# Colorful subgraphs in Kneser-like graphs 

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#### Abstract

Combining Ky Fan's theorem with ideas of Greene and Matoušek we prove a generalization of Dol'nikov's theorem. Using another variant of the Borsuk-Ulam theorem due to Bacon and Tucker, we also prove the presence of all possible completely multicolored $t$-vertex complete bipartite graphs in $t$-colored $t$-chromatic Kneser graphs and in several of their relatives. In particular, this implies a generalization of a recent result of G. Spencer and F. E. Su.


## 1 Introduction

The solution of Kneser's conjecture in 1978 by László Lovász [19] opened up a new area of combinatorics that is usually referred to as the topological method [5. Many results of this area, including the first one by Lovász, belong to one of its by now most developed branches that applies the celebrated Borsuk-Ulam theorem to graph coloring problems. An account of such results and other applications of the Borsuk-Ulam theorem in combinatorics is given in the excellent book of Matoušek [21].

Recently it turned out that a generalization of the Borsuk-Ulam theorem found by Ky Fan [13] in 1952 can give useful generalizations and variants of the Lovász-Kneser theorem. Examples of such results can be found in [23, [26, [28, 29].

In this note our aim is twofold. In Section 2 we show a further application of Ky Fan's theorem. More precisely, we give a generalization of Dol'nikov's theorem, which is itself a generalization of the Lovász-Kneser theorem. The proof will be a simple combination of Ky Fan's result with the simple proof of Dol'nikov's theorem given by Matoušek in [21] that was inspired by Greene's recent proof [15] of the Lovász-Kneser theorem.

In Section 3 we use another variant of the Borsuk-Ulam theorem due to Philip Bacon [3] and A. W. Tucker [32] to show a property of optimal colorings of certain $t$-chromatic graphs, including Kneser graphs, Schrijver graphs, Mycielski graphs, Borsuk graphs, odd chromatic rational complete graphs, and two other types of graphs appearing in [26] that will also be defined in Subsection 3.3. The claimed property (in a somewhat weakened form) is that all the complete bipartite graphs $K_{l, m}$ with $l+m=t$ will have totally multicolored copies in proper $t$-colorings of the above graphs.

When applied to rational complete graphs this implies a new proof of an earlier result about the circular chromatic number (see Subsection (3.3.6). It was originally obtained in [26] and independently in [23] for some special cases.

When applied to Kneser graphs the above property implies a generalization of a recent result due to Gwen Spencer and Francis Edward Su [28, 29] which we will also present in the last subsection.

## 2 A generalization of Dol'nikov's theorem

We recall some concepts and notations from [21]. For any family $\mathcal{F}$ of subsets of a fixed finite set we define the general Kneser $\operatorname{graph} \operatorname{KG}(\mathcal{F})$ by

$$
\begin{aligned}
V(\mathrm{KG}(\mathcal{F})) & =\mathcal{F}, \\
E(\mathrm{KG}(\mathcal{F})) & =\left\{\left\{F, F^{\prime}\right\}: F, F^{\prime} \in \mathcal{F}, F \cap F^{\prime}=\emptyset\right\}
\end{aligned}
$$

We note about our terminology that when referring to a Kneser graph (without the adjective "general") we mean the "usual" Kneser graph $\operatorname{KG}(n, k)$ that is identical to the general Kneser graph of a set system consisting of all $k$-subsets of an $n$-set.

A hypergraph $H$ is $m$-colorable, if its vertices can be colored by (at most) $m$ colors so that no hyperedge becomes monochromatic. The $m$-colorability defect of a set system $\emptyset \notin \mathcal{F}$ (identified with a hypergraph in the obvious way) is

$$
\operatorname{cd}_{m}(\mathcal{F}):=\min \{|Y|:(X \backslash Y,\{F \in \mathcal{F}: F \cap Y=\emptyset\}) \text { is } m \text {-colorable }\}
$$

Dol'nikov's theorem ([9]) For any finite set system $\emptyset \notin \mathcal{F}$, the inequality

$$
\operatorname{cd}_{2}(\mathcal{F}) \leq \chi(\mathrm{KG}(\mathcal{F}))
$$

holds.
This theorem generalizes the Lovász-Kneser theorem, as it is easy to check that if $\mathcal{F}$ consists of all the $k$-subsets of an $n$-set with $k<n / 2$, then $\operatorname{cd}_{2}(\mathcal{F})=n-2 k+2$ which is the true value of $\chi(\operatorname{KG}(\mathcal{F}))$ in this case. On the other hand, as also noted in [21], equality between $\operatorname{cd}_{2}(\mathcal{F})$ and $\chi(\mathrm{KG}(\mathcal{F}))$ does not hold in general.

Recently Greene [15] found a very simple new proof of the Lovász-Kneser theorem. (This proof follows the footsteps of Bárány's [4] earlier simple proof but with a skillful trick it can avoid the use of Gale's theorem.) In [21] Matoušek observed that one can generalize Greene's proof so that it also gives Dol'nikov's theorem. Here we combine this proof with Ky Fan's theorem (see below) to obtain the following generalization.

Theorem 1 Let $\mathcal{F}$ be a finite family of sets, $\emptyset \notin \mathcal{F}$ and $\operatorname{KG}(\mathcal{F})$ its general Kneser graph. Let $r=\operatorname{cd}_{2}(\mathcal{F})$. Then any proper coloring of $\operatorname{KG}(\mathcal{F})$ with colors $1, \ldots, m$ ( $m$ arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r / 2\rceil,\lfloor r / 2\rfloor}$ such that the $r$ different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.

This theorem generalizes Dol'nikov's theorem, because it implies that any proper coloring must use at least $\operatorname{cd}_{2}(\mathcal{F})$ different colors.
Remark 1. Theorem $\mathbb{1}$ is clearly in the spirit of the Zig-zag theorem of 26 the special case of which for Kneser graphs was already established by Ky Fan in [14. This theorem claims that if a specific topological parameter of a graph $G$, the value of which is a lower bound on its chromatic number, is at least $t$ then any proper coloring of the graph must contain a completely multicolored $K_{[t / 27,\lfloor t / 2\rfloor}$ where the colors also alternate on the two sides with respect to their natural order. As we will show in Remark 2 the proof below can be modified to show that the topological parameter mentioned is at least $\mathrm{cd}_{2}(\mathcal{F})$ for any $\operatorname{KG}(\mathcal{F})$. We will say more about this topological parameter in Section 3

To prove the theorem we first have to state Ky Fan's theorem [13]. Just like the Borsuk-Ulam theorem it has several equivalent forms (see [13, (3)). The one fitting best for our purposes is the following. (To find exactly this form, see [3].) For a set $A$ on the unit sphere $S^{h}$ we denote by $-A$ its antipodal set, i.e., $-A=\{-\boldsymbol{x}: \boldsymbol{x} \in A\}$.

Ky Fan's theorem ([13]) Let $A_{1}, \ldots, A_{m}$ be open subsets of the $h$-dimensional sphere $S^{h}$ satisfying that none of them contains antipodal points (i.e., $\forall i$ and $\forall \boldsymbol{x} \in S^{h} \boldsymbol{x} \in A_{i}$ implies $-\boldsymbol{x} \notin A_{i}$ ) and that at least one of $\boldsymbol{x}$ and $-\boldsymbol{x}$ is contained in $\cup_{i=1}^{m} A_{i}$ for all $\boldsymbol{x} \in S^{h}$.

Then there exists an $\boldsymbol{x} \in S^{h}$ and $h+1$ distinct indices $i_{1}<\ldots<i_{h+1}$ such that $\boldsymbol{x} \in A_{i_{1}} \cap-A_{i_{2}} \cap \ldots \cap(-1)^{h} A_{i_{h+1}}$.

Proof of Theorem [1. Let $h=\operatorname{cd}_{2}(\mathcal{F})-1$ and consider the sphere $S^{h}$. We assume without loss of generality that the base set $X:=\cup \mathcal{F}$ is finite and identify its elements with points of $S^{h}$ in general position, i.e., so that at most $h$ of them can be on a common hyperplane through the origin. Consider an arbitrary fixed proper coloring of $\operatorname{KG}(\mathcal{F})$ with colors $1, \ldots, m$. For every $\boldsymbol{x} \in S^{h}$ let $H(\boldsymbol{x})$ denote the open hemisphere centered at $\boldsymbol{x}$. Define the sets $A_{1}, \ldots, A_{m}$ as follows. Set $A_{i}$ will contain exactly those points $\boldsymbol{x} \in S^{h}$ that have the property that $H(\boldsymbol{x})$ contains the points of some $F \in \mathcal{F}$ which is colored by color $i$ in the coloring of $\operatorname{KG}(\mathcal{F})$ considered. The sets $A_{i}$ are all open. None of them contains an antipodal pair of points, otherwise there would be two disjoint open hemispheres both of which contain some element of $\mathcal{F}$ that is colored $i$. But these two elements of $\mathcal{F}$ would be disjoint contradicting the assumption that the coloring was proper. Now we show that there is no $\boldsymbol{x} \in S^{h}$ for which neither $\boldsymbol{x}$ nor $-\boldsymbol{x}$ is in $\cup_{i=1}^{m} A_{i}$. Color the points in $X \cap H(\boldsymbol{x})$ red, the points in $X \cap H(-\boldsymbol{x})$ blue and delete the points of $X$ not colored, i.e., those on the "equator" between $H(\boldsymbol{x})$ and $H(-\boldsymbol{x})$. Since at most $h<\operatorname{cd}_{2}(\mathcal{F})$ points are deleted there exists some $F \in \mathcal{F}$ which became completely red or completely blue. All points of $F$ are either in $H(\boldsymbol{x})$ or in $H(-\boldsymbol{x})$. This implies that $\boldsymbol{x}$ or $-\boldsymbol{x}$ should belong to $A_{i}$ where $i$ is the color of $F$ in our fixed coloring of $\operatorname{KG}(\mathcal{F})$.

Thus our sets $A_{1}, \ldots, A_{m}$ satisfy the conditions and therefore also the conclusion of Ky Fan's theorem. Let $F_{i_{j}} \in \mathcal{F}$ be the set responsible for $\boldsymbol{x} \in A_{i_{j}}$ for $j$ odd and for $-\boldsymbol{x} \in A_{i_{j}}$ for $j$ even in the conclusion of Ky Fan's theorem. Then all the $F_{i_{j}}$ 's with odd $j$ must be disjoint from all the $F_{i_{j}}$ 's with even $j$. Thus they form the complete bipartite graph claimed.

### 2.1 Consequences

We would like to point out that Theorem can really be stronger than Dol'nikov's theorem, especially if looked at as an upper bound for $\operatorname{cd}_{2}(\mathcal{F})$. It is not difficult to see (and is included as an exercise in Matoušek's book [21], cf. page 64, Exercise 3) that every graph $G$ is isomorphic to the general Kneser graph $\operatorname{KG}(\mathcal{F})$ of some set system $\mathcal{F}$.

Consider a graph $G$ with girth at least 5 and high chromatic number. (According to a famous result of Erdős [10] such graphs exist.) Let $\mathcal{F}_{G}$ denote a set system with the property that $\operatorname{KG}\left(\mathcal{F}_{G}\right)=G$. Theorem $⿴$ implies that $\operatorname{cd}_{2}\left(\mathcal{F}_{G}\right) \leq 3$ since $\operatorname{KG}\left(\mathcal{F}_{G}\right)$ does not contain any $K_{2,2}$ subgraph. At the same time, the upper bound Dol'nikov's theorem implies on $\operatorname{cd}_{2}\left(\mathcal{F}_{G}\right)$ is the chromatic number of $G$ which may be one million.

Formulating the above argument about $K_{2,2}$-free graphs in terms of the set system, we have the following.

Corollary 2 Let $\mathcal{F}$ be a set system not containing the empty set and satisfying that for any two distinct sets $A, B \in \mathcal{F}$ there is at most one set in $\mathcal{F}$ that is disjoint from both $A$ and $B$ ．Then $\operatorname{cd}_{2}(\mathcal{F}) \leq 3$ ．

Proof．The condition on $\mathcal{F}$ implies that $K_{2,2} \nsubseteq \operatorname{KG}(\mathcal{F})$ ，thus Theorem 1 implies the statement．

It would be unfair to deny that Corollary 2 can be proven easily in a rather elementary way，too．Just consider sets $A, B \in \mathcal{F}$ for which $A \cup B$ is minimal．By this choice any $C \subseteq A \cup B, C \in \mathcal{F}$ ，should satisfy $(A \backslash B) \cup(B \backslash A) \subseteq C$ ．Therefore deleting a point from $A \backslash B$ and one from $B \backslash A$ if neither of these is empty and coloring the rest of $A \cup B$ red，no completely red element of $\mathcal{F}$ may occur．There is at most one $D \in \mathcal{F}$ with $D \cap(A \cup B)=\emptyset$ ．Delete a point of $D$（if it exists）and color all the remaining points blue． This gives a red－blue coloring that proves the statement in case $A \backslash B \neq \emptyset$ and $B \backslash A \neq \emptyset$ both hold．If，say，$A \backslash B=\emptyset$ ，then we have to delete a point of $A$ instead of one of $A \backslash B$ that is impossible now．Otherwise we can do everything similarly thereby proving the statement also in this other case．

Even though such an elementary proof exists we believe that the condition of Corol－ lary 2 comes to mind much more naturally because of the topological background．Also， Theorem $\square$ implies other statements of the like，that are perhaps more complicated to prove in an elementary way．For example，even if $\operatorname{KG}(\mathcal{F})$ contains one $K_{q, q}$ subgraph for some huge $q$ ，but no other $K_{2,2}$ subgraph（i．e．，no $K_{2,2}$ subgraph apart from those con－ tained in the large $K_{q, q}$ ）then $\mathrm{cd}_{2}(\mathcal{F}) \leq 3$ still holds．This is because the colors appearing on the two sides of the $K_{q, q}$ can be labeled so that the colors on one side all precede the colors of the other side in their ordering and therefore a $K_{2,2}$ required by Theorem $⿴ 囗 十$ $\operatorname{cd}_{2}(\mathcal{F}) \geq 4$ still cannot occur．

## 3 Applying a theorem of Bacon and Tucker

## 3．1 Preliminaries

First we give a very brief introduction of some topological concepts we need．All this can be found in detail，e．g．，in［21］．A $\mathbb{Z}_{2}$－space is a pair $(T, \nu)$ ，where $T$ is a topological space and $\nu: T \rightarrow T$ is an involution，that is，a continuous map satisfying $\nu(\nu(x))=x$ for all $x \in T$ ．A $\mathbb{Z}_{2}$－space $(T, \nu)$ is free if $\nu(x) \neq x$ for every $x \in T$ ．If $\nu$ is clear from the context we write $T$ in place of $(T, \nu)$ ．Accordingly，we write $S^{h}$ for the most important $\mathbb{Z}_{2}$－space we deal with，the $h$－dimensional sphere with the usual antipodal map as involution．

A continuous map $f:(T, \nu) \rightarrow(W, \mu)$ is a $\mathbb{Z}_{2}$－map if it respects the respective invo－ lutions，that is，$f(\nu(x))=\mu(f(x))$ for every $x \in T$ ．We write $(T, \nu) \rightarrow(W, \mu)$ if there exists a $\mathbb{Z}_{2}$－map from $(T, \nu)$ to $(W, \mu)$ ．Two important parameters of a $\mathbb{Z}_{2}$－space are its $\mathbb{Z}_{2}$－index and $\mathbb{Z}_{2}$－coindex that are defined as

$$
\operatorname{ind}(T, \nu):=\min \left\{h \geq 0:(T, \nu) \rightarrow S^{h}\right\}
$$

and

$$
\operatorname{coind}(T, \nu):=\max \left\{h \geq 0: S^{h} \rightarrow(T, \nu)\right\}
$$

respectively. The inequality

$$
\operatorname{coind}(T, \nu) \leq \operatorname{ind}(T, \nu)
$$

always holds and is one of the standard forms of the celebrated Borsuk-Ulam theorem.
In applications of the topological method one often associates so-called box complexes to graphs. These give rise to topological spaces the index and coindex of which can serve to obtain lower bounds for the chromatic number of the graph. Following ideas in earlier works by Alon, Frankl, Lovász [2] and others, the paper [22] introduces several box complexes two of which we also define below.

Definition 1 The box complex $B(G)$ is a simplicial complex on the vertices $V(G) \times\{1,2\}$. For subsets $S, T \subseteq V(G)$ the set $S \uplus T:=S \times\{1\} \cup T \times\{2\}$ forms a simplex if and only if $S \cap T=\emptyset$, the vertices in $S$ have at least one common neighbor and the same is true for $T$, and the complete bipartite graph with sides $S$ and $T$ is a subgraph of $G$. The $\mathbb{Z}_{2}$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B(G)$ making the body $\|B(G)\|$ of the complex a free $\mathbb{Z}_{2}$-space.

It is explained in [22] and [21] that $B(G)$ is a functor, meaning for example, that whenever there exists a homomorphism from a graph $F$ to another graph $G$ then $B(F) \rightarrow$ $B(G)$ is also true. It is not hard to see that $\left\|B\left(K_{n}\right)\right\| \cong S^{n-2}$ with a $\mathbb{Z}_{2}$-homeomorphism (i.e., a homeomorphism which is a $\mathbb{Z}_{2}$-map). For the $\mathbb{Z}_{2}$-index and $\mathbb{Z}_{2}$-coindex of $\|B(G)\|$ we simply write $\operatorname{ind}(B(G))$ and $\operatorname{coind}(B(G))$, respectively, and we will do similarly for the other box complex $B_{0}(G)$ defined below. Since a graph is $t$-colorable if and only if it admits a homomorphism to $K_{t}$, the foregoing implies

$$
\begin{equation*}
\chi(G) \geq \operatorname{ind}(B(G))+2 \geq \operatorname{coind}(B(G))+2 \tag{1}
\end{equation*}
$$

Another box complex $B_{0}(G)$ defined in [22] differs from $B(G)$ only by containing all those simplices $S \uplus T$, too, where one of $S$ or $T$ is empty independently of the existence of common neighbors required in the definition of $B(G)$.

Definition 2 The box complex $B_{0}(G)$ is a simplicial complex on the vertices $V(G) \times$ $\{1,2\}$. For subsets $S, T \subseteq V(G)$ the set $S \uplus T:=S \times\{1\} \cup T \times\{2\}$ forms a simplex if and only if $S \cap T=\emptyset$, and the complete bipartite graph with sides $S$ and $T$ is a subgraph of $G$. The $\mathbb{Z}_{2}$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B_{0}(G)$ making the body $\left\|B_{0}(G)\right\|$ of the complex a free $\mathbb{Z}_{2}$-space.

Csorba 6] proved a strong topological relationship between $B(G)$ and $B_{0}(G)$, namely, that the body of $B_{0}(G)$ is $\mathbb{Z}_{2}$-homotopy equivalent to the suspension of the body of $B(G)$. This extends (11) to the following longer chain of inequalities, cf. [22] and also [26].

$$
\begin{equation*}
\chi(G) \geq \operatorname{ind}(B(G))+2 \geq \operatorname{ind}\left(B_{0}(G)\right)+1 \geq \operatorname{coind}\left(B_{0}(G)\right)+1 \geq \operatorname{coind}(B(G))+2 \tag{2}
\end{equation*}
$$

Note that $B_{0}(G)$ is also a functor, and it is easy to see even without Csorba's result that $\left\|B_{0}\left(K_{n}\right)\right\| \cong S^{n-1}$ with a $\mathbb{Z}_{2}$-homeomorphism.

There are several interesting graph families the members of which satisfy the inequalities in (2) with equality. These include, for example, Kneser graphs, and a longer list is given in Corollary 5 below. (We note that some of the graphs in Corollary 5 give equality only in the first three of the above inequalities, cf. Subsection 3.3.4.)

Remark 2. The topological parameter mentioned in Remark 1 in Section 2 is coind $\left(B_{0}(G)\right)+1$. Thus our claim in Remark 1 is that the proof of Theorem 1 implies $\operatorname{coind}\left(B_{0}(\mathrm{KG}(\mathcal{F}))\right) \geq \mathrm{cd}_{2}(\mathcal{F})-1$ for any $\mathcal{F}$ not containing the empty set. Here we sketch the proof of this claim which is similar to the proof of Proposition 8 in [26]. Assume again without loss of generality that $X=\cup \mathcal{F}$ is finite and identify its elements with points of $S^{h}$ in general position as in the proof of Theorem $\square$ with $h=\operatorname{cd}_{2}(\mathcal{F})-1$. For each vertex $v$ of $\mathrm{KG}(\mathcal{F})$ and $\boldsymbol{x} \in S^{h}$ let $D_{v}(\boldsymbol{x})$ be the smallest distance of a point in $v$ (this point is an element of $X$ ) from the set $S^{h} \backslash H(\boldsymbol{x})$. Notice that $D_{v}(\boldsymbol{x})>0$ iff $H(\boldsymbol{x})$ contains all points of $v$. Set $D(\boldsymbol{x}):=\sum_{v \in \mathcal{F}}\left(D_{v}(\boldsymbol{x})+D_{v}(-\boldsymbol{x})\right)$. The argument in the proof of Theorem ${ }^{\text {implies }}$ $D(\boldsymbol{x})>0$. Therefore the map $f(\boldsymbol{x})=(1 / D(\boldsymbol{x}))\left(\sum_{v} D_{v}(\boldsymbol{x})\|(v, 1)\|+\sum_{v} D_{v}(-\boldsymbol{x})\|(v, 2)\|\right)$ is a $\mathbb{Z}_{2}$-map from $S^{h}$ to $\left\|B_{0}(\operatorname{KG}(\mathcal{F}))\right\|$, thus $\operatorname{coind}\left(B_{0}(\operatorname{KG}(\mathcal{F}))\right) \geq h$ as claimed.

### 3.2 A colorful $K_{l . m}$-theorem

In their recent paper [8] Csorba, Lange, Schurr, and Wassmer proved that if $\operatorname{ind}(B(G)) \geq$ $l+m-2$ then $G$ must contain the complete bipartite graph $K_{l, m}$ as a subgraph. They called this "the $K_{l, m}$-theorem". In case of those graphs that satisfy $\operatorname{coind}\left(B_{0}(G)\right)+1=\chi(G)$ (for a list of such graphs see Corollary (5), the following statement generalizes their result. We use the notation $[t]:=\{1, \ldots, t\}$.

Theorem 3 Let $G$ be a graph for which $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1=t$. Let $c: V(G) \rightarrow[t]$ be a proper coloring of $G$ and let $A, B \subseteq[t]$ form a bipartition of the color set, i.e., $A \cup B=[t]$ and $A \cap B=\emptyset$.

Then there exists a complete bipartite subgraph $K_{l, m}$ of $G$ with sides $L, M$ such that $|L|=l=|A|,|M|=m=|B|$, and $\{c(v): v \in L\}=A$, and $\{c(v): v \in M\}=B$. In particular, all vertices of this $K_{l, m}$ receive different colors at $c$.

For the proof we will use a modified version of the following theorem. It is given in [3] for more general $\mathbb{Z}_{2}$-spaces in place of $S^{h}$ but for our purposes this restricted version will be sufficient.
Bacon-Tucker theorem ([3, 32]) If $C_{1}, \ldots, C_{h+2}$ are closed subsets of $S^{h}$,

$$
\cup_{i=1}^{h+2} C_{i}=S^{h}, \quad \forall i: \quad C_{i} \cap\left(-C_{i}\right)=\emptyset,
$$

and $j \in\{1, \ldots, h+1\}$, then there is an $\boldsymbol{x} \in S^{h}$ such that

$$
\boldsymbol{x} \in \cap_{i=1}^{j} C_{i}, \text { and }-\boldsymbol{x} \in \cap_{i=j+1}^{h+2} C_{i} .
$$

Bacon [3] shows the above theorem to be equivalent to 14 other statements that include standard forms of the Borsuk-Ulam theorem, and also Ky Fan's theorem. Denoting its statement by $A_{h}(T)$ for a more general $\mathbb{Z}_{2}$-space $T$, he writes the following about the origins of this result: A weakened form of $A_{h}\left(S^{h}\right)$ was stated in 1935 (1], Satz X, p. 487); $A_{2}\left(S^{2}\right)$ is stated and the higher dimensional cases hinted at by Tucker in 1945 [32]. (The weakened form in [1] states only that any $h+1$ of the $C_{i}$ 's intersect which would be too weak for our purposes.)

It is a routine matter to see that the Bacon-Tucker theorem also holds for open sets $C_{i}$. (For a collection of open sets $C_{i}$ one can define closed sets $C_{i}^{\prime}$ so that $C_{i}^{\prime} \subseteq C_{i}$ for all $i$ and $\cup_{i=1}^{h+2} C_{i}^{\prime}=\cup_{i=1}^{h+2} C_{i}=S^{h}$. Cf. [1], Satz VII, p. 73, quoted also in [3]. We give a short argument below for the sake of completeness. Denote by $f(\boldsymbol{x})$ the largest $\varepsilon$ for which the open $\varepsilon$-neighborhood of $\boldsymbol{x}$ is contained in some $C_{i}$. Since $f$ is continuous, by compactness, it has a positive minimum $\delta$. Now $C_{i}^{\prime}$ can be the set of all those points whose open $\delta$ neighborhood is in $C_{i}$. The condition about antipodal points is automatically satisfied by $C_{i}^{\prime} \subseteq C_{i}$. Thus the closed set version of the theorem can be applied to the sets $C_{i}^{\prime}$ and the conclusion implies the same conclusion for the sets $C_{i}$, again by $C_{i}^{\prime} \subseteq C_{i}$.)

The modified version we need is the following.
Bacon-Tucker theorem, second form. If $C_{1}, \ldots, C_{h+1}$ are open subsets of $S^{h}$,

$$
\cup_{i=1}^{h+1}\left(C_{i} \cup\left(-C_{i}\right)\right)=S^{h}, \quad \forall i: \quad C_{i} \cap\left(-C_{i}\right)=\emptyset
$$

and $j \in\{0, \ldots, h+1\}$, then there is an $\boldsymbol{x} \in S^{h}$ such that

$$
\boldsymbol{x} \in C_{i} \text { for } i \leq j \text { and }-\boldsymbol{x} \in C_{i} \text { for } i>j .
$$

Proof. Let $D_{h+2}=S^{h} \backslash\left(\cup_{i=1}^{h+1} C_{i}\right)$. Then $D_{h+2} \cap\left(-D_{h+2}\right)=\emptyset$ by the first condition on the sets $C_{i}$. Since $D_{h+2}$ is closed there is some $\varepsilon>0$ bounding the distance of any pair of points $\boldsymbol{x} \in D_{h+2}$ and $\boldsymbol{y} \in-D_{h+2}$ from below. Let $C_{h+2}$ be the open $\frac{\varepsilon}{2}$-neighborhood of $D_{h+2}$. Then the open sets $C_{1}, \ldots, C_{h+2}$ satisfy the conditions (of the open set version) of the Bacon-Tucker theorem. Therefore its conclusion holds. Neglecting the set $C_{h+2}$ in this conclusion the proof is completed for $j>0$. To see the statement for $j=0$ one can take the negative of the value $\boldsymbol{x}$ guaranteed for $j=h+1$.

Proof of Theorem 3. Let $G$ be a graph with $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1=t$ and fix an arbitrary proper $t$-coloring $c: V(G) \rightarrow[t]$. Let $g: S^{t-1} \rightarrow B_{0}(G)$ be a $\mathbb{Z}_{2}$-map that exists by $\operatorname{coind}\left(B_{0}(G)\right)=t-1$.

We define for each color $i \in[t]$ an open set $C_{i}$ on $S^{t-1}$. For $\boldsymbol{x} \in S^{t-1}$ we let $\boldsymbol{x}$ be an element of $C_{i}$ iff the minimal simplex $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}} \in B_{0}(G)$ whose body contains $g(\boldsymbol{x})$ has a vertex $v \in S_{\boldsymbol{x}}$ for which $c(v)=i$. These $C_{i}$ 's are open. If an $\boldsymbol{x} \in S^{t-1}$ is not covered by
any $C_{i}$ then $S_{\boldsymbol{x}}$ must be empty, which also implies $T_{\boldsymbol{x}} \neq \emptyset$. Since $S_{-\boldsymbol{x}}=T_{\boldsymbol{x}}$ this further implies $-\boldsymbol{x} \in \cup_{i=1}^{t} C_{i}$, thus $\cup_{i=1}^{t}\left(C_{i} \cup\left(-C_{i}\right)\right)=S^{t-1}$ follows.

If a set $C_{i}$ contained an antipodal pair $\boldsymbol{x}$ and $-\boldsymbol{x}$ then $S_{-\boldsymbol{x}}=T_{\boldsymbol{x}}$ would contain an $i$-colored vertex $u$, while $S_{\boldsymbol{x}}$ would contain an $i$-colored vertex $v$. Since $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}}$ is a simplex of $B_{0}(G) u$ and $v$ must be adjacent, contradicting that $c$ is a proper coloring. Thus the $C_{i}$ 's satisfy the conditions of (the second form of) the Bacon-Tucker theorem.

Let $j=|A|=l$ and relabel the colors so that colors $1, \ldots, j$ be in $A$, and the others be in $B$. The indices of the $C_{i}$ 's are relabeled accordingly. We apply the second form of the Bacon-Tucker theorem with $h=t-1$. It guarantees the existence of an $\boldsymbol{x} \in S^{t-1}$ with the property $\boldsymbol{x} \in C_{i}$ for $i \in A$ and $-\boldsymbol{x} \in C_{i}$ for $i \in B$. Then $S_{\boldsymbol{x}}$ contains vertices $u_{1}, \ldots, u_{l}$ with $c\left(u_{i}\right)=i$ for all $i \in A$ and $S_{-\boldsymbol{x}}=T_{\boldsymbol{x}}$ contains vertices $v_{1}, \ldots, v_{m}$ with $c\left(v_{i}\right)=l+i$ for all $(l+i) \in B$. Since all vertices of $S_{\boldsymbol{x}}$ are connected to all vertices in $T_{\boldsymbol{x}}$ by the definition of $B_{0}(G)$, they give the required completely multicolored $K_{l, m}$ subgraph.

### 3.3 Graphs that are subject of the colorful $K_{l, m}$ theorem

Let us put the statement of Theorem 3 into the perspective of our earlier work in [26]. There we investigated the local chromatic number of graphs that is defined in [11] as

$$
\psi(G):=\min _{c} \max _{v \in V(G)}|\{c(u): u v \in E(G)\}|+1
$$

where the minimum is taken over all proper colorings $c$ of $G$. That is $\psi(G)$ is the minimum number of colors that must appear in the closed neighborhood of some vertex in any proper coloring.

With similar techniques to those applied in this paper we have shown using Ky Fan's theorem that $\operatorname{coind}\left(B_{0}(G)\right) \geq t-1$ implies that $G$ must contain a completely multicolored $K_{[t / 2\rceil,\lfloor t / 2\rfloor}$ subgraph in any proper coloring with the colors alternating with respect to their natural order on the two sides of this complete bipartite graph. This is the Zig-zag theorem in [26] we already referred to in Remark 1 in Section 2] The Zig-zag theorem implies that any graph satisfying its condition must have $\psi(G) \geq\lceil t / 2\rceil+1$. In [26] we have shown for several graphs $G$ for which $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1=t$ that it can be colored with $t+1$ colors so that no vertex has more than $\lfloor t / 2\rfloor+2$ colors in its closed neighborhood. When $t$ is odd, this established the exact value $\psi(G)=\lceil t / 2\rceil+1$ for these graphs. For odd $t$ this also means that the only type of $K_{l, m}$ subgraph with $l+m=t$ that must appear completely multicolored when using $t+1$ colors is the $K_{\lceil t / 2\rceil,\lfloor t / 2\rfloor}$ subgraph guaranteed by the Zig-zag theorem (apart from the empty graph $K_{t, 0}$ ). If we use only $t$ colors, however, then the situation is quite different. It is true for any graph $F$ that if it is properly colored with $\chi(F)$ colors then each color class must contain a vertex that sees all other colors in its (open) neighborhood. If it were not so, we could completely eliminate a color class by recoloring each of its vertices to a color which is not present in
its neighborhood. In the context of local chromaticity this means that if $\psi(F)<\chi(F)$ then it can only be attained by a coloring that uses strictly more than $\chi(F)$ colors. Now Theorem [3 says that if $G$ satisfies $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1=t$ then all $t$-colorings give rise not only to completely multicolored $K_{[t / 2\rceil,\lfloor t / 2\rfloor}$ 's that are guaranteed by the Zig-zag theorem, and $K_{1, t-1}$ 's that must appear in any optimal coloring (with all possible choices of the color being on the single vertex side), but to all possible completely multicolored complete bipartite graphs on $t$ vertices.

To conclude this subsection we list explicitly some classes of graphs $G$ that satisfy the $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$ condition of Theorem 3] We recall that in [26] we used the term topologically $t$-chromatic for graphs $G$ with $\operatorname{coind}\left(B_{0}(G)\right) \geq t-1$. With this notation we are listing graphs $G$ that are topologically $\chi(G)$-chromatic, i.e., for which this specific lower bound on their chromatic number is tight.

The subtitles below refer to graph classes that have at least some members with the required property. For the precise formulation see Corollary 5 below.

### 3.3.1 Kneser graphs and Schrijver graphs

Kneser graphs $\mathrm{KG}(n, k)$ with $t=n-2 k+2$ form the probably best known class of graphs for which the conditions in Theorem 3 hold, cf., e.g., 21]. Another such class is formed by their vertex color-critical induced subgraphs called Schrijver graphs $\operatorname{SG}(n, k)$, also with $t=n-2 k+2$. The graph $\operatorname{SG}(n, k)$ is just the general Kneser $\operatorname{graph} \operatorname{KG}(\mathcal{F})$ for the set system $\mathcal{F}$ consisting of exactly those $k$-subsets of $[n]$ that contain no cyclically consecutive elements, i.e., neither a pair $\{i, i+1\}$, nor $\{1, n\}$. Schrijver graphs were introduced by Schrijver in [25] and are discussed in detail in several subsequent papers, here we refer to [21] and [26] for further information.

The argument presented in Remark 2 proves that the general Kneser graph $G=$ $\operatorname{KG}(\mathcal{F})$ also satisfies $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$ as long as $\chi(\mathrm{KG}(\mathcal{F}))=\operatorname{cd}_{2}(\mathcal{F})$. Note that while the graphs $\operatorname{KG}(n, k)$ are included in the latter family, the graphs $\operatorname{SG}(n, k)$ are not if $k>1$.

### 3.3.2 Mycielski and generalized Mycielski graphs

Mycielski graphs were introduced by Mycielski [24]. These triangle free graphs are recursively defined starting from $K_{2}$ and their chromatic number increases by 1 at every iteration. It is also true that applying the Mycielski construction to any graph the clique number will not change while the chromatic number increases by 1 .

The generalized Mycielski construction first appeared probably in [30], cf. also 16]. It also reappears in [18, 31]. This construction creates from a graph $G$ its generalized Mycielskian $M_{r}(G)$, where $r$ denotes the number of "levels" in the construction. (The ordinary Mycielski construction is the $r=2$ special case.) For a graph $G$ with vertices $v_{1}, \ldots, v_{n}$ the vertex set of $M_{r}(G)$ is $\left\{v_{1}^{(p)}, \ldots, v_{n}^{(p)}: 0 \leq p \leq r-1\right\} \cup\{z\}$ and the pair
$v_{i}^{(p)} v_{j}^{(q)}$ forms an edge iff $v_{i} v_{j} \in E(G)$ and either $p=q=0$ or $p=q \pm 1$. The additional vertex $z$ is connected to $v_{1}^{(r)}, \ldots, v_{n}^{(r)}$.

When applying this construction to an arbitrary graph, the clique number does not increase (except in the trivial case when $r=1$ ) while the chromatic number may or may not increase. If it does it increases by 1. Generalizing Stiebitz's result 30] (see also in [16, 21]) Csorba [7] proved that $B\left(M_{r}(G)\right)$ is $\mathbb{Z}_{2}$-homotopy equivalent to the suspension of $B(G)$ for every graph $G$. (Csorba's result is in terms of the so-called homomorphism complex $\operatorname{Hom}\left(K_{2}, G\right)$ but this is known to be $\mathbb{Z}_{2}$-homotopy equivalent to $B(G)$ by results in [6, 22, 34].) Together with Csorba's already mentioned other result in [6] stating the $\mathbb{Z}_{2}$-homotopy equivalence of $B_{0}(G)$ and the suspension of $B(G)$, the foregoing implies that if a graph $G$ satisfies $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$, then the analogous equality will also hold for the graph $M_{r}(G)$. In this case the chromatic number does increase by 1 . Thus iterating the construction $d$ times (perhaps with varying parameters $r$ ) we arrive to a graph which has chromatic number $\chi(G)+d$ and still satisfies that its chromatic number is equal to its third (in fact, for $d>0$ also the fourth) lower bound given in (2). For further explanation of these relations we refer to [26].

### 3.3.3 Borsuk graphs

The lower bounds of (2) are also tight for Borsuk graphs $B(n, \alpha)$ with large enough $\alpha<2$ that are defined on $S^{n-1}$ as vertices and edges are formed by those pairs that are at least distance $\alpha$ apart, cf. [12, 20]. The chromatic number of these graphs is $t=n+1$. The paper [20] shows that some finite subgraphs of $B(t-1, \alpha)$ also have the required properties.

All the above examples are more or less standard. There are, however, two more, less known examples the corresponding properties of which are implicit in [26]. Below we define these two types of graphs. Finally, one more family will be given the members of which are well-known graphs, but their relevance in the present context is probably not widely known.

### 3.3.4 Homomorphism universal graphs for local colorings

Definition 3 (11) For positive integers $r \leq m$ the graph $U(m, r)$ is defined as follows.

$$
\begin{aligned}
V(U(m, r)) & =\{(i, A): i \in[m], A \subseteq[m],|A|=r-1, i \notin A\} \\
E(U(m, r)) & =\{\{(i, A),(j, B)\}: i \in B, j \in A\}
\end{aligned}
$$

It is shown in [11] that these graphs characterize local colorability in the following sense: a graph $G$ has an $m$-coloring where no closed neighborhood of any vertex contains more than $r$ colors if and only if $G$ admits a homomorphism to $U(m, r)$. As mentioned above, in [26] we showed for several odd-chromatic graphs satisfying the conditions of

Theorem 3 that their local chromatic number is $\lceil t / 2\rceil+1$ and it is attained with a coloring using $t+1$ colors. It follows that for odd $t$ the $t$-chromatic graph $U\left(t+1, \frac{t+3}{2}\right)$ also satisfies the conditions of Theorem 3. (Indeed, by the functoriality of $B_{0}(G), \operatorname{coind}\left(B_{0}(F)\right) \geq t-1$ and the existence of a homomorphism $F \rightarrow G$ implies $S^{t-1} \rightarrow\left\|B_{0}(F)\right\| \rightarrow\left\|B_{0}(G)\right\|$ and thus coind $\left(B_{0}(G)\right) \geq t-1$. For further details, cf. Remark 3 in [26].)

The graphs $U\left(t+1, \frac{t+2}{2}\right)$ with $t$ even also belong here. It is proven in [27] that $\operatorname{coind}\left(B_{0}\left(U\left(t+1, \frac{t+2}{2}\right)\right)=t-1\right.$, but the way of proof is rather different than the previous argument above. The $t$-colorability of these graphs is also easy to check. We also mention the result from [27] according to which the fourth lower bound on the chromatic number in (22) is not tight for these graphs (while it is for the graphs of the previous paragraphs). This shows that Theorem 3 in its present form is somewhat stronger than it would be with the stronger requirement $\chi(G)=\operatorname{coind}(B(G))+2$ in place of $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$. We needed the second form of the Bacon-Tucker theorem for obtaining this stronger form.

We mention that $\chi\left(U\left(t+1,\left\lfloor\frac{t+3}{2}\right\rfloor\right)\right)=t$ is a special case of Theorem 2.6 in [11.

### 3.3.5 Homomorphism universal graphs for wide colorings

Definition 4 Let $H_{s}$ be the path on the vertices $0,1,2, \ldots, s$ (i and $i-1$ connected for $1 \leq i \leq s)$ with a loop at $s$. We define $W(s, t)$ to be the graph with

$$
\begin{aligned}
V(W(s, t)) & =\left\{\left(x_{1} \ldots x_{t}\right): \forall i x_{i} \in\{0,1, \ldots, s\}, \exists!i x_{i}=0, \exists j x_{j}=1\right\} \\
E(W(s, t)) & =\left\{\left\{x_{1} \ldots x_{t}, y_{1} \ldots y_{t}\right\}: \forall i\left\{x_{i}, y_{i}\right\} \in E\left(H_{s}\right)\right\} .
\end{aligned}
$$

The graphs $W(2, t)$ are defined in [16] in somewhat different terms. It is shown there that a graph can be colored properly with $t$ colors so that the neighborhood of each color class is an independent set if and only if it admits a homomorphism into $W(2, t)$. The described property is equivalent to having a $t$ coloring where no walk of length 3 can connect vertices of the same color. Similarly, a graph $F$ admits a homomorphism into $W(s, t)$ if and only if it can be colored with $t$ colors so that no walk of length $2 s-1$ can connect vertices of the same color. Such colorings are called $s$-wide in [26]. Graphs having the mentioned property of $W(s, t)$ are also defined in [16, though they are not minimal. The graphs $W(s, t)$ are defined and shown to be minimal with respect to the above property in [26]. The $t$-colorability of $W(s, t)$ is obvious: $c\left(x_{1} \ldots x_{t}\right)=i$ if $x_{i}=0$ gives a proper coloring. It is also shown in [26] that several of the above mentioned $t$ chromatic graphs (e.g., $B(t-1, \alpha)$ for $\alpha$ close enough to 2 and $\operatorname{SG}(n, k)$ with $n-2 k+2=t$ and $n, k$ large enough with respect to $s$ and $t$ ) admit a homomorphism to $W(s, t)$. This implies $\operatorname{coind}\left(B_{0}(W(s, t))\right) \geq t-1$ (with equality, since $\chi(W(s, t))=t$ ). Thus the graphs $W(s, t)$ form another family of graphs for which Theorem 3 applies.

### 3.3.6 Rational complete graphs

Our last example of a graph family satisfying the conditions of Theorem 3 consists of certain rational (or circular) complete graphs $K_{p / q}$, as they are called, for example, in
[17. The graph $K_{p / q}$ is defined for positive integers $p \geq 2 q$ on the vertex set $\{0, \ldots, p-1\}$ and $\{i, j\}$ is an edge if and only if $q \leq|i-j| \leq p-q$. The widely investigated chromatic parameter $\chi_{c}(G)$, called the circular chromatic number of graph $G$ (cf. [33], or Section 6.1 in [17]) can be defined as the infimum of those values $p / q$ for which $G$ admits a homomorphism to $K_{p / q}$. It is well known that $\chi(G)-1<\chi_{c}(G) \leq \chi(G)$ for every graph $G$. In [18] it is shown that certain odd-chromatic generalized Mycielski graphs can have their circular chromatic number arbitrarily close to the above lower bound. Building on this we showed similar results also for odd chromatic Schrijver graphs and Borsuk graphs in [26]. As it is also known that $K_{p / q}$ admits a homomorphism into $K_{r / s}$ whenever $r / s \geq p / q$ (see, e.g., as Theorem 6.3 in [17]), the above and the functoriality of $B_{0}(G)$ together imply that $\operatorname{coind}\left(B_{0}\left(K_{p / q}\right)\right)+1=\chi\left(K_{p / q}\right)=\lceil p / q\rceil$ whenever $\lceil p / q\rceil$ is odd.

We remark that the oddness condition is crucial here. It also follows from results in [26] (cf. also [23] for some special cases) that the graphs $K_{p / q}$ with $\lceil p / q\rceil$ even and $p / q$ not integral do not satisfy the conditions of Theorem 3] Here we state more: the conclusion of Theorem 3 does not hold for these graphs. Indeed, let us color the vertex $i$ with the color $\lfloor i / q\rfloor+1$. This is a proper coloring with the minimal number $\lceil p / q\rceil$ of colors, but it does not contain a complete bipartite graph with all the even colors on one side and all the odd colors on the other.

The remaining case of $p / q$ even and integral is not especially interesting as $K_{p / q}$ with $p / q$ integral is homomorphic equivalent to the complete graph on $p / q$ vertices and therefore trivially satisfies the $\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$ condition.

Taking the contrapositive in the above observation we obtain a new proof of Theorem 6 in [26] the special case of which for Kneser graphs and Schrijver graphs was independently obtained by Meunier [23].

Corollary 4 ([26], cf. also [23]) If $\operatorname{coind}\left(B_{0}(G)\right)$ is odd for a graph $G$, then $\chi_{c}(G) \geq$ $\operatorname{coind}\left(B_{0}(G)\right)+1$.

Proof. If a graph $G$ has $\chi_{c}(G)=p / q$, then $G$ admits a homomorphism to $K_{p / q}$, thus by the functoriality of $B_{0}(G)$ we have $\operatorname{coind}\left(B_{0}\left(K_{p / q}\right)\right) \geq \operatorname{coind}\left(B_{0}(G)\right)$. Then $\lceil p / q\rceil=$ $\chi\left(K_{p / q}\right) \geq \operatorname{coind}\left(B_{0}\left(K_{p / q}\right)\right)+1 \geq \operatorname{coind}\left(B_{0}(G)\right)+1$. If in addition coind $\left(B_{0}(G)\right)+1>p / q$, then all the previous inequalities hold with equality by the integrality of the coindex. Then $K_{p / q}$ satisfies the conditions of Theorem 3, thus it must satisfy its conclusion. But we just have seen that this is not so if $\lceil p / q\rceil$ is even and $\mathrm{p} / \mathrm{q}$ is not integral. Thus coind $\left(B_{0}(G)\right)+1$ cannot be even in this case.

We remark that the consequences of the above result include a partial solution of two conjectures about the circular chromatic number that are mentioned in 33. For a detailed discussion of implications and references we refer to [26].

The proof of Corollary 4] in [26] and also the proof in [23] relies on Ky Fan's theorem. The above argument shows that Ky Fan's theorem can be substituted by (the second form of) the Bacon-Tucker theorem in obtaining this result. Nevertheless, it may be
worth noting, that the missing bipartite graph in the above optimal coloring of an evenchromatic $K_{p / q}$ is one the presence of which would also be required by the Zig-zag theorem.

### 3.3.7 The entire collection

Our examples are collected in the following corollary.
Corollary 5 For any proper t-coloring of any member of the folowing t-chromatic families of graphs the property described as the conclusion of Theorem 3 holds.
(i) Kneser graphs $\mathrm{KG}(n, k)$ with $t=n-2 k+2$,
(ii) Schrijver graphs $\operatorname{SG}(n, k)$ with $t=n-2 k+2$,
(iii) Borsuk graphs $B(t-1, \alpha)$ with large enough $\alpha<2$ and some of their finite subgraphs,
(iv) $U\left(t+1,\left\lfloor\frac{t+3}{2}\right\rfloor\right)$, for any $t \geq 2$,
(v) $W(s, t)$ for every $s \geq 1, t \geq 2$,
(vi) Rational complete graphs $K_{p / q}$ for $t=\lceil p / q\rceil$ odd,
(vii) The $t$-chromatic graphs obtained by $1 \leq d \leq t-2$ iterations of the generalized Mycielski construction starting with $a(t-d)$-chromatic version of any graph appearing on the list above.

Proof. The above explanation and references contain the argument implying that all the families of graphs $G$ above satisfy $t=\chi(G)=\operatorname{coind}\left(B_{0}(G)\right)+1$, thus Theorem 3 is applicable.

### 3.4 Generalization of G. Spencer and F. Su's result

Recently Gwen Spencer and Francis Edward Su [28, 29] found an interesting consequence of Ky Fan's theorem. They prove that if the Kneser graph $\operatorname{KG}(n, k)$ is colored optimally, that is, with $t=n-2 k+2$ colors, but otherwise arbitrarily, then the following holds. Given any bipartition of the color set $[t]$ into partition classes $B_{1}$ and $B_{2}$ that are as equal as possible (i.e., $\left|\left|B_{1}\right|-\left|B_{2}\right|\right| \leq 1$ ), there exists a bipartition of the ground set $[n]$ into $E_{1}$ and $E_{2}$, such that, the $k$-subsets of $E_{i}$ as vertices of $\operatorname{KG}(n, k)$ are all colored with colors from $B_{i}$ and every color in $B_{i}$ does occur ( $\mathrm{i}=1,2$ ).

Theorem 3 implies an analogous statement where no special requirement is needed about the sizes of $B_{1}$ and $B_{2}$. It can also be obtained by simply replacing Ky Fan's theorem by the Bacon-Tucker theorem in Spencer and Su's argument.

Corollary 6 Let $t=n-2 k+2$ and fix an arbitrary proper $t$-coloring $c$ of the Kneser graph $\operatorname{KG}(n, k)$ with colors from the color set $[t]$. Let $B_{1}$ and $B_{2}$ form a bipartition of $[t]$, i.e., $B_{1} \cup B_{2}=[t]$ and $B_{1} \cap B_{2}=\emptyset$. Then there exists a bipartition $\left(E_{1}, E_{2}\right)$ of $[n]$ such that for $i=1,2$ we have $\left\{c(v): v \subseteq E_{i}\right\}=B_{i}$.

Proof. Set $A=B_{1}$ and $B=B_{2}$ and consider the complete bipartite graph Theorem 3 returns for this bipartition of the color set. Let the vertices on the two sides of this bipartite graph be $u_{1}, \ldots, u_{|A|}$ and $v_{1}, \ldots, v_{|B|}$. All vertices $u_{i}$ and $v_{j}$ are subsets of $[n]$. Since $u_{i}$ is adjacent to $v_{j}$ for every $i, j$ we have that $E_{1}^{\prime}:=\cup_{i=1}^{|A|} u_{i}$ and $E_{2}^{\prime}:=\cup_{j=1}^{|B|} v_{j}$ are disjoint. If there are elements of $[n]$ that are neither in $E_{1}^{\prime}$ nor in $E_{2}^{\prime}$ then put each such element into either one of the sets $E_{i}^{\prime}$ thus forming the sets $E_{1}$ and $E_{2}$. We show that these sets $E_{i}$ satisfy our requirements. It follows from the construction that $E_{1} \cap E_{2}=\emptyset$ and $E_{1} \cup E_{2}=[n]$. It is also clear that all colors from $B_{1}$ appear as the color of some $k$-subset of $E_{1}$, namely, the $k$-subsets $u_{1}, \ldots, u_{|A|}$ take care of this condition. Since $E_{2}$ is disjoint from $E_{1}$ no $k$-subset of $E_{2}$ can be colored by any of the colors from $B_{1}$. Thus each $k$-subset of $E_{2}$ is colored by a color from $B_{2}$, and all these colors appear on some $k$-subset of $E_{2}$ by the presence of $v_{1}, \ldots, v_{|B|}$. Exchanging the role of $E_{1}$ and $E_{2}$ we get that all $k$-subsets of $E_{1}$ are colored with some color of $B_{1}$ and the proof is complete.
Remark 3. The same argument proves a similar statement for the general Kneser graph $\operatorname{KG}(\mathcal{F})$ in place of $\operatorname{KG}(n, k)$ as long as we have $t=\chi(\operatorname{KG}(\mathcal{F}))=\operatorname{coind}\left(B_{0}(\operatorname{KG}(\mathcal{F}))\right)+1$. Such graphs include the Schrijver graphs $S G(n, k)$ with $t=n-2 k+2$ and (by the argument presented in Remark 2) the graphs $\operatorname{KG}(\mathcal{F})$ with a family $\emptyset \notin \mathcal{F}$ satisfying $t=\chi(\operatorname{KG}(\mathcal{F}))=\operatorname{cd}_{2}(\mathcal{F})$.

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