

DYCK PATHS WITH COLOURED ASCENTS

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ABSTRACT

We introduce a notion of Dyck paths with coloured ascents. For several ways of colouring, we establish bijections between sets of such paths and other combinatorial structures, such as non-crossing trees, dissections of a convex polygon, etc. In some cases enumeration gives new expression for sequences enumerating these structures.

KEYWORDS. Dyck paths, non-crossing graphs, dissections of polygon by diagonals.

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1. INTRODUCTION

1.1. Coloured Dyck paths. A *Dyck path of length $2n$* is a sequence P of letters U and D , such that $\#(U) = \#(D) = n$ (where $\#$ means “number of”) in P , and $\#(U) \geq \#(D)$ in any initial subsequence of P . A Dyck path of length $2n$ is usually represented graphically as a lattice path from the point $(0, 0)$ to the point $(2n, 0)$ that does not pass below the x -axis, where U is the “up step” $(1, 1)$ and D is the “down step” $(1, -1)$. The set of all Dyck paths of length $2n$ will be denoted by $\mathcal{D}(n)$. We shall also denote $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$. It is well known that $|\mathcal{D}(n)|$ is equal to the n -th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$ (see [10, Page 222, Exercise 19(i)]).

A maximal subsequence of k consecutive U ’s (that is, not preceded or followed by another U) in a Dyck path will be called a k -*ascent* and denoted by U^k . Similarly, a maximal subsequence of k consecutive D ’s will be denoted by D^k . The Dyck path $U^k D^k$ will be called *pyramid of length $2k$* and denoted by Λ^k .

In this paper we present a new generalization of Dyck paths. Let $\mathfrak{L} = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \dots\}$ be a sequence of sets, and let $a_k = |\mathcal{L}_k|$. We colour all ascents in a Dyck path, when the set of colours for each k -ascent is \mathcal{L}_k . In this way we obtain *Dyck paths with ascents coloured by \mathfrak{L}* (shortly *Dyck paths coloured by \mathfrak{L}* , or *coloured Dyck paths*). Each Dyck path P produces thus $\prod a_i$ coloured Dyck paths, when the product is taken over the lengths of all ascents in P . Coloured Dyck paths will be denoted by capital letters with “hat”, e. g. \hat{P} . The pyramid Λ^k with U^k coloured by a specified colour $C \in \mathcal{L}(k)$ will be denoted by $\Lambda^k \langle C \rangle$.

The set of all Dyck paths of length $2n$ coloured by members of \mathfrak{L} will be denoted by $\mathcal{D}^{\mathfrak{L}}(n)$. We shall also denote $\mathcal{D}^{\mathfrak{L}} = \bigcup_{n \geq 0} \mathcal{D}^{\mathfrak{L}}(n)$. In order to obtain a general expression enumerating $|\mathcal{D}^{\mathfrak{L}}(n)|$, we note that any Dyck path can be presented uniquely in the form

$$U^k D P_k D P_{k-1} \dots D P_1,$$

where P_k, P_{k-1}, \dots, P_1 are (possibly empty) Dyck paths. Therefore $M(x)$, the generating function for the sequence $\{|\mathcal{D}^\mathfrak{L}(n)|\}_{n \geq 0}$, satisfies

$$M(x) = a_0 + a_1 x M(x) + a_2 x^2 M^2(x) + \dots,$$

or

$$(1) \quad M = A(xM),$$

where $A(x) = \sum_{i \geq 0} a_i x^i$ is the generating function for the sequence $\{|\mathcal{L}_i|\}_{i \geq 0}$.

However, we shall consider rather \mathcal{L}_k 's than merely a_k 's, and we shall establish bijections between $\mathcal{D}^\mathfrak{L}(n)$ for some specific \mathfrak{L} , and other combinatorial structures. For example, in our main result $\mathcal{L}_k = \mathcal{D}(k)$, and thus $a_k = C_k = \frac{1}{k+1} \binom{2k}{k}$. We shall show the bijection between Dyck paths coloured in this way and non-crossing trees to be defined in Section 1.2.

1.2. Non-crossing trees. A *non-crossing tree* (an “NC-tree”) on $[n]$ is a labeled tree which can be represented by a drawing in which the vertices are points on a circle, labeled by $\{1, 2, \dots, n\}$ clockwise, and the edges are non-crossing straight segments. The vertex 1 will be also called *the root*, and we shall depict it as a top vertex. Non-crossing trees have been studied by Chen et al. [2], Deutsch et al. [3, 4], Flajolet et al. [5], Hough [6], Noy et al. [7], and Panholzer et al. [9]. Denote the set of all NC-trees on $[n]$ by $\mathcal{NC}(n)$. It is well known that $|\mathcal{NC}(n+1)| = \frac{1}{2n+1} \binom{3n}{n}$.

We shall use the following notion. If $\{a, b\}$ is an edge of T , we shall rather denote it by (a, b) , always assuming that $a < b$. For a vertex v , we denote $N^-(v) = \{a \in V(T) : (a, v) \in E(T)\}$ (the *in-edges incident to v*), $N^+(v) = \{b \in V(T) : (v, b) \in E(T)\}$ (the *out-edges incident to v*); $d^-(v) = |N^-(v)|$ (the *in-degree of v*), $d^+(v) = |N^+(v)|$ (the *out-degree of v*).

Consider the NC-trees on $[n]$ with the property: For each vertex $v \neq 1$, we have $d^-(v) = 1$. Such trees will be called *non-crossing out-trees* (“NCO-trees”); the set of NCO-trees on $[n]$ will be denoted by $\mathcal{NCO}(n)$.

We also denote $\mathcal{NC} = \bigcup_{n \geq 0} \mathcal{NC}(n+1)$ and $\mathcal{NCO} = \bigcup_{n \geq 0} \mathcal{NCO}(n+1)$.

1.3. The results. We establish bijections between $\mathcal{D}^\mathfrak{L}(n)$ and other combinatorial structures, for a few specific choices of \mathfrak{L} . The main result is the following theorem:

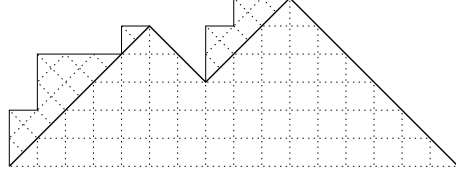
Theorem 1. *There is a bijection between the set of Dyck paths of length $2n$ with k -ascents coloured by Dyck paths of length $2k$ and the set of non-crossing trees on $[n+1]$.*

Other results are special cases and variations of this theorem. Substituting the generation function of \mathfrak{L} in (1) enables us to enumerate easily the combinatorial structures being in bijection with $\mathcal{D}^\mathfrak{L}(n)$.

2. DYCK PATHS COLOURED BY DYCK PATHS

Let $\mathfrak{D} = \{\mathcal{D}(0), \mathcal{D}(1), \mathcal{D}(2), \dots\}$. In this Section we consider $\mathcal{D}^\mathfrak{D}(n)$ – the set of Dyck paths of length $2n$ with k -ascents coloured by Dyck paths of length $2k$, i. e. we take $\mathcal{L}_k = \mathcal{D}(k)$.

First we introduce a convenient way to depict thus coloured Dyck paths. Given a k -ascent U^k coloured by a Dyck path C of length $2k$, we draw a copy of C , rotated by 45° and scaled by $1/\sqrt{2}$, between the endpoints of U^k . Figure 1 presents in this way the Dyck path $U^5 D^2 U^3 D^6$ with U^5 coloured by $UUDUDDDDUD$ and U^3 coloured by $UUDUDD$.

FIGURE 1. A Dyck path with k -ascents coloured by Dyck paths of length $2k$.

2.1. Enumeration. Let us enumerate $\mathcal{D}^{\mathfrak{D}}(n)$. The generating function for $\{|\mathcal{D}(n)|\}_{n \geq 0}$ is

$$\frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + \dots$$

Substituting this in (1), we obtain

$$M = \frac{1 - \sqrt{1 - 4xM}}{2xM}.$$

After simplifications, we have $M - 1 = xM^3$. Denoting $L = M - 1$ and applying Lagrange's inversion formula (see [10, Section 5.4] and [11, Section 5.1]) on $L = x(L + 1)^3$, we get that the coefficient of x^n in L is

$$[x^n]L = \frac{1}{n}[L^{n-1}](L + 1)^{3n} = \frac{1}{n} \binom{3n}{n-1} = \frac{1}{2n+1} \binom{3n}{n}.$$

Thus we have $|\mathcal{D}^{\mathfrak{D}}(n)| = |\mathcal{NC}(n+1)|$.

In this Section we shall construct, for each $n \geq 0$, a bijective function $\varphi_n : \mathcal{D}^{\mathfrak{D}}(n) \rightarrow \mathcal{NC}(n+1)$. It will be presented as a restriction of a bijective function $\varphi : \mathcal{D}^{\mathfrak{D}} \rightarrow \mathcal{NC}$. The function φ will be constructed by the following steps: In Subsection 2.2 we describe a recursive procedure of decomposing a Dyck path into pyramids. In Subsection 2.3 we construct a bijection $\vartheta : \mathcal{D} \rightarrow \mathcal{NCO}$. In Subsection 2.4 we first define φ for coloured pyramids and then, using observations from Subsection 2.2, for all Dyck paths coloured by \mathfrak{D} . All by all, this will give us the function φ , and we shall also show that it is bijective.

2.2. Decomposition of a Dyck path into pyramids. Let P be a Dyck path. Recall that it can be presented uniquely in the form

$$(2) \quad P = U^k D P_k D P_{k-1} \dots D P_1,$$

where P_k, P_{k-1}, \dots, P_1 are (possibly empty) Dyck paths. We say that Λ^k is the *base pyramid* of P and that P_k, P_{k-1}, \dots, P_1 are *appended to* Λ^k , and denote this by $P = \Lambda^k * [P_k, P_{k-1}, \dots, P_1]$. This will be called a [primary] decomposition of P . If $P_k = P_{k-1} = \dots = P_1 = \emptyset$, then P is a pyramid Λ^k , and we stop, identifying $\Lambda^k * [\emptyset, \emptyset, \dots, \emptyset]$ with Λ^k . Otherwise we decompose nonempty paths among P_k, P_{k-1}, \dots, P_1 in the same way. Repeating this process recursively, we obtain the complete decomposition of P . Since P_k, P_{k-1}, \dots, P_1 are shorter than P , the paths participating in the complete decomposition are pyramids and empty paths. Thus we call it *the complete decomposition of P into pyramids*.

The complete decomposition of a Dyck path can be also represented by a rooted tree: Given $P = \Lambda^k * [P_k, P_{k-1}, \dots, P_1]$, we represent it by Λ^k as the root with children P_k, P_{k-1}, \dots, P_1 . Then we do the same for P_k, P_{k-1}, \dots, P_1 and continue recursively, until all the leaves are pyramids or empty paths. A Dyck path is easily restored from its complete decomposition.

An example of complete decomposition is

$$\begin{aligned}
 U^4 DU^2 DU DDDU^2 DDDDU^2 DUD &= \\
 &= \Lambda^4 * [U^2 DU DD, U^2 DD, \emptyset, UDU^2 DDUD] \\
 &= \Lambda^4 * [\Lambda^2 * [UD, \emptyset], \Lambda^2, \emptyset, \Lambda^1 * [U^2 DDUD]] \\
 &= \Lambda^4 * [\Lambda^2 * [\Lambda^1, \emptyset], \Lambda^2, \emptyset, \Lambda^1 * [\Lambda^2 * [\emptyset, UD]]] \\
 &= \Lambda^4 * [\Lambda^2 * [\Lambda^1, \emptyset], \Lambda^2, \emptyset, \Lambda^1 * [\Lambda^2 * [\emptyset, \Lambda^1]]],
 \end{aligned}$$

and it is illustrated on Figures 2 and 3. On Figure 2 paths appended to a pyramid are shaded in a more dark colour than the pyramid.

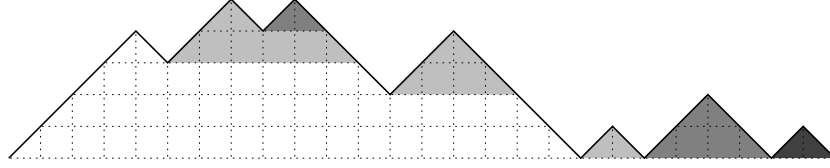


FIGURE 2. Complete decomposition of a Dyck path.

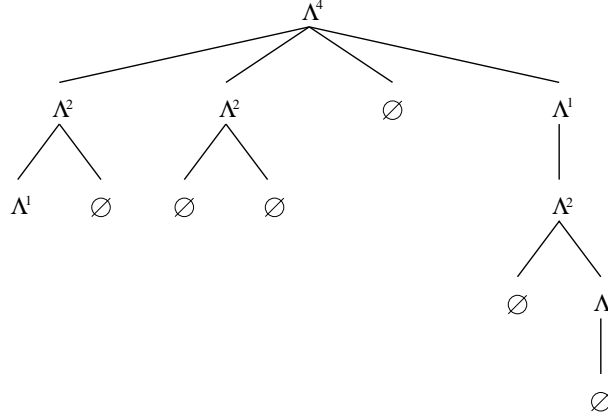


FIGURE 3. Complete decomposition of a Dyck path represented by a rooted tree.

Note that each U^k in a Dyck path results in a Λ^k in the complete decomposition. Therefore the decomposition (2) is valid also for coloured Dyck paths: $\hat{P} = \hat{U}^k D \hat{P}_k D \hat{P}_{k-1} \dots D \hat{P}_1$, and in the complete decomposition of a coloured Dyck path, each U^k coloured by C results in Λ^k coloured by C . It is also clear how to restore the coloured Dyck path from its complete decomposition to coloured pyramids.

We remark that the expression (1) enumerates thus also the following structure: rooted trees with n edges, each vertex v coloured by one of $a_{d(v)}$ colours, where $d(v)$ is the out-degree of the vertex. For instance, if each vertex v is coloured by one of $C_{d(v)}$ colours, there are $\frac{1}{2n+1} \binom{3n}{n}$ such trees.

2.3. A bijection between non-coloured Dyck paths and NCO-trees. We begin with a simple bijection $\vartheta : \mathcal{D} \rightarrow \mathcal{NCO}$. Given $P \in \mathcal{D}$, we construct $\vartheta(P)$ according to the following algorithm. Start with the NC-tree which has one point 1 and no edges. Scan P and do the following: For each U , add

a new edge beginning in the present point (for a while its end is not determined). For each D , add the next point, move to it and let it be the end of the last incomplete edge. See Figure 4.

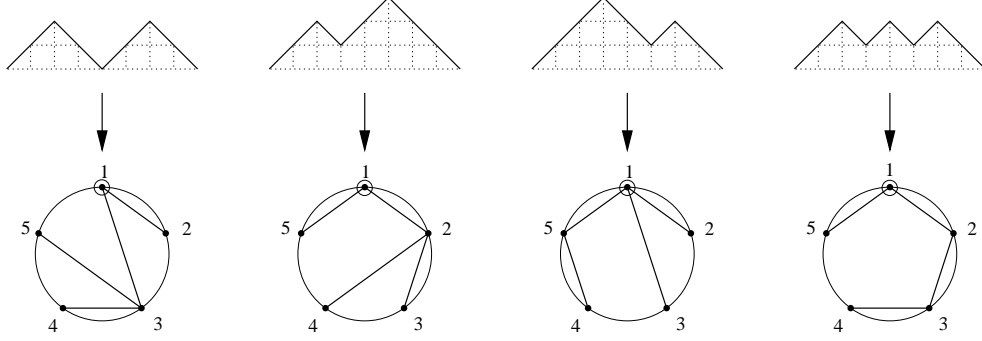


FIGURE 4. The function $\vartheta : \mathcal{D} \rightarrow \mathcal{NCO}$.

It is easy to see that ϑ is well defined (since P is a Dyck path, we never need to complete a non-existing edge, and all edges are completed by the end), and that ϑ is invertible: Given $T \in \mathcal{NCO}$, scan it from the vertex 1 clockwise. Visiting a vertex, first count in-edges incident with it, and then out-edges. For each in-edge add U , for each out-edge add D , and move to the next vertex. It is easy to see that thus obtained Dyck path P satisfies $\vartheta(P) = T$. Besides, if $P \in \mathcal{D}(k)$ then $\vartheta(P) \in \mathcal{NCO}(n+1)$. Thus we have a family of bijections $\vartheta_n : \mathcal{D}(n) \rightarrow \mathcal{NCO}(n+1)$, for all $n \geq 0$, which shows in particular that $|\mathcal{NCO}(n+1)| = C_n$.

2.4. Definition of $\varphi : \mathcal{D}^{\mathfrak{D}} \rightarrow \mathcal{NC}$. First we define φ for coloured pyramids. Consider $\Lambda^k \langle C \rangle$ where $C \in \mathcal{D}$. We define $\varphi(\Lambda^k \langle C \rangle) = \vartheta(C)$.

Now we define φ for all coloured Dyck paths. Let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \dots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{D}}$. Suppose that we know $\varphi(\hat{\Lambda}^k)$ and $\varphi(\hat{P}_i)$ for $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, insert a copy of $\varphi(\hat{P}_i)$ into $\varphi(\hat{\Lambda}^k)$ so that the vertex 1 of $\varphi(\hat{P}_i)$ is mapped to the vertex $i+1$ of $\varphi(\hat{\Lambda}^k)$, and the vertices $2, 3, \dots$ of $\varphi(\hat{P}_i)$ are mapped clockwise to new vertices between i and $i+1$ in $\varphi(\hat{\Lambda}^k)$ (if $\hat{P}_i = \emptyset$ nothing happens). See Figure 5.

The function φ is invertible. Given $T \in \mathcal{NC}$, we want to find $\hat{P} \in \mathcal{D}^{\mathfrak{D}}$ such that $\varphi(\hat{P}) = T$. Take the subtree of T with root 1 obtained by recursive adding only out-edges incident with each reached vertex. After appropriate relabeling of vertices, it forms an NCO-tree V . It corresponds to the base pyramid $\hat{\Lambda}^k$ of \hat{P} , where k is equal to the number of edges in V , and the colouring is $\vartheta^{-1}(V)$. For $i = 1, 2, 3, \dots$, the subtree of T attached to the vertex i of V corresponds to \hat{P}_i which is determined recursively. This allows to restore \hat{P} .

It is easy to see that if $\hat{P} \in \mathcal{D}^{\mathfrak{D}}(n)$ then $\varphi(\hat{P}) \in \mathcal{NC}(n+1)$. Thus we have a family of bijections $\varphi_n : \mathcal{D}^{\mathfrak{D}}(n) \rightarrow \mathcal{NC}(n+1)$, for all $n \geq 0$.

This completes the proof of Theorem 1.

3. DYCK PATHS COLOURED BY DYCK PATHS WITH ASCENTS OF BOUNDED LENGTH

In this section we restrict the Dyck paths used as colours, considering in this role only Dyck paths with ascents of bounded length.

Let $\mathcal{M}^m(n)$ be the set of Dyck paths of length $2n$ with ascents of length $\leq m$. It is known that $|\mathcal{M}^m(n)|$ is equal to the n -th m -generalized Motzkin number. For $m = 2$ we have Motzkin numbers

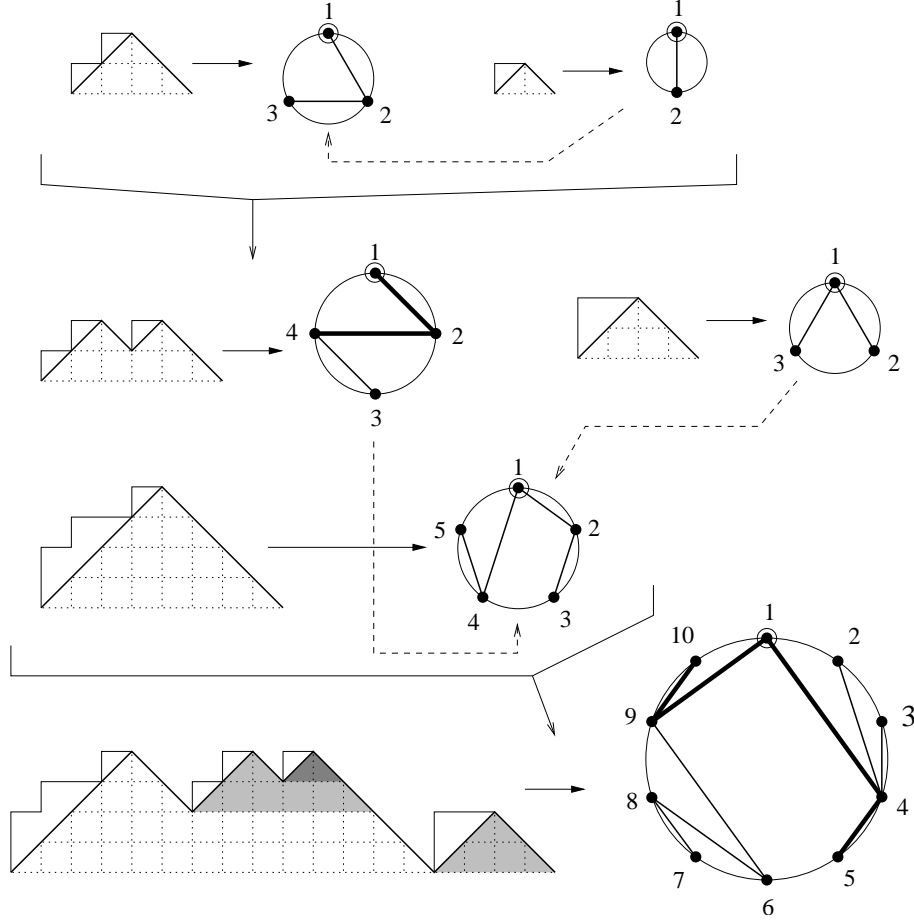


FIGURE 5. The function $\varphi : \mathcal{D}^{\mathfrak{D}} \rightarrow \mathcal{NC}$. The bold edges are those corresponding to the base pyramid.

which enumerate Motzkin paths. For $m \geq n$ we have $|\mathcal{M}^{\geq n}(n)| = C_n$. In this sense the sequence of sequences of m -generalized Motzkin numbers “converges”, with $m \rightarrow \infty$, to the sequence of Catalan numbers.

Among other structures enumerated by $|\mathcal{M}^m(n)|$ we have

- The set of rooted trees on $n + 1$ vertices with degree $\leq m$.
- The set of all partitions of the vertices of a convex labeled n -polygon to $(\leq m)$ -sets with disjoint convex hulls.

Denote $\mathfrak{M}^m = \{\mathcal{M}^m(0), \mathcal{M}^m(1), \mathcal{M}^m(2), \dots\}$. We consider $\mathcal{D}^{\mathfrak{M}^m}(n)$ for fixed m , and prove the following:

Theorem 2. *There is a bijection between $\mathcal{D}^{\mathfrak{M}^m}(n)$ and the set of partitions of the vertices of a labeled convex $(2n)$ -polygon to $(\leq 2m)$ -sets of even size with disjoint convex hulls. The cardinality of both sets is $\sum_{p=0}^{n/m-1} \frac{(-1)^p}{n-mp} \binom{n-mp}{p} \binom{3n-mp-p}{n-mp-1}$.*

3.1. Enumeration. The generating function $A(x)$ for $\{|\mathcal{M}^m(n)|\}_{n \geq 0}$ satisfies

$$(3) \quad A(x) = 1 + xA(x) + x^2A^2(x) + \cdots + x^mA^m(x).$$

Substituting (1) in (3), we obtain that the generating function $h_m(x)$ for $\{|\mathcal{D}^{\mathfrak{M}^m}(n)|\}_{n \geq 0}$ satisfies

$$h_m(x) = 1 + xh_m^2(x) + x^2h_m^4(x) + \cdots + x^mh_m^{2m}(x),$$

which is equivalent to

$$h_m(x) - 1 = xh_m^3(x) - (xh_m^2(x))^{m+1}.$$

Applying the Lagrange inversion formula on

$$h_m(x, a) - 1 = a(xh_m^3(x, a) - (xh_m^2(x, a))^{m+1}),$$

we obtain

$$h_m(x, a) - 1 = \sum_{\ell \geq 1} \frac{a^\ell x^\ell}{\ell} \sum_{i=0}^{\ell-1} \sum_{j=0}^{\ell} (-1)^j x^{mj} \binom{3\ell}{i} \binom{\ell}{j} \binom{(2m-1)j}{\ell-1-i},$$

which implies that the coefficient of x^n in $h_m(x) = h_m(x, 1)$ is

$$(4) \quad \begin{aligned} [x^n](h_m(x)) &= \sum_{p=0}^{n/m-1} \frac{(-1)^p}{n-mp} \binom{n-mp}{p} \sum_{i=0}^{n-mp-1} \binom{3n-3mp}{i} \binom{2mp-p}{n-mp-1-i} = \\ &= \sum_{p=0}^{n/m-1} \frac{(-1)^p}{n-mp} \binom{n-mp}{p} \binom{3n-mp-p}{n-mp-1}. \end{aligned}$$

This is the cardinality of $\mathcal{D}^{\mathfrak{M}^m}(n)$.

3.2. Partitions of convex polygons. Denote by $\mathcal{E}^m(n)$ the set of all partitions of the vertices of a convex labeled $(2n)$ -polygon to $(\leq 2m)$ -sets of even size with disjoint convex hulls. We label the vertices of the $(2n)$ -polygon by $[n] \times \{a, b\} = \{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$ and depict them appearing on a circle clockwise in this order, a_1 being the top point. Denote also $\mathcal{E} = \bigcup_{n \geq 0, m \geq 1} \mathcal{E}^m(n)$.

For all $n \geq 0$, $m \geq 1$ we shall construct a bijection $\rho_{n,m} : \mathcal{D}^{\mathfrak{M}^m}(n) \rightarrow \mathcal{E}^m(n)$. It will be presented as a restriction of a bijection $\rho : \mathcal{D}^{\mathfrak{D}} \rightarrow \mathcal{E}$.

We start with a bijection ψ from \mathcal{D} to $\bar{\mathcal{E}}$, the set of all partitions of the vertices of a convex labeled polygon to sets with disjoint convex hulls. We label the vertices of the polygon by $1, 2, 3, \dots$ and depict them appearing on a circle clockwise in this order, 1 being the top point.

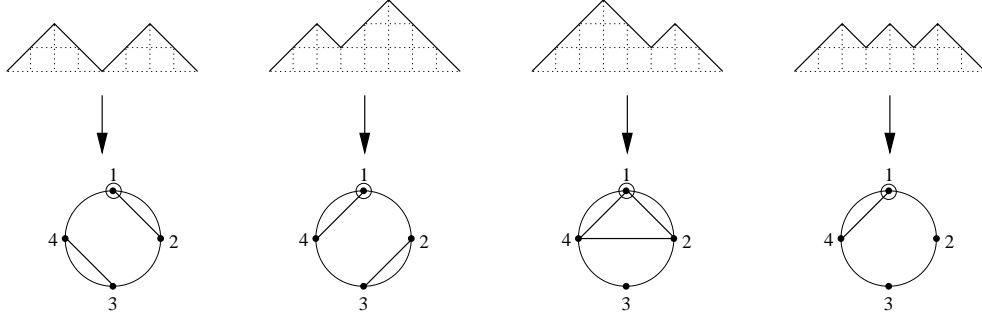
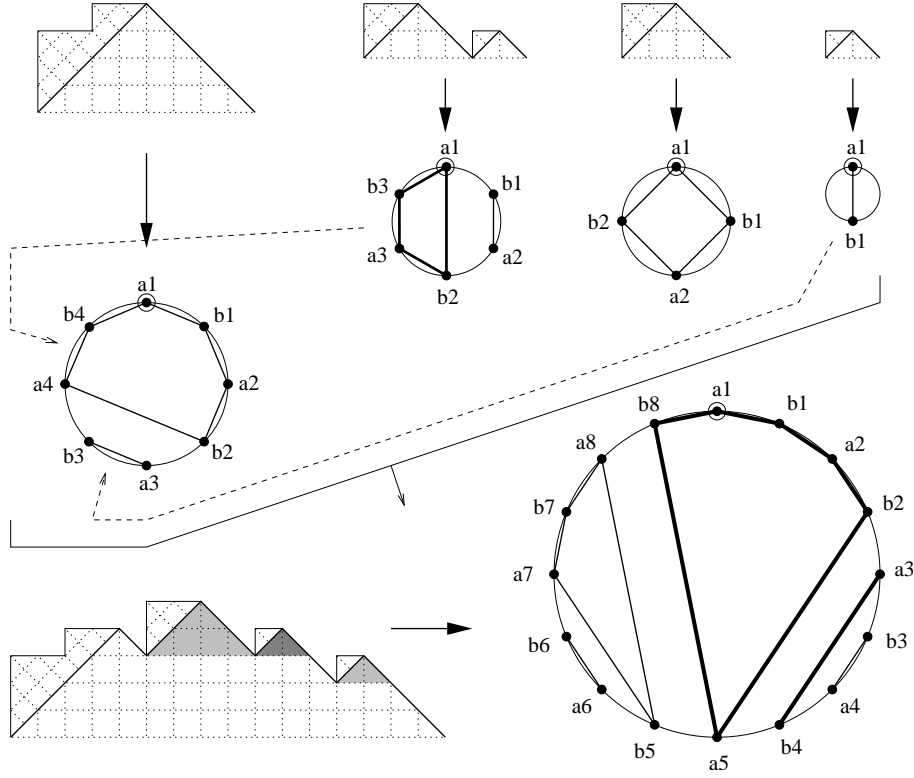
Given $P \in \mathcal{D}$ we construct $\psi(P)$ according to the following algorithm. Start with the circle without points. Scan P and do the following: For each $U^k D$ add the next point which will be the first vertex of a k -polygon (the other vertices of this polygon are determined later). For each D not preceded by U , add the next point, move to it and let it be a new vertex of the last incomplete polygon.

It is easy to see that ψ is well defined and invertible – similarly to the function ϑ . Indeed, comparing Figures 4 and 6, the reader will easily construct a bijection between \mathcal{NCO} and $\bar{\mathcal{E}}$.

Now we define $\rho : \mathcal{D}^{\mathfrak{D}} \rightarrow \mathcal{E}$. First we define it for coloured pyramids.

Let $M \in \mathcal{D}$. We define $\rho(\Lambda^k \langle M \rangle)$ to be a “duplicated $\psi(M)$ ”, i. e. for each polygon with vertices x_1, x_2, x_3, \dots in $\psi(M)$, we have a polygon with vertices $a_{x_1}, b_{x_1}, a_{x_2}, b_{x_2}, a_{x_3}, b_{x_3} \dots$ in $\rho(\Lambda^k \langle M \rangle)$.

Now let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \dots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{M}^m}(n)$, and suppose we know $\rho(\hat{\Lambda}^k)$ and $\rho(\hat{P}_i)$ for $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, insert a copy of $\rho(\hat{P}_i)$ into $\rho(\hat{\Lambda}^k)$ so that all the points of $\rho(\hat{P}_i)$ are mapped clockwise to new points between a_i and b_i . The obtained partition is $\rho(\hat{P})$. See Figure 7 for an illustration.

FIGURE 6. The function $\psi : \mathcal{D} \rightarrow \bar{\mathcal{E}}$.FIGURE 7. The function $\rho : \mathcal{D}^{\mathcal{D}} \rightarrow \mathcal{E}$.

The function ρ is invertible. Given $T \in \mathcal{E}$, we want to find \hat{P} such that $\rho(\hat{P}) = T$. Consider T as the union of polygons and choose points of T , beginning from a_1 and moving clockwise as follows: from a_i pass to the vertex connected to it (by an edge of a polygon in the partition), from b_i pass to a_{i+1} . Denote by V the union of polygons formed by the chosen points after appropriate relabeling of vertices (these polygons have bold edges in Figure 7). It corresponds to the base pyramid of \hat{P} with colouring determined by joining points a_i and b_i into one point i and then applying ψ^{-1} . For

$i = 1, 2, 3, \dots$, the part of T between the points a_i and b_i of V corresponds to P_i which are determined recursively. This allows to restore \hat{P} .

It is easy to see that if $\hat{P} \in \mathcal{D}^{\mathfrak{M}^m}(n)$ then $\rho(\hat{P}) \in \mathcal{E}^m(n)$. In particular, each k -ascent in \hat{P} results in a $(2k)$ -polygon in $\rho(\hat{P})$. Thus we have a family of bijections $\rho_{n,m} : \mathcal{D}^{\mathfrak{M}^m}(n) \rightarrow \mathcal{E}^m(n)$, for all $n \geq 0, m \geq 1$, and this completes the proof of Theorem 2.

3.3. Two special cases. We consider two special cases: $m = 1$ and $m = n$.

1. Let $m = 1$. Substituting this in (4), we get

$$\sum_{p=0}^{n-1} \frac{(-1)^p}{n-p} \binom{n-p}{p} \binom{3n-2p}{n-p-1},$$

which is equal to C_n : since $|\mathcal{M}^1(k)| = 1$ for each k , we have $|\mathcal{D}^{\mathfrak{M}^1}(n)| = |\mathcal{D}(n)| = C_n$.

The partitions of $[2n]$ to even sets with disjoint convex hulls corresponding to the members of $\mathcal{D}^{\mathfrak{M}^1}(n)$ in the bijection ρ are those in which each set in partition has two members (all the ways to connect pairs of points of $[2n]$ in convex position by disjoint segments).

2. Let $m = n$. Substitute this in (4). The only relevant value of p is 0, and we get therefore

$$\frac{1}{n} \binom{3n}{n-1} = \frac{1}{2n+1} \binom{3n}{n},$$

which is expected: we have $|\mathcal{M}^n(n)| = C_n$ and thus $\mathcal{D}^{\mathfrak{M}^n}(n) = \mathcal{D}^{\mathfrak{D}}(n)$.

The corresponding partitions of $[2n]$ are all possible partitions into even polygons.

4. DYCK PATHS COLOURED BY FIBONACCI PATHS

In this Section we consider a further restriction of Dyck paths taken as colours.

Let $\mathcal{F}^m(n)$ be the set of Dyck paths of length $2n$ which have the form $\Lambda^{k_1} \Lambda^{k_2} \Lambda^{k_3} \dots$ with $k_i \leq m$ – a concatenation of pyramids of length no more than $2m$. Note that $\mathcal{F}^m(n) \subset \mathcal{M}^m(n)$. It is known that $|\mathcal{F}^2(n)|$ is equal to the $(n+1)$ -st Fibonacci number; as a generalization $|\mathcal{F}^m(n)|$ is the $(n+1)$ -st m -generalized Fibonacci number (see [10, A092921]). The members of $\mathcal{F}^m(n)$ will be therefore called *m-generalized Fibonacci paths*. Besides, denote $\mathcal{F}(n) = \mathcal{F}^n(n)$. We have $|\mathcal{F}(n)| = 2^{n-1}$ for $n > 0$, and $|\mathcal{F}(0)| = 1$.

Denote $\mathfrak{F}^m = \{\mathcal{F}_0^m, \mathcal{F}_1^m, \mathcal{F}_2^m, \dots\}$ and $\mathfrak{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots\}$. We consider $\mathcal{D}^{\mathfrak{F}^m}(n)$ for fixed m , and $\mathcal{D}^{\mathfrak{F}}(n)$, and prove the following:

Theorem 3. *There is a bijection between $\mathcal{D}^{\mathfrak{F}^m}(n)$ and the set of diagonal dissections of a labeled convex $(n+2)$ -polygon into 3-, 4-, \dots , $(m+2)$ -polygons. The cardinality of both sets is*

$$\sum_{\ell=0}^{n-1} \frac{1}{\ell+1} \binom{n+\ell+1}{\ell} \sum_{i=0}^{\ell+1} (-1)^i \binom{n-1-mi}{\ell} \binom{\ell+1}{i}.$$

4.1. Enumeration. The generating function of the sequence $\{|\mathcal{F}^m(n)|\}_{n \geq 0}$ is

$$\sum_{n \geq 0} |\mathcal{F}(n)| x^n = \frac{1}{1 - x - x^2 - \dots - x^m}.$$

Substituting this in (1), we obtain that the generating function $g_m(x)$ for the sequence $\{|\mathcal{D}^{\mathfrak{F}^m}(n)|\}_{n \geq 0}$ satisfies

$$g_m(x) = \frac{1}{1 - xg_m(x) - \dots - (xg_m(x))^m}$$

which is equivalent to

$$g_m(x) - 1 = x(g_m(x))^2 \frac{1 - x^m (g_m(x))^m}{1 - xg_m(x)}.$$

Applying the Lagrange inversion formula on

$$g_m(x, a) - 1 = ax(g_m(x, a))^2 \frac{1 - x^m(g_m(x, a))^m}{1 - xg_m(x, a)},$$

we obtain

$$g_m(x, a) - 1 = \sum_{\ell \geq 0} \frac{a^{\ell+1}}{\ell+1} \sum_{j \geq 0} \sum_{i=0}^{\ell+1} (-1)^i x^{\ell+j+mi+1} \binom{\ell+j}{j} \binom{\ell+1}{i} \binom{2\ell+2+j+mi}{\ell},$$

which implies that the coefficient of x^n in $g_m(x) = g_m(x, 1)$ is

$$(5) \quad \sum_{\ell=0}^{n-1} \frac{1}{\ell+1} \binom{n+\ell+1}{\ell} \sum_{i=0}^{\ell+1} (-1)^i \binom{n-1-mi}{\ell} \binom{\ell+1}{i}.$$

This is the cardinality of $\mathcal{D}^{\mathfrak{s}^m}(n)$.

4.2. Dissections of a convex polygon. Denote by $\mathcal{R}_m(n)$ the set of all dissections of a labeled convex n -polygon into i -polygons with $i = 3, 4, \dots, m$ by non-crossing diagonals. We shall label the vertices of the n -polygon by $\alpha, 0, 1, \dots, n$ clockwise, the top vertex being α . Denote also $\mathcal{R} = \bigcup_{n \geq 0, m \geq 1} \mathcal{R}_{m+2}(n+2)$.

For all $n \geq 0, m \geq 1$ we construct a bijection $\sigma_{n,m} : \mathcal{D}^{\mathfrak{s}^m}(n) \rightarrow \mathcal{R}_{m+2}(n+2)$. It will be presented as a restriction of a bijection $\sigma : \mathcal{D}^{\mathfrak{s}} \rightarrow \mathcal{R}$.

Consider $\Lambda^k \langle F \rangle$, a pyramid of length $2k$ coloured by a generalized Fibonacci path $F \in \mathcal{F}(k)$. We define $\sigma(\Lambda^k \langle F \rangle)$ to be the dissection of the convex polygon with $k+2$ vertices, taking a diagonal (α, i) if and only if the path F touches the x -axis at point $(i, 0)$ (see Figure 8).

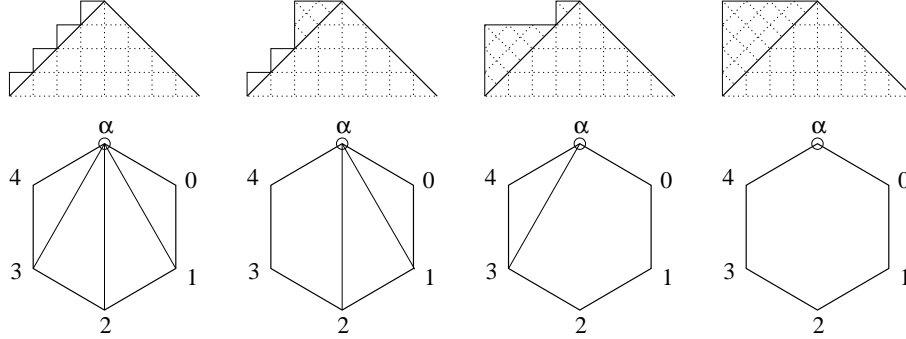
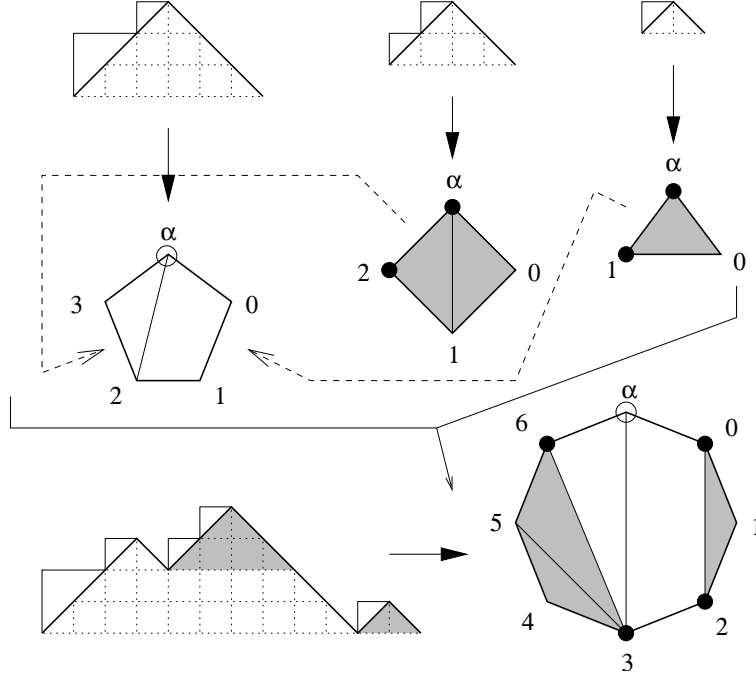


FIGURE 8. Definition of the function σ on pyramids.

Let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \dots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{s}}$. Suppose that we know dissections $\sigma(\hat{\Lambda}^k)$ and $\sigma(\hat{P}_i)$ for $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, insert a copy of $\sigma(\hat{P}_i)$ into $\sigma(\hat{\Lambda}^k)$ so that the vertex α of $\sigma(\hat{P}_i)$ is mapped to the vertex $i-1$ of $\sigma(\hat{\Lambda}^k)$, the last vertex of $\sigma(\hat{P}_i)$ is mapped to the vertex i of $\sigma(\hat{\Lambda}^k)$, and the vertices $1, 2, \dots$ of $\sigma(\hat{P}_i)$ are mapped clockwise to new vertices between $i-1$ and i of $\sigma(\hat{\Lambda}^k)$. After relabeling the vertices we obtain a dissection $\sigma(\hat{P})$. See Figure 9 for an example.

This function σ is invertible: Let $T \in \mathcal{R}$ and we want to find $\hat{P} \in \mathcal{D}^{\mathfrak{s}}$ such that $\sigma(\hat{P}) = T$. Let V be the union of all the polygons in the dissection that have α as a vertex corresponds, after the appropriate relabeling of its vertices. V corresponds to the base pyramid of \hat{P} which is restored

FIGURE 9. The function $\sigma : \mathcal{D}^{\delta} \rightarrow \mathcal{R}$.

immediately. The part attached to V along the edge $(i-1, i)$ corresponds to \hat{P}_i which is restored recursively.

Each pyramid of length $2j$ in F (a colouring of Λ^k) results in a $(j+2)$ -polygon in the dissection of $(k+2)$ -polygon. Therefore if $\hat{P} \in \mathcal{D}^{\delta^m}(n)$ then $\sigma(\hat{P}) \in \mathcal{R}_{m+2}(n+2)$. Thus we have a family of bijections $\sigma_{n,m} : \mathcal{D}^{\delta^m}(n) \rightarrow \mathcal{R}_{m+2}(n+2)$, for all $n \geq 0, m \geq 1$, and this completes the proof of Theorem 3.

4.3. Two special cases. As in Section 3.3, we consider two special cases: $m = 1$ and $m = n$.

1. Let $m = 1$. Substitute this in (5). It can be shown that

$$\sum_{i=0}^{\ell+1} (-1)^i \binom{n-1-i}{\ell} \binom{\ell+1}{i} = 0$$

for $l \in \{0, 1, \dots, n-2\}$. Therefore the whole expression is equal to

$$\frac{1}{n} \binom{2n}{n-1} \sum_{i=0}^n (-1)^i \binom{n-1-i}{n-1} \binom{n}{i} = \frac{1}{n} \binom{2n}{n-1} = C_n,$$

as expected, since $\mathcal{F}^1(n) = \mathcal{M}^1(n)$ and thus the corresponding NC-trees are as in Section 3.3. This agrees with a well-known fact that $|\mathcal{R}_3(n+2)|$, i. e. the number of dissections of $n+2$ -polygon into triangles, is C_n (see [10, Page 221, Exercise 19(a)]).

2. Let $m = n$. Substitute this in (5). It is clear that

$$(-1)^i \binom{n-1-ni}{\ell} \binom{\ell+1}{i} = 0$$

for $i > 0$. Therefore the whole expression is equal to

$$\sum_{\ell=0}^{n-1} \frac{1}{\ell+1} \binom{n+\ell+1}{\ell} \binom{n-1}{\ell} = \frac{1}{n} \sum_{\ell=0}^{n-1} \binom{n+\ell+1}{\ell+1} \binom{n-1}{\ell}.$$

This expression defines the sequence of “Little Schröder numbers” [8, A001003]. Indeed, it is well known that it enumerates $\mathcal{R}_n(n+2)$, i. e. all possible dissections of $n+2$ -polygon. See [1] for a recent related result.

Let us enumerate $\mathcal{D}^{\mathfrak{S}}(n)$ in another way. The generating function for $\{|\mathcal{F}(n)|\}_{n \geq 0}$ is

$$\sum_{n \geq 0} |\mathcal{F}(n)| x^n = \frac{1-x}{1-2x} = 1 + x + 2x^2 + 4x^3 + \dots$$

Substituting this in (1), we get

$$M = \frac{1-xM}{1-2xM},$$

which is equivalent to $M-1 = xM(2M-1)$. Taking $L = M-1$, we have $L = x(L+1)(2L+1)$, and by Lagrange’s inversion formula,

$$[x^n]L = \frac{1}{n} [L^{n-1}] ((L+1)(2L+1))^n = \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} \binom{n}{i+1} 2^i.$$

This is the cardinality of $\mathcal{D}^{\mathfrak{S}}(n)$ and thus another expression for Little Schröder numbers.

The sequence of Little Schröder numbers enumerates “Little Schröder paths”, see [8, A001003]. A *Little Schröder path of length $2n$* is a lattice path from $(0,0)$ to $(2n,0)$ with moves $U = (1,1)$, $D = (1,-1)$, $L = (2,0)$, which does not pass below the x -axis and does not contain an L -step on the x -axis.

Denote the set of all Little Schröder paths of length $2n$ by $\mathcal{LS}(n)$, and $\mathcal{LS} = \bigcup_{n \geq 0} \mathcal{LS}(n)$. According to our result, $|\mathcal{D}^{\mathfrak{S}}(n)| = |\mathcal{LS}(n)|$. We construct a simple direct bijection between these sets:

Observation 4. *There is a bijection between $\mathcal{D}^{\mathfrak{S}}(n)$ and $\mathcal{LS}(n)$. The cardinality of both sets is the n -th Little Schröder number.*

Let $F \in \mathcal{F}(k)$. Represent it by a $\{0,1\}$ -sequence $(x_1 x_2 \dots x_{k-1})$: consider F as a lattice path from $(0,0)$ to $(2k,0)$ and let $x_i = 1$ if F touches the x -axis at the point $(2i,0)$, and $x_i = 0$ otherwise.

Let $\hat{P} \in \mathcal{D}^{\mathfrak{S}}$. Consider the complete decomposition of \hat{P} . Each pyramid $\hat{\Lambda}_k$ in this decomposition is coloured by the members of $\{0,1\}^{k-1}$. Replace $\Lambda_k \langle (x_1 x_2 \dots x_{k-1}) \rangle$ with $U^{\beta+1} A_{k-1} A_{k-2} \dots A_2 A_1 D$ where β is the number of 1’s in $(x_1 x_2 \dots x_{k-1})$, $A_i = D$ if $x_i = 1$, $A_i = L$ if $x_i = 0$. In this way a Little Schröder path is obtained, see Figure 10 for an example.

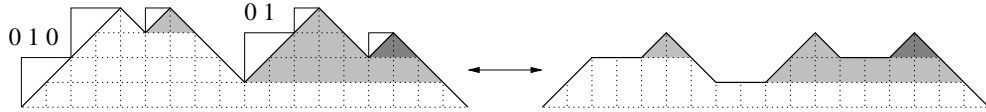


FIGURE 10. The bijection $\mathcal{D}^{\mathfrak{S}}(n) \leftrightarrow \mathcal{LS}(n)$.

This function is easily seen to be invertible: this is based on the fact that any Little Schröder path P may be written in a unique way as $P = U^\ell X_k P_k X_{k-1} P_{k-1} \dots X_2 P_2 D P_1$, where each X_i is D or L , and each P_i is a (possibly empty) Little Schröder path.

Besides, the members of $\mathcal{D}^{\mathfrak{S}}(n)$ correspond to the members of $\mathcal{LS}(n)$.

5. DYCK PATHS COLOURED BY SCHRÖDER PATHS

Finally, we take \mathcal{L}_k to be the set of all Schröder paths of length $2k$. A *Schröder path of length $2n$* is a lattice path from $(0, 0)$ to $(2n, 0)$ with moves $U = (1, 1)$, $D = (1, -1)$, $L = (2, 0)$, which does not pass below the x -axis. The set of all Schröder paths of length $2n$ will be denoted by $\mathcal{S}(n)$. Schröder sequences are enumerated by [8, A006318]. Let $\mathfrak{S} = \{\mathcal{S}(0), \mathcal{S}(1), \mathcal{S}(2), \dots\}$.

Denote by $\mathcal{T}(n)$ the set of all lattice paths from $(0, 0)$ to $(3n, 0)$ with moves $H = (1, 2)$, $G = (2, 1)$, and $D = (1, -1)$, that do not pass below the x -axis. It is known that $\mathcal{T}(n)$ is enumerated by [8, A027307]. Denote $t_n = |\mathcal{T}(n)|$, for all $n \geq 0$.

Observation 5. *There is a bijection between $\mathcal{D}^{\mathfrak{S}}(n)$ and $\mathcal{T}(n)$. The cardinality of both sets is t_n .*

We enumerate $\mathcal{D}^{\mathfrak{S}}(n)$ as follows. The generating function for $\{|\mathcal{S}(n)|\}_{n \geq 0}$ is

$$\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

Substituting this in (1) we get

$$M = \frac{1 - xM - \sqrt{1 - 6xM + x^2M^2}}{2xM},$$

or, after simplifications, $M - 1 = x(M^3 + M^2)$. Denoting $L = M - 1$ and applying Lagrange inversion formula on $L = x(L + 1)^2(L + 2)$, we finally get

$$\begin{aligned} [x^n](L) &= \frac{1}{n} [L^{n-1}]((L + 1)^2(L + 2))^n \\ &= \frac{1}{n} \sum_{i,j \geq 0} \binom{n}{i} \binom{n}{j} \binom{n}{i+j+1} 2^{i+j+1} = \frac{1}{n} \sum_{k=0}^{n-1} \binom{2n}{k} \binom{n}{k+1} 2^{k+1}. \end{aligned}$$

This is the cardinality of $\mathcal{D}^{\mathfrak{S}}(n)$, and it is known to be an expression for t_n , see [8, A027307].

We show a simple direct bijection between $\mathcal{D}^{\mathfrak{S}}(n)$ and $\mathcal{T}(n)$. It can be even said that $\mathcal{T}(n)$ is another representation of $\mathcal{D}^{\mathfrak{S}}(n)$.

Consider a member of $\mathcal{D}^{\mathfrak{S}}(n)$. Replace each k -ascent by the Schröder path which colours it, rotated by 45° and scaled by $1/\sqrt{2}$ (here we formalize what we already did in illustrations, see Figure 1). We obtain a path with steps $U = (1, 1)$, $N = (0, 1)$, $E = (1, 0)$, $D = (1, -1)$.

Replacing $N \rightarrow H$, $U \rightarrow G$, $E \rightarrow D$, $D \rightarrow D$, we obtain a member of $\mathcal{T}(n)$. See Figure 11 for an illustration.

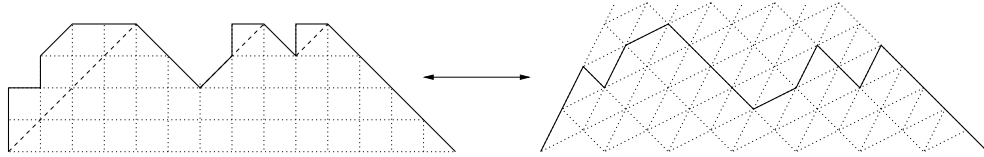


FIGURE 11. The bijection $\mathcal{D}^{\mathfrak{S}}(n) \leftrightarrow \mathcal{T}(n)$.

This correspondence is easily seen to be invertible: in the inverse correspondence, given a member of $\mathcal{T}(n)$, we replace $H \rightarrow N$, $G \rightarrow U$, $D \rightarrow E$ or $D \rightarrow D$ according to the following rule: For each H , define its *match* to be the closest (from right) D such that the number of H 's and of D 's between them is equal; if D is the match of an H then $D \rightarrow E$, otherwise $D \rightarrow D$.

A restriction of this correspondence is that between Dyck paths with ascents coloured by Dyck paths and the members of $\mathcal{T}(n)$ with only moves $H = (1, 2)$ and $D = (1, -1)$.

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