Parabolic Kazhdan-Lusztig polynomials, plethysm and gereralized Hall-Littlewood functions for classical types

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Abstract

We use power sums plethysm operators to introduce H functions which interpolate between the Weyl characters and the Hall-Littlewood functions Q' corresponding to classical Lie groups. The coefficients of these functions on the basis of Weyl characters are parabolic Kazhdan-Lusztig polynomials and thus, are nonnegative. We prove that they can be regarded as quantizations of branching coefficients obtained by restriction to certain Levi subgroups. The H functions associated to linear groups coincide with the polynomials introduced by Lascoux Leclerc and Thibon in [7].

1 Introduction

Given μ a partition with at most n parts, the Hall Littlewood function Q'_{μ} can be defined by

$$Q'_{\mu} = \sum_{\lambda} K_{\lambda,\mu}(q) s_{\lambda}$$

where the sum runs over the partitions of length at most n, $K_{\lambda,\mu}(q)$ is the Lusztig q-analogue of weight multiplicity associated to (λ,μ) and s_{λ} the Schur function indexed by λ , that is the Weyl character of the irreducible finite dimensional GL_n -module $V(\lambda)$. Since $K_{\lambda,\mu}(1)$ is equal to the dimension of the weight space μ in $V(\lambda)$, Q'_{μ} can be regarded as a quantization of the homogeneous function h_{μ} . In [7], Lascoux, Leclerc and Thibon have introduced a new family of symmetric functions H^{ℓ}_{μ} depending on a fixed nonnegative integer ℓ which interpolate between the Schur functions s_{μ} and the Hall-Littlewood functions Q'_{μ} . The polynomials H^{ℓ}_{μ} can be combinatorially described in terms of the spin statistic on certain generalized Young tableaux called ℓ -ribbon tableaux. These Ribbon tableaux naturally appear in the description of the action of the power sum plethysm ψ_{ℓ} on symmetric functions. Recall that for any symmetric function f, $\psi_{\ell}(f)$ is obtained by replacing in f each variable x_i by x_i^{ℓ} . In particular ψ_{ℓ} multiplies the degrees by ℓ . The space of symmetric functions is endowed with an inner product $\langle \cdot, \cdot \rangle$ which makes the basis of Schur functions orthonormal. Then φ_{ℓ} , the adjoint operator of ψ_{ℓ} , divides the degrees by ℓ . It is well known that $\varphi_{\ell}(s_{\mu})$ can be computed from Jacobi-Trudi determinantal identity. Namely we have

$$\varphi_{\ell}(s_{\mu}) = 0 \text{ or } \varphi_{\ell}(s_{\mu}) = \varepsilon(\sigma_0) s_{\mu^{(0)}} \cdots s_{\mu^{(\ell-1)}}$$
 (1)

where $\varepsilon(\sigma_0) = \pm 1$ is the signature of a permutation $\sigma_0 \in S_n$ and $(\mu^{(0)}, ..., \mu^{(\ell-1)})$ a ℓ -tuple of partitions defined by ℓ and μ . By expanding $\varphi_{\ell}(s_{\mu})$ on the basis of Schur functions we obtain then

$$\varphi_{\ell}(s_{\mu}) = \varepsilon(\sigma_0) \sum_{\lambda} c_{\mu^{(0)}, \dots, \mu^{(\ell-1)}}^{\lambda} s_{\lambda}$$
(2)

where $c_{\mu^{(0)},...,\mu^{(\ell-1)}}^{\lambda}$ is the Littlewood-Richardson coefficient giving the multiplicity of $V(\lambda)$ in the tensor product $V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(\ell-1)})$. When $\ell = 1$, one has $\varphi_{\ell}(s_{\ell\mu}) = s_{\mu}$ and when $\ell > n$ one can prove that $\varphi_{\ell}(s_{\ell\mu}) = h_{\mu}$. Thus the functions $h_{\mu}^{(\ell)} = \varepsilon(\sigma_0)\varphi_{\ell}(s_{\ell\mu})$ interpolate between the functions s_{μ} and h_{μ} and have nonnegative coefficients on the basis of Schur functions.

In [7], the authors have interpreted the algebra of symmetric functions as the bosonic Fock space representation of the quantum affine Lie algebra $U_q(\widehat{sl}_n)$. This permits to introduce a natural quantization $\psi_{q,\ell}$ of the power sum plethysm ψ_{ℓ} . Let $\varphi_{q,\ell}$ be the adjoint operator of $\psi_{q,\ell}$ with respect to $\langle \cdot, \cdot \rangle$. The function H^{ℓ}_{μ} is then defined as a simple renormalization of $\varphi_{q,\ell}(s_{\ell\mu})$. This gives an identity of the form

$$H^{\ell}_{\mu} = \sum_{\lambda} c^{\lambda}_{\mu^{(0)},\dots,\mu^{(\ell-1)}}(q) s_{\lambda}$$

where the polynomial $c_{\mu^{(0)},\dots,\mu^{(\ell-1)}}^{\lambda}(q)$ is a q-analogue of $c_{\mu^{(0)},\dots,\mu^{(\ell-1)}}^{\lambda}$.

Lusztig q-analogues $K_{\lambda,\mu}(q)$ are particular affine Kazhdan-Lusztig polynomials. These polynomials arise in affine Hecke algebra theory as the entries of the transition matrix between the basis of generators and a special basis defined by Lusztig. By replacing the affine Hecke algebra \hat{H} by one of its parabolic module $\hat{H}\nu$ (ν being a weight of the affine root system considered), Deodhar has introduced analogues of the Kazhdan-Lusztig polynomials. In [8], it is shown that the family constituted by these parabolic Kazhdan-Lusztig polynomials contains in particular the q-analogues $c_{\mu^{(0)},\dots,\mu^{(\ell-1)}}^{\lambda}(q)$. By a result of Kashiwara and Tanisaki [6], this implies notably that the coefficients of the polynomial $c_{\mu^{(0)},\dots,\mu^{(\ell-1)}}^{\lambda}(q)$ are nonnegative integers.

The aim of the paper is to introduce analogues of the polynomials H^{ℓ}_{μ} for the classical Lie groups $G = SO_{2n+1}, Sp_{2n}$ and SO_{2n} which interpolate between the Weyl characters and the Hall-Littlewood functions associated to G. Write also s_{λ} for the Weyl character of the irreducible G-module $V(\lambda)$ of highest weight λ . We define the plethysm operator φ_{ℓ} and its dual ψ_{ℓ} on the \mathbb{Z} -algebra generated by these Weyl characters. Given a Levi subgroup L of G and γ one of its highest weight, we denote by $[V(\lambda):V_L(\gamma)]$ the multiplicity of the irreducible L-module $V_L(\gamma)$ of highest weight γ in the restriction of $V(\lambda)$ to L. Then, providing that ℓ is odd when $G = Sp_{2n}$ or SO_{2n} , we establish for any Weyl character s_{μ} such that $\varphi_{\ell}(s_{\mu}) \neq 0$, a formula of the type

$$\varphi_{\ell}(s_{\mu}) = \varepsilon(w_0) \sum_{\lambda} [V(\lambda) : V_L \begin{pmatrix} \mu \\ \ell \end{pmatrix}] s_{\lambda}$$
(3)

where $\varepsilon(w_0)$ is the signature of an element $w_0 \in W$ the Weyl group of G, L a Levi subgroup of G and $\binom{\mu}{\ell}$ a dominant weight associated to L. The procedure which yields w_0, L and $\binom{\mu}{\ell}$ from ℓ and μ can be regarded as an analogue of the algorithm computing the ℓ -quotient of a partition which implicitly appears in (1). The identity (2) can also be rewritten as in (3). Indeed, take $L = GL_{r_0} \times \cdots \times GL_{r_{\ell-1}}$ where for any $k = 1, ..., \ell-1, r_k$ is the length of $\mu^{(k)}$. Then $\binom{\mu}{\ell} = (\mu^{(0)}, ..., \mu^{(\ell-1)})$ can be interpreted as a dominant weight for the Levi subgroup L of GL_n and we have the duality $c^{\lambda}_{\mu^{(0)}, ..., \mu^{(\ell-1)}} = [V(\lambda) : V_L\binom{\mu}{\ell}]$. The surprising constraint ℓ odd when $G = Sp_{2n}$ or SO_{2n} appearing in (3) follows from the fact that the procedure giving w_0, L and $\binom{\mu}{\ell}$ mentioned above depends not only on the Lie group considered but also on the parity of the integer ℓ . For $G = SO_{2n+1}$ the coefficients of $\varepsilon(w_0)\varphi_{\ell}(s_{\mu})$ on the basis of Weyl characters are always branching coefficients corresponding to restriction to L. For $G = Sp_{2n}$ or SO_{2n} this is only true when ℓ is odd. Otherwise the coefficients of the expansion of $\varphi_{\ell}(s_{\mu})$ may have opposite signs and their absolute value cannot be identified with branching coefficients. To define the functions H^{ℓ}_{μ} in type B, C or D, we prove the equalities

$$|\langle \psi_{\ell}(s_{\lambda}), s_{\mu} \rangle| = |\langle s_{\lambda}, \varphi(s_{\mu}) \rangle| = P_{\mu+o,\ell\lambda+o}^{-}(1)$$

which show that the coefficients of the expansion of $\psi_{\ell}(s_{\lambda})$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at q = 1. By using (3) this gives, providing ℓ is odd for $G = Sp_{2n}$ or SO_{2n}

$$[V(\lambda): V_L \binom{\mu}{\ell}] = P_{\mu+\rho,\ell\lambda+\rho}^-(1).$$

We then introduce the functions

$$G^{\ell}_{\mu} = \sum_{\lambda} [V(\lambda) : V_L \binom{\mu}{\ell}]_q s_{\lambda}$$

where $[V(\lambda):V_L\binom{\mu}{\ell}]_q = P^-_{\mu+\rho,\ell\lambda+\rho}(q)$. This yields nonnegative q-analogues of the branching coefficients $[V(\lambda):V_L\binom{\mu}{\ell}]$. The functions H^ℓ_μ are then defined by setting $H^\ell_\mu = G^\ell_{\ell\mu}$. We obtain the identities $H^1_\mu = s_\mu$ and $H^\ell_\mu = Q'_\mu$ when ℓ is sufficiently large. Thus the functions H^ℓ_μ interpolate between the Weyl characters and the Hall-Littlewood functions associated to G.

The paper is organized as follows. In Section 2 we recall the necessary background on classical root systems, Weyl characters, Levi subgroups and their corresponding branching coefficients. In Section 3, we define the plethysm operators ψ_{ℓ} and their dual operators φ_{ℓ} . By abuse of notation, we also denote by φ_{ℓ} the linear operator on the group algebra $\mathbb{Z}[\mathbb{Z}^n]$ with basis the formal exponentials (e^{β}) such that

$$\varphi_{\ell}(e^{\beta}) = \begin{cases} e^{\beta/\ell} \text{ if } \beta \in (\ell \mathbb{Z})^n \\ 0 \text{ otherwise} \end{cases} \text{ for any } \beta \in \mathbb{Z}^n.$$

We then show how the determination of $\varphi_{\ell}(s_{\mu})$ can be reduced to the computation of the polynomial

$$\varphi_{\ell}(e^{\mu} \prod_{\alpha \in R_{+}} (1 - e^{\alpha}))$$

where R_+ is the set of positive roots corresponding to the Lie group G. This permits to establish formulas (3) providing ℓ is odd when $G = Sp_{2n}$ or SO_{2n} . For completion we have also included the case $G = GL_n$ and showed why (3) cannot hold when ℓ is even and $G = Sp_{2n}$ or SO_{2n} . To make the paper self contained, we have summarized in Section 4 some necessary results on affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials. Section 5 is devoted to the definition of the polynomials G^{ℓ}_{μ} and H^{ℓ}_{μ} and to their links with the Weyl characters and the Hall-Littlewood functions. Finally we briefly discuss in Section 6 the problem of defining nonnegative q-analogues of tensor product multiplicities when $G \neq GL_n$. We add also a few remarks concerning the exceptional root systems.

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2 Background

2.1 Classical root systems

In the sequel G is one of the complex Lie groups GL_n , Sp_{2n} , SO_{2n+1} or SO_{2n} and \mathfrak{g} its Lie algebra. We follow the convention of [6] to realize G as a subgroup of GL_N and \mathfrak{g} as a subalgebra of \mathfrak{gl}_N where

$$N = \begin{cases} n \text{ when } G = GL_n \\ 2n \text{ when } G = Sp_{2n} \text{ or } SO_{2n} \\ 2n + 1 \text{ when } G = SO_{2n+1} \end{cases}.$$

With this convention the maximal torus T of G and the Cartan subalgebra \mathfrak{h} of \mathfrak{g} coincide respectively with the subgroup and the subalgebra of diagonal matrices of G and \mathfrak{g} . Similarly the Borel subgroup G of G and the Borel subalgebra \mathfrak{h} of \mathfrak{g} coincide respectively with the subgroup and subalgebra of upper triangular matrices of G and \mathfrak{g} .

Let d_N be the linear subspace of \mathfrak{gl}_N consisting of the diagonal matrices. For any $i \in I_n = \{1, ..., n\}$, write ε_i for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_{n-i+1}$ for any diagonal matrix D whose (i,i)-coefficient is δ_i . Then $(\varepsilon_1, ..., \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$ (the real part of \mathfrak{h}^*). Let (\cdot, \cdot) be the corresponding nondegenerate symmetric bilinear form defined on $\mathfrak{h}_{\mathbb{R}}^*$. Write R for the root system associated to R. For any R0 we set R1 we set R2 we set R3 admits the diagonal decomposition R3 admits the diagonal decomposition R3. We take for the set of positive roots:

$$\begin{cases} R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } A_{n-1} \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{cases}$$

Let ρ be the half sum of positive roots. Set $J_n = \{\overline{n} < \cdots < \overline{1} < 1 < \cdots < n\}$ where, for each integer i = 1, ..., n, we have written \overline{i} for the negative integer -i. For any $x \in J_n$ we have $\overline{\overline{x}} = x$ and we set |x| = x if x > 0, $|x| = \overline{x}$ otherwise. Given a subset $U \subset J_n$, we define $|U| = \{|x| \mid x \in U\}$ and $\overline{U} = \{\overline{x} \mid x \in U\}$.

The Weyl group of GL_n is the symmetric group S_n and for $G = SO_{2n+1}$, Sp_{2n} or SO_{2n} , the Weyl group W of the Lie group G is the subgroup of the permutation group of J_n generated by the permutations

$$\begin{cases} s_i = (i, i+1)(\overline{i}, \overline{i+1}), \ i=1, ..., n-1 \text{ and } s_n = (n, \overline{n}) \text{ for the root systems } B_n \text{ and } C_n \\ s_i = (i, i+1)(\overline{i}, \overline{i+1}), \ i=1, ..., n-1 \text{ and } s_n' = (n, \overline{n-1})(n-1, \overline{n}) \text{ for the root system } D_n \end{cases}$$

where for $a \neq b$ (a, b) is the simple transposition which switches a and b. For types B_n and C_n , W is the group of signed permutations. It is the subgroup of the permutation group of J_n containing the permutations w such that $w(\overline{i}) = \overline{w(i)}$. For type D_n , the elements of W verify the additional constraint $\operatorname{card}\{i \in I_n \mid w(i) < 0\} \in 2\mathbb{N}$. We identify the subgroup of W generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, i=1,...,n-1 with the symmetric group S_n . The signature ε of $w \in W$ is defined by $\varepsilon(w) = (-1)^{l(w)}$ where l is the length function corresponding to the above sets of generators. Consider the increasing sequence $K = (\overline{i}_p, ..., \overline{i}_1, i_1, ..., i_p) \subset J_n$. For X = B, D set

$$W_{X,K} = \{ w \in W \text{ of type } X_n \mid w(x) = x \text{ for any } x \notin K \}.$$

Then, $W_{X,K}$ is isomorphic to the Weyl group of type X_p . Let $\varepsilon_{X,K}$ be the corresponding signature.

Lemma 2.1.1 Consider X = B, D and $w \in W_{X,K}$. Then we have $\varepsilon_{X,K}(w) = \varepsilon(w)$.

Proof. Suppose X = B. The generators of the Weyl group $W_{X,K}$ are the $t_k = (i_k, i_{k+1})(\overline{i_k}, \overline{i_{k+1}})$, k = 1, ..., p-1 and $s_n = (\overline{i_p}, i_p)$. One verifies easily that, considered as elements of W, they have an odd length. We proceed similarly when X = D.

The action of $w \in W$ on $\beta = (\beta_1, ..., \beta_n) \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta_1^w, \dots, \beta_n^w) \tag{4}$$

where $\beta_i^w = \beta_{w(i)}$ if $\sigma(i) \in I_n$ and $\beta_i^w = -\beta_{w(\overline{i})}$ otherwise. The dot action of W on $\beta = (\beta_1, ..., \beta_n) \in \mathfrak{h}_{\mathbb{R}}^*$ is defined by

$$w \circ \beta = w \cdot (\beta + \rho) - \rho. \tag{5}$$

The fundamental weights of \mathfrak{g} belong to $\left(\frac{\mathbb{Z}}{2}\right)^n$. More precisely we have $\omega_i = (0^i, 1^i) \in \mathbb{N}^n$ for i < n-1 and also i = n-1 for $\mathfrak{g} \neq \mathfrak{g}_{D_{\infty}}$, $\omega_n^{C_n} = (1^n)$, $\omega_n^{B_n} = \omega_n^{D_n} = (\frac{1}{2}^n)$ and $\omega_{n-1}^{D_n} = (-\frac{1}{2}, \frac{1}{2}^{n-1})$. The weight lattice P of \mathfrak{g} can be considered as the \mathbb{Z} -sublattice of $\left(\frac{\mathbb{Z}}{2}\right)^n$ generated by the ω_i , $i \in I$. For any $\beta = (\beta_1, ..., \beta_n) \in P$, we set $|\beta| = \beta_1 + \cdots + \beta_n$. Write P^+ for the cone of dominant weights of G. With our convention, a partition of length m is a weakly increasing sequence of m nonnegative integers. Denote by \mathcal{P}_n the set of partitions with at most n parts. Each partition $\lambda = (\lambda_1, ..., \lambda_m) \in \mathcal{P}_n$ will be identified with the dominant weight $\sum_{i=1}^m \lambda_i \varepsilon_i$. Then the irreducible finite dimensional polynomial representations of G are parametrized by the partitions of \mathcal{P}_n . For any $\lambda \in \mathcal{P}_n$, denote by $V(\lambda)$ the irreducible finite dimensional representation of G of highest weight λ . We will also need the irreducible rational representations of GL_n . They are indexed by the n-tuples

$$(\gamma^{-}, \gamma^{+}) = (-\gamma_{q}^{-}, ..., -\gamma_{1}^{-}, \gamma_{1}^{+}, \gamma_{2}^{+}, ..., \gamma_{p}^{+})$$

$$(6)$$

where $\gamma^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_p^+)$ and $\gamma^- = (\gamma_1^-, ..., \gamma_q^-)$ are partitions of length p and q such that p+q=n. Write $\widetilde{\mathcal{P}}_n$ for the set of such n-tuples and denote also by $V(\gamma)$ the irreducible rational representation of GL_n of highest weight $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$.

In the sequel, our computations will also make appear root subsystems of the root systems R described above. Suppose that G is of type X_n with $X_n = A_{n-1}, B_n, C_n$ or D_n . Let $I = (i_1, ..., i_r)$ be an increasing sequence of integers belonging to I_n , that is $i_k \in I_n$ for any k = 1, ..., r and $i_1 < \cdots < i_r$. Then

$$R_I = \{ \alpha \in R \cap \bigoplus_{i \in I} \mathbb{Z} \varepsilon_i \}$$

is a root subsystem of R of type X_r . Write R_I^+ for the set of positive roots in R_I . Then we have $R_I^+ = R_I \cap R^+$. The dominant weights associated to R_I have the form $\lambda = (\lambda_1, ..., \lambda_n)$ where $\lambda_i \neq 0$ only if $i \in I$ and $\lambda^{(I)} = (\lambda_{i_1}, ..., \lambda_{i_r}) \in \mathcal{P}_r$. We slightly abuse the notation by identifying λ with $\lambda^{(I)}$. Consider an increasing sequence $X = (x_1, ..., x_r)$ of integers belonging to J_n such that $|x_k| = |x_{k'}|$ if and only if k = k'. For any integer i = 1, ..., n, set $\varepsilon_{\overline{i}} = -\varepsilon_i$. Then

$$R_{A,X} = \{ \pm (\varepsilon_{x_i} - \varepsilon_{x_i}) \mid 1 \le i < j \le r \}$$

is a root subsystem of R of type A_{r-1} . To see this, consider the linear map $\theta_X: \mathbb{Z}^r \to \mathbb{Z}^n$ such that $\theta_X(\varepsilon_i) = \varepsilon_{x_i}$. The map θ is injective and preserves the scalar product in \mathbb{Z}^r and \mathbb{Z}^n . Moreover the root system $\{\pm(\varepsilon_j - \varepsilon_i) \mid 1 \le i < j \le r\} \subset \mathbb{Z}^r$ of type A_r is sent on $R_{A,X}$ by θ_X . The set of positive roots in $R_{A,X}$ is equal to $R_{A,X}^+ = R_{A,X} \cap R^+$. Denote by $s \in \{1, ..., r\}$ the maximal integer such that $x_s < 0$. We associate to X, the increasing sequence of indices $I \subset I_n$ defined by

$$I = (\overline{x}_s, ..., \overline{x}_1, x_{s+1}, ..., x_r). \tag{7}$$

It will be useful to consider the weights corresponding to $R_{A,X}$ as the r-tuples $\beta = (\beta_{x_1}, ..., \beta_{x_r})$ with coordinates indexed by X. The coordinates $(\beta'_1, ..., \beta'_n)$ of β on the initial basis $(\varepsilon_1, ..., \varepsilon_n)$ are such that $\beta'_i = \beta_{x_a}$ if $i = x_a \in X$, $\beta'_i = -\beta_{x_a}$ if $\overline{i} = x_a \in X$ and $\beta'_i = 0$ otherwise. With this convention the dominant weights for $R_{A,X}$ have the form

$$\lambda^{(X)} = (\lambda_{x_1}, ..., \lambda_{x_r}) \in \widetilde{\mathcal{P}}_r. \tag{8}$$

This simply means that we have chosen to expand the weights of $R_{A,X}$ on the basis $\{\varepsilon_x \mid x \in X\}$ rather than on the basis $\{\varepsilon_i \mid i \in I\}$ to preserve the identification of the dominant weights with the nondecreasing r-tuples of integers.

Example 2.1.2 *Take* $G = Sp_{10}$.

• For I = (2, 4, 5) we have

$$R_I^+ = \{ \varepsilon_5 \pm \varepsilon_4, \varepsilon_5 \pm \varepsilon_2, \varepsilon_4 \pm \varepsilon_2, 2\varepsilon_2, 2\varepsilon_4, 2\varepsilon_5 \}$$

which is the set of positive roots of a root system of type C_3 . The weight $\lambda = (1, 2, 2)$ is dominant for G_I . Considered as a weight of Sp_{10} , we have $\lambda = (0, 1, 0, 2, 2)$.

• For $X = (\overline{5}, \overline{2}, 1, 4)$ we have

$$R_{A,X}^+ = \{ \varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_5, \varepsilon_4 + \varepsilon_2, \varepsilon_4 + \varepsilon_5 \}$$

which is the set of positive roots of a root system of type A_3 . The weight $\gamma = (-3, -1, 4, 5)$ is dominant for G_X . Considered as a weight of Sp_{10} , we have $\gamma = (4, 1, 0, 5, 3)$.

2.2 Levi subgroups

Consider $p \ge 1$ an integer. Let $I^{(0)} = (i_1^{(0)}, ..., i_{r_0}^{(0)})$ be an increasing sequence of integers in I_n . For k = 1, ..., p, consider increasing sequences $X^{(k)} = (x_1^{(k)}, ..., x_{r_k}^{(k)}) \subset J_n$ such that $\operatorname{card}(X^{(k)}) = r_k$. Let s_k be maximal in $\{1, ..., r_k\}$ such that $x_{s_k}^{(k)} < 0$. Set

$$I^{(k)} = (\overline{x}_{s_k}^{(k)}, ..., \overline{x}_1^{(k)}, x_{s_k+1}^{(k)}, ..., x_{r_k}^{(k)}) \subset I_n.$$

$$(9)$$

We suppose that the sets $I^{(k)}$, k=0,...,p are pairwise disjoint and verify $\bigcup_{k=0}^{p} I^{(k)} = I_n$. Set $\mathcal{I} = \{I^{(0)}, X^{(1)}, ..., X^{(p)}\}$ and

$$R_{\mathcal{I}} = R_{I^{(0)}} \cup \bigcup_{k=1}^{p} R_{A,X^{(k)}}$$

Then $\mathfrak{g}_{\mathcal{I}} = \mathfrak{h} \oplus \coprod_{\alpha \in R_{\mathcal{I}}} \mathfrak{g}_{\alpha}$ is a Lie subalgebra of \mathfrak{g} . Its corresponding Lie group $G_{\mathcal{I}}$ is a Levi subgroup. More precisely we have

$$G_{\mathcal{I}} \simeq \begin{cases} GL_{r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = GL_n \\ SO_{2r_{0+1}} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n+1} \\ Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = Sp_{2n} \\ SO_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n} \end{cases}.$$

The root system associated to $G_{\mathcal{I}}$ is $R_{\mathcal{I}}$. Denote by $P_{\mathcal{I}}^+$ its cone of dominant weights. The weight lattice of $G_{\mathcal{I}}$ coincides with that of G since the Lie algebras $\mathfrak{g}_{\mathcal{I}}$ and \mathfrak{g} have the same Cartan subalgebra. The elements of $P_{\mathcal{I}}^+$ are the (p+1)-tuples $\lambda=(\lambda^{(0)},\lambda^{(1)},...,\lambda^{(p)})$ where $\lambda^{(0)}=(\lambda_i\mid i\in I^{(0)})$ is a dominant weight of $R_{G,I^{(0)}}$ and for any $k=1,...,p,\ \lambda^{(k)}=(\lambda_i\mid i\in X^{(k)})$ is a dominant weight of $R_{G,X^{(k)}}$. For any $\lambda\in P_{\mathcal{I}}^+$, we denote by $V_{\mathcal{I}}(\lambda)$ the irreducible finite dimensional $G_{\mathcal{I}}$ -module of highest weight λ . Each weight $\beta=(\beta^{(0)},\beta^{(1)},...,\beta^{(p)})\in P_{\mathcal{I}}$ can be considered as a weight $\beta=(\beta'_1,...,\beta'_n)$ of P. With the convention (8) we have then $\beta'_i=\beta_{i_a^{(0)}}$ if $i=i_a^{(0)}\in I^{(0)}$ and for any $k=1,...,p,\ \beta'_i=\beta_{i_a^{(k)}}$ if $i=i_a^{(k)}\in X^{(k)},\ \beta'_i=-\beta_{i_a^{(k)}}$ if $i=i_a^{(k)}\in X^{(k)}$. In the sequel we identify the two expressions

$$\beta = (\beta^{(0)}, \beta^{(1)}, ..., \beta^{(p)}) \text{ and } \beta = (\beta'_1, ..., \beta'_n)$$
 (10)

of the weights of $P_{\mathcal{I}}$.

2.3 Weyl characters and dual bases

We refer the reader to [10] and [12] for a detailed exposition of the results used in this paragraph. We use for a basis of the group algebra $\mathbb{Z}[\mathbb{Z}^n]$, the formal exponentials $(e^{\beta})_{\beta \in \mathbb{Z}^n}$ satisfying the relations $e^{\beta_1}e^{\beta_2} = e^{\beta_1+\beta_2}$. We furthermore introduce n independent indeterminates $x_1, ..., x_n$ in order to identify $\mathbb{Z}[\mathbb{Z}^n]$ with the ring of polynomials $\mathbb{Z}[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}]$ by writing $e^{\beta} = x_1^{\beta_1} \cdot \cdot \cdot \cdot x_n^{\beta_n} = x^{\beta}$ for any $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$. Define the action of the Weyl group W on $\mathbb{Z}[\mathbb{Z}^n]$ by $w \cdot x^{\beta} = x^{w(\beta)}$. The Weyl character s_{β} is defined by

$$s_{\beta} = \frac{a_{\beta+\rho}}{a_{\rho}}$$
 where $a_{\beta} = \sum_{w \in W} (-1)^{l(\sigma)} (w \cdot x^{\beta})$

For any $\beta \in \mathbb{Z}^n$ we have

$$s_{\beta} = \begin{cases} (-1)^{l(w)} s_{\lambda} \text{ if there exists } w \in W \text{ and } \lambda \in \mathcal{P}_n \text{ such that } \lambda = w \circ \beta \\ 0 \text{ otherwise} \end{cases}$$
 (11)

Let A be the \mathbb{Z} -algebra generated by the characters $s_{\lambda}, \lambda \in \mathcal{P}_n$. For any $\beta \in \mathbb{Z}^n$, denote by W_{β} the stabilizer of β under the action of the Weyl group W and W^{β} a set of representatives in W/W_{β} with minimal length. Then the functions

$$m_{\beta} = \sum_{w \in W^{\beta}} w \cdot x^{\beta}$$

belong to A. Moreover $\{m_{\lambda} \mid \lambda \in \mathcal{P}_n\}$ is a basis of A. We have the decomposition

$$s_{\lambda} = \sum_{\mu \in \mathcal{P}_n} K_{\lambda,\mu} m_{\mu} \tag{12}$$

where $K_{\lambda,\mu}$ is equal to the dimension of the weight space μ in the irreducible representation $V(\lambda)$. There exists an inner product $\langle \cdot, \cdot \rangle$ on A which makes the characters s_{λ} orthonormal. We denote by $\{h_{\mu} \mid \mu \in \mathcal{P}_n\}$ the dual basis of $\{m_{\lambda} \mid \lambda \in \mathcal{P}_n\}$ with respect to $\langle \cdot, \cdot \rangle$. The homogeneous functions h_{μ} are given in terms of the Weyl characters by the decomposition

$$h_{\mu} = \sum_{\lambda \in \mathcal{P}_{n}} K_{\lambda,\mu} s_{\lambda}. \tag{13}$$

This decomposition is infinite in general when $G \neq GL_n$. Nevertheless, by embedding A in the ring \widehat{A} of universal characters defined by Koike and Terada [6], it makes sense to consider formal series in the characters $s_{\lambda}, \lambda \in \mathcal{P}_n$. Note that the function h_{μ} is not the character of the representation $V(\mu_1\omega_1) \otimes \cdots \otimes V(\mu_n\omega_1)$ when $G \neq GL_n$, . For any $\beta \in \mathbb{Z}^n$ we define the function h_{β} by

$$h_{\beta} = h_{\mu} \tag{14}$$

where μ is the unique dominant weight contained in the orbit $W \cdot \beta$.

2.4 Jacobi-Trudi identities

Denote by $\mathcal{L}_n = \mathbb{K}[[x^{\beta}]]$ the vector space of formal series in the monomials x^{β} with $\beta \in \mathbb{Z}$. We identify the ring of polynomials $\mathcal{F}_n = \mathbb{K}[x^{\beta}]$ with the sub-space of \mathcal{L}_n containing the finite formal series. The vector space \mathcal{L}_n is not a ring since $\beta \in \mathbb{Z}$. More precisely, the product $F_1 \cdots F_r$ of the formal series

 $F_i = \sum_{\beta \in E_i} x^{\beta^{(i)}} \ i = 1, ..., r$ is defined if and only if, for any $\gamma \in \mathbb{Z}^n$, the number N_{γ} of decompositions $\gamma = \beta^{(1)} + \cdots + \beta^{(r)}$ such that $\beta^{(i)} \in E_i$ is finite and in this case we have

$$F_1 \cdots F_r = \sum_{\gamma \in \mathbb{Z}^n} N_{\gamma} x^{\gamma}.$$

In particular the product $P \cdot F$ with $P \in \mathcal{P}_n$ and $F \in \mathcal{L}_n$ is well defined. Set

$$\nabla = \prod_{\alpha \in R_+} \frac{1}{(1 - x^{\alpha})}$$
 and $\Delta = \prod_{\alpha \in R_+} (1 - x^{\alpha}).$

Then $\Delta \in \mathcal{F}_n$ and $\nabla \in \mathcal{L}_n$. We define two linear maps

$$S: \left\{ \begin{array}{c} \mathcal{L}_n \to \widehat{A} \\ x^{\beta} \mapsto s_{\beta} \end{array} \right. \text{ and } H: \left\{ \begin{array}{c} \mathcal{L}_n \to \widehat{A} \\ x^{\beta} \mapsto h_{\beta} \end{array} \right..$$

From Theorem 2.14 of [12] we obtain

Proposition 2.4.1 For any $\beta \in \mathbb{Z}^n$, $s_{\beta} = \sum_{w \in W} (-1)^{l(w)} h_{\beta + \rho - w \cdot \rho}$

By using the identity

$$\Delta = x^{\rho} \sum_{w \in W} (-1)^{l(w)} x^{-w \cdot \rho} \tag{15}$$

the previous proposition is equivalent to the following identity:

$$S(x^{\beta}) = H(\Delta \times x^{\beta}). \tag{16}$$

Proposition 2.4.2 For any $\beta \in \mathbb{Z}^n$ we have $H(x^{\beta}) = S(\nabla \times x^{\beta})$.

Proof. Denote by χ_{Δ} and χ_{∇} the linear maps defined on \mathcal{L}_n by setting $\chi_{\Delta}(x^{\beta}) = \Delta \times x^{\beta}$ and $\chi_{\nabla}(x^{\beta}) = \nabla \times x^{\beta}$ respectively. By (16) we have $S = H \circ \chi_{\Delta}$. Moreover for any $\beta \in \mathbb{Z}^n$, $\chi_{\Delta} \circ \chi_{\nabla}(x^{\beta}) = x^{\beta}$. This gives $S(\nabla \times x^{\beta}) = S \circ \chi_{\nabla}(x^{\beta}) = H \circ \chi_{\Delta} \circ \chi_{\nabla}(x^{\beta}) = H(x^{\beta})$.

2.5 Branching coefficients for the restriction to Levi subgroups

Consider $\mathcal{I} = \{I_0, X_1, ..., X_p\}$ as in 2.2. The set \mathcal{I} characterizes a Levi subgroup $G_{\mathcal{I}}$ of G. Set

$$\Delta_{\mathcal{I}} = \prod_{\alpha \in R_{\mathcal{I}}^+} (1 - x^{\alpha}) \text{ and } \nabla_{\mathcal{I}} = \prod_{\alpha \in R^+ - R_{\mathcal{I}}^+} \frac{1}{(1 - x^{\alpha})}$$

Then $\Delta_{\mathcal{I}} \in \mathcal{F}_n$ and $\nabla_{\mathcal{I}} \in \mathcal{L}_n$. Note that $\nabla_{\mathcal{I}} = \nabla \times \Delta_{\mathcal{I}}$. As a formal series, $\nabla_{\mathcal{I}}$ can be expanded on the form

$$\nabla_{\mathcal{I}} = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_{\mathcal{I}}(\gamma) x^{\gamma}. \tag{17}$$

Consider $\lambda \in \mathcal{P}_n$ and $\mu = (\mu^{(0)}, ..., \mu^{(p)})$ a dominant weight associated to $G_{\mathcal{I}}$. We denote by $[V(\lambda) : V_{\mathcal{I}}(\mu)]$ the multiplicity of the irreducible representation $V_{\mathcal{I}}(\mu)$ in the restriction of $V(\lambda)$ from G to $G_{\mathcal{I}}$. The proposition below follows from Theorem 8.2.1 in [1]:

Proposition 2.5.1 Consider $\lambda \in \mathcal{P}_n$ and $\mu = (\mu^{(0)}, ..., \mu^{(p)})$ a dominant weight of $P_{\mathcal{I}}^+$. Then

$$[V(\lambda):V_{\mathcal{I}}(\mu)] = \sum_{w \in W} (-1)^{l(w)} \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \mu).$$

Define the linear map

$$\left\{ \begin{array}{c} S_{\mathcal{I}} : \mathcal{L}_n \to \widehat{A} \\ x^{\beta} \mapsto H(\Delta_{\mathcal{I}} \times x^{\beta}) \end{array} \right..$$

For any dominant weight $\mu \in P_{\mathcal{I}}^+$, set

$$S_{\mu,\mathcal{I}} = \mathcal{H}(\Delta_{\mathcal{I}} \times x^{\mu}) = \mathcal{S}_{\mathcal{I}}(x^{\mu}). \tag{18}$$

Proposition 2.5.2 With the above notations we have

$$S_{\mu,\mathcal{I}} = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_{\mathcal{I}}(\mu)] s_{\lambda}. \tag{19}$$

Proof. For any $\beta \in \mathbb{Z}^n$, we have obtained in the proof of Proposition 2.4.2, the identity $H(x^{\beta}) = S \circ \chi_{\nabla}(x^{\beta})$. Denote by $\chi_{\Delta,\mathcal{I}}$ the linear map defined on \mathcal{L}_n by setting $\chi_{\Delta,\mathcal{I}}(x^{\beta}) = \Delta_{\mathcal{I}} \times x^{\beta}$. We obtain $S_{\mathcal{I}}(x^{\beta}) = H(\Delta_{\mathcal{I}} \times x^{\beta}) = S \circ \chi_{\nabla} \circ \chi_{\Delta,\mathcal{I}}(x^{\beta}) = S(\nabla_{\mathcal{I}} \times x^{\beta}) \text{since } \nabla_{\mathcal{I}} = \nabla \times \Delta_{\mathcal{I}}$. Thus by (17) this yields $S_{\mathcal{I}}(x^{\beta}) = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_{\mathcal{I}}(\gamma) s_{\beta+\gamma}$. For any γ , we know by (11) that $s_{\beta+\gamma} = 0$ or there exists $\lambda \in \mathcal{P}_n$, $w \in W$ such that $\lambda = w \circ (\beta + \gamma)$ and $s_{\beta+\gamma} = (-1)^{l(w)} s_{\lambda}$. This permits to write

$$S_{\mathcal{I}}(x^{\beta}) = \sum_{\lambda \in \mathcal{P}_n} \sum_{w \in W} (-1)^{l(w)} \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \beta) s_{\lambda}.$$

When $\beta = \mu$ is a dominant weight of $P_{\mathcal{I}}^+$, we obtain the desired identity by using Proposition 2.5.1.

Remarks:

(i): When $G = G_{\mathcal{I}}$ that is, when $r_0 = n$ and $r_1 = \cdots = r_p = 0$, we have $\mu = \mu^{(0)}$, $\Delta_{\mathcal{I}} = \Delta$ and $H_{\mathcal{I}} = H$. Thus $S_{\mu,\mathcal{I}} = s_{\mu^{(0)}}$. This can be recover by using (19) since in this case $[V(\lambda) : V_{\mathcal{I}}(\mu)] = 0$ except when $\lambda = \mu^{(0)}$.

(ii): When $G_{\mathcal{I}} = H$ the maximal torus of G, that is when n = p + 1 and $r_k = 1$ for any k = 0, ..., p, we have $\mu_i = \mu^{(i-1)}$ for any i = 1, ..., n, $\Delta_{\mathcal{I}} = 1$ and $H_{\mathcal{I}}(x^{\beta}) = h_{\beta}$ for any $\beta \in \mathbb{Z}^n$. Hence $S_{\mu,\mathcal{I}} = h_{\mu}$. In this case $[V(\lambda): V_{\mathcal{I}}(\mu)] = K_{\lambda,\mu}$ for any $\lambda \in \mathcal{P}_n$. Thus (19) reduces to (13).

(iii): By (i) and (ii) the functions $S_{\mu,\mathcal{I}}$ interpolates between the Weyl characters s_{μ} and the homogeneous functions h_{μ} .

(iv): When $G = GL_n$, we have the duality

$$[V(\lambda):V_{\mathcal{I}}(\mu)] = c^{\lambda}_{\mu^{(0)},\dots,\mu^{(p)}}$$

where $c_{\mu^{(0)},...,\mu^{(p)}}^{\lambda}$ is the Littlewood-Richardson coefficient associated to the multiplicity of $V(\lambda)$ in the tensor product $V_{\mu} = V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(p)})$. Thus we can write $S_{\mu,\mathcal{I}} = \sum_{\lambda \in \mathcal{P}_n} c_{\mu^{(0)},...,\mu^{(p)}}^{\lambda} s_{\lambda}$. This means that $S_{\mu,\mathcal{I}}$ is the character of V_{μ} . Such a duality does not exist for $G = Sp_{2n}, SO_{2n+1}$ or SO_{2n} , (but see Section 6).

3 Plethysm on Weyl characters

3.1 The operators Ψ_{ℓ} and φ_{ℓ}

Consider ℓ a positive integer. The power sum plethysm operator Ψ_{ℓ} is defined on A be setting $\Psi_{\ell}(m_{\beta}) = m_{\ell\beta}$ for any $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$ where $\ell\beta = (\ell\beta_1, ..., \ell\beta_n)$. Since $\{m_{\lambda} \mid \lambda \in \mathcal{P}_n\}$ and $\{h_{\lambda} \mid \lambda \in \mathcal{P}_n\}$ are dual bases for the inner product $\langle \cdot, \cdot \rangle$, the adjoint operator φ_{ℓ} of Ψ_{ℓ} verifies

$$\varphi_{\ell}(h_{\beta}) = \begin{cases} h_{\beta/\ell} \text{ if } \beta \in (\ell \mathbb{Z})^n \\ 0 \text{ otherwise} \end{cases}$$
 (20)

where $\beta/\ell = (\beta_1/\ell, ..., \beta_n/\ell)$ when $\beta \in (\ell \mathbb{Z})^n$.

By abuse of notation, we also denote by Ψ_{ℓ} and φ_{ℓ} the linear operators respectively defined on \mathcal{L}_n by setting

$$\Psi_{\ell}(x^{\beta}) = x^{\ell\beta} \text{ and } \varphi_{\ell}(x^{\beta}) = \begin{cases} x^{\beta/\ell} \text{ if } \beta \in (\ell \mathbb{Z})^n \\ 0 \text{ otherwise} \end{cases} \text{ for any } \beta \in \mathbb{Z}^n.$$
 (21)

Remark: Since $\Psi_{\ell}(x^{\beta} \times x^{\beta'}) = \Psi_{\ell}(x^{\beta}) \times \Psi_{\ell}(x^{\beta'})$ for any $\beta, \beta' \in \mathbb{Z}^n$ the map Ψ_{ℓ} is a morphism of algebra. This is not true for φ_{ℓ} . Nevertheless, if $\{i_1, ..., i_r\}$ and $\{j_1, ..., j_s\}$ are disjoint subsets of I_n , $\iota = (\iota_1, ..., \iota_r) \in \mathbb{Z}^r$ and $\gamma = (\gamma_1, ..., \gamma_s) \in \mathbb{Z}^s$ we have

$$\varphi_{\ell}(x_{i_1}^{\iota_1}\cdots x_{i_r}^{\iota_r}\times x_{j_1}^{\gamma_1}\cdots x_{j_r}^{\gamma_r}) = \varphi_{\ell}(x_{i_1}^{\iota_1}\cdots x_{i_r}^{\iota_r})\times \varphi_{\ell}(x_{j_1}^{\gamma_1}\cdots x_{j_r}^{\gamma_r}). \tag{22}$$

For any $\lambda \in \mathcal{P}_n$, $\Psi_{\ell}(s_{\lambda})$ belongs to A thus decomposes on the basis $\{s_{\mu} \mid \mu \in \mathcal{P}_n\}$. Let us write

$$\Psi_{\ell}(s_{\lambda}) = \sum_{\mu \in \mathcal{P}_n} n_{\lambda,\mu} s_{\mu}.$$

Since Ψ_{ℓ} and φ_{ℓ} are dual operators with respect to $\langle \cdot, \cdot \rangle$, we can write $n_{\lambda,\mu} = \langle \Psi_{\ell}(s_{\lambda}), s_{\mu} \rangle = \langle s_{\lambda}, \varphi_{\ell}(s_{\mu}) \rangle$. So we have

$$\varphi_{\ell}(s_{\mu}) = \sum_{\lambda \in \mathcal{P}_n} n_{\lambda,\mu} s_{\lambda}.$$

By (16) and Proposition 2.4.1, we obtain the identity

$$s_{\mu} = \sum_{w \in W} (-1)^{l(w)} h_{\mu+\rho-w \cdot \rho} = \mathcal{H}(\Delta \times x^{\mu}).$$

Thus from (20) and (21) we derive $\varphi_{\ell}(s_{\mu}) = H(\varphi_{\ell}(\Delta \times x^{\mu}))$. Set $P_{\mu} = \Delta \times x^{\mu}$. From the previous arguments the coefficients $n_{\lambda,\mu}$ are determined by the computation of $\varphi_{\ell}(P_{\mu})$.

3.2 Computation of $\varphi_{\ell}(P_{\mu})$

For any $i \in \{\overline{n}, ..., \overline{1}\}$ we set $x_i = \frac{1}{x_i}$. This permits to consider also variables indexed by negative integers. Given $X = (i_1, ..., i_r)$ an increasing sequence contained in J_n and $\beta = (\beta_1, ..., \beta_r) \in \mathbb{Z}^r$, we set $x_X^{\beta} = x_{i_1}^{\beta_1} \cdots x_{i_r}^{\beta_r}$. We also denote by S_X the group of permutations of the set X. Each $\sigma \in S_X$ determinates a unique permutation σ^* of the set $\{1, ..., r\}$ defined by

$$\sigma(i_p) = i_{\sigma^*(p)} \text{ for any } p = 1, \dots r.$$
(23)

In the sequel, we identify for short σ and σ^* . Similarly, given $Z = (\overline{u}_r, ..., \overline{u}_1, u_1, ..., u_r)$ an increasing sequence such that $\{u_1, ..., u_r\} \subset I_n$, each signed permutation w defined on Z will be identified with the signed permutation w^* defined on J_r by $w(u_p) = u_{w^*(p)}$ for any p = 1, ... r.

Set $\rho_n = (1, 2, ..., n)$. For any $w \in W$ we have $w \cdot \rho_n = (w(1), ..., w(n))$. This permits to write

$$\sum_{w \in W} (-1)^{l(w)} x^{-w \cdot \rho_n} = \sum_{w \in W} (-1)^{l(w)} x_1^{-w(1)} \cdots x_n^{-w(n)}.$$
(24)

3.2.1 For $G = GL_n$

Set $\kappa_n = (1, ..., 1) \in \mathbb{Z}^n$. Since $\sigma(\kappa_n) = \kappa_n$ for any $\sigma \in S_n$, one can replace ρ by $\rho_n = (1, 2, ..., n)$ in (15). By using (24) we can write

$$P_{\mu} = x_1^{(\mu_1 + 1)} \cdots x_n^{(\mu_n + n)} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)}$$

where $\mu = (\mu_1, ..., \mu_n)$. For any $k \in \{0, ..., \ell - 1\}$ consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \operatorname{mod} \ell) \text{ and } J^{(k)} = (i \in I_n \mid i \equiv k \operatorname{mod} \ell).$$
 (25)

Set $r_k = \text{card}(I^{(k)})$ and write $I^{(k)} = (i_1^{(k)}, ..., i_{r_k}^{(k)})$. Then

$$\widehat{\mu}^{(k)} = \left(\frac{\mu_i + i + \ell - k}{\ell} \mid i \in I^{(k)}\right) \in \mathbb{Z}^{r_k}$$

We derive

$$P_{\mu} = x_{I^{(0)}}^{\ell \widehat{\mu}^{(0)}} x_{I^{(1)}}^{\ell \widehat{\mu}^{(1)}} \cdots x_{I^{(\ell-1)}}^{\ell \widehat{\mu}^{(\ell-1)}} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)} \times \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-(\ell-k)}.$$

This gives

$$\varphi_{\ell}(P_{\mu}) = x_{I(0)}^{\widehat{\mu}^{(0)}} x_{I(1)}^{\widehat{\mu}^{(1)}} \cdots x_{I(\ell-1)}^{\widehat{\mu}^{(\ell-1)}} \sum_{\sigma \in S_{r}} (-1)^{l(\sigma)} \varphi_{\ell} \left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\sigma(i_{a}^{(k)}) - (\ell-k)} \right). \tag{26}$$

The contribution of a fixed permutation $\sigma \in S_n$ in the above sum is nonzero if and only if for any $k = 0, ..., \ell - 1$

$$i \in I^{(k)} \Longrightarrow \sigma(i) \equiv k \mod \ell.$$

Thus we must have $\sigma(I^{(k)}) \subset J^{(k)}$ for any $k = 0, ..., \ell - 1$. Since σ is a bijection, $I^{(k)} \cap I^{(k')} = J^{(k)} \cap J^{(k')} = \emptyset$ if $k \neq k'$ and $\bigcup_{0 \leq k \leq \ell - 1} I^{(k)} = \bigcup_{0 \leq k \leq \ell - 1} J^{(k)} = I_n$, the restriction of σ on I_k is a bijection from $I^{(k)}$ to $J^{(k)}$. In particular $\operatorname{card}(J^{(k)}) = \operatorname{card}(I^{(k)}) = r_k$. This means that we have the equivalencies

$$\varphi_{\ell} \left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)}) - (\ell - k)} \right) \neq 0 \iff \sigma(I^{(k)}) = J^{(k)} \text{ for any } k = 0, ..., \ell - 1.$$
 (27)

Write $J^{(k)}=(k,k+\ell,...,k+(r_k-1)\ell)$. Denote by $\sigma_0\in S_n$ the permutation verifying

$$\sigma_0(i_a^{(k)}) = k + (a-1)\ell \tag{28}$$

for any $k=0,...,\ell-1$ and any $a=1,...,r_k$. Let $S_{I^{(k)}}$ be the permutation group of the set $I^{(k)}$. The permutations σ which verify the right hand side of (27) can be written $\sigma=\sigma_0\tau$ where $\tau=(\tau^{(0)},\cdot\cdot\cdot,\tau^{(\ell-1)})$ belongs to the direct product $S_{I^{(0)}}\times\cdots\times S_{I^{(\ell-1)}}$. We have then $(-1)^{l(\sigma)}=(-1)^{l(\sigma_0)}(-1)^{l(\tau^{(0)})}\times\cdots\times (-1)^{l(\tau^{(p)})}$

For any $k \in \{0, ..., \ell - 1\}$ set

$$P_k = \sum_{\tau^{(k)} \in S_{I^{(k)}}} (-1)^{l(\tau)} \varphi_{\ell} \left(\prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma_0 \tau^{(k)} (i_a^{(k)}) - (\ell - k)} \right).$$

From (22) and (26) we derive

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(\sigma_0)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\widehat{\mu}^{(k)}} P_k.$$

Since $\sigma_0(i_a^{(k)}) = k + (a-1)\ell$, we can write by (23) $\sigma_0 \tau^{(k)}(i_a^{(k)}) = k + (\tau^{(k)}(a) - 1)\ell$. Thus we obtain

$$P_k = \sum_{\tau^{(k)} \in S_{I^{(k)}}} (-1)^{l(\tau)} x_{i_1^{(k)}}^{-\tau^{(k)}(1)} \cdots x_{i_{r_k}^{(k)}}^{-\tau^{(k)}(r_k)} = x_{I^{(k)}}^{-\rho_{r_k}} \Delta_{I^{(k)}}$$

where $\rho_{r_k} = (1, 2, ..., r_k)$ and $\Delta_{I^{(k)}} = \prod_{i < j} (1 - x_j/x_i)$. Finally, this gives

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(\sigma_0)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\widehat{\mu}^{(k)} - \rho_{r_k}} \Delta_{I^{(k)}} = (-1)^{l(\sigma_0)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu^{(k)}} \Delta_{I^{(k)}}$$

where for any $k = 0, ..., \ell - 1$,

$$\mu^{(k)} = \left(\frac{\mu_i + i + \ell - k}{\ell} \mid i \in I^{(k)}\right) - (1, 2, ..., r_k) \in \mathbb{Z}^{r_k}.$$
 (29)

Theorem 3.2.1 Consider a partition μ of length n and ℓ a positive integer. For any $k = 0, ..., \ell - 1$ define the sets $I^{(k)}$ and $J^{(k)}$ as in (25).

- If there exists $k \in \{0, ..., \ell 1\}$ such that $\operatorname{card}(I^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, for any $k = 0, ..., \ell 1$, set $r_k = \operatorname{card}(I^{(k)}) = \operatorname{card}(J^{(k)})$ and define σ_0 as in (28). Then each r_k -tuple defined by (29) is a partition and we have

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(\sigma_0)} S_{(\mu),\mathcal{I}} = (-1)^{l(\sigma_0)} \operatorname{char}(V_{\mu})$$

where $\mathcal{I} = \{I^{(0)}, ..., I^{(\ell-1)}\}, \ \binom{\mu}{\ell} = (\mu^{(0)}, ..., \mu^{(\ell-1)})$ and $\operatorname{char}(V_{\mu})$ is the character of the GL_n module $V_{\mu} = V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(\ell-1)})$.

Proof. One verifies easily from (29) that each $\mu^{(k)}$ is a partition. By the previous computation, we obtain

$$\varphi_{\ell}(\Delta \times x^{\mu}) = (-1)^{l(\sigma_0)} \Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}$$

(with the notation of 2.5). By definition of φ_{ℓ} we have also

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(\sigma_0)} \mathbf{H} \circ \varphi_{\ell}(\Delta \times x^{\mu}) = (-1)^{l(\sigma_0)} \mathbf{H}(\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}) = (-1)^{l(\sigma_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}$$

where the last equality follows from (18).

Remark: The Levi subgroup $G_{\mathcal{I}}$ appearing in Theorem 3.2.1 is characterized by $\mathcal{I} = \{I^{(0}, ..., I^{(\ell-1)}\}$. This means that for type A, we have $X^{(k)} = I^{(k)}$ for any k > 0 with the notation of 2.2, that is the sets $X^{(k)}$ contain only positive indices.

Example 3.2.2

Consider $\mu = (1, 2, 3, 4, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14)$. Thus $I^{(0)} = \{3, 5\}, I^{(1)} = \{2, 6, 7\}, I^{(2)} = \{1, 4, 8\}$ and $J^{(0)} = \{3, 6\}, J^{(1)} = \{1, 4, 7\}, J^{(2)} = \{2, 5, 8\}$. Then $\mu^{(0)} = (1, 1), \mu^{(1)} = (1, 2, 2)$ and $\mu^{(2)} = (0, 1, 2)$.

3.2.2 For $G = Sp_{2n}$

We have $\rho = \rho_n = (1, 2, ..., n)$. By using (24) we deduce the identity:

$$P_{\mu} = x_1^{(\mu_1 + 1)} \cdots x_n^{(\mu_n + n)} \sum_{w \in W} (-1)^{l(w)} x_1^{-w(1)} \cdots x_n^{-w(n)}$$
(30)

where W is the group of signed permutations defined on $J_n = \{\overline{n}, ..., \overline{1}, 1, ..., n\}$, that is the subgroup of permutations $w \in S_{J_n}$ verifying $w(\overline{x}) = \overline{w(x)}$ for any $x \in J_n$. Given $k \in \{0, ..., \ell - 1\}$ consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \operatorname{mod} \ell) \text{ and } J^{(k)} = (x \in J_n \mid x \equiv k \operatorname{mod} \ell).$$
(31)

Set $p = \frac{\ell}{2}$ if ℓ is even and $p = \frac{\ell-1}{2}$ otherwise.

The odd case $\ell = 2p-1$ Set $r_0 = \operatorname{card}(I^{(0)})$ and for any k = 1, ..., p-1, $s_k = \operatorname{card}(I_k)$, $r_k = \operatorname{card}(I_k) + \operatorname{card}(I_{\ell-k})$. Write $X^{(k)}, k = 1, ..., p$ for the increasing reordering of $\overline{I}_k \cup I_{\ell-k}$. Set $I^{(0)} = (i_1^{(0)}, ..., i_{r_0}^{(0)})$ and for k > 0

$$X^{(k)} = (i_1^{(k)}, ..., i_{r_k}^{(k)}). (32)$$

This means that $I^{(k)} = (\overline{i}_{s_k}^{(k)}, ..., \overline{i}_1^{(k)})$ and $I^{(\ell-k)} = (i_{s_{k+1}}^{(k)}, ..., i_{r_k}^{(k)})$. To simplify the computation, we are going to use the indices and the variables $x_i, i \in X^{(k)}$ rather than the variables $x_i, i \in I^{(k)} \cup I^{(\ell-k)}$ when $k \in \{1, ..., p-1\}$.

Consider

$$\widehat{\mu}^{(0)} = \left(\frac{\mu_i + i}{\ell} \mid i \in I^{(0)}\right) \in \mathbb{Z}^{r_0} \text{ and for } k > 0, \ \widehat{\mu}^{(k)} = \left(\text{sign}(i) \frac{\mu_{|i|} + |i| + \text{sign}(i)k}{\ell} \mid i \in X^{(k)}\right) \in \mathbb{Z}^{r_k}$$

where for any $i \in J_n$, $\operatorname{sign}(i) = 1$ if i > 0 and -1 otherwise. For any $i \in I$, we have $x_i^{-w(i)} = x_{\overline{i}}^{-w(\overline{i})}$. Thus

$$\prod_{i \in X^{(k)}} x_i^{-w(i)} = \prod_{i \in I^{(k)}} x_i^{-w(i)} \prod_{i \in I^{(\ell-k)}} x_i^{-w(i)} \text{ and } x_1^{(\mu_1+1)} \cdot \cdot \cdot \cdot x_n^{(\mu_n+n)} = x_{I^{(0)}}^{\ell \widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell \widehat{\mu}^{(k)}} \prod_{i \in X^{(k)}} x_i^{-k}$$

by definition of the $\widehat{\mu}^{(k)}$'s. Then (30) can be rewritten

$$P_{\mu} = x_{I^{(0)}}^{\ell \widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell \widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)} \times \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-k}.$$

This gives

$$\varphi_{\ell}(P_{\mu}) = x_{I^{(0)}}^{\widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \varphi_{\ell} \left(\prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k} \right)$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if and only if

$$\begin{cases} i \in I^{(0)} \Longrightarrow w(i) \equiv 0 \mod \ell \\ i \in X^{(k)} \Longrightarrow w(i) \equiv -k \mod \ell \text{ for any } k = 1, ..., p - 1 \end{cases}$$
(33)

Thus we must have $w(\overline{I}^{(0)} \cup I^{(0)}) \subset J^{(0)}$ and for any k = 1, ..., p - 1, $w(X^{(k)}) \subset J^{(\ell-k)}$. Recall that $\overline{J}^{(0)} = J^{(0)}$ and $\overline{J}^{(\ell-k)} = J^{(k)}$ for k = 1, ..., p - 1. Moreover

$$I^{(0)} \cup \overline{I}^{(0)} \bigcup_{k=1}^{p-1} X^{(k)} \cup \overline{X}^{(k)} = J_n \text{ and } J^{(0)} \cup \bigcup_{k=1}^{p-1} J^{(k)} \cup J^{(\ell-k)} = J_n.$$

Since the sets appearing in the left hand side of these two equalities are pairwise disjoint, we must have $w(\overline{I}^{(0)} \cup I^{(0)}) = J^{(0)}$, and for k = 1, ..., p-1, $w(X^{(k)}) = J^{(\ell-k)}$. In particular $\operatorname{card}(J^{(0)}) = 2\operatorname{card}(I^{(0)}) = 2r_0$ and $\operatorname{card}(J^{(\ell-k)}) = \operatorname{card}(X^{(k)}) = r_k$ for any k = 1, ..., p-1. We have the equivalencies

$$\varphi_{\ell}\left(\prod_{i\in I^{(0)}} x_i^{-w(i)} \prod_{k=1}^{p-1} \prod_{i\in X^{(k)}} x_i^{-w(i)+k}\right) \neq 0 \iff \begin{cases} \text{ (i)} : w(I^{(0)} \cup \overline{I}^{(0)}) = J^{(0)} \\ \text{ (ii)} : w(X^{(k)}) = J^{(\ell-k)} \text{ for any } k = 1, ..., p-1 \end{cases}$$
(34)

Note that condition (ii) can be rewritten: $w(\overline{X}^{(k)}) = J^{(k)}$ for any k = 1, ..., p - 1. We can set $J^{(0)} = (-r_0\ell, ..., r_0\ell)$ and for k = 1, ..., p - 1,

$$J^{(\ell-k)} = (-k - \alpha_k \ell, ..., -k + \beta_k \ell), J^{(k)} = (k - \beta_k \ell, ..., k + \alpha_k \ell)$$

with $\alpha_k + \beta_k + 1 = r_k$.

Consider $w_0 \in W$ defined by

$$w_0(i_a^{(0)}) = a\ell \text{ for } a \in \{1, ..., r_0\}$$

$$w_0(i_a^{(k)}) = -k - \alpha_k \ell + (a-1)\ell \text{ for any } k = 1, ..., p-1.$$
(35)

Denote by \mathcal{W} the set of signed permutations w which verify (i) and (ii) in (34). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$ where $v = (v^{(0)}, \tau^{(1)}, ..., \tau^{(p-1)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$. Here $W_{I^{(0)}}$ is the group of signed permutations defined on $\overline{I}^{(0)} \cup I^{(0)}$ and for k = 1, ..., p - 1, $S_{X^{(k)}}$ is the group of signed permutations $\tau^{(k)}$ defined on $\overline{X}^{(k)} \cup X^{(k)}$ and verifying $\tau^{(k)}(X^{(k)}) = X^{(k)}$. Indeed if $\tau^{(k)}(x) \in \overline{X}^{(k)}$ and $x \in X^{(k)}$, we would have $w(x) \in J^{(k)}$ and $x \in X^{(k)}$ which contradicts (ii). This means that $S_{X^{(k)}}$ is in fact isomorphic to the symmetric group S_{r_k} . Since the sets $I^{(0)}$ and $X^{(k)}, k = 1, ..., p - 1$ are increasing subsequences of J_n , we have by Lemma 2.1.1 $(-1)^{l(w)} = (-1)^{l(w_0)}(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p-1)})}$. Set

$$\begin{split} P_0 &= \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \varphi_{\ell} \left(\prod_{i \in I^{(0)}} x_i^{-w_0 v^{(0)}(i)} \right) \text{ and } \\ P_k &= \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_{\ell} \left(\prod_{i \in X^{(k)}} x_i^{-w_0 \tau^{(k)}(i) - k} \right), k \in \{1, ..., p - 1\}. \end{split}$$

We obtain

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}} P_0 \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_k.$$

From (23) and (35), we have $w_0 v^{(0)}(i_a^{(0)}) = v^{(0)}(a)\ell$ for any $a = 1, ..., r_0$ and

$$w_0 \tau^{(k)}(i_a^{(k)}) = -k - \alpha_k \ell + (\tau^{(k)}(a) - 1)\ell$$
 for any $a = 1, ..., r_k$.

This yields

$$P_0 = \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-v^{(0)}(a)} = x_{I^{(0)}}^{-\rho_{r_0}} \Delta_{I^{(0)}} \text{ and}$$

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a) + (\alpha_k + 1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}}$$

where for any $k = 1, ..., p - 1, \eta_{r_k} = -\rho_{r_k} + (\alpha_k + 1, ..., \alpha_k + 1) \in \mathbb{Z}^{r_k}$,

$$\Delta_{I^{(0)}} = \prod_{i < j \ i, j \in I^{(0)}} (1 - \frac{x_j}{x_i}) \prod_{r \le s \ r, s \in I^{(0)}} (1 - x_r x_s)$$
 and
$$\Delta_{X^{(k)}} = \prod_{i < j \ i, j \in X^{(k)}} (1 - \frac{x_j}{x_i}) \text{ for any } k = 1, ..., p - 1.$$

Finally, this gives

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\widehat{\mu}^{(0)} - \rho_{r_0}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(k)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{I^{(0)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(k)}} \Delta_{I^{(0)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(k)}}$$

where

$$\mu^{(0)} = \left(\frac{\mu_i + i}{\ell} \mid i \in I^{(0)}\right) - (1, ..., r_0) \in \mathbb{Z}^{r_0}$$
(36)

and for any k = 1, ..., p - 1

$$\widehat{\mu}^{(k)} = \left(\text{sign}(i) \frac{\mu_{|i|} + |i| + \text{sign}(i)k}{\ell} \mid i \in X^{(k)}\right) - (1, ..., r_k) + (\alpha_k + 1, ..., \alpha_k + 1) \in \mathbb{Z}^{r_k}.$$
(37)

Recall that the weights corresponding to the Levi subgroup $G_{\mathcal{I}}$ are written following the convention (8).

Theorem 3.2.3 Consider a partition μ of length n and $\ell = 2p-1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (31). For any k = 1, ..., p-1 define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (32).

- If $\operatorname{card}(I^{(0)}) \neq \frac{1}{2}\operatorname{card}(J^{(0)})$ or if there exists $k \in \{1, ..., p-1\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, set $r_0 = \operatorname{card}(I^{(0)})$ and for any $k = 1, ..., p-1, r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ be as in (35). Consider $\binom{\mu}{\ell} = (\mu^{(0)}, \mu^{(1)}, ..., \mu^{(p-1)})$ where the $\mu^{(k)}$'s are defined by (36) and (37). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{I^{(0)}, X^{(1)}, ..., X^{(p-1)}\}$ and we have

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Proof. The proof is essentially the same as in Theorem 3.2.1. We obtain

$$\varphi_{\ell}(\Delta \times x^{\mu}) = (-1)^{l(w_0)} \Delta_{\mathcal{I}} \times x^{{\mu \choose \ell}}$$

where in the right hand side of the preceding equality $\binom{\mu}{\ell}$ is expressed on the basis $\{\varepsilon_i \mid i \in I_n\}$ (see (10)). This permits to write as in the case $G = GL_n$

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} \mathbf{H} \circ \varphi_{\ell}(\Delta \times x^{\mu}) = (-1)^{l(w_0)} \mathbf{H}(\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}) = (-1)^{l(w_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.4

Consider $\mu = (1, 2, 3, 4, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14)$. Thus $I^{(0)} = \{3, 5\}, X^{(1)} = \{\overline{7}, \overline{6}, \overline{2}, 1, 4, 8\}$ and $J^{(0)} = \{\overline{6}, \overline{3}, 3, 6\}, J^{(1)} = \{\overline{8}, \overline{5}, \overline{2}, 1, 4, 7\}, J^{(2)} = \{\overline{7}, \overline{4}, \overline{1}, 2, 5, 8\}$. In particular $\alpha_1 = 2$. Then $\mu^{(0)} = (1, 1)$ and $\mu^{(1)} = (1, 1)$

$$\left(-\frac{13-1}{3}-1+3, -\frac{10-1}{3}-2+3, -\frac{4-1}{3}-3+3, \frac{2+1}{3}-4+3, \frac{8+1}{3}-5+3, \frac{14+1}{3}-6+3\right) = (-2, -2, -1, 0, 1, 2)$$

with the convention (8).

The even case $\ell = 2p$ With the same notation as in the odd case, (30) can be rewritten

$$P_{\mu} = x_{I^{(0)}}^{\ell \widehat{\mu}^{(0)}} x_{I^{(p)}}^{\ell \widehat{\mu}^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell \widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{i \in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k}.$$

where

$$\widehat{\mu}^{(p)} = \left(\frac{\mu_i + i + p}{\ell} \mid i \in I^{(p)}\right).$$

This gives

$$\varphi_{\ell}(P_{\mu}) = x_{I^{(0)}}^{\widehat{\mu}^{(0)}} x_{I^{(p)}}^{\widehat{\mu}^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(p)}} \times \sum_{w \in W} (-1)^{l(w)} \varphi_{\ell} \left(\prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{i \in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k} \right)$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if conditions (34) are verified and

$$i \in I^{(p)} \Longrightarrow w(i) \in J^{(p)}.$$

Since $p \equiv -p \mod \ell$ we have $J^{(p)} = \overline{J}^{(p)} = \{-p - \alpha_p \ell, ..., -p, p, ..., p + \alpha_p \ell\}$. This implies that $w(I^{(p)} \cup \overline{I}^{(p)}) = J^{(p)}$ and thus $\operatorname{card}(I^{(p)}) = \frac{1}{2}\operatorname{card}(J^{(p)})$. We then define w_0 by requiring (35) and $w_0(i_a^{(p)}) = p + (a-1)\ell$ for $a \in \{1, ..., r_p\}$. By using similar arguments as in the odd case, we obtain that w can be written $w = w_0 v$ where $\tau = (v^{(0)}, \tau^{(1)}, ..., \tau^{(p-1)}, v^{(p)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(\ell-1)}} \times W_{I^{(p)}}$ where $W_{I^{(p)}}$ is the group of signed permutations defined on $\overline{I^{(p)}} \cup I^{(p)}$. Note that $W_{I^{(p)}}$ is a Weyl group of type B_{r_p} . By Lemma 2.1.1 we have also $(-1)^{l(w)} = (-1)^{l(w_0)}(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p)})} \times (-1)^{l(v^{(p)})}$. We obtain

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}} P_0 \times x_{I^{(p)}}^{\widehat{\mu}^{(p)}} P_p \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_k \text{ where } P_p = \sum_{v^{(p)} \in W_{I^{(p)}}} (-1)^{l(v^{(p)})} \varphi_{\ell} \left(\prod_{i \in I^{(p)}} x_i^{-w_0 v^{(p)}(i) - p} \right).$$

The functions $P_k, k = 0, ..., p-1$ can be computed as in the odd case. For P_p , observe that each $v^{(p)} \in W_{I^{(p)}}$ can be written $v^{(p)} = \zeta \sigma$ according to the decomposition of $W_{I^{(p)}}$ as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{r_p} \propto S_{I^{(p)}}$. We have then for any $a = 1, ..., r_p, w_0 v^{(p)}(i_a^{(p)}) = \xi(a)(p + (\sigma(a) - 1)\ell)$. This yields

$$P_p = \sum_{v^{(p)} \in W_{I(p)}} (-1)^{l(v^{(p)})} \varphi_\ell \left(\prod_{a=1}^{r_p} x_i^{-\xi(a)(p+(\sigma(a)-1)\ell)-p} \right) = \sum_{v^{(p)} \in W_{I(p)}} (-1)^{l(v^{(p)})} \prod_{a=1}^{r_p} x_{i_a}^{-\frac{1-\xi(a)}{2}-\xi\sigma(a)}.$$

Thus

$$P_p = \prod_{i \in I^{(p)}} x_i^{-1/2} \sum_{v^{(p)} \in W_{I^{(p)}}} \left(-1 \right)^{l(v^{(p)})} \left(v^{(p)} \cdot \prod_{a=1}^{r_p} x_{i_a^{(p)}}^{-(a-\frac{1}{2})} \right) = x_{I^{(p)}}^{-\rho_{r_p}} \Delta_{I^{(p)}, B_{r_p}}$$

where

$$\Delta_{I^{(p)},B_{r_p}} = \prod_{i < j \ i,j \in I^{(p)}} (1 - \frac{x_j}{x_i}) \prod_{r < s \ r,s \in I^{(p)}} (1 - x_r x_s) \prod_{i \in I^{(p)}} (1 - x_i).$$

Indeed the half sum of positive roots is equal to $(\frac{1}{2},...,r_p-\frac{1}{2})$ in type B_{r_p} . This means that when ℓ is even

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} \times x_{I^{(p)}}^{\mu^{(p)}} \Delta_{I^{(p)}, B_{r_p}}$$

where $\mu^{(p)} = \widehat{\mu}^{(p)} - (1, ..., r_p)$. In particular the computation of $\varphi_{\ell}(P_{\mu})$ makes appear positive roots corresponding to a root system of type B_{r_p} . These roots do not belong to the root lattice associated to Sp_{2n} . Hence, there cannot exist an analogue of Theorem 3.2.3 when ℓ is even. With the previous notation, we only obtain:

Proposition 3.2.5 Suppose $G = SP_{2n}$ and $\ell = 2p$.

- If $\operatorname{card}(I^{(0)}) \neq \frac{1}{2}\operatorname{card}(J^{(0)})$, $\operatorname{card}(I^{(p)}) \neq \frac{1}{2}\operatorname{card}(J^{(p)})$ or there exists $k \in \{1, ..., p-1\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, the coefficients appearing in the decomposition of $\varphi_{\ell}(s_{\mu})$ on the basis of Weyl characters cannot be interpreted as branching coefficients and have signs alternatively positive and negative.

3.2.3 For $G = SO_{2n}$

As for $G = Sp_{2n}$, the coefficients appearing in the decomposition of $\varphi_{\ell}(s_{\mu})$ with $\ell = 2p$ on the basis of Weyl characters cannot be interpreted as branching coefficients. Note that there is an additional difficulty in this case. Indeed, $\varphi_{\ell}(P_{\mu})$ cannot be factorized as a product of polynomials $(1 - x^{\beta})$ where $\beta \in \mathbb{Z}^n$. For example, we have for SO_4

$$\varphi_2(P_{(0,0)}) = \varphi_2\left(\left(1 - \frac{x_2}{x_1}\right)\left(1 - x_1x_2\right)\right) = 1 + x_2.$$

This is due to the incompatibility between the signatures defined on the Weyl groups of types B and D when they are realized as subgroups of the permutation group S_{J_n} .

So we will suppose $\ell=2p-1$ in this paragraph. Recall that the elements of W are the signed permutations w defined on $J_n=\{\overline{n},...,\overline{1},1,...,n\}$ such that $\operatorname{card}(\{i\in I_n\mid w(i)<0\})$ is even. Set $K_n=\{\overline{n-1},...,\overline{1},0,1,...,n-1\}$. Each $w\in W$ can be written $w=\zeta\sigma$ according to the decomposition of W as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{n-1}\propto S_n$. For any $x\in J_n$, we have then $\xi(x)=1$ if w(x)>0 and $\xi(x)=-1$ otherwise. Given $w\in W$ we define $\widehat{w}:J_n\to K_n$ such that $\widehat{w}(x)=w(x)-\xi(x)$ for any $x\in J_n$. Then $\widehat{w}(\overline{x})=\overline{\widehat{w}(x)}$.

For type D_n , we have $\rho = \rho'_n = (0, 1, ..., n - 1) = \rho_n - (1, ..., 1)$. Hence

$$w \cdot \rho'_n = w \cdot \rho_n - (\xi(1), ..., \xi(n)) = (\widehat{w}(1), ..., \widehat{w}(n)) = \widehat{w} \cdot \rho_n.$$

Then we obtain

$$P_{\mu} = x_1^{(\mu_1 + 0)} \cdots x_n^{(\mu_n + n - 1)} \sum_{w \in W} (-1)^{l(w)} x_1^{-\widehat{w}(1)} \cdots x_n^{-\widehat{w}(n)}.$$

For any $k = 0, ..., \ell - 1$ set

$$I^{(k)} = (i \in I_n \mid \mu_i + i - 1 \equiv k \mod \ell) \text{ and } J^{(k)} = (x \in K_n \mid x \equiv k \mod \ell).$$
 (38)

We then proceed essentially as in 3.2.2 by using \widehat{w} instead of w and $\rho'_n = (0, 1, ..., n-1)$ instead of $\rho_n = (1, ..., n)$. We only sketch below the main steps of the computation. Set $r_0 = \operatorname{card}(I^{(0)})$ and for any k = 1, ..., p-1, $s_k = \operatorname{card}(I_k)$, $r_k = \operatorname{card}(I_k) + \operatorname{card}(I_{\ell-k})$. For

 $k=1,...,p-1,\,X^{(k)}$ is defined as the increasing reordering of $\overline{I}^{(k)}\cup I^{(\ell-k)}$. Consider

$$\widehat{\mu}^{(0)} = \left(\frac{\mu_i + i - 1}{\ell} \mid i \in I^{(0)}\right) \in \mathbb{Z}^{r_0} \text{ and for } k > 0,$$

$$\widehat{\mu}^{(k)} = \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| - 1 + \operatorname{sign}(i)k}{\ell} \mid i \in X^{(k)}\right) \in \mathbb{Z}^{r_k}.$$

We obtain

$$\varphi_{\ell}(P_{\mu}) = x_{I^{(0)}}^{\widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(p)}} \times \sum_{w \in W} (-1)^{l(w)} \varphi_{\ell} \left(\prod_{i \in I^{(0)}} x_i^{-\widehat{w}(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-\widehat{w}(i)-k} \right).$$

We also have the equivalencies

$$\varphi_{\ell}\left(\prod_{i\in I^{(0)}} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i\in X^{(k)}}^{s_{k}} x_{i}^{-\widehat{w}(i)+k}\right) \neq 0 \Longleftrightarrow \begin{cases} (\mathbf{i}) : \widehat{w}(I^{(0)} \cup \overline{I}^{(0)}) = J^{(0)} \\ (\mathbf{ii}) : \widehat{w}(X^{(k)}) = J^{(\ell-k)} \text{ for any } k = 1, ..., p-1 \end{cases}$$
(39)

We can write $J^{(0)} = (-(r_0 - 1)\ell, ..., 0, ..., (r_0 - 1)\ell)$ and for k = 1, ..., p,

$$J^{(\ell-k)} = (-k - \alpha_k \ell, ..., -k + \beta_k \ell), J^{(k)} = (k - \beta_k \ell, ..., k + \alpha_k \ell)$$

with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by

$$\widehat{w}_0(i_a^{(0)}) = (a-1)\ell \text{ for } a \in \{1, ..., r_0\}$$

$$\widehat{w}_0(i_a^{(k)}) = -k - \alpha_k \ell + (a-1)\ell \text{ for any } k = 1, ..., p-1.$$

$$(40)$$

Denote by \mathcal{W} the set of signed permutations $w \in W$ which verify (i) and (ii) in (39). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$ where $v = (v^{(0)}, \tau^{(1)}, ..., \tau^{(p-1)})$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$ where $W_{I^{(0)}}$ is the Weyl group of type D_{r_0} defined on $\overline{I}^{(0)} \cup I^{(0)}$. We have by Lemma 2.1.1 $(-1)^{l(w)} = (-1)^{l(w_0)} (-1)^{l(v^{(0)})} (-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p-1)})}$. Set

$$\begin{split} P_0 &= \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \varphi_{\ell} \left(\prod_{i \in X^{(0)}} x_i^{-\widehat{w}_0 v^{(0)}(i)} \right) \\ P_k &= \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_{\ell} \left(\prod_{i \in X^{(k)}} x_i^{-\widehat{w}_0 v^{(k)}(i) - k} \right) \text{ for any } k \in \{1, ..., p - 1\}. \end{split}$$

We obtain

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}} P_0 \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_k.$$

Given $v^{(0)} \in W_{I^{(0)}}$, we define $\widehat{v}^{(0)} = v^{(0)} - \xi_v$ where $\xi_v(i_a) = 1$ if $v(i_a) > 0$ and -1 otherwise. By (40), we have for any $a = 1, ..., r_0$, $\widehat{w}_0 v^{(0)}(i_a^{(0)}) = \widehat{v}^{(0)}(a)\ell$. Moreover since $\tau^{(k)} \in S_{X^{(k)}}$

$$\widehat{w}_0 \tau^{(k)}(i_a^{(k)}) = -k - \alpha_k \ell + (\tau^{(k)}(a) - 1)\ell$$
 for any $a = 1, ..., r_k$.

This yields

$$\begin{split} P_0 &= \sum_{v^{(0)} \in W_{I^{(0)}}} (-1)^{l(v^{(0)})} \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-\widehat{v}^{(0)}(a)} = x_{I^{(0)}}^{-\rho'_{r_0}} \Delta_{I^{(0)}} \text{ and} \\ P_k &= \sum_{\tau^{(k)} \in S_{X(k)}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a) + (\alpha_k + 1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}} \end{split}$$

where for any $k = 1, ..., p - 1, \eta_{r_k} = -\rho'_{r_k} + (\alpha_k, ..., \alpha_k) = -\rho_{r_k} + (\alpha_k + 1, ..., \alpha_k + 1) \in \mathbb{Z}^{r_k}$,

$$\Delta_{I^{(0)}} = \prod_{i < j, i, j \in I^{(0)}} (1 - \frac{x_j}{x_i}) \prod_{r < s, r, s \in I^{(0)}} (1 - x_r x_s) \text{ and}$$

$$\Delta_{X^{(k)}} = \prod_{i < j, i, j \in X^{(k)}} (1 - \frac{x_j}{x_i}) \text{ for any } k = 1, ..., p - 1.$$

This gives

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(0)}}^{\widehat{\mu}^{(0)} - \rho'_{r_0}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{I^{(0)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(k)}} \Delta_{I^{(0)}} = (-1)^{l(w_0)} x_{I^{(0)}}^{\mu^{(k)}$$

where

$$\mu^{(0)} = \left(\frac{\mu_i + i - 1}{\ell} \mid i \in I^{(0)}\right) - (0, ..., r_0 - 1) \in \mathbb{Z}^{r_0},\tag{41}$$

and for any k = 1, ..., p - 1,

$$\widehat{\mu}^{(k)} = |i \in X^{(k)} \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| - 1 + \operatorname{sign}(i)k}{\ell} \mid i \in X^{(k)} \right) - (0, ..., r_k - 1) + (\alpha_k, ..., \alpha_k) \in \mathbb{Z}^{r_k}.$$
(42)

Note that these formulas are essentially the same as for $G = Sp_{2n}$, except that we use $\rho'_n = (0, ..., n-1)$ instead of $\rho_n = (1, ..., n)$ for the half sum of positive roots. This gives the following theorem whose proof is identical to that of Theorem 3.2.3:

Theorem 3.2.6 Consider a partition μ of length n and $\ell = 2p - 1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (38). For any k = 0, ..., p - 1 define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (38).

- If $\operatorname{card}(I^{(0)}) \neq \frac{1}{2}(\operatorname{card}(J^{(0)}) + 1)$ or if there exists $k \in \{1, ..., p 1\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, set $r_0 = \operatorname{card}(I^{(0)})$ and for any k = 0, ..., p 1, $r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ be as in (40). Consider $\binom{\mu}{\ell} = (\mu^{(0)}, ..., \mu^{(\ell-1)})$ where the $\mu^{(k)}$'s are defined by (41) and (42). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{I^{(0)}, X^{(1)}, ..., X^{(p-1)}\}$ and we have

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.7

Consider $\mu = (1, 2, 3, 4, 4, 4, 6, 6)$ and take $\ell = 3$. We have $\mu + \rho'_8 = (1, 3, 5, 7, 8, 9, 12, 13)$. Thus $I^{(0)} = \{2, 6, 7\}, X^{(1)} = \{\overline{8}, \overline{4}, \overline{1}, 3, 5\}$ and $J^{(0)} = \{\overline{6}, \overline{3}, 0, 3, 6\}, J^{(1)} = \{\overline{5}, \overline{2}, 1, 4, 7\}$ and $J^{(2)} = \{\overline{7}, \overline{4}, \overline{1}, 2, 5\}$. In particular $\alpha_1 = 2$. Then $\mu^{(0)} = (1, 2, 2)$ and

$$\mu^{(1)} = \left(-\frac{13-1}{3} - 1 + 3, -\frac{7-1}{3} - 2 + 3, -\frac{1-1}{3} - 3 + 3, \frac{5+1}{3} - 4 + 3, \frac{8+1}{3} - 5 + 3 \right) = (-2, -1, 0, 1, 1)$$

3.2.4 For $G = SO_{2n+1}$

Set $L_n = \{\overline{n-1}, ..., \overline{1}, 0, 1, ..., n\}$. Each $w \in W$ can be written $w = \zeta \sigma$ according to the decomposition of W as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \propto S_n$. Given $w \in W$ we define $\widetilde{w} : J_n \to L_n$ such that $\widetilde{w}(x) = w(x) + \frac{1}{2}(1 - \xi(x))$ for any $x \in J_n$. For any $y \in L_n$, set $y^* = \overline{y} + 1$. We have then $\widetilde{w}(\overline{x}) = (w(x))^* = \overline{w(x)} + 1$.

Observe that $\rho = \rho_n'' = (\frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2}) = \rho_n - (\frac{1}{2}, ..., \frac{1}{2})$. Thus

$$w \cdot \rho_n'' = w \cdot \rho_n - \frac{1}{2}(\xi(1), ..., \xi(n)) = (\widetilde{w}(1), ..., \widetilde{w}(n)) - \frac{1}{2}(1, ..., 1).$$

This permits to write

$$P_{\mu} = x_1^{(\mu_1 + 1)} \cdots x_n^{(\mu_n + n)} \sum_{w \in W} (-1)^{l(w)} x_1^{-\widetilde{w}(1)} \cdots x_n^{-\widetilde{w}(n)}.$$

$$(43)$$

For any $k = 1, ..., \ell$ set

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \mod \ell) \text{ and } J^{(k)} = (x \in L_n \mid x \equiv k \mod \ell).$$

$$(44)$$

Note that $(J^{(k)})^* = J^{(l-k+1)}$. We then proceed essentially as in 3.2.2 by using \widetilde{w} instead of w. We are going to see that for $G = SO_{2n+1}$, there exists an analogue of Theorem 3.2.3 whatever the parity of ℓ .

The even case $\ell = 2p$ For any k = 1, ..., p, set $s_k = \operatorname{card}(I^{(k)}), r_k = \operatorname{card}(I^{(k)}) + \operatorname{card}(I^{(\ell-k+1)})$ and define $X^{(k)}$ as the increasing reordering of $\overline{I}^{(k)} \cup I^{(\ell-k+1)}$. Set $I^{(0)} = \{i_1^{(0)}, ..., i_{r_0}^{(0)}\}$ and

$$X^{(k)} = (i_1^{(k)}, ..., i_{r_k}^{(k)}). (45)$$

For k = 1, ..., p consider the r_k -tuple $\widehat{\mu}^{(k)}$ such that

$$\widehat{\mu}^{(k)} = \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| + \operatorname{sign}(i)k - \frac{1 + \operatorname{sign}(i)}{2}}{\ell} \mid i \in X^{(k)}\right) \in \mathbb{Z}^{r_k}.$$

For any $i\in I^{(k)}$ with k=1,...,p, we have $x_i^{-\widetilde{w}(i)-1}=x_{\overline{i}}^{-\widetilde{w}(\overline{i})}.$ Thus

$$\prod_{i \in X^{(k)}} x_i^{-\widetilde{w}(i)} = \prod_{i \in \overline{I}^{(k)}} x_i^{-\widetilde{w}(i)} \prod_{i \in I^{(\ell-k+1)}} x_i^{-\widetilde{w}(i)} = \prod_{i \in I^{(k)}} x_i^{-\widetilde{w}(i)-1} \prod_{i \in I^{(\ell-k+1)}} x_i^{-\widetilde{w}(i)}$$

and by definition of the $\widehat{\mu}^{(k)}$'s, (43) can be rewritten

$$P_{\mu} = \prod_{k=1}^{p} x_{X^{(k)}}^{\ell \widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)} \times \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-k+1}.$$

We obtain

$$\varphi_{\ell}(P_{\mu}) = \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \varphi_{\ell} \left(\prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)-k+1} \right)$$

We deduce the equivalencies

$$\varphi_{\ell}\left(\prod_{k=0}^{p}\prod_{i\in X^{(k)}}^{s_{k}}x_{i}^{-w(i)+k-1}\right)\neq0\iff\widetilde{w}(X^{(k)})=J^{(\ell-k+1)}\text{ for any }k=1,...,p.$$
(46)

In particular we must have $\operatorname{card}(J^{(\ell-k+1)}) = \operatorname{card}(J^{(k)}) = r_k$. We can write

$$J^{(\ell-k+1)} = (-k+1-\alpha_k\ell, ..., -k+1+\beta_k\ell)$$
 and $J^{(k)} = (k-\beta_k\ell, ..., k+\alpha_k\ell)$

with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by

$$\widetilde{w}_0(i_a^{(k)}) = -k + 1 - \alpha_k \ell + (a-1)\ell \text{ for any } k = 1, ..., p.$$
 (47)

Denote by \mathcal{W} the set of signed permutations $w \in W$ which verify the right hand side of (46). We have $w_0 \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w = w_0 v$ where $\tau = (\tau^{(1)}, ..., \tau^{(p)})$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}}$. We have also by Lemma 2.1.1 $(-1)^{l(w)} = (-1)^{l(w_0)} (-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p)})}$. For any k = 1, ..., p, set

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_\ell \left(\prod_{i \in X^{(k)}} x_i^{-\widetilde{w}_0 \tau^{(k)}(i) - k + 1} \right).$$

We obtain

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_k.$$

By (47) we have

$$w_0 \tau^{(k)}(i_a^{(k)}) = -k + 1 - \alpha_k \ell + (\tau^{(k)}(a) - 1)\ell$$
 for any $a = 1, ..., r_k$.

This yields

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a) + (\alpha_k + 1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}}$$

where for any k = 1, ..., p, $\eta_{r_k} = -\rho_{r_k} + (\alpha_k + 1, ..., \alpha_k + 1) \in \mathbb{Z}^{r_k}$ and

$$\Delta_{X^{(k)}} = \prod_{i < j \ i, j \in X^{(k)}} (1 - \frac{x_j}{x_i}).$$

Note that the computation only makes appear root systems of type A in this case. This gives

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} \prod_{k=1}^{p} x_{X^{(k)}}^{\hat{\mu}^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = (-1)^{l(w_0)} \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}$$

where for any k = 1, ..., p,

$$\widehat{\mu}^{(k)} = \left(\operatorname{sign}(i) \frac{\mu_{|i|} + |i| + \operatorname{sign}(i)k - \frac{1 + \operatorname{sign}(i)}{2}}{\ell} \mid i \in X^{(k)} \right) - (1, ..., r_k) + (\alpha_{k+1}, ..., \alpha_{k+1}) \in \mathbb{Z}^{r_k}.$$
(48)

Similarly to Theorem 3.2.3 we obtain:

Theorem 3.2.8 Consider a partition μ of length n and $\ell = 2p$ a positive integer. For any k = 1, ..., p define the sets $X^{(k)}, J^{(k)}$ by (44) and (45).

- If there exists $k \in \{1, ..., p\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, for any k = 1, ..., p, set $r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ be as in (47). Consider $\binom{\mu}{\ell} = (\mu^{(1)}, ..., \mu^{(p)})$ where the $\mu^{(k)}$'s are defined by (48). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{X^{(1)}, ..., X^{(p)}\}$ and we have

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.9 Consider $\mu = (1, 5, 5, 6, 7, 9)$ and take $\ell = 2$. We have $\mu + \rho_6 = (2, 7, 8, 10, 12, 15)$. Thus $X^{(1)} = \{\overline{6}, \overline{2}, 1, 3, 4, 5\}$ and $J^{(2)} = \{\overline{4}, \overline{2}, 0, 2, 4, 6\}, J^{(1)} = \{\overline{5}, \overline{3}, \overline{1}, 1, 3, 5\}$. In particular $\alpha_0 = 2$. Then

$$\mu^{(1)} = \left(-\frac{15-1}{2} - 1 + 3, -\frac{7-1}{2} - 2 + 3, \frac{2}{2} - 3 + 3, \frac{8}{2} - 4 + 3, \frac{10}{2} - 5 + 3, \frac{12}{2} - 6 + 3 \right) = (-5, -2, 1, 3, 3, 3)$$

The case $\ell=2p+1$ In addition to the sets $X^{(k)}, k=1,...,p$ defined in (45), we have also to consider $I^{(p+1)}=\{i_1^{(p+1)},...,i_{r_{p+1}}^{(p+1)}\}$. This yields to defined

$$\widehat{\mu}^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \mid i \in I^{(p+1)}\right)$$

We have

$$\varphi_{\ell}(P_{\mu}) = x_{I(p+1)}^{\widehat{\mu}^{(p+1)}} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} \times \sum_{w \in W} (-1)^{l(w)} \varphi_{\ell} \left(\prod_{i \in I^{(p+1)}} x_{i}^{-\widetilde{w}(i)-p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)-k-1} \right)$$

and the equivalence

$$\varphi_{\ell} \left(\prod_{i \in I^{(p+1)}} x_i^{-\widetilde{w}(i)-p} \prod_{k=1}^p \prod_{i \in X^{(k)}} x_i^{-w(i)+k-1} \right) \neq 0 \iff \begin{cases} \widetilde{w}(X^{(k)}) = J^{(\ell-k+1)} \text{ for any } k = 1, ..., p \\ \widetilde{w}(I^{(p+1)} \cup \overline{I}^{(p+1)}) = J^{(\ell-p)} = J^{(p+1)} \end{cases}$$
(49)

Indeed $(J^{(p+1)})^* = J^{(p+1)}$. In particular we must have $card(J^{(p+1)}) = 2card(I^{(p+1)}) = 2r_{p+1}$. Thus we can set $J^{(p+1)} = (-p - (r_{p+1} - 1)\ell, ..., -p + r_{p+1}\ell)$. Consider $w_0 \in W$ defined by (47) and

$$\widetilde{w}_0(i_a^{(p+1)}) = -p + a\ell \text{ for any } a = 1, ..., r_{p+1}.$$
 (50)

Denote by W the set of signed permutations $w \in W$ which verify the right hand side of (49). We have $w_0 \in W$. Each $w \in W$ can be written $w = w_0 v$ where $v = (\tau^{(1)}, ..., \tau^{(p)}, v^{(p+1)})$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}} \times W_{I^{(p+1)}}$. We have also $(-1)^{l(w)} = (-1)^{l(w_0)}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p)})}(-1)^{l(v^{(p+1)})}$. This permits to write

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} x_{I^{(p+1)}}^{\widehat{\mu}^{(p+1)}} P_{p+1} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_k \text{ where}$$

$$P_{p+1} = \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{l(v^{(p+1)})} \varphi_{\ell} \left(\prod_{i \in I^{(p+1)}} x_i^{-\widetilde{w}_0 v^{(p+1)}(i) - p} \right).$$

The functions $P_k, k = 1, ..., p$ can be computed as in the even case. For P_{p+1} , observe that each $v^{(p+1)} \in W_{I^{(p+1)}}$ can be written $v^{(p+1)} = \zeta \sigma$ with $\sigma \in S_{I^{(p+1)}}$. According to this decomposition we have for any $a = 1, ..., r_{p+1}, \widetilde{w}_0 v^{(p+1)} (i_a^{(p+1)}) = \xi(a)(-p + \sigma(a)\ell)$.

$$\begin{split} P_{p+1} &= \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{l(v^{(p+1)})} \varphi_{\ell} \left(\prod_{a=1}^{r_{p+1}} x_{i_{a}^{(p+1)}}^{-\xi(a)(-p+\sigma(a)\ell)-p} \right) = \\ &\qquad \qquad \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{l(v^{(p+1)})} \prod_{a=1}^{r_{p+1}} x_{i_{a}^{(p+1)}}^{-\frac{1-\xi(a)}{2} - \xi\sigma(a)}. \end{split}$$

Thus

$$\begin{split} P_{p+1} &= \prod_{i \in I^{(p+1)}} x_i^{-1/2} \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} \left(-1 \right)^{l(v^{(p+1)})} \left(\nu^{(p+1)} \cdot \prod_{a=1}^{r_{p+1}} x_{i_a^{(p+1)}}^{-(a-\frac{1}{2})} \right) = x_{I^{(p+1)}}^{-\rho_{r_{p+1}}} \Delta_{I^{(p+1)}} \\ \Delta_{I^{(p+1)}} &= \prod_{i < j \ i, j \in I^{(p+1)}} (1 - \frac{x_j}{x_i}) \prod_{r < s \ r, s \in I^{(p+1)}} (1 - x_r x_s) \prod_{i \in I^{(p+1)}} (1 - x_i). \end{split}$$

This means that when ℓ is odd

$$\varphi_{\ell}(P_{\mu}) = (-1)^{l(w_0)} \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} \times x_{I^{(p+1)}}^{\mu^{(p+1)}} \Delta_{I^{(p+1)}}$$

where

$$\mu_{i_a}^{(p+1)} = \left(\frac{\mu_i + i + p}{\ell} \mid i \in I^{(p+1)}\right) - (1, ..., r_{p+1}) \in \mathbb{Z}^{r_{p+1}}.$$
 (51)

This gives the following theorem:

Theorem 3.2.10 Consider a partition μ of length n and $\ell = 2p+1$ a positive integer. Define $X^{(k)}, J^{(k)}$ k = 1, ..., p and $I^{(p+1)}, J^{(p+1)}$ by (44) and (45).

- If $\operatorname{card}(I^{(p+1)}) \neq \frac{1}{2}\operatorname{card}(J^{(p+1)})$ or if there exists $k \in \{1, ..., p\}$ such that $\operatorname{card}(X^{(k)}) \neq \operatorname{card}(J^{(k)})$ then $\varphi_{\ell}(s_{\mu}) = 0$.
- Otherwise, set $r_{p+1} = \operatorname{card}(I^{(p+1)})$ and for any k = 1, ..., p, $r_k = \operatorname{card}(X^{(k)})$. Let $w_0 \in W$ verifying (47) and (50). Consider $\binom{\mu}{\ell} = (\mu^{(p+1)}, \mu^{(1)}, ..., \mu^{(p)})$ where the $\mu^{(k)}$'s are defined by (48) and (51). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^+$ with $\mathcal{I} = \{I^{(p+1)}, X^{(1)}, ..., X^{(p)}\}$ and we have

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} S_{\binom{\mu}{\ell}, \mathcal{I}}.$$

Example 3.2.11 Consider $\mu = (1, 5, 5, 6, 7, 9)$ and take $\ell = 3$. We have $\mu + \rho_6 = (2, 7, 8, 10, 12, 15)$. Thus $X^{(1)} = \{\overline{4}, \overline{2}, 5, 6\}, I^{(2)} = \{1, 3\}$ and $J^{(1)} = \{\overline{5}, \overline{2}, 1, 4\}, J^{(2)} = \{\overline{4}, \overline{1}, 2, 5\}, J^{(3)} = \{\overline{3}, 0, 3, 6\}$. In particular $\alpha_0 = 1$. Then

$$\mu^{(1)} = \left(-\frac{10-1}{3} - 1 + 1, -\frac{7-1}{3} - 2 + 1, \frac{12}{3} - 3 + 1, \frac{15}{3} - 4 + 1 \right) = (-2, -2, 3, 3)$$

and
$$\mu^{(2)} = (\frac{2+1}{3} - 1, \frac{8+1}{3} - 2) = (0, 1).$$

4 Parabolic Kazhdan-Lusztig polynomials

We recall briefly in this section, some basics on Affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials associated to classical root systems. The reader is referred to [11] and [13] for detailed expositions. Note that the definition of the Hecke algebra used in [11] coincides with that used in [8] and [13] (with generators H_w) up to the change $q \to q^{-1}$.

4.1 Extended affine Weyl group

Consider a root system of type A_{n-1}, B_n, C_n or D_n . For any $\beta \in P$, we denote by t_β the translation defined in $\mathfrak{h}_{\mathbb{R}}^*$ by $\gamma \longmapsto \gamma + \beta$. The extended affine Weyl group \widehat{W} is the group

$$\widehat{W} = \{ wt_{\beta} \mid w \in W, \beta \in P \}$$

with multiplication determined by the relations $t_{\beta}t_{\gamma} = t_{\beta+\gamma}$ and $wt_{\beta} = t_{w\cdot\beta}w$. The group \widehat{W} is not a Coxeter group but contains the affine Weyl group \widetilde{W} generated by reflections through the affine hyperplanes $H_{\alpha,k} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid (\beta, \alpha^{\vee}) = k\}$. It makes sense to define a length function on \widehat{W} verifying

$$l(wt_{\beta}) = \sum_{\alpha \in R_{+}} \left| (\beta, \alpha^{\vee}) + 1_{R_{-}}(w \cdot \alpha) \right|$$
(52)

where for any $w \in W$, $1_{R_{-}}(w \cdot \alpha) = 0$ if $w \cdot \alpha \in R_{+}$ and $1_{R_{-}}(w \cdot \alpha) = 1$ if $w \cdot \alpha \in -R_{+} = R_{-}$. Write n_{β} for the element of maximal length in $Wt_{\beta}W$. It follows from (52) that for any $\lambda \in P_{+}$ we have $l(wt_{\lambda}) = l(w) + l(t_{\lambda})$. This gives

$$n_{\lambda} = w_0 t_{\lambda} \tag{53}$$

where w_0 denotes the longest element of W. There exists a unique element $\eta \in R_+$ such that the fundamental alcove

$$\mathcal{A} = \{ \beta \in \mathfrak{h}_{\mathbb{R}}^* \mid (\beta, \alpha^{\vee}) \ge 0 \ \forall \alpha \in R_+ \ \mathrm{and} \ (\beta, \eta^{\vee}) < 1 \}$$

is a fundamental region for the action of \widetilde{W} on $\mathfrak{h}_{\mathbb{R}}^*$. This means that, for any $\beta \in \mathfrak{h}_{\mathbb{R}}^*$, the orbit $\widetilde{W} \cdot \beta$ intersects \mathcal{A} in a unique point. Each $w \in \widehat{W}$ can be written on the form $w = w_{\mathcal{A}}w_{\text{aff}}$ where $w_{\text{aff}} \in \widetilde{W}$ and $w_{\mathcal{A}}$ belongs to the stabilizer of \mathcal{A} under the action of \widehat{W} . This implies that \mathcal{A} is also a fundamental domain for the action of \widehat{W} on $\mathfrak{h}_{\mathbb{R}}^*$. The Bruhat ordering on \widehat{W} is defined by taking the transitive closure of the relations

$$w < sw$$
 whenever $l(w) < l(sw)$

for all $w \in \widehat{W}$ and all (affine) reflections $s \in \widetilde{W}$.

In fact the natural action of \widehat{W} on the weight lattice P obtained by considering P as a sublattice of $\mathfrak{h}_{\mathbb{R}}^*$ is not that which is relevant for our purpose. For any integer $m \in \mathbb{Z}^*$ we obtain a faithful representation π_m of \widehat{W} on P by setting for any $\beta, \gamma \in P, w \in W$

$$\pi_m(w) \cdot \gamma = w \cdot \gamma \text{ and } \pi_m(t_\beta) \cdot \gamma = \gamma + m\beta.$$

Warning: In the sequel, the extended affine Weyl group \widehat{W} acts on the weight lattice P via $\pi_{-\ell}$ where ℓ is a fixed nonnegative integer.

We write for simplicity $wt_{\beta} \cdot \gamma$ rather that $\pi_{-\ell}(wt_{\beta}) \cdot \gamma$. Hence for any $w \in W$ and any $\beta \in P$, we have $wt_{\beta} \cdot \gamma = w \cdot \gamma - \ell w \cdot \beta$. The fundamental region for this new action of \widehat{W} on P is the alcove \mathcal{A}_{ℓ} obtained by expanding \mathcal{A} with the factor $-\ell$. This gives

$$\mathcal{A}_{\ell} = \begin{cases} \{ \nu = (\nu_1, ..., \nu_n) \mid 0 \ge \nu_1 \ge \cdots \ge \nu_n > -\ell \} \text{ for types } A, B, C \\ \{ \nu = (\nu_1, ..., \nu_n) \mid 0 \ge -|\nu_1| \ge \nu_2 \ge \cdots \ge \nu_n > -\ell \} \text{ for type } D \end{cases}$$

Consider a weight $\beta \in P$. Then its orbit intersects \mathcal{A}_{ℓ} in a unique weight ν . Then there is a unique $w(\beta) \in \widehat{W}$ of minimal length such that $w(\beta) \cdot \nu = \beta$. We denote by W_{ν} the stabilizer of $\nu \in \mathcal{A}_{\ell}$ in \widehat{W} . Since $\nu \in \mathcal{A}_{\ell}$, W_{ν} is in fact a subgroup of W.

Lemma 4.1.1 Consider $\lambda \in P^+$ and suppose $\ell > n$. Then

- 1. $w(\ell\lambda + \rho) = n_{\lambda^*}\tau^{-n+1}$ with $\lambda^* = -w_0(\lambda)$ and $\tau = s_1s_2\cdots s_{r-1}t_{\varepsilon_1}$ for type A.
- 2. $w(\ell\lambda + \rho) = n_{\lambda}$ for types B, C and D.

Proof. 1 : See Lemma 2.3 in [8].

2: Observe first that $w_0 \cdot \rho = -\rho$ belongs to \mathcal{A}_{ℓ} for types B, C, D since $\ell > n$. We have

$$\ell\lambda + \rho = t_{-\lambda} \cdot \rho = t_{-\lambda} w_0 \cdot (w_0 \cdot \rho) = t_{-\lambda} w_0 \cdot (-\rho).$$

Moreover $W_{-\rho} = \{1\}$. Since $-\rho \in \mathcal{A}_{\ell}$ this means that $w(\ell\lambda + \rho) = t_{-\lambda}w_0 = w_0t_{w_0\cdot(-\lambda)} = w_0t_{\lambda} = n_{\lambda}$ where the last equality follows from (53).

4.2 Affine Hecke algebra and K-L polynomials

The Hecke algebra associated to the root system R of type A_n, B_n, C_n or D_n is the $\mathbb{Z}[q, q^{-1}]$ -algebra defined by the generators $T_w, w \in \widehat{W}$ and relations

$$T_{w_1}T_{w_2} = T_{w_1}T_{w_2}$$
 if $l(w_1w_2) = l(w_1) + l(w_2)$,
 $T_{s_i}T_w = (q^{-1} - q)T_w + T_{s_iw}$ if $l(s_iw) < l(w)$ and $i \in I_n$.

In particular we have $T_i^2 = (q^{-1} - q)T_i + 1$ for any $i \in I_n$. The bar involution on \widehat{H} is the \mathbb{Z} -linear automorphism defined by

$$\overline{q} = q^{-1}$$
 and $\overline{T}_w = T_{w^{-1}}^{-1}$ for any $w \in \widehat{W}$.

Kazhdan and Lusztig have proved that there exists a unique basis $\{C'_w \mid w \in \widehat{W}\}\$ of \widehat{H} such that

$$\overline{C}'_w = C'_w$$
 and $C'_w = \sum_{y \le w} p_{y,w} T_y$

where $p_{w,w} = 1$ and $p_{y,w} \in q\mathbb{Z}[q]$ for any y < w. We will refer to the polynomials $p_{y,w}(q)$ as Kazhdan-Lusztig polynomials. They are renormalizations of the polynomials $P_{y,w}$ originally introduced by Kazhdan and Lusztig in [5]. Namely we have $p_{y,w} = q^{l(w)-l(y)}P_{y,w}$. Let us define the q-partition function \mathcal{P}_q by

$$\prod_{\alpha \in R_{+}} \frac{1}{1 - qx^{\alpha}} = \sum_{\beta \in \mathbb{Z}^{n}} \mathcal{P}_{q}(\beta) x^{\beta}.$$

Given λ and μ in P, the Lusztig q-analogue $K_{\lambda,\mu}(q)$ is defined by

$$K_{\lambda,\mu}(q) = \sum_{w \in W} (-1)^{l(w)} \mathcal{P}_q(w \circ \lambda - \mu).$$

Then one has the following theorem due to Lusztig

Theorem 4.2.1 Suppose λ, μ are dominant weights. Then $K_{\lambda,\mu}(q) = p_{n_{\mu},n_{\mu}}(q)$.

One defines the action of the bar involution on the parabolic module $P_{\nu} = \widehat{H}\nu$, $\nu \in P$, by setting $\overline{q} = q^{-1}$ and $\overline{w \cdot \nu} = \overline{w} \cdot \nu$. Deodhar has proved that there exist two bases $\{C_{\lambda}^{+} \mid \lambda \in \widehat{W} \cdot \nu\}$ and $\{C_{\lambda}^{-} \mid \lambda \in \widehat{W} \cdot \nu\}$ of P_{ν} belonging respectively to

$$L_{\nu}^{+} = \bigoplus_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}[q]\lambda \text{ and } L_{\nu}^{-} = \bigoplus_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}[q^{-1}]\lambda$$

characterized by

$$\left\{ \begin{array}{l} \overline{C}_{\lambda}^{+} = C_{\lambda}^{+} \\ C_{\lambda}^{+} \equiv \lambda \operatorname{mod} q L_{\nu}^{+} \end{array} \right. \text{ and } \left\{ \begin{array}{l} \overline{C}_{\lambda}^{-} = C_{\lambda}^{-} \\ C_{\lambda}^{-} \equiv \lambda \operatorname{mod} q^{-1} L_{\nu}^{-} \end{array} \right. .$$

We will only need the basis $\{C_{\lambda}^- \mid \lambda \in \widehat{W} \cdot \nu\}$ in the sequel. The parabolic Kazhdan-Lusztig polynomials $P_{\lambda,\mu}^-$ are then defined by the expansion

$$C_{\lambda}^{-} = \sum_{\mu \in \widehat{W} \cdot \lambda} (-1)^{l(w(\lambda)) + l(w(\mu))} P_{\mu,\lambda}^{-}(q^{-1})\mu$$

(see [13] Theorem 3.5). In particular they belong to $\mathbb{Z}[q]$. Their expansion in terms of the ordinary Kazhdan-Lusztig polynomials is given by the Following theorem due to Deodhar

Theorem 4.2.2 Consider $\nu \in P$ and $\lambda \in \widehat{W} \cdot \nu$. Then for any $\mu \in \widehat{W} \cdot \lambda$ we have

$$P_{\lambda,\mu}^{-}(q) = \sum_{z \in W_{\nu}} (-q)^{l(z)} p_{w(\mu)z,w(\lambda)}(q)$$

with the notation of 4.1.

Remark: When ν is regular, that is $W_{\nu} = \{1\}$, we have $P_{\lambda,\mu}^{-}(q) = p_{w(\mu),w(\lambda)}(q)$.

5 Generalized Hall-Littlewood functions

5.1 Plethysm and parabolic K-L polynomials

Consider ℓ a nonnegative integer and $\zeta \in \mathbb{C}$ such that ζ^2 is a primitive ℓ -th root of 1. We briefly recall in this paragraph the arguments of [8] which establish that the coefficients of the plethysm $\psi_{\ell}(s_{\lambda})$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at q = 1.

For any $\lambda \in P_+$, denote by $V_q(\lambda)$ the finite dimensional $U_q(\mathfrak{g})$ -module of highest weight λ . Its character is also the Weyl character s_{λ} . Let $U_{q,\mathbb{Z}}(\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$ -subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$E_i^{(k)} = \frac{E_i^{(k)}}{[k]_i!}, F_i^{(k)} = \frac{F_i^{(k)}}{[k]_i!} \text{ and } K_i^{\pm 1}$$

where $E_i, F_i, K_i^{\pm 1}$, $i \in I_n$ are the generators of $U_q(\mathfrak{g})$. The indeterminate q can be specialized at ζ in $U_{q,\mathbb{Z}}(\mathfrak{g})$. Thus it makes sense to define $U_{\zeta}(\mathfrak{g}) = U_{q,\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ where $\mathbb{Z}[q,q^{-1}]$ acts on \mathbb{C} by $q \mapsto \zeta$. Fix a highest weight vector v_{λ} in $V_q(\lambda)$. We have $V_q(\lambda) = U_q(\mathfrak{g}) \cdot v_{\lambda}$. Similarly $V_{\zeta}(\lambda) = U_{\zeta}(\mathfrak{g}) \cdot v_{\lambda}$ is a $U_{\zeta}(\mathfrak{g})$ module called a Weyl module and one has $\operatorname{char}(V_{\zeta}(\lambda)) = s_{\lambda}$. The module $V_{\zeta}(\lambda)$ is not simple but admits a unique simple quotient denoted by $L(\lambda)$.

From results due to Kazhdan-Lusztig and Kashiwara-Tanisaki one obtains the following decomposition of $\operatorname{char}(L(\lambda))$ on the basis of Weyl characters:

Theorem 5.1.1 Consider $\lambda \in P_+$.

1. The character of $L(\lambda)$ decomposes on the form

$$char(L(\lambda)) = \sum_{\mu} (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho,\lambda+\rho}^{-}(1) s_{\mu}$$
(54)

where the sum runs over the dominant weights $\mu \in P_+$ such that $\mu + \rho \in \widehat{W} \cdot (\lambda + \rho)$.

2. The parabolic Kazhdan-Lusztig polynomials $P_{\mu+\rho,\lambda+\rho}^-(q)$ have nonnegative integer coefficients.

Remark: The decomposition (54) has been conjectured by Kazhdan-Lusztig and proved by Kashiwara-Tanisaki. In [4], Kashiwara and Tanisaki have also obtained that the parabolic Kazhdan-Lusztig polynomials have nonnegative integer coefficients as soon as the Coxeter system considered corresponds to the Weyl group of a Kac-Moody Lie algebra, thus in the particular context of this paper.

Consider a nonnegative integer ℓ . The Frobenius map Fr_{ℓ} is the algebra homomorphism defined from $U_{\zeta}(\mathfrak{g})$ to $U(\mathfrak{g})$ by $\operatorname{Fr}_{\ell}(K_i) = 1$ and

$$\operatorname{Fr}_{\ell}(E_i^{(k)}) = \left\{ \begin{array}{l} e_i^{(k/\ell)} \text{ if } \ell \text{ divides } k \\ 0 \text{ otherwise} \end{array} \right. \text{ and } \operatorname{Fr}_{\ell}(F_i^{(k)}) = \left\{ \begin{array}{l} f_i^{(k/\ell)} \text{ if } \ell \text{ divides } k \\ 0 \text{ otherwise} \end{array} \right.$$

where $e_i, f_i, i \in I_n$ are the Chevalley generators of the enveloping algebra $U(\mathfrak{g})$. This permits to endow each $U(\mathfrak{g})$ -module M with the structure of a $U_{\zeta}(\mathfrak{g})$ -module $M^{\operatorname{Fr}_{\ell}}$. Then we have

$$\operatorname{char}(M^{\operatorname{Fr}_{\ell}}) = \psi_{\ell}(\operatorname{char}(M))$$

in particular for any $\lambda \in P_+$, $\operatorname{char}(V(\lambda)^{\operatorname{Fr}_{\ell}}) = \psi_{\ell}(s_{\lambda})$.

Each dominant weight $\lambda \in P_+$, can be uniquely decomposed on the form $\lambda = \stackrel{r}{\lambda} + \ell \stackrel{q}{\lambda}$ where $\stackrel{r}{\lambda}, \stackrel{q}{\lambda} \in P_+$ and $\stackrel{r}{\lambda} = (\stackrel{r}{\lambda}_1, ..., \stackrel{r}{\lambda}_n)$ verifies $0 \le \stackrel{r}{\lambda}_{i+1} - \stackrel{r}{\lambda}_i < \ell$ for any $i \in I_n$.

Theorem 5.1.2 (Lusztig) The simple $U_{\zeta}(\mathfrak{g})$ -module $L(\lambda)$ is isomorphic to the tensor product

$$L(\lambda) \simeq L(\overset{r}{\lambda}) \otimes V(\overset{q}{\lambda})^{\operatorname{Fr}_{\ell}}.$$

By replacing λ by $\ell\lambda$ in the previous theorem, we have $\stackrel{r}{\lambda} = 0$ and $\stackrel{q}{\lambda} = \lambda$. Thus $L(\ell\lambda) \simeq V(\lambda)^{\operatorname{Fr}_{\ell}}$. Then one deduces from (54) the equality

$$\psi_{\ell}(s_{\lambda}) = \operatorname{char}(L(\ell\lambda)) = \sum_{\mu+\rho \in \widehat{W} \cdot (\ell\lambda+\rho)} (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho,\ell\lambda+\rho}^{-}(1) s_{\mu}$$

which shows that the coefficients of the expansion of $\psi_{\ell}(s_{\lambda})$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at q=1. This gives

$$|\langle \psi_{\ell}(s_{\lambda}), s_{\mu} \rangle| = |\langle s_{\lambda}, \varphi(s_{\mu}) \rangle| = P_{\mu+\rho,\ell\lambda+\rho}^{-}(1).$$

By definition of the action of \widehat{W} on P we have $\widehat{W} \cdot (\ell \lambda + \rho) = \widehat{W} \cdot \rho$. This implies the

Corollary 5.1.3 (of Theorems 4.2.2 and 5.1.2). For any nonnegative integer ℓ

$$\psi_{\ell}(s_{\lambda}) = \sum_{\mu+\rho \in \widehat{W} \cdot \rho} (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho,\ell\lambda+\rho}^{-}(1) s_{\mu}.$$

In particular $\varphi(s_{\mu}) \neq 0$ if and only if $\mu + \rho \in \widehat{W} \cdot \rho$, that is $\mu + \rho = w \cdot \rho - \ell \beta$ with $w \in W$ and $\beta \in P$.

Remark: The equivalence $\varphi(s_{\mu}) \neq 0 \iff \mu + \rho \in \widehat{W} \cdot \rho$ can also be obtained more elementary from algorithms described in 3.2.

5.2 Parabolic K-L polynomials and branching coefficients

Warning: In the sequel of the paper we will suppose that ℓ is odd when the Lie groups considered are of type C or D.

Under this hypothesis we have for any $\mu \in \mathcal{P}_n \ \varphi_{\ell}(s_{\mu}) = 0$ or

$$\varphi_{\ell}(s_{\mu}) = (-1)^{l(w_0)} S_{\binom{\mu}{\delta}, \mathcal{I}} \tag{55}$$

according to the results of 3.2.

Remark: Following the algorithms described in 3.2, when $\varphi_{\ell}(s_{\mu}) \neq 0$, the cardinalities of the sets $I^{(k)}$ or $X^{(k)}$ contained in \mathcal{I} are determined by those of the sets $J^{(k)}$. In particular they depend only on n and ℓ and not on the partition μ considered. Thus in (55), the underlying Levi subgroup $G_{\mathcal{I}}$ is, up to isomorphism, independent on μ .

By using Proposition 2.5.2 and Theorems 3.2.1, 3.2.3, 3.2.6, 3.2.8, 3.2.10 we deduce from Corollary 5.1.3 the

Theorem 5.2.1 For any $\lambda, \mu \in \mathcal{P}_n$ such that $\mu + \rho \in \widehat{W} \cdot \rho$

$$P_{\mu+\rho,\ell\lambda+\rho}^{-}(1) = [V(\lambda) : V_{\mathcal{I}} {\mu \choose \ell}]$$

where $\binom{\mu}{\ell}$ and $\mathcal I$ are obtained from μ and ℓ by applying the algorithms described in 3.2.

5.3 The functions H^{ℓ}_{μ}

For any $\mu \in \mathcal{P}_n$, we define the function G_{μ}^{ℓ} by setting

$$G_{\mu}^{\ell} = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_{\mathcal{I}} {\mu \choose \ell}]_q s_{\lambda}$$
 (56)

where for any $\lambda \in \mathcal{P}_n$, $[V(\lambda):V_{\mathcal{I}}\binom{\mu}{\ell}]_q = P_{\mu+\rho,\ell\lambda+\rho}^-(q)$. We also consider the function H_{μ}^{ℓ} such that

$$H^{\ell}_{\mu} = G^{\ell}_{\ell\mu}.\tag{57}$$

Theorem 5.3.1 Consider a partition $\mu \in \mathcal{P}_n$.

- 1. The coefficients of H^{ℓ}_{μ} on the basis of Weyl characters are polynomials in q with nonnegative integer coefficients.
- 2. We have $H_{\mu}^{1} = s_{\mu}$
- 3. For ℓ sufficiently large $H^{\ell}_{\mu} = Q'_{\mu}$, that is H^{ℓ}_{μ} coincide with the Hall-Littlewood function associated to μ .

To prove our theorem we need the following Lemma:

Lemma 5.3.2 Consider $\beta \in P$.

- In type A_{n-1} , suppose $\ell > n$. Then the weight $\ell\beta + \rho$ is regular.
- In type B_n, C_n or D_n , suppose $\ell > 2n$. Then the weight $\ell\beta + \rho$ is regular.

Proof. Consider $w \in W$ and t_{γ} such that $t_{\gamma}w \cdot (\ell\beta + \rho) = \ell\beta + \rho$. Then $\delta = \ell\beta + \rho - w \cdot (\ell\beta + \rho) \in (\ell\mathbb{Z})^{\ell}$. Set $\beta = (\beta_1, ..., \beta_n)$. For any i = 1, ..., n, the i-th coordinate of δ is $\delta_i = \ell\beta_i + \rho_i - \ell\beta_{w(i)} - \rho_{w(i)}$. Since $\delta_i \in \ell\mathbb{Z}$, we must have $|\rho_i - \rho_{w(i)}| \in \ell\mathbb{Z}$. One verifies easily that for type A_{n-1} , $|\rho_i - \rho_{w(i)}| \leq n-1$ and for types $B_n, C_n, D_n \mid \rho_i - \rho_{w(i)} \mid \leq 2n$. Hence when the conditions of the lemma are verified, $|\rho_i - \rho_{w(i)}| = 0$ for any i = 1, ..., n. This gives w = 1. The equality $t_{\gamma}w \cdot (\ell\beta + \rho) = \ell\beta + \rho$ implies then that $\gamma = 0$. Thus the stabilizer of $\ell\beta + \rho$ is reduced to $\{1\}$, that is $\ell\beta + \rho$ is regular.

Proof. (of Theorem 5.3.1)

- 1: Follows from Theorem 5.1.1 and (56).
- 2: When $\ell = 1$, we have seen that $G = G_{\mathcal{I}}$ and $\binom{\mu}{\ell} = \mu$. Thus $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q \neq 0$ only if $\lambda = \mu$. In this case $H^1_{\mu} = s_{\mu}$ for $[V(\lambda) : V_{\mathcal{I}}\binom{\mu}{\ell}]_q = [V(\lambda) : V(\lambda)]_q = 1$.
- 3: Suppose ℓ as in the previous lemma. We have $[V(\lambda):V_{\mathcal{I}}\binom{\ell\mu}{\ell}]_q = P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q)$. Since $\ell\lambda+\rho$ is regular for the action of \widehat{W} , we obtain by Theorem 4.2.2, $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = p_{w(\ell\mu+\rho),w(\ell\lambda+\rho)}(q)$. By using Lemma 4.1.1, we deduce $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = p_{n\mu,n_{\lambda}}(q)$. Now by Theorem 4.2.1 this gives $P^-_{\ell\mu+\rho,\ell\lambda+\rho}(q) = K_{\lambda,\mu}(q)$. Finally

$$H_{\mu}^{\ell} = \sum_{\lambda \in \mathcal{P}_{n}} [V(\lambda) : V_{\mathcal{I}} \binom{\ell \mu}{\ell}]_{q} s_{\lambda} = \sum_{\lambda \in \mathcal{P}_{n}} K_{\lambda,\mu}(q) s_{\lambda} = Q_{\mu}'. \tag{58}$$

Remarks:

(i): By the previous theorem the functions H^{ℓ}_{μ} interpolate between the Weyl characters and the Hall-Littlewood functions.

(ii): When ℓ is even for types C and D, one can also define the functions G^{ℓ}_{μ} and H^{ℓ}_{μ} by setting $G^{\ell}_{\mu} = \sum_{\lambda \in \mathcal{P}_n} P^-_{\mu+\rho,\ell\lambda+\rho}(q) s_{\lambda}$ and $H^{\ell}_{\mu} = G^{\ell}_{\ell\mu}$, respectively. When $\ell > 2n$ we have yet $H^{\ell}_{\mu} = Q'_{\mu}$, but the polynomials $P^-_{\mu+\rho,\ell\lambda+\rho}(q)$ cannot be interpreted as quantizations of branching coefficients.

(iii): The conditions $\ell > n$ for type A_{n-1} and $\ell > 2n$ for types B_n, C_n, D_n appear also naturally in the algorithms of 3.2. When they are fulfilled, one has $\varphi_{\ell}(s_{\ell\mu}) = 0$, or $J_k = I_k$ for any k = 1, ..., n and $J_k = I_k = \emptyset$ for $k \notin \{1, ..., n\}$. Then $[(\ell\mu)/\ell] = \mu$ and $G_{\mathcal{I}} = H$. Hence $[V(\lambda) : V_{\mathcal{I}}(\mu)] = K_{\lambda,\mu}$ for any $\lambda \in \mathcal{P}_n$. This yields equality (58) specialized at q = 1.

6 Further remarks

6.1 Quantization of tensor product coefficients

Consider $\mu \in \mathcal{P}_n$ and set $\mu = (\mu^{(0)}, ..., \mu^{(\ell-1)})$ as in Theorem 3.2.1. For $G = GL_n$, the duality $c^{\lambda}_{(\mu^{(0)}, ..., \mu^{(\ell-1)})} = [V(\lambda) : V_{\mathcal{I}}(^{\mu}_{\ell})]$ yields a q-analogue of the Littlewood-Richardson coefficient $c^{\lambda}_{(\mu^{(0)}, ..., \mu^{(\ell-1)})}$ defined by setting

$$c_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}^{\lambda}(q) = [V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}]_q = P_{\mu+\rho,\ell\lambda+\rho}^-(q).$$
 (59)

By Theorem 5.1.1, $c^{\lambda}_{(\mu^{(0)},\dots,\mu^{(\ell-1)})}(q)$ have then nonnegative integer coefficients.

In [9], we have shown that there also exists a duality between tensor product coefficients for types B, C, D defined as the analogues of the Littlewood-Richardson coefficients by counting the multiplicities of the isomorphic irreducible components in a tensor product of irreducible representations and branching coefficients. These branching coefficients correspond to the restriction of SO_{2n} to subgroups of the form $SO_{2r_0} \times \cdots SO_{2r_p}$ where the r_i 's are positive integers summing n. These subgroups are not Levi subgroups, thus the Littlewood-Richardson coefficients for types B, C, D cannot be quantified as in (59) by using parabolic Kazhdan-Lusztig polynomials.

For $G = SO_{2n+1}$, Sp_{2n} or SO_{2n} and $\lambda \in \mathcal{P}_n$, denote by $\mathfrak{V}(\lambda)$ the restriction of the irreducible finite dimensional GL_N -module of highest weight λ to G. Consider a p-tuple $(\mu^{(0)}, ..., \mu^{(p-1)})$ of partitions such that $\mu^{(k)} \in \mathcal{P}_{r_k}$ for any k = 0, ..., p - 1. One can define the coefficients $\mathfrak{D}^{\lambda}_{\mu^{(0)}, ..., \mu^{(p-1)}}$ as the multiplicity of $V(\lambda)$ in $\mathfrak{V}(\mu^{(0)}) \otimes \cdots \otimes \mathfrak{V}(\mu^{(p-1)})$, that is such that

$$\mathfrak{V}(\mu^{(0)}) \otimes \cdots \otimes \mathfrak{V}(\mu^{(p-1)}) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V(\lambda)^{\oplus \mathfrak{D}^{\lambda}_{\mu^{(0)}, \dots, \mu^{(p-1)}}}.$$

We have also obtained in [9] a duality result between the coefficients $\mathfrak{D}_{\mu^{(0)},...,\mu^{(p-1)}}^{\lambda}$ and branching coefficients corresponding to the restriction of G to the Levi subgroup $GL_{r_0} \times \cdots \times GL_{r_{p-1}}$. The coefficients $\mathfrak{D}_{\mu^{(0)},...,\mu^{(p-1)}}^{\lambda}$ can be expressed by using a partition function similarly to Proposition 2.5.1. By quantifying this partition function, one shows that they admit nonnegative q-analogues. It is conjectured that stable one-dimensional sums defined in [3] from affine crystals obtained by considering the affinizations of the classical root systems are special cases of the q-analogues obtained in this way. Recall that the Levi subgroups $G_{\mathcal{I}}$ obtained in the theorems of 3.2 are, up to isomorphism, determined only by G and ℓ . This implies that there exist Levi subgroups L in G which are not isomorphic to a subgroup $G_{\mathcal{I}}$. This is in particular the case when $G = Sp_{2n}$ for the Levi subgroups $G_{\mathcal{I}} \simeq GL_{r_0} \times \cdots \times GL_{r_{p-1}}$ such that $r_k > 1$ for any k = 0, ..., p - 1. Indeed, by Theorem 3.2.3, when $r_0 = \operatorname{card}(I^{(0)}) > 1$, $G_{\mathcal{I}}$ is isomorphic to

$$Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_{n-1}}$$
.

This implies that one cannot obtain in general a quantization of the tensor product coefficients $\mathfrak{D}_{\mu^{(0)},\dots,\mu^{(p-1)}}^{\lambda}$ by using parabolic Kazhdan-Lusztig polynomials as in (59).

6.2 Combinatorial description of the functions G_u^{ℓ}

When $G = GL_n$, the functions G^{ℓ}_{μ} defined in (56) admit the following combinatorial description

$$G^{\ell}_{\mu} = \sum_{T \in \text{Tab}_{\ell}(\mu)} q^{s(T)} x^{T}$$

where $\operatorname{Tab}_{\ell}(\mu)$ is the set of ℓ -ribbon tableaux of shape μ on I_n and s the spin statistic defined on ribbon tableaux (see [7] page 1057). Recently, Haglund, Haiman and Loehr have obtained the expansion of the Macdonald polynomials in terms of simple renormalizations of the LLT polynomials G_{μ}^{ℓ} . This expansion yields a combinatorial formula for the Macdonald polynomials [2]. This suggests to investigate the following combinatorial problem:

Problem 6.2.1 Find a combinatorial description of the polynomials G^{ℓ}_{μ} and the q-analogues $[V(\lambda):V_{\mathcal{I}}({}^{\mu}_{\ell})]_q$ related to the roots systems of type B,C or D.

6.3 Exceptional root systems

It is also possible to define the plethysm ψ_{ℓ} and the dual plethysm φ_{ℓ} for exceptional root systems. Consider such an exceptional root system R and μ a dominant weight for R. Denote also by s_{μ} the Weyl character of the irreducible finite dimensional module of highest weight λ . When ℓ is sufficiently large (the bound depends on R), we have $\varphi_{\ell}(s_{\mu}) = s_{\mu}$. For the other values of ℓ , one shows that the polynomial $\varphi_{\ell}(e^{\mu}\prod_{\alpha\in R_{+}}(1-e^{\alpha}))$ do not factorize in general as a product of factors $(1-x^{\beta})$ where β is a positive root. This implies that one cannot define generalized Hall-Littlewood functions for exceptional types by proceeding as in (57).

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