$$

that is

$$
0<r \leq d-1
$$

contradicting the hypothesis (2.8).
We stress that the number $b(r)$ will play a key role in the sequel of the paper.

## 3. Some upper and Lower bounds for $\gamma(n, d, r)$

In this Section we establish an upper bound for $\gamma(n, d, r)$, when $r$ satisfies (2.8). We also provide a lower bound for $\gamma(n, d, r)$ under one additional hypothesis.

Proposition 3.1. Let $r$ be a positive integer that satisfies

$$
d \leq r \leq \frac{d-1}{d} n,
$$

then

$$
\begin{equation*}
\gamma(n, d, r) \leq \sum_{j=0}^{\min \{b(r), d-1\}}\binom{b(r)}{j}\binom{r}{d-j} . \tag{3.1}
\end{equation*}
$$

Proof. Since $1 \leq b(r) \leq n-r-1$, we construct a weight function $f \in W_{n}(\mathbb{R})$, with $f^{+}=r$, such that

$$
\begin{equation*}
\phi(f, d)=\sum_{j=0}^{\min \{b(r), d-1\}}\binom{b(r)}{j}\binom{r}{d-j} . \tag{3.2}
\end{equation*}
$$

This is sufficient to prove the thesis.
Let $h=\min \{b(r), d-1\}$. Let $\alpha$ be a positive real number. In order to simplify the notation, we call $\beta$ the number $\frac{r+b(r)(-\alpha)}{n-r-b(r)}$, in such a way that

$$
r+b(r)(-\alpha)+(n-r-b(r))(-\beta)=0
$$

holds.
At this point we define the function

$$
\begin{array}{ccccccccc}
1 & \cdots & r & r+1 & \cdots & r+b(r) & r+(b(r)+1) & \cdots & r+(n-r)  \tag{3.3}\\
f_{\alpha}: & \downarrow & \cdots & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots \\
1 & \cdots & 1 & -\alpha & \cdots & -\alpha & -\beta & \cdots & -\beta
\end{array} .
$$

We now show that for $\alpha$ sufficiently small, that is

$$
\begin{equation*}
0<\alpha<\min \left\{\frac{r}{b(r)}, \frac{d}{h}-1, \frac{d}{b(r)}\left(r-\frac{d-1}{d}(n-b(r))\right)\right\} \tag{3.4}
\end{equation*}
$$

$f_{\alpha}$ is a weight function that satisfies (3.2).
In fact:
a) the denominator of $\beta$, due to Lemma [2.2, is a positive number. Furthermore the numerator of $\beta$ is a positive number if and only if $\alpha<\frac{r}{b(r)}$. Therefore (3.4) and the definition of $\beta$ assure that $f_{\alpha}$ is a weight function.
b) having $\alpha<d / h-1$ is equivalent to require

$$
\begin{equation*}
\underbrace{1+\ldots+1}_{d-h \text { times }}+\underbrace{(-\alpha)+\ldots+(-\alpha)}_{h \text { times }} \geq 0 \tag{3.5}
\end{equation*}
$$

This condition assures that the subsets of the type

$$
\begin{array}{cccc}
{[1, \ldots, r]_{d}^{+}} & {[r+1, \ldots, r+b(r)]_{0}^{-},} & \text {in total } & \binom{b(r)}{0}\binom{r}{d} \\
{[1, \ldots, r]_{d-1}^{+}} & {[r+1, \ldots, r+b(r)]_{1}^{-},} & \text {in total } & \binom{b(r)}{1}\binom{r}{d-1}  \tag{3.6}\\
\vdots & \vdots & \vdots & \vdots \\
{[1, \ldots, r]_{d-h}^{+}} & {[r+1, \ldots, r+b(r)]_{h}^{-},} & \text {in total } & \binom{b(r)}{h}\binom{r}{d-h}
\end{array}
$$

are $\left(d^{+}, n\right)$-subsets of $f$
c) firstly we note that the requirement

$$
\alpha<\frac{d}{b(r)}\left(r-\frac{d-1}{d}(n-b(r))\right)
$$

is equivalent to require

$$
\begin{equation*}
\frac{d-1}{d}(n-b(r))+\alpha \frac{b(r)}{d}<r . \tag{3.7}
\end{equation*}
$$

Lemma 2.2 assures the existence of a such $\alpha$. Note that (3.7) is equivalent to

$$
\begin{equation*}
\underbrace{1+\ldots+1}_{d-1 \text { times }}+(-\beta)<0, \tag{3.8}
\end{equation*}
$$

that assures that the $\left(d^{+}, n\right)$-strings of $f$ are only of the type (3.6). Therefore we have constructed a weight function $f$ with $r$ non-negative elements that satisfies (3.2).

Corollary 3.2. Let $r$ be a positive integer such that $r \geq d$ and $\frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n$. Then

$$
\begin{equation*}
\gamma(n, d, r) \leq\binom{ r}{d}+\binom{r}{d-1} \tag{3.9}
\end{equation*}
$$

Proof. The result follows directly from Proposition 3.1] since $b(r)=1$.
Proposition 3.3. Let $r$ a positive integer such that $d \leq r \leq \frac{d-1}{d} n$. Let $f \in W_{n}(\mathbb{R})$, with $f^{+}=r$, as in (2.1). If

$$
\begin{equation*}
x_{1}+y_{n-r} \geq 0 \tag{3.10}
\end{equation*}
$$

then

$$
\begin{equation*}
\phi(f, d) \geq\binom{ r-1}{d-2}(n-r)+\binom{r}{d} \geq\binom{ r}{d}+\binom{r}{d-1} . \tag{3.11}
\end{equation*}
$$

Proof. We can consider the $d$-strings of $\{1, \ldots, n\}$ of type

$$
\begin{equation*}
1 i_{1} \ldots i_{d-2} \mid(n-r) \tag{3.12}
\end{equation*}
$$

where $i_{1} \ldots i_{d-2}$ are chosen in $\{2, \ldots, r\}$.
By virtue of (3.10), each string of the type (3.12) is a $\left(d^{+}, n\right)$-string of $f$.
On the other hand, since $y_{1} \geq y_{2} \geq \ldots \geq y_{n-r}$, each string of type

$$
\begin{equation*}
1 i_{1} \ldots i_{d-2} \mid k \tag{3.13}
\end{equation*}
$$

where $i_{1} \ldots i_{d-2}$ are chosen in $\{2, \ldots, r\}$ and $k$ in $\{1, \ldots, n-r\}$, will be a $\left(d^{+}, n\right)$-string of $f$.

The distinct strings of the type (3.13) are exactly $\binom{r-1}{d-2}(n-r)$. There are moreover all the $\left(d^{+}, n\right)$-strings of $f$ that are the $d$-strings on $\{1, \ldots, r\}$. This proves the first inequality in (3.11). Moreover, since $r \leq \frac{d-1}{d} n$, we also have $n-r \geq \frac{r}{d-1}$. Therefore

$$
\binom{r-1}{d-2}(n-r) \geq\binom{ r-1}{d-2} \frac{r}{d-1}=\binom{r}{d-1}
$$

Thus the second inequality also holds.
As a direct consequence of Corollary 3.2 and Proposition 3.3 it follows that if $r$ is a positive integer with $r \geq d$ such that $\frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n$, then

$$
\begin{equation*}
\min \left\{\phi(f, d): f \in W_{n}(\mathbb{R}), f^{+}=r, x_{1}+y_{n-r} \geq 0\right\}=\binom{r}{d}+\binom{r}{d-1} \tag{3.14}
\end{equation*}
$$

Remark 3.4. We conjecture that

$$
\begin{equation*}
\gamma(n, d, r)=\binom{r}{d}+\binom{r}{d-1} \tag{3.15}
\end{equation*}
$$

when $r \geq d$ and $\frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n$.
In Section 5 we give a partial answer to this conjecture. Note that, in order to prove (3.15), by Corollary 3.2 it is sufficient to show

$$
\begin{equation*}
\gamma(n, d, r) \geq\binom{ r}{d}+\binom{r}{d-1} \tag{3.16}
\end{equation*}
$$

Moreover, by virtue of (3.14), the inequality (3.16) is equivalent to the following:

$$
\begin{align*}
& \min \left\{\phi(f, d): f \in W_{n}(\mathbb{R}), f^{+}=r, x_{k}+y_{n-r}<0,\right.  \tag{3.17}\\
& \qquad \text { for every } k=1, \ldots, r\} \geq\binom{ r}{d}+\binom{r}{d-1} .
\end{align*}
$$

In Section 5 we shall prove this inequality in the special case $n=2 d+2$.
We close this section providing a simple combinatorial interpretation of the inequalities

$$
\frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n .
$$

For this purpose let us note that the last inequalities are equivalent to the following:

$$
\begin{equation*}
(n-r-1)(d-1)<r \leq(n-r)(d-1) \tag{3.18}
\end{equation*}
$$

Let now $r$ be a positive integer that satisfies (3.18) and $f \in W_{n}(\mathbb{R})$, with $f^{+}=r$, as in (2.1). Let us consider the following representation

$$
\begin{array}{lll}
\lrcorner+\llcorner \lrcorner+ & \ldots & +\llcorner \lrcorner+k_{1} \\
\lrcorner+\llcorner \lrcorner+ & \ldots & +\llcorner \lrcorner+k_{2}  \tag{3.19}\\
& \ldots & \\
\lrcorner+\llcorner \lrcorner+ & \ldots & +\llcorner \lrcorner+k_{n-r-1} \\
\lrcorner+\llcorner \lrcorner+ & \ldots & +\llcorner \lrcorner+k_{n-r}
\end{array}
$$

where every $\lrcorner$ can be seen as a "box" initially empty and every row contains $d-1$ boxes. Every of such boxes can be occupied by at most one non-negative element of $f$. Thus (3.18) is equivalent to state that $n-r-1$ rows in (3.19) must be completely occupied, whereas the last row must contain at least a non-empty box and, furthermore, the number of non-negative elements of $f$ cannot exceed the number of empty boxes in (3.19). This combinatorial interpretation of (3.18) suggests to examine firstly the $\left(d^{+}, n\right)$-strings of $f$ of the form $+\ldots+-$, that is a subset with $d-1$ non-negative elements and only one negative.

## 4. An application of Hall's Theorem

In this Section we use Hall's theorem on distinct representatives to determine some biunivocal functions between $q$-subsets of a set with $2 q+1$ elements. The results of this Section are used in Section [5 to determine $\gamma(n, d, r)$ when $r=2 d-1$ and $n=2 d+2$.

We now introduce some definitions and notation useful in the sequel.
Let $\Omega=\{1,2, \ldots, 2 q, 2 q+1\}$, where $q$ is a fixed positive integer.

Given a $q$-string $a_{1} \ldots a_{q} \in \Omega^{(q)}$, for notation convenience we denote by $\mathfrak{C}_{q}\left(a_{1} \ldots a_{q}\right)$ the family of all the $q$-strings on $\Omega \backslash\left\{a_{1}, \ldots, a_{q}\right\}$, that is

$$
\mathfrak{C}_{q}\left(a_{1} \ldots a_{q}\right)=\left(\Omega \backslash\left\{a_{1}, \ldots, a_{q}\right\}\right)^{(q)}=\left\{b_{1} \ldots b_{q}: b_{1}, \ldots, b_{q} \in \Omega, b_{i} \neq a_{j}, i, j=1, \ldots, q\right\} .
$$

Note that the family $\mathfrak{C}_{q}\left(a_{1} \ldots a_{q}\right)$ has exactly $\binom{q+1}{q}=q+1$ distinct $q$-strings.
A $q$-string in $\mathfrak{C}_{q}\left(a_{1} \ldots a_{q}\right)$ will be called a $q$-almost-complementary (or $q$-AC) of $a_{1}, \ldots a_{q}$.
From now on we call $p$ the number of the distinct $q$-strings of $\Omega^{(q)}$, that is $p=\binom{2 q+1}{q}$. We denote by $A_{1}, \ldots, A_{p}$ all the $q$-strings of $\Omega^{(q)}$ such that

$$
A_{1} \prec \ldots \prec A_{p},
$$

where $\prec$ is the usual lexicographic order.
Definition 4.1. A $q$-pairing of almost-complementaries on $\Omega$ (or $q$-PAC on $\Omega$ ) is a biunivocal function $\varphi: \Omega^{(q)} \rightarrow \Omega^{(q)}$ such that $\varphi\left(A_{i}\right)$ is a $q-\mathrm{AC}$ of $A_{i}$ for $i=1, \ldots p$, that is

$$
\varphi\left(A_{i}\right) \in \mathfrak{C}_{q}\left(A_{i}\right),
$$

for $i=1, \ldots, p$.
Let us set now $\mathfrak{F}_{q}=\left\{\mathfrak{C}_{q}\left(A_{1}\right), \ldots, \mathfrak{C}_{q}\left(A_{p}\right)\right\}$.
We recall that the family $\mathfrak{F}_{q}$ has a system of distinct representatives (SDR), say $\left(C_{1}, \ldots, C_{p}\right)$, if $C_{1} \in \mathfrak{C}_{q}\left(A_{1}\right), \ldots, C_{p} \in \mathfrak{C}_{q}\left(A_{p}\right)$ and $C_{i} \neq C_{j}$ for $i, j \in\{1, \ldots, p\}$ with $i \neq j$.

Proposition 4.2. The family $\mathfrak{F}_{q}$ has a SDR if and only if there exists a $q-P A C$ on $\Omega$.
Proof. Sufficiency. Let $\left(C_{1}, \ldots, C_{p}\right)$ be a SDR for $\mathfrak{F}_{q}$. This means that all the $C_{i}$ are distinct $q$-strings and that $C_{i} \in \mathfrak{C}_{q}\left(A_{i}\right)$ for $i=1, \ldots, p$. Thus the function

$$
\varphi: \Omega^{(q)} \rightarrow \Omega^{(q)}
$$

defined by

$$
\varphi\left(A_{i}\right)=C_{i} \in \mathfrak{C}_{q}\left(A_{i}\right),
$$

for $i=1, \ldots, p$, is a $q$-PAC on $\Omega$.
Necessity. If $\varphi$ is a $q$-PAC on $\Omega$, then $\varphi$ is a bijection such that $\varphi\left(A_{i}\right) \in \mathfrak{C}_{q}\left(A_{i}\right)$, for $i=1, \ldots, p$. Since $\varphi$ is a bijection, $\varphi\left(A_{1}\right), \ldots, \varphi\left(A_{p}\right)$ is a SDR for $\mathfrak{F}_{q}$.

Proposition 4.3. For every positive integer $q$ there exists a $q-P A C$ on $\Omega$.

Proof. By virtue of Proposition 4.2 it is sufficient to show that the family $\mathfrak{F}_{q}$ has a SDR, i.e. that the well-known Hall's condition holds:

$$
\begin{equation*}
\text { for every } I \subset\{1, \ldots, p\} \text {, we have }\left|\bigcup_{i \in I} \mathfrak{C}_{q}\left(A_{i}\right)\right| \geq|I| \text {. } \tag{4.1}
\end{equation*}
$$

Therefore, let $I=\left\{i_{1}, \ldots, i_{k}\right\}$ be an arbitrary subset of indices $i_{1}, \ldots, i_{k} \in\{1, \ldots, p\}$. Let $Y:=\mathfrak{C}_{q}\left(A_{i_{1}}\right) \cup \cdots \cup \mathfrak{C}_{q}\left(A_{i_{1} k}\right)=\left\{C_{1}, \ldots, C_{a}\right\}$. With this notation (4.1) is equivalent to $a \geq k$, therefore we shall prove now this last inequality. Set $\mathfrak{A}:=\left\{\mathfrak{C}_{q}\left(A_{i_{1}}\right), \ldots, \mathfrak{C}_{q}\left(A_{i_{k}}\right)\right\}$. For all $C_{l} \in Y$ we denote by $d_{\mathfrak{A}}\left(C_{l}\right)$ the degree of $C_{l}$ respect to the family $\mathfrak{A}$, that is the number of distinct sets $\mathfrak{C}_{q}\left(A_{i_{j}}\right)$ that contain $C_{l}$. We have previously observed that $\left|\mathfrak{C}_{q}\left(A_{i_{j}}\right)\right|=q+1$ for all $A_{i_{j}}$, moreover, by a classical double counting principle we also have

$$
\sum_{l=1}^{a} d_{\mathfrak{A}}\left(C_{l}\right)=\sum_{j=1}^{k}\left|\mathfrak{C}_{q}\left(A_{i_{j}}\right)\right|
$$

hence

$$
\begin{equation*}
\sum_{l=1}^{a} d_{\mathfrak{A}}\left(C_{l}\right)=k(q+1) \tag{4.2}
\end{equation*}
$$

On the other hand, every $C_{l}$ is a $q$-string, let us say $C_{l}=c_{1} \ldots c_{q}$, which belongs to the sets $\mathfrak{C}_{q}\left(a_{1} \ldots a_{q}\right)$, where $a_{1} \ldots a_{q}$ is a $q$ - AC of $c_{1} \ldots c_{q}$. Since the number of the distinct $q$-AC strings of $c_{1} \ldots c_{q}$ is $\binom{q+1}{q}$, it follows that every $C_{l}$ belongs exactly to $q+1$ subsets $\mathfrak{C}_{q}\left(A_{s}\right)$, with $s=1, \ldots, p$; therefore $d_{\mathfrak{A}}\left(C_{l}\right) \leq q+1$ for $l=1, \ldots, a$. By (4.2) we obtain then

$$
k(q+1) \leq a(q+1)
$$

i.e. $k \leq a$.

The Proposition 4.3 does not provide an explicit construction of a $q-\mathrm{PAC}$ on $\Omega$. In order to construct a $q$-PAC on $\Omega$ we suggest the following Algorithm:

## $q-$ PAC Algorithm

Input: a positive integer $q$
Output: a $q-\mathrm{PAC}$ on $\Omega$
Step 1: Write all the $q$-strings of $\Omega^{(q)}$ ordered in increasing way with respect to the lexicographic order

$$
B_{1} \prec B_{2} \prec \ldots \prec B_{p}
$$

and put them in an array $\operatorname{Dom}[\mathrm{p}]$ of $q$-strings, that has $p$ positions, where $p=$ $\binom{2 q+1}{q}$.

Step 2: For all $i=1, \ldots, p$ write all the $q$-strings of $\mathfrak{C}_{q}\left(B_{i}\right)$ in decreasing lexicographic order

$$
C_{i_{1}} \succ \ldots \succ C_{i_{q+1}} .
$$

Step 3: Set up an array $\operatorname{Im}[\mathrm{p}]$ of $q$-strings, that has $p$ positions, and initialize every position with the $q$-string with all zero entries.
Step 4: For all $i=1, \ldots, p$ examine in sequence the $q$-strings $C_{i_{1}}, \ldots, C_{i_{q+1}}$ and put the first of such $q$-strings that does not appear in $\operatorname{Im}[1], \ldots, \operatorname{Im}[i]$ in the position Im[i].

Then the correspondence $\operatorname{Dom}[\mathrm{i}] \longmapsto \operatorname{Im}[\mathrm{i}](i=1, \ldots, p)$ provides a $q-\mathrm{PAC}$ on $\Omega$.

For small values of $q$ we have implemented the previous algorithm in Java. For example, if $q=3$ then $p=\binom{2 q+1}{q}=\binom{7}{3}=35$. In this case Dom [35] and $\operatorname{Im}$ [35] are two arrays with 35 position, both containing all the 3 -strings on $\{1, \ldots, 7\}$. The execution of our program for $q=3$ provides the following result (the strings on the left of ---> are those of Dom [35], the strings on the right of ---> are those of $\operatorname{Im}$ [35] ):

$$
\begin{aligned}
& 123 \text {---> } 567 \text {; } 124 \text {---> } 367 \text {; } 125 \text {---> } 467 \text {; } 126 \text {---> } 457 \text {; } 127 \text {---> } 456 \text {; } \\
& 134 \text {---> 267; } 135 \text {---> 247; } 136 \text {---> 257; } 137 \text {---> } 256 \text {; } \\
& 145 \text {---> } 237 \text {; } 146 \text {---> } 357 \text {; } 147 \text {---> } 356 \text {; } \\
& 156 \text {---> } 347 \text {; } 157 \text {---> } 346 \text {; } \\
& 167 \text {---> 345; } \\
& 234 \text {---> 167; } 235 \text {---> 147; } 236 \text {---> 157; } 237 \text {---> 156; } \\
& 245 \text {---> 137; } 246 \text {---> 135; } 247 \text {---> 136; } \\
& 256 \text {---> 134; } 257 \text {---> 146; } \\
& 267 \text {---> 145; } \\
& 345 \text {---> 127; } 346 \text {---> 125; } 347 \text {---> 126; } \\
& 356 \text {---> 124; } 357 \text {---> 246; } \\
& 367 \text {---> 245; } \\
& 456 \text {---> 123; } 457 \text {---> 236; } \\
& 467 \text {---> 235; } \\
& 567 \text {---> 234; }
\end{aligned}
$$

## 5. The case $d \leq r, \frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n, n=2 d+2$

In this Section we shall assume that $n=2 d+2$ and that $r$ is a positive integer such that $r \geq d, \frac{d-1}{d}(n-1)<r \leq \frac{d-1}{d} n$. Under such hypotheses we can apply the Proposition 2.1 to the case $k=1$, obtaining

$$
r=\left\lfloor\frac{d-1}{d} n\right\rfloor=\left\lfloor\frac{d-1}{d}(2 d+2)\right\rfloor=2 d-1=n-3 .
$$

For such values of $r$ and $n$ we determine the value of $\gamma(n, d, r)$. This result implies that in this case the (MS)-conjecture does not hold. We also compare our results with the ones in [6].

Theorem 5.1. If $n=2 d+2$ and $r=2 d-1=n-3$ then

$$
\begin{equation*}
\gamma(n, d, r)=\binom{r}{d}+\binom{r}{d-1} \tag{5.1}
\end{equation*}
$$

Proof. Due to Remark 3.4 we only need to show (3.17) when $n-r=3$. Thus take $f \in W_{n}(\mathbb{R})$, with $f^{+}=r$, as in (2.1) and suppose that $x_{k}+y_{3}<0$ for every $k=1, \ldots, r$.

Take $q=d-1$ (and therefore $r=2 q+1$ ). By Proposition 4.3 there exists a $q-\mathrm{PAC}$ on $\Omega$, where $\Omega=\{1, \ldots, r\}=\{1, \ldots, 2 q+1\}$. We use the notation introduced in Section 4. Take $\Omega^{(q)}=\left\{A_{1}, \ldots, A_{p}\right\}$ with the lexicographic order:

$$
A_{1} \prec \ldots \prec A_{p}
$$

where $p=\binom{2 q+1}{q}$. Let $C_{s}=\varphi\left(A_{s}\right)$ for $s=1, \ldots, p$.
Since $A_{s}$ and $C_{s}$ are $q$-strings with no common elements there exists in $\Omega$ a unique element, say $i_{s}$, that is not an element of the $q$-string $A_{s}$ and nor an element of the $q-$ string $C_{s}$. We point out that the elements $i_{1}, \ldots, i_{p}$ are not distinct between them, since $p>r$.

If $A=t_{1} \ldots t_{d-1} \in \Omega^{(q)}$ and $k \in\{1,2\}$, with the notation $A \mid k$ we mean the $d$-string $t_{1} \ldots t_{d-1} \mid k$ and with $i_{s} \mid 3$ the 2 -string with the non-negative element $i_{s}$ and with the negative element 3 . We now consider the following configuration:

$$
\begin{array}{ccc}
A_{1} \mid 1 & C_{1} \mid 2 & i_{1} \mid 3 \\
A_{2} \mid 1 & C_{2} \mid 2 & i_{2} \mid 3  \tag{5.2}\\
& \ldots & \\
A_{p} \mid 1 & C_{p} \mid 2 & i_{p} \mid 3
\end{array}
$$

Since $\varphi$ is a bijection, the $q$-strings $C_{1}\left|2, \ldots, C_{p}\right| 2$ are themselves distinct. Moreover, since $\varphi$ is a $q-\mathrm{PAC}$ on $\Omega$, each row in (5.2) contains all the elements (non-negative and negative) of $f$. Since the function $f$ is a weight function and from the hypothesis we have $x_{i_{s}}+y_{3}<0$ (that is each $i_{s} \mid 3$ corresponds to a negative sum), in every sth-row of the configuration (5.2) at least one $d$-string between $A_{s} \mid 1$ and $C_{s} \mid 2$ must be a $\left(d^{+}, n\right)$-string for $f$.

This shows that the number of the distinct $\left(d^{+}, n\right)$-strings for $f$ is at least equal to the number of the rows in (5.2), that is $p=\binom{2 q+1}{q}=\binom{r}{d-1}$.

The remaining $\left(d^{+}, n\right)$-strings for $f$ that we need in order to obtain (3.17) are all the $d$-strings of $\Omega$, which are $\binom{r}{d}$. This shows that $\phi(f, d) \geq\binom{ r}{d}+\binom{r}{d-1}$.

Since $f$ is arbitrary, (3.17) follows.
From this result we deduce the following consequence on the (MS)-conjecture.
Corollary 5.2. The (MS)-conjecture does not hold when $n=2 d+2$ and $d \geq 3$.
Proof. We take $r=2 d-1$. Then from (1.1) and Theorem 5.1 we have

$$
\psi(n, d) \leq\binom{ r}{d}+\binom{r}{d-1}
$$

and for such values of $r$ and $n$ we have

$$
\binom{r}{d}+\binom{r}{d-1}=\frac{2(2 d-1)!}{d!(d-1)!}<\frac{(2 d+1)!}{(d-1)!(d+2)!}=\binom{n-1}{d-1},
$$

if $d \geq 3$.
When $n=2 d+2$ and $r=2 d-1$, from Theorem 5.1] it follows that

$$
\phi(f, d) \geq\binom{ 2 d-1}{d}+\binom{2 d-1}{d-1}>\binom{2 d-1}{d-1}=\binom{(2 d+2)-2-1}{d-1}
$$

that is

$$
\begin{equation*}
\phi(f, d)>\binom{n-k-1}{d-1} \tag{5.3}
\end{equation*}
$$

This inequality improves the estimate

$$
\begin{equation*}
\phi(f, d) \geq\binom{ n-k-1}{d-1} \tag{5.4}
\end{equation*}
$$

obtained in [6] under the additional hypotheses
(i) $\sum_{x \in I_{n}} f(x)=0$,
(ii) $\sum_{y \in Y} f(y) \neq 0$ for all $Y \subseteq I_{n}$ such that $|Y|=d$.

$$
\text { 6. THE CASE } r=\frac{d-1}{d} n
$$

There is also another case when we can prove

$$
\gamma(n, d, r)=\binom{r}{d}+\binom{r}{d-1}
$$

This is the case $r=\frac{d-1}{d} n$. In such case $d \mid n$ and therefore $r \geq d$ if $d \geq 2$.
Theorem 6.1. Let $r=\frac{d-1}{d} n$. Then

$$
\gamma(n, d, r)=\binom{r}{d}+\binom{r}{d-1}
$$

Proof. The condition $r=\frac{d-1}{d} n$ is equivalent to $r=(d-1)(n-r)$. Take $f \in W_{n}(\mathbb{R})$ with $f^{+}=r$. Then we can build partitions $\mathfrak{S}$ of the set $\{1, \ldots, r, r+1, \ldots, n\}$ of the type $\mathfrak{S}=\left\{C_{1}, \ldots, C_{n-r}\right\}$, where

$$
\begin{align*}
C_{1} & =i_{1,1} \ldots i_{1, d-1} \mid k_{1} \\
C_{2} & =i_{2,1} \ldots i_{2, d-1} \mid k_{2}  \tag{6.1}\\
& \ldots \\
C_{n-r} & =i_{n-r, 1} \ldots i_{n-r, d-1} \mid k_{n-r}
\end{align*}
$$

with $i_{s, t} \in\{1, \ldots, r\}, k_{l} \in\{1, \ldots, n-r\}$, for $1 \leq s \leq n-r, 1 \leq t \leq d-1,1 \leq l \leq n-r$. By means of a technique similar to the one used by Bier and Manickam in the proof of Lemma 1 of [6], we can claim that there exist exactly $\binom{r-1}{d-2}(n-r)$ disjoint partition of type (6.1). Since $f$ is a weight function, at least a row in (6.1) is $\left(d^{+}, n\right)$-string for $f$.

Since the partitions are disjoint, if we extract from each of them at least a $\left(d^{+}, n\right)$-string for $f$, we get at least $\binom{r-1}{d-2}(n-r)$ distinct $\left(d^{+}, n\right)$-string for $f$.

Since $r=\frac{d-1}{d} n$, we have

$$
\binom{r-1}{d-2}(n-r)=\binom{r-1}{d-2} \frac{r}{d-1}=\binom{r}{d-1}
$$

Therefore

$$
\phi(f, d) \geq\binom{ r}{d}+\binom{r}{d-1}
$$

This show that

$$
\gamma(n, d, r) \geq\binom{ r}{d}+\binom{r}{d-1}
$$

The equality follows form Corollary (3.2).

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