NEW RESULTS RELATED TO A CONJECTURE OF MANICKAM AND SINGHI

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ABSTRACT. In 1998 Manickam and Singhi conjectured that for every positive integer dand every $n \ge 4d$, every set of n real numbers whose sum is nonnegative contains at least $\binom{n-1}{d-1}$ subsets of size d whose sums are nonnegative. In this paper we establish new results related to this conjecture. We also prove that the conjecture of Manickam and Singhi does not hold for n = 2d + 2.

1. INTRODUCTION

In this paper we establish new results related to a conjecture of Manickam and Singhi (from now on, (MS)-conjecture). In order to illustrate the (MS)-conjecture and our results we need to introduce the following notation. Let $n \in \mathbb{N}$ and let I_n be the set $\{1, 2, \ldots, n\}$. A function $f: I_n \to \mathbb{R}$ is called a *n*-weight function if

$$\sum_{x \in I_n} f(x) \ge 0.$$

Let $W_n(\mathbb{R})$ denote the set of all *n*-weight functions. If $f \in W_n(\mathbb{R})$ we set

$$f^+ := |\{x \in I_n : f(x) \ge 0\}|$$

If d is an integer with $1 \leq d \leq n$ and Y is a subset of I_n having d elements such that

$$\sum_{y \in Y} f(y) \ge 0,$$

we call Y a (d^+, n) -subset of f. If $f \in W_n(\mathbb{R})$, we denote by $\phi(f, d)$ the number of distinct (d^+, n) -subsets of f.

Furthermore, we set

$$\psi(n,d) := \min\{\phi(f,d) : f \in W_n(\mathbb{R})\}.$$

In 1988, Manickam and Singhi [14] stated the following conjecture:

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(MS)-Conjecture : If d is a positive integer and f is a n-weight function with $n \ge 4d$, then

$$\psi(n,d) \ge \binom{n-1}{d-1}.$$

We remark that, as previously observed in [8], the conjecture is equivalent to require that $\psi(n,d) = \binom{n-1}{d-1}$.

This conjecture is interesting for several reasons. It is deeply related with the first distribution invariant of the Johnson-scheme [6, 12, 13, 14]. The distribution invariants were introduced by Bier [5] and later investigated in [7, 11, 12, 14]. Manickam and Singhi [14] claim that this conjecture is, in some sense, dual to the theorem of Erdös–Ko–Rado [9]. Also, as pointed out by Srinivasan [17], this conjecture settles some cases of another conjecture on multiplicative functions by Alladi, Erdös and Vaaler [1].

In general the conjecture of Manickam and Singhi still remains open. So far the following partial results have been achieved:

- (1) The (MS)-conjecture is true if n = ud, $u \ge 4$ (Corollary 1 of [6]).
- (2) The (MS)-conjecture is true if d = 2 (Corollary 3 of [6]).
- (3) If we set n in the form n = ud + v, where $u \ge 4$ and $v = 1, \ldots, d-1$, the (MS)-conjecture is true if $r \le \min\left\{\frac{n-v-1}{(d-1)(d-2)}, \frac{n-v}{d}\right\}$ (Lemma 2 of [6]).
- (4) The (MS)-conjecture is true if d = 3 and $n \ge 93$ (Theorem of [6]).
- (5) The (MS)-conjecture is true if $r \le d \le n/2$ (Proposition 2 of [8]).
- (6) The (MS)-conjecture is true if it is true when $d < r \leq \frac{d-1}{d}n$ (Proposition 5 of [8]).
- (7) The (MS)-conjecture is true if d = 3 (Section 3 of [16]).
- (8) The (MS)-conjecture is true if $n \ge 2^d d^{d+1} + 2d^3$ (Theorem 3 of [15]).
- (9) The (MS)-conjecture is true if d > 3 and $n \ge d(d-1)^d(d-2)^d + d(d-1)^2(d-2) + d[n]_k$, where $[n]_k$ denotes the smallest positive integer congruent to $n \pmod{k}$ (Main Theorem in [6]).
- (10) The (MS)-conjecture is true if $n \ge 2^{d+1}e^d d^{d+1}$ (Theorem 1 of [3]).

We point out that for d > 4 the best estimate between (8), (9) and (10) is (8).

Different techniques have been used to attempt to tackle the (MS)-conjecture. In [6, 8, 15] the approach is combinatorial. In particular Bier and Manickam [6] use a result of Baranyai (see for example [2, 18]). Manickam and Miklos [15] use a circle permutation method, previously utilized by Katona [10] for a simpler proof of the theorem of Erdös–Ko–Rado. The approach in [3, 4, 16] is somewhat different. In fact the techniques in [16] are analytical-combinatorial and in [3, 4] are probabilistic.

A natural question arises when one studies the (MS)-conjecture:

What is the value of $\psi(n, d)$ for each $d \leq n$?

In order to provide an answer to this question, in [8] the following numbers were introduced:

$$\gamma(n,d,r) = \min\{\phi(f,d) : f \in W_n(\mathbb{R}), f^+ = r\},\$$

where $r, d \in \mathbb{N}$, with $r, d \leq n$.

It is clear that a complete computation of these numbers would also provide a complete determination of the numbers $\psi(n, d)$, since

(1.1)
$$\psi(n,d) = \min_{1 \le r \le n} \gamma(n,d,r).$$

In particular, the knowledge of $\gamma(n, d, r)$ when $n \ge 4d$ and r is an arbitrary integer such that $r \le n$, would supply an answer to the (MS)-conjecture.

Remark 1.1. In general the computation of $\gamma(n, d, r)$, started in [8], is not an easy task. For some values of n, d, r, this has been done in [8]. Nevertheless we stress that there is a gap in the proof of Proposition 2 of [8]. In particular, this means that it is not clear whether the identity

$$\gamma(n, d, r) = \binom{n-1}{d-1}$$

holds or not for $r \leq d \leq n/2$.

In this paper we continue the study of the numbers $\gamma(n, d, r)$. Here we establish some lower and upper bounds for $\gamma(n, d, r)$ when $d \leq r \leq \frac{d-1}{d}n$. From these inequalities we obtain that

(1.2)
$$\gamma(n,d,r) = \binom{r}{d} + \binom{r}{d-1},$$

when n = 2d + 2 with r = 2d - 1 and when $r = \frac{d-1}{d}n$. Combining our results with the ones in [8] we obtain the following values of $\gamma(n, d, r)$:

$$\gamma(n,d,r) = \begin{cases} \binom{n-1}{d-1} & \text{if } r \leq d \leq \frac{n}{2} \quad (\star) \\ \binom{n-r}{d-r} & \text{if } r \leq d < n \quad \text{and} \quad r < \frac{n}{n-d} \\ \binom{r}{d} & \text{if } d < r < n \quad \text{and} \quad r > \frac{d-1}{d}n \\ \binom{n-1}{d-1} & \text{if } r = 1 \\ \binom{r}{d} + \binom{r-1}{d-1} & \text{if } n = 2d+2 \quad \text{and} \quad r = 2d-1 \\ \binom{r}{d} + \binom{r-1}{d-1} & \text{if } r \geq d \quad \text{and} \quad r = \frac{d-1}{d}n, \end{cases}$$

where (\star) in the first row means that the equality in that case is uncertain (see Remark 1.1). We stress that the determination of the numbers $\gamma(n, d, r)$ in general is an open problem.

A straightforward consequence of (1.2) is that the (MS)-conjecture does not hold if n = 2d + 2. This provides another range of values of n, when n < 4d, for which the (MS)-conjecture does not hold. Note that Bier and Manickam [6] already proved that (MS)-conjecture does not hold in general. In particular they proved that the (MS)-conjecture does not hold for n = 2d + 1, with $d \ge 2$, and for n = 3d + 1, with $d \ge 3$.

We also prove that, for n = 2d + 2 and r = 2d - 1, (1.2) improves the results of Lemma 1 of [6].

A key tool in our paper is Hall's Theorem, as far as we know, used here for the first time in this context. We use this Theorem to determine, in a non constructive way, certain biunivocal functions between complementary q-subsets of a set with 2q + 1 elements. Such functions are important to compute the numbers $\gamma(n, d, r)$ in the case n = 2d + 2and r = 2d - 1.

We also suggest a new algorithm to determine the previous functions also in a constructive way.

2. Preliminaries

In this Section we introduce some notation and prove some elementary arithmetical preliminaries useful in the sequel of this paper.

We shall assume that a generic weight function $f \in W_n(\mathbb{R})$, with $f^+ = r$, has the form

with

$$x_1 \ge x_2 \ge \ldots \ge x_r \ge 0 > y_1 \ge y_2 \ge \ldots \ge y_{n-r}$$

Let us call the indexes $1, \ldots, r$ the non-negative elements of f and the indexes $r+1, \ldots, n$ the negative elements of f. The real numbers $x_1, \ldots x_r$ are said to be the non-negative values of f and the numbers $y_1, \ldots y_{n-r}$ are said to be the negative values of f.

If i_1, \ldots, i_{α} are non-negative elements of f and j_1, \ldots, j_{β} are negative elements of f, with $i_1 < \ldots < i_{\alpha}$ and $j_1 < \ldots < j_{\beta}$, a subset A of $\{1, \ldots, n\}$ is said to be of type

(2.2)
$$[i_1,\ldots,i_{\alpha}]^+_a [j_1,\ldots,j_{\beta}]^-_b$$

if A is made of a elements chosen in $\{i_1, \ldots, i_{\alpha}\}$ and b elements chosen in $\{j_1, \ldots, j_{\beta}\}$.

Let X be a finite set of integers. If q is an integer less or equal than |X|, we call q-string on X a sequence $a_1 \ldots a_q$, where a_1, \ldots, a_q are distinct elements of X such that $a_1 < \ldots < a_q$. The family of all the q-strings on X will be denoted by $X^{(q)}$. In this paper, each subset Y of X with q elements will be identified with the q-string of his elements ordered in an increasing way. When i_1, \ldots, i_k are non-negative elements of f and j_1, \ldots, j_l are negative elements of f, with $i_1 < \ldots < i_k < j_1 < \ldots < j_l$, the (k+l)-string $i_1 \ldots i_k j_1 \ldots j_l$ will be written in the form

$$i_1\ldots i_k|(j_1-r)\ldots (j_l-r),$$

(thus $j_1 - r, \ldots, j_l - r \in \{1, \ldots, n - r\}$).

For example, if n = 10 and r = 7, the 4-string 1269 will be written in the form 126|2.

Using the string-terminology instead of the set-terminology, in the sequel we call a (d^+, n) -subset of f a (d^+, n) -string of f.

Let us consider now the partition \mathcal{P} of the real interval $(0, \frac{d-1}{d}n]$:

(2.3)
$$\mathcal{P} = \left\{0, \frac{d-1}{d}, 2\frac{d-1}{d}, \dots, (n-1)\frac{d-1}{d}, n\frac{d-1}{d}\right\}.$$

The following Proposition establishes when an interval determined by \mathcal{P} contains an integer.

Proposition 2.1. If k = 1, ..., n and if $n - k \not\equiv_d 0$, there exists a unique integer r such that

(2.4)
$$\frac{d-1}{d}(n-k) < r \le \frac{d-1}{d}(n-k+1),$$

and r coincides with $\lfloor \frac{d-1}{d}(n-k+1) \rfloor$. Furthermore if $n-k \equiv_d 0$ no integer r satisfies (2.4).

Proof. Let $k \in \{1, \ldots, n\}$ and set m = n - k + 1. Since the interval $(\frac{d-1}{d}(m-1), \frac{d-1}{d}m]$ has length $\frac{d-1}{d} < 1$, there is at most one integer r that satisfies (2.4). Let us now write m in the form

$$(2.5) m = \tilde{q}d + s_1$$

where \tilde{q}, s are integers such that $\tilde{q} \ge 0, 1 \le s \le d$. Let us suppose now that $n - k \not\equiv_d 0$, that is $m \not\equiv_d 1$; then we have $2 \le s \le d$.

Let $r = \lfloor \frac{d-1}{d}m \rfloor$. We show that r satisfies (2.4).

Firstly, the second inequality of (2.4) is straightforward; secondly, for the first inequality we observe

$$\frac{d-1}{d}(m-1) = \frac{d-1}{d}(\tilde{q}d+s-1) = \tilde{q}(d-1) + (s-1)\frac{d-1}{d}.$$

Furthermore

(2.6)
$$r = \lfloor \frac{d-1}{d} (\tilde{q}d+s) \rfloor = \lfloor \tilde{q}(d-1) + s - \frac{s}{d} \rfloor = \tilde{q}(d-1) + (s-1).$$

Therefore $r > \frac{d-1}{d}(m-1)$, since $s \ge 2$.

If $n - k \equiv_d 0$, that is $m \equiv_d 1$, in (2.5) we have s = 1 and (2.4) becomes

(2.7)
$$\tilde{q}(d-1) < r \le \tilde{q}(d-1) + \frac{d-1}{d}$$

Note that (2.7) has no integer solutions.

Lemma 2.2. Let r be a positive integer such that

$$(2.8) d \le r \le \frac{d-1}{d}n$$

Then there exists a unique positive integer $b(r) \in \{1, \ldots, n-r-1\}$ that satisfies

(2.9)
$$\frac{d-1}{d}(n-b(r)) < r \le \frac{d-1}{d}(n-b(r)+1),$$

Proof. By construction of partition \mathcal{P} , as in (2.3), there exists a unique $b(r) \in \{1, \ldots, n\}$ such that (2.9) holds.

We now show that b(r) cannot exceed n - r - 1.

Firstly, we suppose that b(r) > n - r. Then, we write b(r) in the form $b(r) = n - r + \zeta$, with ζ integer such that $1 \leq \zeta \leq r$. Since r satisfies (2.9), we have

$$\frac{d-1}{d}(r-\zeta) < r \le \frac{d-1}{d}(r-\zeta+1),$$

that is

(2.10)
$$\zeta(1-d) < r \le (d-1)(1-\zeta).$$

Since $\zeta(1-d) < 0$ and $(d-1)(1-\zeta) \leq 0$, there is no positive integer r that satisfies (2.10).

Secondly, if b(r) = n - r, (2.9) becomes

$$\frac{d-1}{d}r < r \le \frac{d-1}{d}(r+1),$$

that is

$$0 < r \le d - 1,$$

contradicting the hypothesis (2.8).

We stress that the number b(r) will play a key role in the sequel of the paper.

3. Some upper and lower bounds for $\gamma(n, d, r)$

In this Section we establish an upper bound for $\gamma(n, d, r)$, when r satisfies (2.8). We also provide a lower bound for $\gamma(n, d, r)$ under one additional hypothesis.

Proposition 3.1. Let r be a positive integer that satisfies

$$d \le r \le \frac{d-1}{d}n,$$

then

(3.1)
$$\gamma(n,d,r) \leq \sum_{j=0}^{\min\{b(r),\,d-1\}} \binom{b(r)}{j} \binom{r}{d-j}.$$

Proof. Since $1 \leq b(r) \leq n - r - 1$, we construct a weight function $f \in W_n(\mathbb{R})$, with $f^+ = r$, such that

(3.2)
$$\phi(f,d) = \sum_{j=0}^{\min\{b(r), d-1\}} {b(r) \choose j} {r \choose d-j}.$$

This is sufficient to prove the thesis.

Let $h = \min\{b(r), d-1\}$. Let α be a positive real number. In order to simplify the notation, we call β the number $\frac{r+b(r)(-\alpha)}{n-r-b(r)}$, in such a way that

$$r + b(r)(-\alpha) + (n - r - b(r))(-\beta) = 0$$

holds.

At this point we define the function

We now show that for α sufficiently small, that is

(3.4)
$$0 < \alpha < \min\left\{\frac{r}{b(r)}, \frac{d}{h} - 1, \frac{d}{b(r)}\left(r - \frac{d-1}{d}(n-b(r))\right)\right\},$$

 f_{α} is a weight function that satisfies (3.2).

In fact:

- a) the denominator of β , due to Lemma 2.2, is a positive number. Furthermore the numerator of β is a positive number if and only if $\alpha < \frac{r}{b(r)}$. Therefore (3.4) and the definition of β assure that f_{α} is a weight function.
- b) having $\alpha < d/h 1$ is equivalent to require

(3.5)
$$\underbrace{1+\ldots+1}_{d-h \text{ times}} + \underbrace{(-\alpha)+\ldots+(-\alpha)}_{h \text{ times}} \ge 0.$$

This condition assures that the subsets of the type

(3.6)
$$\begin{bmatrix} [1, \dots, r]_{d}^{+} & [r+1, \dots, r+b(r)]_{0}^{-}, \text{ in total } \begin{pmatrix} b(r) \\ 0 \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} \\ [1, \dots, r]_{d-1}^{+} & [r+1, \dots, r+b(r)]_{1}^{-}, \text{ in total } \begin{pmatrix} b(r) \\ 1 \end{pmatrix} \begin{pmatrix} r \\ d-1 \end{pmatrix} \\ \vdots & \vdots & \vdots \\ [1, \dots, r]_{d-h}^{+} & [r+1, \dots, r+b(r)]_{h}^{-}, \text{ in total } \begin{pmatrix} b(r) \\ h \end{pmatrix} \begin{pmatrix} r \\ d-h \end{pmatrix}$$

are (d^+, n) -subsets of f

c) firstly we note that the requirement

$$\alpha < \frac{d}{b(r)} \Big(r - \frac{d-1}{d} (n - b(r)) \Big)$$

is equivalent to require

(3.7)
$$\frac{d-1}{d}(n-b(r)) + \alpha \frac{b(r)}{d} < r.$$

Lemma 2.2 assures the existence of a such α . Note that (3.7) is equivalent to

(3.8)
$$\underbrace{1+\ldots+1}_{d-1 \text{ times}} + (-\beta) < 0,$$

that assures that the (d^+, n) -strings of f are only of the type (3.6). Therefore we have constructed a weight function f with r non-negative elements that satisfies (3.2).

Corollary 3.2. Let r be a positive integer such that $r \ge d$ and $\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n$. Then

(3.9)
$$\gamma(n,d,r) \le \binom{r}{d} + \binom{r}{d-1}.$$

Proof. The result follows directly from Proposition 3.1 since b(r) = 1.

Proposition 3.3. Let r a positive integer such that $d \leq r \leq \frac{d-1}{d}n$. Let $f \in W_n(\mathbb{R})$, with $f^+ = r$, as in (2.1). If

$$(3.10) x_1 + y_{n-r} \ge 0,$$

then

(3.11)
$$\phi(f,d) \ge \binom{r-1}{d-2}(n-r) + \binom{r}{d} \ge \binom{r}{d} + \binom{r}{d-1}.$$

Proof. We can consider the *d*-strings of $\{1, \ldots, n\}$ of type

$$(3.12) 1i_1 \dots i_{d-2} | (n-r),$$

where $i_1 \ldots i_{d-2}$ are chosen in $\{2, \ldots, r\}$.

By virtue of (3.10), each string of the type (3.12) is a (d^+, n) -string of f.

On the other hand, since $y_1 \ge y_2 \ge \ldots \ge y_{n-r}$, each string of type

(3.13)
$$1i_1 \dots i_{d-2}|k|$$

where $i_1 \ldots i_{d-2}$ are chosen in $\{2, \ldots, r\}$ and k in $\{1, \ldots, n-r\}$, will be a (d^+, n) -string of f.

The distinct strings of the type (3.13) are exactly $\binom{r-1}{d-2}(n-r)$. There are moreover all the (d^+, n) -strings of f that are the d-strings on $\{1, \ldots, r\}$. This proves the first inequality in (3.11). Moreover, since $r \leq \frac{d-1}{d}n$, we also have $n - r \geq \frac{r}{d-1}$. Therefore

$$\binom{r-1}{d-2}(n-r) \ge \binom{r-1}{d-2}\frac{r}{d-1} = \binom{r}{d-1}$$

Thus the second inequality also holds.

As a direct consequence of Corollary 3.2 and Proposition 3.3 it follows that if r is a positive integer with $r \ge d$ such that $\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n$, then

(3.14)
$$\min\{\phi(f,d): f \in W_n(\mathbb{R}), f^+ = r, x_1 + y_{n-r} \ge 0\} = \binom{r}{d} + \binom{r}{d-1}.$$

Remark 3.4. We conjecture that

(3.15)
$$\gamma(n,d,r) = \binom{r}{d} + \binom{r}{d-1},$$

when $r \ge d$ and $\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n$.

In Section 5 we give a partial answer to this conjecture. Note that, in order to prove (3.15), by Corollary 3.2 it is sufficient to show

(3.16)
$$\gamma(n,d,r) \ge \binom{r}{d} + \binom{r}{d-1}.$$

Moreover, by virtue of (3.14), the inequality (3.16) is equivalent to the following:

(3.17)
$$\min\{\phi(f,d): f \in W_n(\mathbb{R}), f^+ = r, x_k + y_{n-r} < 0,$$

for every $k = 1, \dots, r$ $\geq \binom{r}{d} + \binom{r}{d-1}$.

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In Section 5 we shall prove this inequality in the special case n = 2d + 2.

We close this section providing a simple combinatorial interpretation of the inequalities

$$\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n.$$

For this purpose let us note that the last inequalities are equivalent to the following:

(3.18)
$$(n-r-1)(d-1) < r \le (n-r)(d-1),$$

Let now r be a positive integer that satisfies (3.18) and $f \in W_n(\mathbb{R})$, with $f^+ = r$, as in (2.1). Let us consider the following representation

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where every \square can be seen as a "box" initially empty and every row contains d-1 boxes. Every of such boxes can be occupied by at most one non-negative element of f. Thus (3.18) is equivalent to state that n - r - 1 rows in (3.19) must be completely occupied, whereas the last row must contain at least a non-empty box and, furthermore, the number of non-negative elements of f cannot exceed the number of empty boxes in (3.19). This combinatorial interpretation of (3.18) suggests to examine firstly the (d^+, n) -strings of fof the form $+ \ldots + -$, that is a subset with d - 1 non-negative elements and only one negative.

4. An application of Hall's Theorem

In this Section we use Hall's theorem on distinct representatives to determine some biunivocal functions between q-subsets of a set with 2q + 1 elements. The results of this Section are used in Section 5 to determine $\gamma(n, d, r)$ when r = 2d - 1 and n = 2d + 2.

We now introduce some definitions and notation useful in the sequel.

Let $\Omega = \{1, 2, \dots, 2q, 2q+1\}$, where q is a fixed positive integer.

Given a q-string $a_1 \ldots a_q \in \Omega^{(q)}$, for notation convenience we denote by $\mathfrak{C}_q(a_1 \ldots a_q)$ the family of all the q-strings on $\Omega \setminus \{a_1, \ldots, a_q\}$, that is

$$\mathfrak{C}_q(a_1...a_q) = (\Omega \setminus \{a_1,...,a_q\})^{(q)} = \{b_1...b_q : b_1,...,b_q \in \Omega, b_i \neq a_j, i, j = 1,...,q\}.$$

Note that the family $\mathfrak{C}_q(a_1 \ldots a_q)$ has exactly $\binom{q+1}{q} = q+1$ distinct q-strings.

A q-string in $\mathfrak{C}_q(a_1 \dots a_q)$ will be called a q-almost-complementary (or q-AC) of a_1, \dots, a_q .

From now on we call p the number of the distinct q-strings of $\Omega^{(q)}$, that is $p = \binom{2q+1}{q}$. We denote by A_1, \ldots, A_p all the q-strings of $\Omega^{(q)}$ such that

$$A_1 \prec \ldots \prec A_p,$$

where \prec is the usual lexicographic order.

Definition 4.1. A *q*-pairing of almost-complementaries on Ω (or *q*-PAC on Ω) is a biunivocal function $\varphi : \Omega^{(q)} \to \Omega^{(q)}$ such that $\varphi(A_i)$ is a *q*-AC of A_i for $i = 1, \ldots p$, that is

$$\varphi(A_i) \in \mathfrak{C}_q(A_i),$$

for i = 1, ..., p.

Let us set now $\mathfrak{F}_q = \{\mathfrak{C}_q(A_1), \ldots, \mathfrak{C}_q(A_p)\}.$

We recall that the family \mathfrak{F}_q has a system of distinct representatives (SDR), say (C_1, \ldots, C_p) , if $C_1 \in \mathfrak{C}_q(A_1), \ldots, C_p \in \mathfrak{C}_q(A_p)$ and $C_i \neq C_j$ for $i, j \in \{1, \ldots, p\}$ with $i \neq j$.

Proposition 4.2. The family \mathfrak{F}_q has a SDR if and only if there exists a q-PAC on Ω .

Proof. Sufficiency. Let (C_1, \ldots, C_p) be a SDR for \mathfrak{F}_q . This means that all the C_i are distinct q-strings and that $C_i \in \mathfrak{C}_q(A_i)$ for $i = 1, \ldots, p$. Thus the function

$$\varphi: \Omega^{(q)} \to \Omega^{(q)}$$

defined by

$$\varphi(A_i) = C_i \in \mathfrak{C}_q(A_i),$$

for $i = 1, \ldots, p$, is a q-PAC on Ω .

Necessity. If φ is a q-PAC on Ω , then φ is a bijection such that $\varphi(A_i) \in \mathfrak{C}_q(A_i)$, for $i = 1, \ldots, p$. Since φ is a bijection, $\varphi(A_1), \ldots, \varphi(A_p)$ is a SDR for \mathfrak{F}_q .

Proposition 4.3. For every positive integer q there exists a q-PAC on Ω .

Proof. By virtue of Proposition 4.2 it is sufficient to show that the family \mathfrak{F}_q has a SDR, i.e. that the well-known Hall's condition holds:

(4.1) for every
$$I \subset \{1, \dots, p\}$$
, we have $\left| \bigcup_{i \in I} \mathfrak{C}_q(A_i) \right| \ge |I|$.

Therefore, let $I = \{i_1, \ldots, i_k\}$ be an arbitrary subset of indices $i_1, \ldots, i_k \in \{1, \ldots, p\}$. Let $Y := \mathfrak{C}_q(A_{i_1}) \cup \cdots \cup \mathfrak{C}_q(A_{i_1k}) = \{C_1, \ldots, C_a\}$. With this notation (4.1) is equivalent to $a \ge k$, therefore we shall prove now this last inequality. Set $\mathfrak{A} := \{\mathfrak{C}_q(A_{i_1}), \ldots, \mathfrak{C}_q(A_{i_k})\}$. For all $C_l \in Y$ we denote by $d_{\mathfrak{A}}(C_l)$ the degree of C_l respect to the family \mathfrak{A} , that is the number of distinct sets $\mathfrak{C}_q(A_{i_j})$ that contain C_l . We have previously observed that $|\mathfrak{C}_q(A_{i_j})| = q + 1$ for all A_{i_j} , moreover, by a classical double counting principle we also have

$$\sum_{l=1}^{a} d_{\mathfrak{A}}(C_l) = \sum_{j=1}^{k} |\mathfrak{C}_q(A_{i_j})|,$$

hence

(4.2)
$$\sum_{l=1}^{a} d_{\mathfrak{A}}(C_l) = k(q+1).$$

On the other hand, every C_l is a q-string, let us say $C_l = c_1 \dots c_q$, which belongs to the sets $\mathfrak{C}_q(a_1 \dots a_q)$, where $a_1 \dots a_q$ is a q-AC of $c_1 \dots c_q$. Since the number of the distinct q-AC strings of $c_1 \dots c_q$ is $\binom{q+1}{q}$, it follows that every C_l belongs exactly to q+1 subsets $\mathfrak{C}_q(A_s)$, with $s = 1, \dots, p$; therefore $d_{\mathfrak{A}}(C_l) \leq q+1$ for $l = 1, \dots, a$. By (4.2) we obtain then

$$k(q+1) \le a(q+1),$$

i.e. $k \leq a$.

The Proposition 4.3 does not provide an explicit construction of a q-PAC on Ω . In order to construct a q-PAC on Ω we suggest the following Algorithm:

q-PAC Algorithm

Input: a positive integer q

Output: a q-PAC on Ω

Step 1: Write all the q-strings of $\Omega^{(q)}$ ordered in increasing way with respect to the lexicographic order

$$B_1 \prec B_2 \prec \ldots \prec B_p,$$

and put them in an array Dom[p] of q-strings, that has p positions, where $p = \binom{2q+1}{q}$.

Step 2: For all i = 1, ..., p write all the q-strings of $\mathfrak{C}_q(B_i)$ in decreasing lexicographic order

$$C_{i_1} \succ \ldots \succ C_{i_{q+1}}$$

- Step 3: Set up an array Im[p] of q-strings, that has p positions, and initialize every position with the q-string with all zero entries.
- Step 4: For all i = 1, ..., p examine in sequence the q-strings $C_{i_1}, ..., C_{i_{q+1}}$ and put the first of such q-strings that does not appear in Im[1], ..., Im[i] in the position Im[i].

Then the correspondence $Dom[i] \mapsto Im[i]$ (i = 1, ..., p) provides a q-PAC on Ω .

For small values of q we have implemented the previous algorithm in Java. For example, if q = 3 then $p = \binom{2q+1}{q} = \binom{7}{3} = 35$. In this case Dom[35] and Im[35] are two arrays with 35 position, both containing all the 3-strings on $\{1, \ldots, 7\}$. The execution of our program for q = 3 provides the following result (the strings on the left of ---> are those of Dom[35], the strings on the right of ---> are those of Im[35]):

5. The case
$$d \le r$$
, $\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n$, $n = 2d + 2$

In this Section we shall assume that n = 2d + 2 and that r is a positive integer such that $r \ge d$, $\frac{d-1}{d}(n-1) < r \le \frac{d-1}{d}n$. Under such hypotheses we can apply the Proposition 2.1 to the case k = 1, obtaining

$$r = \lfloor \frac{d-1}{d}n \rfloor = \lfloor \frac{d-1}{d}(2d+2) \rfloor = 2d - 1 = n - 3.$$

For such values of r and n we determine the value of $\gamma(n, d, r)$. This result implies that in this case the (MS)-conjecture does not hold. We also compare our results with the ones in [6].

Theorem 5.1. If n = 2d + 2 and r = 2d - 1 = n - 3 then

(5.1)
$$\gamma(n,d,r) = \binom{r}{d} + \binom{r}{d-1}$$

Proof. Due to Remark 3.4 we only need to show (3.17) when n - r = 3. Thus take $f \in W_n(\mathbb{R})$, with $f^+ = r$, as in (2.1) and suppose that $x_k + y_3 < 0$ for every $k = 1, \ldots, r$.

Take q = d - 1 (and therefore r = 2q + 1). By Proposition 4.3 there exists a q-PAC on Ω , where $\Omega = \{1, \ldots, r\} = \{1, \ldots, 2q + 1\}$. We use the notation introduced in Section 4. Take $\Omega^{(q)} = \{A_1, \ldots, A_p\}$ with the lexicographic order:

$$A_1 \prec \ldots \prec A_p$$

where $p = \binom{2q+1}{q}$. Let $C_s = \varphi(A_s)$ for $s = 1, \dots, p$.

Since A_s and C_s are q-strings with no common elements there exists in Ω a unique element, say i_s , that is not an element of the q-string A_s and nor an element of the q-string C_s . We point out that the elements i_1, \ldots, i_p are not distinct between them, since p > r.

If $A = t_1 \dots t_{d-1} \in \Omega^{(q)}$ and $k \in \{1, 2\}$, with the notation A|k we mean the *d*-string $t_1 \dots t_{d-1}|k$ and with $i_s|3$ the 2-string with the non-negative element i_s and with the negative element 3. We now consider the following configuration:

(5.2)
$$\begin{array}{c} A_1|1 \quad C_1|2 \quad i_1|3\\ A_2|1 \quad C_2|2 \quad i_2|3\\ & & \\ & & \\ A_p|1 \quad C_p|2 \quad i_p|3 \end{array}$$

Since φ is a bijection, the *q*-strings $C_1|2, \ldots, C_p|2$ are themselves distinct. Moreover, since φ is a *q*-PAC on Ω , each row in (5.2) contains all the elements (non-negative and negative) of *f*. Since the function *f* is a weight function and from the hypothesis we have $x_{i_s} + y_3 < 0$ (that is each $i_s|3$ corresponds to a negative sum), in every sth-row of the configuration (5.2) at least one *d*-string between $A_s|1$ and $C_s|2$ must be a (d^+, n) -string for *f*.

This shows that the number of the distinct (d^+, n) -strings for f is at least equal to the number of the rows in (5.2), that is $p = \binom{2q+1}{q} = \binom{r}{d-1}$.

The remaining (d^+, n) -strings for f that we need in order to obtain (3.17) are all the *d*-strings of Ω , which are $\binom{r}{d}$. This shows that $\phi(f, d) \ge \binom{r}{d} + \binom{r}{d-1}$.

Since f is arbitrary, (3.17) follows.

From this result we deduce the following consequence on the (MS)-conjecture.

Corollary 5.2. The (MS)-conjecture does not hold when n = 2d + 2 and $d \ge 3$.

Proof. We take r = 2d - 1. Then from (1.1) and Theorem 5.1 we have

$$\psi(n,d) \le \binom{r}{d} + \binom{r}{d-1},$$

and for such values of r and n we have

$$\binom{r}{d} + \binom{r}{d-1} = \frac{2(2d-1)!}{d!(d-1)!} < \frac{(2d+1)!}{(d-1)!(d+2)!} = \binom{n-1}{d-1},$$

if $d \geq 3$.

When n = 2d + 2 and r = 2d - 1, from Theorem 5.1 it follows that

$$\phi(f,d) \ge \binom{2d-1}{d} + \binom{2d-1}{d-1} > \binom{2d-1}{d-1} = \binom{(2d+2)-2-1}{d-1},$$

that is

(5.3)
$$\phi(f,d) > \binom{n-k-1}{d-1}$$

This inequality improves the estimate

(5.4)
$$\phi(f,d) \ge \binom{n-k-1}{d-1},$$

obtained in [6] under the additional hypotheses

(i) $\sum_{x \in I_n} f(x) = 0$, (ii) $\sum_{y \in Y} f(y) \neq 0$ for all $Y \subseteq I_n$ such that |Y| = d.

6. The case
$$r = \frac{d-1}{d}n$$

There is also another case when we can prove

$$\gamma(n, d, r) = \binom{r}{d} + \binom{r}{d-1}.$$

This is the case $r = \frac{d-1}{d}n$. In such case d|n and therefore $r \ge d$ if $d \ge 2$.

Theorem 6.1. Let $r = \frac{d-1}{d}n$. Then

$$\gamma(n, d, r) = \binom{r}{d} + \binom{r}{d-1}$$

Proof. The condition $r = \frac{d-1}{d}n$ is equivalent to r = (d-1)(n-r). Take $f \in W_n(\mathbb{R})$ with $f^+ = r$. Then we can build partitions \mathfrak{S} of the set $\{1, \ldots, r, r+1, \ldots, n\}$ of the type $\mathfrak{S} = \{C_1, \ldots, C_{n-r}\}$, where

(6.1)

$$C_{1} = i_{1,1} \dots i_{1,d-1} | k_{1}$$

$$C_{2} = i_{2,1} \dots i_{2,d-1} | k_{2}$$

$$\dots$$

$$C_{n-r} = i_{n-r,1} \dots i_{n-r,d-1} | k_{n-r}$$

with $i_{s,t} \in \{1, \ldots, r\}$, $k_l \in \{1, \ldots, n-r\}$, for $1 \le s \le n-r$, $1 \le t \le d-1$, $1 \le l \le n-r$. By means of a technique similar to the one used by Bier and Manickam in the proof of Lemma 1 of [6], we can claim that there exist exactly $\binom{r-1}{d-2}(n-r)$ disjoint partition of type (6.1). Since f is a weight function, at least a row in (6.1) is (d^+, n) -string for f.

Since the partitions are disjoint, if we extract from each of them at least a (d^+, n) -string for f, we get at least $\binom{r-1}{d-2}(n-r)$ distinct (d^+, n) -string for f.

Since $r = \frac{d-1}{d}n$, we have

$$\binom{r-1}{d-2}(n-r) = \binom{r-1}{d-2}\frac{r}{d-1} = \binom{r}{d-1}.$$

Therefore

$$\phi(f,d) \ge \binom{r}{d} + \binom{r}{d-1}.$$

This show that

$$\gamma(n, d, r) \ge \binom{r}{d} + \binom{r}{d-1}$$

The equality follows form Corollary (3.2).

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