# Parabolic Kazhdan-Lusztig polynomials, plethysm and gereralized Hall-Littlewood functions for classical types 

Cédric Lecouvey<br>Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville<br>B.P. 69962228 Calais Cedex<br>Cedric.Lecouvey@lmpa.univ-littoral.fr


#### Abstract

We use power sums plethysm operators to introduce $H$ functions which interpolate between the Weyl characters and the Hall-Littlewood functions $Q^{\prime}$ corresponding to classical Lie groups. The coefficients of these functions on the basis of Weyl characters are parabolic Kazhdan-Lusztig polynomials and thus, are nonnegative. We prove that they can be regarded as quantizations of branching coefficients obtained by restriction to certain Levi subgroups. The $H$ functions associated to linear groups coincide with the polynomials introduced by Lascoux Leclerc and Thibon in [7].


## 1 Introduction

Given $\mu$ a partition with at most $n$ parts, the Hall Littlewood function $Q_{\mu}^{\prime}$ can be defined by

$$
Q_{\mu}^{\prime}=\sum_{\lambda} K_{\lambda, \mu}(q) s_{\lambda}
$$

where the sum runs over the partitions of length at most $n, K_{\lambda, \mu}(q)$ is the Lusztig $q$-analogue of weight multiplicity associated to $(\lambda, \mu)$ and $s_{\lambda}$ the Schur function indexed by $\lambda$, that is the Weyl character of the irreducible finite dimensional $G L_{n}$-module $V(\lambda)$. Since $K_{\lambda, \mu}(1)$ is equal to the dimension of the weight space $\mu$ in $V(\lambda), Q_{\mu}^{\prime}$ can be regarded as a quantization of the homogeneous function $h_{\mu}$. In [7], Lascoux, Leclerc and Thibon have introduced a new family of symmetric functions $H_{\mu}^{\ell}$ depending on a fixed nonnegative integer $\ell$ which interpolate between the Schur functions $s_{\mu}$ and the Hall-Littlewood functions $Q_{\mu}^{\prime}$. The polynomials $H_{\mu}^{\ell}$ can be combinatorially described in terms of the spin statistic on certain generalized Young tableaux called $\ell$-ribbon tableaux. These Ribbon tableaux naturally appear in the description of the action of the power sum plethysm $\psi_{\ell}$ on symmetric functions. Recall that for any symmetric function $f, \psi_{\ell}(f)$ is obtained by replacing in $f$ each variable $x_{i}$ by $x_{i}^{\ell}$. In particular $\psi_{\ell}$ multiplies the degrees by $\ell$. The space of symmetric functions is endowed with an inner product $\langle\cdot, \cdot\rangle$ which makes the basis of Schur functions orthonormal. Then $\varphi_{\ell}$, the adjoint operator of $\psi_{\ell}$, divides the degrees by $\ell$. It is well known that $\varphi_{\ell}\left(s_{\mu}\right)$ can be computed from Jacobi-Trudi determinantal identity. Namely we have

$$
\begin{equation*}
\varphi_{\ell}\left(s_{\mu}\right)=0 \text { or } \varphi_{\ell}\left(s_{\mu}\right)=\varepsilon\left(\sigma_{0}\right) s_{\mu^{(0)}} \cdots s_{\mu^{(\ell-1)}} \tag{1}
\end{equation*}
$$

where $\varepsilon\left(\sigma_{0}\right)= \pm 1$ is the signature of a permutation $\sigma_{0} \in S_{n}$ and $\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)$ a $\ell$-tuple of partitions defined by $\ell$ and $\mu$. By expanding $\varphi_{\ell}\left(s_{\mu}\right)$ on the basis of Schur functions we obtain then

$$
\begin{equation*}
\varphi_{\ell}\left(s_{\mu}\right)=\varepsilon\left(\sigma_{0}\right) \sum_{\lambda} c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda} s_{\lambda} \tag{2}
\end{equation*}
$$

where $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda}$ is the Littlewood-Richardson coefficient giving the multiplicity of $V(\lambda)$ in the tensor product $V\left(\mu^{(0)}\right) \otimes \cdots \otimes V\left(\mu^{(\ell-1}\right)$. When $\ell=1$, one has $\varphi_{\ell}\left(s_{\ell \mu}\right)=s_{\mu}$ and when $\ell>n$ one can prove that $\varphi_{\ell}\left(s_{\ell \mu}\right)=h_{\mu}$. Thus the functions $h_{\mu}^{(\ell)}=\varepsilon\left(\sigma_{0}\right) \varphi_{\ell}\left(s_{\ell \mu}\right)$ interpolate between the functions $s_{\mu}$ and $h_{\mu}$ and have nonnegative coefficients on the basis of Schur functions.
In [7], the authors have interpreted the algebra of symmetric functions as the bosonic Fock space representation of the quantum affine Lie algebra $U_{q}\left(\widehat{s l_{n}}\right)$. This permits to introduce a natural quantization $\psi_{q, \ell}$ of the power sum plethysm $\psi_{\ell}$. Let $\varphi_{q, \ell}$ be the adjoint operator of $\psi_{q, \ell}$ with respect to $<\cdot, \cdot>$. The function $H_{\mu}^{\ell}$ is then defined as a simple renormalization of $\varphi_{q, \ell}\left(s_{\ell_{\mu}}\right)$. This gives an identity of the form

$$
H_{\mu}^{\ell}=\sum_{\lambda} c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}(q) s_{\lambda}
$$

where the polynomial $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}(q)$ is a $q$-analogue of $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda}$.
Lusztig $q$-analogues $K_{\lambda, \mu}(q)$ are particular affine Kazhdan-Lusztig polynomials. These polynomials arise in affine Hecke algebra theory as the entries of the transition matrix between the basis of generators and a special basis defined by Lusztig. By replacing the affine Hecke algebra $\widehat{H}$ by one of its parabolic module $\widehat{H} \nu$ ( $\nu$ being a weight of the affine root system considered), Deodhar has introduced analogues of the Kazhdan-Lusztig polynomials. In [8], it is shown that the family constituted by these parabolic Kazhdan-Lusztig polynomials contains in particular the $q$-analogues $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda}(q)$. By a result of Kashiwara and Tanisaki [6], this implies notably that the coefficients of the polynomial $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda}(q)$ are nonnegative integers.

The aim of the paper is to introduce analogues of the polynomials $H_{\mu}^{\ell}$ for the classical Lie groups $G=S O_{2 n+1}, S p_{2 n}$ and $S O_{2 n}$ which interpolate between the Weyl characters and the Hall-Littlewood functions associated to $G$. Write also $s_{\lambda}$ for the Weyl character of the irreducible $G$-module $V(\lambda)$ of highest weight $\lambda$. We define the plethysm operator $\varphi_{\ell}$ and its dual $\psi_{\ell}$ on the $\mathbb{Z}$-algebra generated by these Weyl characters. Given a Levi subgroup $L$ of $G$ and $\gamma$ one of its highest weight, we denote by $\left[V(\lambda): V_{L}(\gamma)\right]$ the multiplicity of the irreducible $L$-module $V_{L}(\gamma)$ of highest weight $\gamma$ in the restriction of $V(\lambda)$ to $L$. Then, providing that $\ell$ is odd when $G=S p_{2 n}$ or $S O_{2 n}$, we establish for any Weyl character $s_{\mu}$ such that $\varphi_{\ell}\left(s_{\mu}\right) \neq 0$, a formula of the type

$$
\begin{equation*}
\varphi_{\ell}\left(s_{\mu}\right)=\varepsilon\left(w_{0}\right) \sum_{\lambda}\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right] s_{\lambda} \tag{3}
\end{equation*}
$$

where $\varepsilon\left(w_{0}\right)$ is the signature of an element $w_{0} \in W$ the Weyl group of $G, L$ a Levi subgroup of $G$ and $\binom{\mu}{\ell}$ a dominant weight associated to $L$. The procedure which yields $w_{0}, L$ and $\binom{\mu}{\ell}$ from $\ell$ and $\mu$ can be regarded as an analogue of the algorithm computing the $\ell$-quotient of a partition which implicitly appears in (11). The identity (2) can also be rewritten as in (3). Indeed, take $L=G L_{r_{0}} \times \cdots \times G L_{r_{\ell-1}}$ where for any $k=1, \ldots, \ell-1, r_{k}$ is the length of $\mu^{(k)}$. Then $\binom{\mu}{\ell}=\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)$ can be interpreted as a dominant weight for the Levi subgroup $L$ of $G L_{n}$ and we have the duality $c_{\mu^{(0)}, \ldots, \mu^{(\ell-1)}}^{\lambda}=\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right]$. The surprising constraint $\ell$ odd when $G=S p_{2 n}$ or $S O_{2 n}$ appearing in (31) follows from the fact that the procedure giving $w_{0}, L$ and $\binom{\mu}{\ell}$ mentioned above depends not only on the Lie group considered but also on the parity of the integer $\ell$. For $G=S O_{2 n+1}$ the coefficients of $\varepsilon\left(w_{0}\right) \varphi_{\ell}\left(s_{\mu}\right)$ on the basis of Weyl characters are always branching coefficients corresponding to restriction to $L$. For $G=S p_{2 n}$ or $S O_{2 n}$ this is only true when $\ell$ is odd. Otherwise the coefficients of the expansion of $\varphi_{\ell}\left(s_{\mu}\right)$ may have opposite signs and their absolute value cannot be identified with branching coefficients.
To define the functions $H_{\mu}^{\ell}$ in type $B, C$ or $D$, we prove the equalities

$$
\left|<\psi_{\ell}\left(s_{\lambda}\right), s_{\mu}>\left|=\left|<s_{\lambda}, \varphi\left(s_{\mu}\right)>\right|=P_{\mu+\rho, \ell \lambda+\rho}^{-}(1)\right.\right.
$$

which show that the coefficients of the expansion of $\psi_{\ell}\left(s_{\lambda}\right)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q=1$. By using (3) this gives, providing $\ell$ is odd for $G=S p_{2 n}$ or $S O_{2 n}$

$$
\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right]=P_{\mu+\rho, \ell \lambda+\rho}^{-}(1)
$$

We then introduce the functions

$$
G_{\mu}^{\ell}=\sum_{\lambda}\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right]_{q} s_{\lambda}
$$

where $\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right]_{q}=P_{\mu+\rho, \ell \lambda+\rho}^{-}(q)$. This yields nonnegative $q$-analogues of the branching coefficients $\left[V(\lambda): V_{L}\binom{\mu}{\ell}\right]$. The functions $H_{\mu}^{\ell}$ are then defined by setting $H_{\mu}^{\ell}=G_{\ell \mu}^{\ell}$. We obtain the identities $H_{\mu}^{1}=s_{\mu}$ and $H_{\mu}^{\ell}=Q_{\mu}^{\prime}$ when $\ell$ is sufficiently large. Thus the functions $H_{\mu}^{\ell}$ interpolate between the Weyl characters and the Hall-Littlewood functions associated to $G$.

The paper is organized as follows. In Section 2 we recall the necessary background on classical root systems, Weyl characters, Levi subgroups and their corresponding branching coefficients. In Section 3 , we define the plethysm operators $\psi_{\ell}$ and their dual operators $\varphi_{\ell}$. By abuse of notation, we also denote by $\varphi_{\ell}$ the linear operator on the group algebra $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ with basis the formal exponentials ( $e^{\beta}$ ) such that

$$
\varphi_{\ell}\left(e^{\beta}\right)=\left\{\begin{array}{l}
e^{\beta / \ell} \text { if } \beta \in(\ell \mathbb{Z})^{n} \\
0 \text { otherwise }
\end{array} \quad \text { for any } \beta \in \mathbb{Z}^{n}\right.
$$

We then show how the determination of $\varphi_{\ell}\left(s_{\mu}\right)$ can be reduced to the computation of the polynomial

$$
\varphi_{\ell}\left(e^{\mu} \prod_{\alpha \in R_{+}}\left(1-e^{\alpha}\right)\right)
$$

where $R_{+}$is the set of positive roots corresponding to the Lie group $G$. This permits to establish formulas (3) providing $\ell$ is odd when $G=S p_{2 n}$ or $S O_{2 n}$. For completion we have also included the case $G=G L_{n}$ and showed why (3) cannot hold when $\ell$ is even and $G=S p_{2 n}$ or $S O_{2 n}$. To make the paper self contained, we have summarized in Section 4 some necessary results on affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials. Section 5 is devoted to the definition of the polynomials $G_{\mu}^{\ell}$ and $H_{\mu}^{\ell}$ and to their links with the Weyl characters and the Hall-Littlewood functions. Finally we briefly discuss in Section 6 the problem of defining nonnegative $q$-analogues of tensor product multiplicities when $G \neq G L_{n}$. We add also a few remarks concerning the exceptional root systems.

Acknowledgments: The author want to thank B. Leclerc for very helpful and stimulating discussions on the results of [7] and [8].

## 2 Background

### 2.1 Classical root systems

In the sequel $G$ is one of the complex Lie groups $G L_{n}, S p_{2 n}, S O_{2 n+1}$ or $S O_{2 n}$ and $\mathfrak{g}$ its Lie algebra. We follow the convention of [6] to realize $G$ as a subgroup of $G L_{N}$ and $\mathfrak{g}$ as a subalgebra of $\mathfrak{g l}_{N}$ where

$$
N=\left\{\begin{array}{l}
n \text { when } G=G L_{n} \\
2 n \text { when } G=S p_{2 n} \text { or } S O_{2 n} \\
2 n+1 \text { when } G=S O_{2 n+1}
\end{array} .\right.
$$

With this convention the maximal torus $T$ of $G$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of $G$ and $\mathfrak{g}$. Similarly the Borel subgroup $B$ of $G$ and the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of $G$ and $\mathfrak{g}$.
Let $d_{N}$ be the linear subspace of $\mathfrak{g l}_{N}$ consisting of the diagonal matrices. For any $i \in I_{n}=\{1, \ldots, n\}$, write $\varepsilon_{i}$ for the linear map $\varepsilon_{i}: d_{N} \rightarrow \mathbb{C}$ such that $\varepsilon_{i}(D)=\delta_{n-i+1}$ for any diagonal matrix $D$ whose $(i, i)$-coefficient is $\delta_{i}$. Then $\left(\varepsilon_{1}, \ldots, \varepsilon_{n}\right)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^{*}$ (the real part of $\mathfrak{h}^{*}$ ). Let $(\cdot, \cdot)$ be the corresponding nondegenerate symmetric bilinear form defined on $\mathfrak{h}_{\mathbb{R}}^{*}$. Write $R$ for the root system associated to $G$. For any $\alpha \in R$ we set $\alpha^{\vee}=\frac{\alpha}{(\alpha, \alpha)}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g}=\mathfrak{h} \oplus \coprod_{\alpha \in R} \mathfrak{g}_{\alpha}$. We take for the set of positive roots:

$$
\left\{\begin{array}{l}
R^{+}=\left\{\varepsilon_{j}-\varepsilon_{i}, \varepsilon_{j}+\varepsilon_{i} \text { with } 1 \leq i<j \leq n\right\} \text { for the root system } A_{n-1} \\
R^{+}=\left\{\varepsilon_{j}-\varepsilon_{i}, \varepsilon_{j}+\varepsilon_{i} \text { with } 1 \leq i<j \leq n\right\} \cup\left\{\varepsilon_{i} \text { with } 1 \leq i \leq n\right\} \text { for the root system } B_{n} \\
R^{+}=\left\{\varepsilon_{j}-\varepsilon_{i}, \varepsilon_{j}+\varepsilon_{i} \text { with } 1 \leq i<j \leq n\right\} \cup\left\{2 \varepsilon_{i} \text { with } 1 \leq i \leq n\right\} \text { for the root system } C_{n} \\
R^{+}=\left\{\varepsilon_{j}-\varepsilon_{i}, \varepsilon_{j}+\varepsilon_{i} \text { with } 1 \leq i<j \leq n\right\} \text { for the root system } D_{n}
\end{array} .\right.
$$

Let $\rho$ be the half sum of positive roots. Set $J_{n}=\{\bar{n}<\cdots<\overline{1}<1<\cdots<n\}$ where, for each integer $i=1, \ldots, n$, we have written $\bar{i}$ for the negative integer $-i$. For any $x \in J_{n}$ we have $\overline{\bar{x}}=x$ and we set $|x|=x$ if $x>0,|x|=\bar{x}$ otherwise. Given a subset $U \subset J_{n}$, we define $|U|=\{|x| \mid x \in U\}$ and $\bar{U}=\{\bar{x} \mid x \in U\}$.
The Weyl group of $G L_{n}$ is the symmetric group $S_{n}$ and for $G=S O_{2 n+1}, S p_{2 n}$ or $S O_{2 n}$, the Weyl group $W$ of the Lie group $G$ is the subgroup of the permutation group of $J_{n}$ generated by the permutations

$$
\left\{\begin{array}{l}
s_{i}=(i, i+1)(\bar{i}, \overline{i+1}), i=1, \ldots, n-1 \text { and } s_{n}=(n, \bar{n}) \text { for the root systems } B_{n} \text { and } C_{n} \\
s_{i}=(i, i+1)(\bar{i}, \overline{i+1}), i=1, \ldots, n-1 \text { and } s_{n}^{\prime}=(n, \overline{n-1})(n-1, \bar{n}) \text { for the root system } D_{n}
\end{array}\right.
$$

where for $a \neq b(a, b)$ is the simple transposition which switches $a$ and $b$. For types $B_{n}$ and $C_{n}, W$ is the group of signed permutations. It is the subgroup of the permutation group of $J_{n}$ containing the permutations $w$ such that $w(\bar{i})=\overline{w(i)}$. For type $D_{n}$, the elements of $W$ verify the additional constraint $\operatorname{card}\left\{i \in I_{n} \mid w(i)<0\right\} \in 2 \mathbb{N}$. We identify the subgroup of $W$ generated by $s_{i}=(i, i+1)(\bar{i}, \overline{i+1})$, $i=1, \ldots, n-1$ with the symmetric group $S_{n}$. The signature $\varepsilon$ of $w \in W$ is defined by $\varepsilon(w)=(-1)^{l(w)}$ where $l$ is the length function corresponding to the above sets of generators. Consider the increasing sequence $K=\left(\bar{i}_{p}, \ldots, \bar{i}_{1}, i_{1}, \ldots, i_{p}\right) \subset J_{n}$. For $X=B, D$ set

$$
W_{X, K}=\left\{w \in W \text { of type } X_{n} \mid w(x)=x \text { for any } x \notin K\right\}
$$

Then, $W_{X, K}$ is isomorphic to the Weyl group of type $X_{p}$. Let $\varepsilon_{X, K}$ be the corresponding signature.
Lemma 2.1.1 Consider $X=B, D$ and $w \in W_{X, K}$. Then we have $\varepsilon_{X, K}(w)=\varepsilon(w)$.
Proof. Suppose $X=B$. The generators of the Weyl group $W_{X, K}$ are the $t_{k}=\left(i_{k}, i_{k+1}\right)\left(\bar{i}_{k}, \bar{i}_{k+1}\right)$, $k=1, \ldots, p-1$ and $s_{n}=\left(\bar{i}_{p}, i_{p}\right)$. One verifies easily that, considered as elements of $W$, they have an odd length. We proceed similarly when $X=D$.

The action of $w \in W$ on $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathfrak{h}_{\mathbb{R}}^{*}$ is defined by

$$
\begin{equation*}
w \cdot\left(\beta_{1}, \ldots, \beta_{n}\right)=\left(\beta_{1}^{w}, \ldots, \beta_{n}^{w}\right) \tag{4}
\end{equation*}
$$

where $\beta_{i}^{w}=\beta_{w(i)}$ if $\sigma(i) \in I_{n}$ and $\beta_{i}^{w}=-\beta_{w(\bar{i})}$ otherwise. The dot action of $W$ on $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathfrak{h}_{\mathbb{R}}^{*}$ is defined by

$$
\begin{equation*}
w \circ \beta=w \cdot(\beta+\rho)-\rho \tag{5}
\end{equation*}
$$

The fundamental weights of $\mathfrak{g}$ belong to $\left(\frac{\mathbb{Z}}{2}\right)^{n}$. More precisely we have $\omega_{i}=\left(0^{i}, 1^{i}\right) \in \mathbb{N}^{n}$ for $i<n-1$ and also $i=n-1$ for $\mathfrak{g} \neq \mathfrak{g}_{D_{\infty}}, \omega_{n}^{C_{n}}=\left(1^{n}\right), \omega_{n}^{B_{n}}=\omega_{n}^{D_{n}}=\left(\frac{1}{2}^{n}\right)$ and $\omega_{n-1}^{D_{n}}=\left(-\frac{1}{2}, \frac{1}{2}^{n-1}\right)$. The weight lattice $P$ of $\mathfrak{g}$ can be considered as the $\mathbb{Z}$-sublattice of $\left(\frac{\mathbb{Z}}{2}\right)^{n}$ generated by the $\omega_{i}, i \in I$. For any $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in P$, we set $|\beta|=\beta_{1}+\cdots+\beta_{n}$. Write $P^{+}$for the cone of dominant weights of $G$. With our convention, a partition of length $m$ is a weakly increasing sequence of $m$ nonnegative integers. Denote by $\mathcal{P}_{n}$ the set of partitions with at most $n$ parts. Each partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in \mathcal{P}_{n}$ will be identified with the dominant weight $\sum_{i=1}^{m} \lambda_{i} \varepsilon_{i}$. Then the irreducible finite dimensional polynomial representations of $G$ are parametrized by the partitions of $\mathcal{P}_{n}$. For any $\lambda \in \mathcal{P}_{n}$, denote by $V(\lambda)$ the irreducible finite dimensional representation of $G$ of highest weight $\lambda$. We will also need the irreducible rational representations of $G L_{n}$. They are indexed by the $n$-tuples

$$
\begin{equation*}
\left(\gamma^{-}, \gamma^{+}\right)=\left(-\gamma_{q}^{-}, \ldots,-\gamma_{1}^{-}, \gamma_{1}^{+}, \gamma_{2}^{+}, \ldots, \gamma_{p}^{+}\right) \tag{6}
\end{equation*}
$$

where $\gamma^{+}=\left(\gamma_{1}^{+}, \gamma_{2}^{+}, \ldots, \gamma_{p}^{+}\right)$and $\gamma^{-}=\left(\gamma_{1}^{-}, \ldots, \gamma_{q}^{-}\right)$are partitions of length $p$ and $q$ such that $p+q=n$. Write $\widetilde{\mathcal{P}}_{n}$ for the set of such $n$-tuples and denote also by $V(\gamma)$ the irreducible rational representation of $G L_{n}$ of highest weight $\gamma=\left(\gamma^{-}, \gamma^{+}\right) \in \widetilde{\mathcal{P}}_{n}$.

In the sequel, our computations will also make appear root subsystems of the root systems $R$ described above. Suppose that $G$ is of type $X_{n}$ with $X_{n}=A_{n-1}, B_{n}, C_{n}$ or $D_{n}$. Let $I=\left(i_{1}, \ldots, i_{r}\right)$ be an increasing sequence of integers belonging to $I_{n}$, that is $i_{k} \in I_{n}$ for any $k=1, . ., r$ and $i_{1}<\cdots<i_{r}$. Then

$$
R_{I}=\left\{\alpha \in R \cap \oplus_{i \in I} \mathbb{Z} \varepsilon_{i}\right\}
$$

is a root subsystem of $R$ of type $X_{r}$. Write $R_{I}^{+}$for the set of positive roots in $R_{I}$. Then we have $R_{I}^{+}=R_{I} \cap R^{+}$. The dominant weights associated to $R_{I}$ have the form $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ where $\lambda_{i} \neq 0$ only if $i \in I$ and $\lambda^{(I)}=\left(\lambda_{i_{1}}, \ldots, \lambda_{i_{r}}\right) \in \mathcal{P}_{r}$. We slightly abuse the notation by identifying $\lambda$ with $\lambda^{(I)}$. Consider an increasing sequence $X=\left(x_{1}, \ldots, x_{r}\right)$ of integers belonging to $J_{n}$ such that $\left|x_{k}\right|=\left|x_{k^{\prime}}\right|$ if and only if $k=k^{\prime}$. For any integer $i=1, \ldots, n$, set $\varepsilon_{\bar{i}}=-\varepsilon_{i}$. Then

$$
R_{A, X}=\left\{ \pm\left(\varepsilon_{x_{j}}-\varepsilon_{x_{i}}\right) \mid 1 \leq i<j \leq r\right\}
$$

is a root subsystem of $R$ of type $A_{r-1}$. To see this, consider the linear map $\theta_{X}: \mathbb{Z}^{r} \rightarrow \mathbb{Z}^{n}$ such that $\theta_{X}\left(\varepsilon_{i}\right)=\varepsilon_{x_{i}}$. The map $\theta$ is injective and preserves the scalar product in $\mathbb{Z}^{r}$ and $\mathbb{Z}^{n}$. Moreover the root system $\left\{ \pm\left(\varepsilon_{j}-\varepsilon_{i}\right) \mid 1 \leq i<j \leq r\right\} \subset \mathbb{Z}^{r}$ of type $A_{r}$ is sent on $R_{A, X}$ by $\theta_{X}$. The set of positive roots in $R_{A, X}$ is equal to $R_{A, X}^{+}=R_{A, X} \cap R^{+}$. Denote by $s \in\{1, \ldots, r\}$ the maximal integer such that $x_{s}<0$. We associate to $X$, the increasing sequence of indices $I \subset I_{n}$ defined by

$$
\begin{equation*}
I=\left(\bar{x}_{s}, \ldots, \bar{x}_{1}, x_{s+1}, \ldots, x_{r}\right) . \tag{7}
\end{equation*}
$$

It will be useful to consider the weights corresponding to $R_{A, X}$ as the $r$-tuples $\beta=\left(\beta_{x_{1}}, \ldots, \beta_{x_{r}}\right)$ with coordinates indexed by $X$. The coordinates $\left(\beta_{1}^{\prime}, \ldots, \beta_{n}^{\prime}\right)$ of $\beta$ on the initial basis $\left(\varepsilon_{1}, \ldots, \varepsilon_{n}\right)$ are such that $\beta_{i}^{\prime}=\beta_{x_{a}}$ if $i=x_{a} \in X, \beta_{i}^{\prime}=-\beta_{x_{a}}$ if $\bar{i}=x_{a} \in X$ and $\beta_{i}^{\prime}=0$ otherwise. With this convention the dominant weights for $R_{A, X}$ have the form

$$
\begin{equation*}
\lambda^{(X)}=\left(\lambda_{x_{1}}, \ldots, \lambda_{x_{r}}\right) \in \widetilde{\mathcal{P}}_{r} . \tag{8}
\end{equation*}
$$

This simply means that we have chosen to expand the weights of $R_{A, X}$ on the basis $\left\{\varepsilon_{x} \mid x \in X\right\}$ rather than on the basis $\left\{\varepsilon_{i} \mid i \in I\right\}$ to preserve the identification of the dominant weights with the nondecreasing $r$-tuples of integers.

Example 2.1.2 Take $G=S p_{10}$.

- For $I=(2,4,5)$ we have

$$
R_{I}^{+}=\left\{\varepsilon_{5} \pm \varepsilon_{4}, \varepsilon_{5} \pm \varepsilon_{2}, \varepsilon_{4} \pm \varepsilon_{2}, 2 \varepsilon_{2}, 2 \varepsilon_{4}, 2 \varepsilon_{5}\right\}
$$

which is the set of positive roots of a root system of type $C_{3}$. The weight $\lambda=(1,2,2)$ is dominant for $G_{I}$. Considered as a weight of $S p_{10}$, we have $\lambda=(0,1,0,2,2)$.

- For $X=(\overline{5}, \overline{2}, 1,4)$ we have

$$
R_{A, X}^{+}=\left\{\varepsilon_{4}-\varepsilon_{1}, \varepsilon_{5}-\varepsilon_{2}, \varepsilon_{1}+\varepsilon_{2}, \varepsilon_{1}+\varepsilon_{5}, \varepsilon_{4}+\varepsilon_{2}, \varepsilon_{4}+\varepsilon_{5}\right\}
$$

which is the set of positive roots of a root system of type $A_{3}$. The weight $\gamma=(-3,-1,4,5)$ is dominant for $G_{X}$. Considered as a weight of $S p_{10}$, we have $\gamma=(4,1,0,5,3)$.

### 2.2 Levi subgroups

Consider $p \geq 1$ an integer. Let $I^{(0)}=\left(i_{1}^{(0)}, \ldots, i_{r_{0}}^{(0)}\right)$ be an increasing sequence of integers in $I_{n}$. For $k=1, \ldots, p$, consider increasing sequences $X^{(k)}=\left(x_{1}^{(k)}, \ldots, x_{r_{k}}^{(k)}\right) \subset J_{n}$ such that $\operatorname{card}\left(X^{(k)}\right)=r_{k}$. Let $s_{k}$ be maximal in $\left\{1, \ldots, r_{k}\right\}$ such that $x_{s_{k}}^{(k)}<0$. Set

$$
\begin{equation*}
I^{(k)}=\left(\bar{x}_{s_{k}}^{(k)}, \ldots, \bar{x}_{1}^{(k)}, x_{s_{k}+1}^{(k)}, \ldots, x_{r_{k}}^{(k)}\right) \subset I_{n} . \tag{9}
\end{equation*}
$$

We suppose that the sets $I^{(k)}, k=0, \ldots, p$ are pairwise disjoint and verify $\cup_{k=0}^{p} I^{(k)}=I_{n}$. Set $\mathcal{I}=$ $\left\{I^{(0)}, X^{(1)}, \ldots, X^{(p)}\right\}$ and

$$
R_{\mathcal{I}}=R_{I^{(0)}} \cup \bigcup_{k=1}^{p} R_{A, X^{(k)}}
$$

Then $\mathfrak{g}_{\mathcal{I}}=\mathfrak{h} \oplus \coprod_{\alpha \in R_{\mathcal{I}}} \mathfrak{g}_{\alpha}$ is a Lie subalgebra of $\mathfrak{g}$. Its corresponding Lie group $G_{\mathcal{I}}$ is a Levi subgroup. More precisely we have

$$
G_{\mathcal{I}} \simeq\left\{\begin{array}{l}
G L_{r_{0}} \times G L_{r_{1}} \times \cdots \times G L_{r_{p}} \text { for } G=G L_{n} \\
S O_{2 r_{0+1}} \times G L_{r_{1}} \times \cdots \times G L_{r_{p}} \text { for } G=S O_{2 n+1} \\
S p_{2 r_{0}} \times G L_{r_{1}} \times \cdots \times G L_{r_{p}} \text { for } G=S p_{2 n} \\
S O_{2 r_{0}} \times G L_{r_{1}} \times \cdots \times G L_{r_{p}} \text { for } G=S O_{2 n}
\end{array} .\right.
$$

The root system associated to $G_{\mathcal{I}}$ is $R_{\mathcal{I}}$. Denote by $P_{\mathcal{I}}^{+}$its cone of dominant weights. The weight lattice of $G_{\mathcal{I}}$ coincides with that of $G$ since the Lie algebras $\mathfrak{g}_{\mathcal{I}}$ and $\mathfrak{g}$ have the same Cartan subalgebra. The elements of $P_{\mathcal{I}}^{+}$are the $(p+1)$-tuples $\lambda=\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(p)}\right)$ where $\lambda^{(0)}=\left(\lambda_{i} \mid i \in I^{(0)}\right)$ is a dominant weight of $R_{G, I^{(0)}}$ and for any $k=1, \ldots, p, \lambda^{(k)}=\left(\lambda_{i} \mid i \in X^{(k)}\right)$ is a dominant weight of $R_{G, X^{(k)}}$. For any $\lambda \in P_{\mathcal{I}}^{+}$, we denote by $V_{\mathcal{I}}(\lambda)$ the irreducible finite dimensional $G_{\mathcal{I}}$-module of highest weight $\lambda$. Each weight $\beta=\left(\beta^{(0)}, \beta^{(1)}, \ldots, \beta^{(p)}\right) \in P_{\mathcal{I}}$ can be considered as a weight $\beta=\left(\beta_{1}^{\prime}, \ldots, \beta_{n}^{\prime}\right)$ of $P$. With the convention (8) we have then $\beta_{i}^{\prime}=\beta_{i_{a}^{(0)}}$ if $i=i_{a}^{(0)} \in I^{(0)}$ and for any $k=1, \ldots, p, \beta_{i}^{\prime}=\beta_{i_{a}^{(k)}}$ if $i=i_{a}^{(k)} \in X^{(k)}, \beta_{i}^{\prime}=-\beta_{i_{a}^{(k)}}$ if $\bar{i}=i_{a}^{(k)} \in X^{(k)}$. In the sequel we identify the two expressions

$$
\begin{equation*}
\beta=\left(\beta^{(0)}, \beta^{(1)}, \ldots, \beta^{(p)}\right) \text { and } \beta=\left(\beta_{1}^{\prime}, \ldots, \beta_{n}^{\prime}\right) \tag{10}
\end{equation*}
$$

of the weights of $P_{\mathcal{I}}$.

### 2.3 Weyl characters and dual bases

We refer the reader to [10 and [12] for a detailed exposition of the results used in this paragraph. We use for a basis of the group algebra $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$, the formal exponentials $\left(e^{\beta}\right)_{\beta \in \mathbb{Z}^{n}}$ satisfying the relations $e^{\beta_{1}} e^{\beta_{2}}=e^{\beta_{1}+\beta_{2}}$. We furthermore introduce $n$ independent indeterminates $x_{1}, \ldots, x_{n}$ in order to identify $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ with the ring of polynomials $\mathbb{Z}\left[x_{1}, \ldots, x_{n}, x_{1}^{-1}, \ldots, x_{n}^{-1}\right]$ by writing $e^{\beta}=x_{1}^{\beta_{1}} \cdots x_{n}^{\beta_{n}}=x^{\beta}$ for any $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{Z}^{n}$. Define the action of the Weyl group $W$ on $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ by $w \cdot x^{\beta}=x^{w(\beta)}$. The Weyl character $s_{\beta}$ is defined by

$$
s_{\beta}=\frac{a_{\beta+\rho}}{a_{\rho}} \text { where } a_{\beta}=\sum_{w \in W}(-1)^{l(\sigma)}\left(w \cdot x^{\beta}\right)
$$

For any $\beta \in \mathbb{Z}^{n}$ we have

$$
s_{\beta}=\left\{\begin{array}{l}
(-1)^{l(w)} s_{\lambda} \text { if there exists } w \in W \text { and } \lambda \in \mathcal{P}_{n} \text { such that } \lambda=w \circ \beta  \tag{11}\\
0 \text { otherwise }
\end{array}\right.
$$

Let $A$ be the $\mathbb{Z}$-algebra generated by the characters $s_{\lambda}, \lambda \in \mathcal{P}_{n}$. For any $\beta \in \mathbb{Z}^{n}$, denote by $W_{\beta}$ the stabilizer of $\beta$ under the action of the Weyl group $W$ and $W^{\beta}$ a set of representatives in $W / W_{\beta}$ with minimal length. Then the functions

$$
m_{\beta}=\sum_{w \in W^{\beta}} w \cdot x^{\beta}
$$

belong to $A$. Moreover $\left\{m_{\lambda} \mid \lambda \in \mathcal{P}_{n}\right\}$ is a basis of $A$. We have the decomposition

$$
\begin{equation*}
s_{\lambda}=\sum_{\mu \in \mathcal{P}_{n}} K_{\lambda, \mu} m_{\mu} \tag{12}
\end{equation*}
$$

where $K_{\lambda, \mu}$ is equal to the dimension of the weight space $\mu$ in the irreducible representation $V(\lambda)$. There exists an inner product $\langle\cdot, \cdot\rangle$ on $A$ which makes the characters $s_{\lambda}$ orthonormal. We denote by $\left\{h_{\mu} \mid \mu \in \mathcal{P}_{n}\right\}$ the dual basis of $\left\{m_{\lambda} \mid \lambda \in \mathcal{P}_{n}\right\}$ with respect to $\langle\cdot, \cdot>$. The homogeneous functions $h_{\mu}$ are given in terms of the Weyl characters by the decomposition

$$
\begin{equation*}
h_{\mu}=\sum_{\lambda \in \mathcal{P}_{n}} K_{\lambda, \mu} s_{\lambda} . \tag{13}
\end{equation*}
$$

This decomposition is infinite in general when $G \neq G L_{n}$. Nevertheless, by embedding $A$ in the ring $\widehat{A}$ of universal characters defined by Koike and Terada [6], it makes sense to consider formal series in the characters $s_{\lambda}, \lambda \in \mathcal{P}_{n}$. Note that the function $h_{\mu}$ is not the character of the representation $V\left(\mu_{1} \omega_{1}\right) \otimes \cdots \otimes V\left(\mu_{n} \omega_{1}\right)$ when $G \neq G L_{n}$, For any $\beta \in \mathbb{Z}^{n}$ we define the function $h_{\beta}$ by

$$
\begin{equation*}
h_{\beta}=h_{\mu} \tag{14}
\end{equation*}
$$

where $\mu$ is the unique dominant weight contained in the orbit $W \cdot \beta$.

### 2.4 Jacobi-Trudi identities

Denote by $\mathcal{L}_{n}=\mathbb{K}\left[\left[x^{\beta}\right]\right]$ the vector space of formal series in the monomials $x^{\beta}$ with $\beta \in \mathbb{Z}$. We identify the ring of polynomials $\mathcal{F}_{n}=\mathbb{K}\left[x^{\beta}\right]$ with the sub-space of $\mathcal{L}_{n}$ containing the finite formal series. The vector space $\mathcal{L}_{n}$ is not a ring since $\beta \in \mathbb{Z}$. More precisely, the product $F_{1} \cdots F_{r}$ of the formal series
$F_{i}=\sum_{\beta \in E_{i}} x^{\beta^{(i)}} i=1, \ldots, r$ is defined if and only if, for any $\gamma \in \mathbb{Z}^{n}$, the number $N_{\gamma}$ of decompositions $\gamma=\beta^{(1)}+\cdots+\beta^{(r)}$ such that $\beta^{(i)} \in E_{i}$ is finite and in this case we have

$$
F_{1} \cdots F_{r}=\sum_{\gamma \in \mathbb{Z}^{n}} N_{\gamma} x^{\gamma}
$$

In particular the product $P \cdot F$ with $P \in \mathcal{P}_{n}$ and $F \in \mathcal{L}_{n}$ is well defined.
Set

$$
\nabla=\prod_{\alpha \in R_{+}} \frac{1}{\left(1-x^{\alpha}\right)} \text { and } \Delta=\prod_{\alpha \in R_{+}}\left(1-x^{\alpha}\right) .
$$

Then $\Delta \in \mathcal{F}_{n}$ and $\nabla \in \mathcal{L}_{n}$. We define two linear maps

$$
\mathrm{S}:\left\{\begin{array} { c } 
{ \mathcal { L } _ { n } \rightarrow \widehat { A } } \\
{ x ^ { \beta } \mapsto s _ { \beta } }
\end{array} \text { and H: } \left\{\begin{array}{c}
\mathcal{L}_{n} \rightarrow \widehat{A} \\
x^{\beta} \mapsto h_{\beta}
\end{array} .\right.\right.
$$

From Theorem 2.14 of [12] we obtain
Proposition 2.4.1 For any $\beta \in \mathbb{Z}^{n}$, $s_{\beta}=\sum_{w \in W}(-1)^{l(w)} h_{\beta+\rho-w \cdot \rho}$.
By using the identity

$$
\begin{equation*}
\Delta=x^{\rho} \sum_{w \in W}(-1)^{l(w)} x^{-w \cdot \rho} \tag{15}
\end{equation*}
$$

the previous proposition is equivalent to the following identity:

$$
\begin{equation*}
\mathrm{S}\left(x^{\beta}\right)=\mathrm{H}\left(\Delta \times x^{\beta}\right) . \tag{16}
\end{equation*}
$$

Proposition 2.4.2 For any $\beta \in \mathbb{Z}^{n}$ we have $\mathrm{H}\left(x^{\beta}\right)=\mathrm{S}\left(\nabla \times x^{\beta}\right)$.
Proof. Denote by $\chi_{\Delta}$ and $\chi_{\nabla}$ the linear maps defined on $\mathcal{L}_{n}$ by setting $\chi_{\Delta}\left(x^{\beta}\right)=\Delta \times x^{\beta}$ and $\chi_{\nabla}\left(x^{\beta}\right)=\nabla \times x^{\beta}$ respectively. By (16) we have $\mathrm{S}=\mathrm{H} \circ \chi_{\Delta}$. Moreover for any $\beta \in \mathbb{Z}^{n}, \chi_{\Delta} \circ \chi_{\nabla}\left(x^{\beta}\right)=x^{\beta}$. This gives $\mathrm{S}\left(\nabla \times x^{\beta}\right)=\mathrm{S} \circ \chi_{\nabla}\left(x^{\beta}\right)=\mathrm{H} \circ \chi_{\Delta} \circ \chi_{\nabla}\left(x^{\beta}\right)=\mathrm{H}\left(x^{\beta}\right)$.

### 2.5 Branching coefficients for the restriction to Levi subgroups

Consider $\mathcal{I}=\left\{I_{0}, X_{1}, \ldots, X_{p}\right\}$ as in 2.2. The set $\mathcal{I}$ characterizes a Levi subgroup $G_{\mathcal{I}}$ of $G$. Set

$$
\Delta_{\mathcal{I}}=\prod_{\alpha \in R_{\mathcal{I}}^{+}}\left(1-x^{\alpha}\right) \text { and } \nabla_{\mathcal{I}}=\prod_{\alpha \in R^{+}-R_{\mathcal{I}}^{+}} \frac{1}{\left(1-x^{\alpha}\right)}
$$

Then $\Delta_{\mathcal{I}} \in \mathcal{F}_{n}$ and $\nabla_{\mathcal{I}} \in \mathcal{L}_{n}$. Note that $\nabla_{\mathcal{I}}=\nabla \times \Delta_{\mathcal{I}}$.
As a formal series, $\nabla_{\mathcal{I}}$ can be expanded on the form

$$
\begin{equation*}
\nabla_{\mathcal{I}}=\sum_{\gamma \in \mathbb{Z}^{n}} \mathcal{P}_{\mathcal{I}}(\gamma) x^{\gamma} \tag{17}
\end{equation*}
$$

Consider $\lambda \in \mathcal{P}_{n}$ and $\mu=\left(\mu^{(0)}, \ldots, \mu^{(p)}\right)$ a dominant weight associated to $G_{\mathcal{I}}$. We denote by $[V(\lambda)$ : $\left.V_{\mathcal{I}}(\mu)\right]$ the multiplicity of the irreducible representation $V_{\mathcal{I}}(\mu)$ in the restriction of $V(\lambda)$ from $G$ to $G_{\mathcal{I}}$. The proposition below follows from Theorem 8.2.1 in [1]:

Proposition 2.5.1 Consider $\lambda \in \mathcal{P}_{n}$ and $\mu=\left(\mu^{(0)}, \ldots, \mu^{(p)}\right)$ a dominant weight of $P_{\mathcal{I}}^{+}$. Then

$$
\left[V(\lambda): V_{\mathcal{I}}(\mu)\right]=\sum_{w \in W}(-1)^{l(w)} \mathcal{P}_{\mathcal{I}}(w \circ \lambda-\mu)
$$

Define the linear map

$$
\left\{\begin{array}{c}
\mathrm{S}_{\mathcal{I}}: \mathcal{L}_{n} \rightarrow \widehat{A} \\
x^{\beta} \mapsto \mathrm{H}\left(\Delta_{\mathcal{I}} \times x^{\beta}\right)
\end{array} .\right.
$$

For any dominant weight $\mu \in P_{\mathcal{I}}^{+}$, set

$$
\begin{equation*}
S_{\mu, \mathcal{I}}=\mathrm{H}\left(\Delta_{\mathcal{I}} \times x^{\mu}\right)=\mathrm{S}_{\mathcal{I}}\left(x^{\mu}\right) . \tag{18}
\end{equation*}
$$

Proposition 2.5.2 With the above notations we have

$$
\begin{equation*}
S_{\mu, \mathcal{I}}=\sum_{\lambda \in \mathcal{P}_{n}}\left[V(\lambda): V_{\mathcal{I}}(\mu)\right] s_{\lambda} . \tag{19}
\end{equation*}
$$

Proof. For any $\beta \in \mathbb{Z}^{n}$, we have obtained in the proof of Proposition 2.4.2 the identity $\mathrm{H}\left(x^{\beta}\right)=$ $\mathrm{S} \circ \chi_{\nabla}\left(x^{\beta}\right)$. Denote by $\chi_{\Delta, \mathcal{I}}$ the linear map defined on $\mathcal{L}_{n}$ by setting $\chi_{\Delta, \mathcal{I}}\left(x^{\beta}\right)=\Delta_{\mathcal{I}} \times x^{\beta}$. We obtain $\mathrm{S}_{\mathcal{I}}\left(x^{\beta}\right)=\mathrm{H}\left(\Delta_{\mathcal{I}} \times x^{\beta}\right)=\mathrm{S} \circ \chi_{\nabla} \circ \chi_{\Delta, \mathcal{I}}\left(x^{\beta}\right)=\mathrm{S}\left(\nabla_{\mathcal{I}} \times x^{\beta}\right)$ since $\nabla_{\mathcal{I}}=\nabla \times \Delta_{\mathcal{I}}$. Thus by (17) this yields $\mathrm{S}_{\mathcal{I}}\left(x^{\beta}\right)=\sum_{\gamma \in \mathbb{Z}^{n}} \mathcal{P}_{\mathcal{I}}(\gamma) s_{\beta+\gamma}$. For any $\gamma$, we know by (11) that $s_{\beta+\gamma}=0$ or there exists $\lambda \in \mathcal{P}_{n}, w \in W$ such that $\lambda=w \circ(\beta+\gamma)$ and $s_{\beta+\gamma}=(-1)^{l(w)} s_{\lambda}$. This permits to write

$$
\mathrm{S}_{\mathcal{I}}\left(x^{\beta}\right)=\sum_{\lambda \in \mathcal{P}_{n}} \sum_{w \in W}(-1)^{l(w)} \mathcal{P}_{\mathcal{I}}(w \circ \lambda-\beta) s_{\lambda} .
$$

When $\beta=\mu$ is a dominant weight of $P_{\mathcal{I}}^{+}$, we obtain the desired identity by using Proposition 2.5.1,

## Remarks:

(i) : When $G=G_{\mathcal{I}}$ that is, when $r_{0}=n$ and $r_{1}=\cdots=r_{p}=0$, we have $\mu=\mu^{(0)}, \Delta_{\mathcal{I}}=\Delta$ and $\mathrm{H}_{\mathcal{I}}=\mathrm{H}$. Thus $S_{\mu, \mathcal{I}}=s_{\mu(0)}$. This can be recover by using (19) since in this case $\left[V(\lambda): V_{\mathcal{I}}(\mu)\right]=0$ except when $\lambda=\mu^{(0)}$.
(ii) : When $G_{\mathcal{I}}=H$ the maximal torus of $G$, that is when $n=p+1$ and $r_{k}=1$ for any $k=0, \ldots, p$, we have $\mu_{i}=\mu^{(i-1)}$ for any $i=1, \ldots, n, \Delta_{\mathcal{I}}=1$ and $\mathrm{H}_{\mathcal{I}}\left(x^{\beta}\right)=h_{\beta}$ for any $\beta \in \mathbb{Z}^{n}$. Hence $S_{\mu, \mathcal{I}}=h_{\mu}$. In this case $\left[V(\lambda): V_{\mathcal{I}}(\mu)\right]=K_{\lambda, \mu}$ for any $\lambda \in \mathcal{P}_{n}$. Thus (19) reduces to (131).
(iii) : By (i) and (ii) the functions $S_{\mu, \mathcal{I}}$ interpolates between the Weyl characters $s_{\mu}$ and the homogeneous functions $h_{\mu}$.
(iv) : When $G=G L_{n}$, we have the duality

$$
\left[V(\lambda): V_{\mathcal{I}}(\mu)\right]=c_{\mu^{(0)}, \ldots, \mu^{(p)}}^{\lambda}
$$

where $c_{\mu^{(0)}, \ldots, \mu^{(p)}}^{\lambda}$ is the Littlewood-Richardson coefficient associated to the multiplicity of $V(\lambda)$ in the tensor product $V_{\mu}=V\left(\mu^{(0)}\right) \otimes \cdots \otimes V\left(\mu^{(p)}\right)$. Thus we can write $S_{\mu, \mathcal{I}}=\sum_{\lambda \in \mathcal{P}_{n}} c_{\mu^{(0)}, \ldots, \mu^{(p)}}^{\lambda} s_{\lambda}$. This means that $S_{\mu, \mathcal{I}}$ is the character of $V_{\mu}$. Such a duality does not exist for $G=S p_{2 n}, S O_{2 n+1}$ or $S O_{2 n}$, (but see Section (6).

## 3 Plethysm on Weyl characters

### 3.1 The operators $\Psi_{\ell}$ and $\varphi_{\ell}$

Consider $\ell$ a positive integer. The power sum plethysm operator $\Psi_{\ell}$ is defined on $A$ be setting $\Psi_{\ell}\left(m_{\beta}\right)=$ $m_{\ell \beta}$ for any $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \in \mathbb{Z}^{n}$ where $\ell \beta=\left(\ell \beta_{1}, \ldots, \ell \beta_{n}\right)$. Since $\left\{m_{\lambda} \mid \lambda \in \mathcal{P}_{n}\right\}$ and $\left\{h_{\lambda} \mid \lambda \in \mathcal{P}_{n}\right\}$ are dual bases for the inner product $\langle\cdot, \cdot\rangle$, the adjoint operator $\varphi_{\ell}$ of $\Psi_{\ell}$ verifies

$$
\varphi_{\ell}\left(h_{\beta}\right)=\left\{\begin{array}{l}
h_{\beta / \ell} \text { if } \beta \in(\ell \mathbb{Z})^{n}  \tag{20}\\
0 \text { otherwise }
\end{array}\right.
$$

where $\beta / \ell=\left(\beta_{1} / \ell, \ldots, \beta_{n} / \ell\right)$ when $\beta \in(\ell \mathbb{Z})^{n}$.
By abuse of notation, we also denote by $\Psi_{\ell}$ and $\varphi_{\ell}$ the linear operators respectively defined on $\mathcal{L}_{n}$ by setting

$$
\Psi_{\ell}\left(x^{\beta}\right)=x^{\ell \beta} \text { and } \varphi_{\ell}\left(x^{\beta}\right)=\left\{\begin{array}{l}
x^{\beta / \ell} \text { if } \beta \in(\ell \mathbb{Z})^{n}  \tag{21}\\
0 \text { otherwise }
\end{array} \quad \text { for any } \beta \in \mathbb{Z}^{n}\right.
$$

Remark: Since $\Psi_{\ell}\left(x^{\beta} \times x^{\beta^{\prime}}\right)=\Psi_{\ell}\left(x^{\beta}\right) \times \Psi_{\ell}\left(x^{\beta^{\prime}}\right)$ for any $\beta, \beta^{\prime} \in \mathbb{Z}^{n}$ the map $\Psi_{\ell}$ is a morphism of algebra. This is not true for $\varphi_{\ell}$. Nevertheless, if $\left\{i_{1}, \ldots, i_{r}\right\}$ and $\left\{j_{1}, \ldots, j_{s}\right\}$ are disjoint subsets of $I_{n}$, $\iota=\left(\iota_{1}, \ldots, \iota_{r}\right) \in \mathbb{Z}^{r}$ and $\gamma=\left(\gamma_{1}, \ldots, \gamma_{s}\right) \in \mathbb{Z}^{s}$ we have

$$
\begin{equation*}
\varphi_{\ell}\left(x_{i_{1}}^{\iota_{1}} \cdots x_{i_{r}}^{\iota_{r}} \times x_{j_{1}}^{\gamma_{1}} \cdots x_{j_{r}}^{\gamma_{r}}\right)=\varphi_{\ell}\left(x_{i_{1}}^{\iota_{1}} \cdots x_{i_{r}}^{\iota_{r}}\right) \times \varphi_{\ell}\left(x_{j_{1}}^{\gamma_{1}} \cdots x_{j_{r}}^{\gamma_{r}}\right) . \tag{22}
\end{equation*}
$$

For any $\lambda \in \mathcal{P}_{n}, \Psi_{\ell}\left(s_{\lambda}\right)$ belongs to $A$ thus decomposes on the basis $\left\{s_{\mu} \mid \mu \in \mathcal{P}_{n}\right\}$. Let us write

$$
\Psi_{\ell}\left(s_{\lambda}\right)=\sum_{\mu \in \mathcal{P}_{n}} n_{\lambda, \mu} s_{\mu}
$$

Since $\Psi_{\ell}$ and $\varphi_{\ell}$ are dual operators with respect to $<\cdot, \cdot>$, we can write $n_{\lambda, \mu}=<\Psi_{\ell}\left(s_{\lambda}\right), s_{\mu}>=<$ $s_{\lambda}, \varphi_{\ell}\left(s_{\mu}\right)>$. So we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=\sum_{\lambda \in \mathcal{P}_{n}} n_{\lambda, \mu} s_{\lambda} .
$$

By (16) and Proposition [2.4.1, we obtain the identity

$$
s_{\mu}=\sum_{w \in W}(-1)^{l(w)} h_{\mu+\rho-w \cdot \rho}=\mathrm{H}\left(\Delta \times x^{\mu}\right) .
$$

Thus from (20) and (21) we derive $\varphi_{\ell}\left(s_{\mu}\right)=\mathrm{H}\left(\varphi_{\ell}\left(\Delta \times x^{\mu}\right)\right)$. Set $P_{\mu}=\Delta \times x^{\mu}$. From the previous arguments the coefficients $n_{\lambda, \mu}$ are determined by the computation of $\varphi_{\ell}\left(P_{\mu}\right)$.

### 3.2 Computation of $\varphi_{\ell}\left(P_{\mu}\right)$

For any $i \in\{\bar{n}, \ldots, \overline{1}\}$ we set $x_{i}=\frac{1}{x_{\bar{i}}}$. This permits to consider also variables indexed by negative integers. Given $X=\left(i_{1}, \ldots, i_{r}\right)$ an increasing sequence contained in $J_{n}$ and $\beta=\left(\beta_{1}, \ldots, \beta_{r}\right) \in \mathbb{Z}^{r}$, we set $x_{X}^{\beta}=x_{i_{1}}^{\beta_{1}} \cdots x_{i_{r}}^{\beta_{r}}$. We also denote by $S_{X}$ the group of permutations of the set $X$. Each $\sigma \in S_{X}$ determinates a unique permutation $\sigma^{*}$ of the set $\{1, \ldots, r\}$ defined by

$$
\begin{equation*}
\sigma\left(i_{p}\right)=i_{\sigma^{*}(p)} \text { for any } p=1, \ldots r . \tag{23}
\end{equation*}
$$

In the sequel, we identify for short $\sigma$ and $\sigma^{*}$. Similarly, given $Z=\left(\bar{u}_{r}, \ldots, \bar{u}_{1}, u_{1}, \ldots, u_{r}\right)$ an increasing sequence such that $\left\{u_{1}, \ldots, u_{r}\right\} \subset I_{n}$, each signed permutation $w$ defined on $Z$ will be identified with the signed permutation $w^{*}$ defined on $J_{r}$ by $w\left(u_{p}\right)=u_{w^{*}(p)}$ for any $p=1, \ldots r$.

Set $\rho_{n}=(1,2, \ldots, n)$. For any $w \in W$ we have $w \cdot \rho_{n}=(w(1), \ldots, w(n))$. This permits to write

$$
\begin{equation*}
\sum_{w \in W}(-1)^{l(w)} x^{-w \cdot \rho_{n}}=\sum_{w \in W}(-1)^{l(w)} x_{1}^{-w(1)} \cdots x_{n}^{-w(n)} . \tag{24}
\end{equation*}
$$

### 3.2.1 For $G=G L_{n}$

Set $\kappa_{n}=(1, \ldots, 1) \in \mathbb{Z}^{n}$. Since $\sigma\left(\kappa_{n}\right)=\kappa_{n}$ for any $\sigma \in S_{n}$, one can replace $\rho$ by $\rho_{n}=(1,2, \ldots, n)$ in (15). By using (24) we can write

$$
P_{\mu}=x_{1}^{\left(\mu_{1}+1\right)} \cdots x_{n}^{\left(\mu_{n}+n\right)} \sum_{\sigma \in S_{n}}(-1)^{l(\sigma)} x_{1}^{-\sigma(1)} \cdots x_{n}^{-\sigma(n)}
$$

where $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right)$. For any $k \in\{0, \ldots, \ell-1\}$ consider the ordering sequences

$$
\begin{equation*}
I^{(k)}=\left(i \in I_{n} \mid \mu_{i}+i \equiv k \bmod \ell\right) \text { and } J^{(k)}=\left(i \in I_{n} \mid i \equiv k \bmod \ell\right) . \tag{25}
\end{equation*}
$$

Set $r_{k}=\operatorname{card}\left(I^{(k)}\right)$ and write $I^{(k)}=\left(i_{1}^{(k)}, \ldots, i_{r_{k}}^{(k)}\right)$. Then

$$
\widehat{\mu}^{(k)}=\left(\left.\frac{\mu_{i}+i+\ell-k}{\ell} \right\rvert\, i \in I^{(k)}\right) \in \mathbb{Z}^{r_{k}}
$$

We derive

$$
P_{\mu}=x_{I^{(0)}}^{\ell \hat{\mu}^{(0)}} x_{I^{(1)}}^{\ell \widehat{\mu}^{(1)}} \cdots x_{I^{\ell(1)}}^{\ell \widehat{\Lambda}^{(\ell-1)}} \sum_{\sigma \in S_{n}}(-1)^{l(\sigma)} x_{1}^{-\sigma(1)} \cdots x_{n}^{-\sigma(n)} \times \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-(\ell-k)} .
$$

This gives

$$
\begin{equation*}
\varphi_{\ell}\left(P_{\mu}\right)=x_{I^{(0)}}^{\widehat{\mu}^{(0)}} x_{I^{(1)}}^{\widehat{\mu}^{(1)}} \cdots x_{I^{(\ell-1)}}^{\widehat{\mu}^{(\ell-1)}} \sum_{\sigma \in S_{n}}(-1)^{l(\sigma)} \varphi_{\ell}\left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\sigma\left(i_{a}^{(k)}\right)-(\ell-k)}\right) . \tag{26}
\end{equation*}
$$

The contribution of a fixed permutation $\sigma \in S_{n}$ in the above sum is nonzero if and only if for any $k=0, \ldots, \ell-1$

$$
i \in I^{(k)} \Longrightarrow \sigma(i) \equiv k \bmod \ell
$$

Thus we must have $\sigma\left(I^{(k)}\right) \subset J^{(k)}$ for any $k=0, \ldots, \ell-1$. Since $\sigma$ is a bijection, $I^{(k)} \cap I^{\left(k^{\prime}\right)}=$ $J^{(k)} \cap J^{\left(k^{\prime}\right)}=\emptyset$ if $k \neq k^{\prime}$ and $\cup_{0 \leq k \leq \ell-1} I^{(k)}=\cup_{0 \leq k \leq \ell-1} J^{(k)}=I_{n}$, the restriction of $\sigma$ on $I_{k}$ is a bijection from $I^{(k)}$ to $J^{(k)}$. In particular $\operatorname{card}\left(J^{(k)}\right)=\operatorname{card}\left(\bar{I}^{(k)}\right)=r_{k}$. This means that we have the equivalencies

$$
\begin{equation*}
\varphi_{\ell}\left(\prod_{k=0}^{\ell-1} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\sigma\left(i_{a}^{(k)}\right)-(\ell-k)}\right) \neq 0 \Longleftrightarrow \sigma\left(I^{(k)}\right)=J^{(k)} \text { for any } k=0, \ldots, \ell-1 . \tag{27}
\end{equation*}
$$

Write $J^{(k)}=\left(k, k+\ell, \ldots, k+\left(r_{k}-1\right) \ell\right)$. Denote by $\sigma_{0} \in S_{n}$ the permutation verifying

$$
\begin{equation*}
\sigma_{0}\left(i_{a}^{(k)}\right)=k+(a-1) \ell \tag{28}
\end{equation*}
$$

for any $k=0, \ldots, \ell-1$ and any $a=1, \ldots, r_{k}$. Let $S_{I^{(k)}}$ be the permutation group of the set $I^{(k)}$. The permutations $\sigma$ which verify the right hand side of (27) can be written $\sigma=\sigma_{0} \tau$ where $\tau=\left(\tau^{(0)},\right.$. $\cdot, \tau^{(l-1)}$ ) belongs to the direct product $S_{I^{(0)}} \times \cdots \times S_{I^{(\ell-1)}}$. We have then $(-1)^{l(\sigma)}=(-1)^{l\left(\sigma_{0}\right)}(-1)^{l\left(\tau^{(0)}\right)} \times$ $\cdots \times(-1)^{l\left(\tau^{(p)}\right)}$

For any $k \in\{0, \ldots, \ell-1\}$ set

$$
P_{k}=\sum_{\tau^{(k)} \in S_{I^{(k)}}}(-1)^{l(\tau)} \varphi_{\ell}\left(\prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\sigma_{0} \tau^{(k)}\left(i_{a}^{(k)}\right)-(\ell-k)}\right)
$$

From (22) and (26) we derive

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\widehat{\mu}^{(k)}} P_{k}
$$

Since $\sigma_{0}\left(i_{a}^{(k)}\right)=k+(a-1) \ell$, we can write by (23) $\sigma_{0} \tau^{(k)}\left(i_{a}^{(k)}\right)=k+\left(\tau^{(k)}(a)-1\right) \ell$. Thus we obtain

$$
P_{k}=\sum_{\tau^{(k)} \in S_{I^{(k)}}}(-1)^{l(\tau)} x_{i_{1}^{(k)}}^{-\tau^{(k)}(1)} \cdots x_{i_{r_{k}}^{(k)}}^{-\tau^{(k)}\left(r_{k}\right)}=x_{I^{(k)}}^{-\rho_{r_{k}}} \Delta_{I^{(k)}}
$$

where $\rho_{r_{k}}=\left(1,2, \ldots, r_{k}\right)$ and $\Delta_{I^{(k)}}=\prod_{i<j i, j \in I^{(k)}}\left(1-x_{j} / x_{i}\right)$. Finally, this gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\hat{\mu}^{(k)}-\rho_{r_{k}}} \Delta_{I^{(k)}}=(-1)^{l\left(\sigma_{0}\right)} \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu^{(k)}} \Delta_{I^{(k)}}
$$

where for any $k=0, \ldots, \ell-1$,

$$
\begin{equation*}
\mu^{(k)}=\left(\left.\frac{\mu_{i}+i+\ell-k}{\ell} \right\rvert\, i \in I^{(k)}\right)-\left(1,2, \ldots, r_{k}\right) \in \mathbb{Z}^{r_{k}} \tag{29}
\end{equation*}
$$

Theorem 3.2.1 Consider a partition $\mu$ of length $n$ and $\ell$ a positive integer. For any $k=0, \ldots, \ell-1$ define the sets $I^{(k)}$ and $J^{(k)}$ as in (25).

- If there exists $k \in\{0, \ldots, \ell-1\}$ such that $\operatorname{card}\left(I^{(k)}\right) \neq \operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, for any $k=0, \ldots, \ell-1$, set $r_{k}=\operatorname{card}\left(I^{(k)}\right)=\operatorname{card}\left(J^{(k)}\right)$ and define $\sigma_{0}$ as in (28). Then each $r_{k}$-tuple defined by (2.9) is a partition and we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}}=(-1)^{l\left(\sigma_{0}\right)} \operatorname{char}\left(V_{\mu}\right)
$$

where $\mathcal{I}=\left\{I^{(0}, \ldots, I^{(\ell-1)}\right\},\binom{\mu}{\ell}=\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)$ and $\operatorname{char}\left(V_{\mu}\right)$ is the character of the $G L_{n}$ module $V_{\mu}=V\left(\mu^{(0)}\right) \otimes \cdots \otimes V\left(\mu^{(\ell-1)}\right)$.

Proof. One verifies easily from (29) that each $\mu^{(k)}$ is a partition. By the previous computation, we obtain

$$
\varphi_{\ell}\left(\Delta \times x^{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} \Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}
$$

(with the notation of 2.5). By definition of $\varphi_{\ell}$ we have also

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} \mathrm{H} \circ \varphi_{\ell}\left(\Delta \times x^{\mu}\right)=(-1)^{l\left(\sigma_{0}\right)} \mathrm{H}\left(\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}\right)=(-1)^{l\left(\sigma_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}}
$$

where the last equality follows from (18).
Remark: The Levi subgroup $G_{\mathcal{I}}$ appearing in Theorem 3.2.1 is characterized by $\mathcal{I}=\left\{I^{(0}, \ldots, I^{(\ell-1)}\right\}$. This means that for type $A$, we have $X^{(k)}=I^{(k)}$ for any $k>0$ with the notation of 2.2 that is the sets $X^{(k)}$ contain only positive indices.

## Example 3.2.2

Consider $\mu=(1,2,3,4,4,4,6,6)$ and take $\ell=3$. We have $\mu+\rho_{8}=(2,4,6,8,9,10,13,14)$. Thus $I^{(0)}=\{3,5\}, I^{(1)}=\{2,6,7\}, I^{(2)}=\{1,4,8\}$ and $J^{(0)}=\{3,6\}, J^{(1)}=\{1,4,7\}, J^{(2)}=\{2,5,8\}$. Then $\mu^{(0)}=(1,1), \mu^{(1)}=(1,2,2)$ and $\mu^{(2)}=(0,1,2)$.

### 3.2.2 $\quad$ For $G=S p_{2 n}$

We have $\rho=\rho_{n}=(1,2, \ldots, n)$. By using (24) we deduce the identity:

$$
\begin{equation*}
P_{\mu}=x_{1}^{\left(\mu_{1}+1\right)} \cdots x_{n}^{\left(\mu_{n}+n\right)} \sum_{w \in W}(-1)^{l(w)} x_{1}^{-w(1)} \cdots x_{n}^{-w(n)} \tag{30}
\end{equation*}
$$

where $W$ is the group of signed permutations defined on $J_{n}=\{\bar{n}, \ldots, \overline{1}, 1, \ldots, n\}$, that is the subgroup of permutations $w \in S_{J_{n}}$ verifying $w(\bar{x})=\overline{w(x)}$ for any $x \in J_{n}$. Given $k \in\{0, \ldots, \ell-1\}$ consider the ordering sequences

$$
\begin{equation*}
I^{(k)}=\left(i \in I_{n} \mid \mu_{i}+i \equiv k \bmod \ell\right) \text { and } J^{(k)}=\left(x \in J_{n} \mid x \equiv k \bmod \ell\right) \tag{31}
\end{equation*}
$$

Set $p=\frac{\ell}{2}$ if $\ell$ is even and $p=\frac{\ell-1}{2}$ otherwise.
The odd case $\ell=2 p-1$ Set $r_{0}=\operatorname{card}\left(I^{(0)}\right)$ and for any $k=1, \ldots, p-1, s_{k}=\operatorname{card}\left(I_{k}\right), r_{k}=$ $\operatorname{card}\left(I_{k}\right)+\operatorname{card}\left(I_{\ell-k}\right)$. Write $X^{(k)}, k=1, \ldots, p$ for the increasing reordering of $\bar{I}_{k} \cup I_{\ell-k}$. Set $I^{(0)}=$ $\left(i_{1}^{(0)}, \ldots, i_{r_{0}}^{(0)}\right)$ and for $k>0$

$$
\begin{equation*}
X^{(k)}=\left(i_{1}^{(k)}, \ldots, i_{r_{k}}^{(k)}\right) \tag{32}
\end{equation*}
$$

This means that $I^{(k)}=\left(\bar{i}_{s_{k}}^{(k)}, \ldots, \bar{i}_{1}^{(k)}\right)$ and $I^{(\ell-k)}=\left(i_{s_{k+1}}^{(k)}, \ldots, i_{r_{k}}^{(k)}\right)$. To simplify the computation, we are going to use the indices and the variables $x_{i}, i \in X^{(k)}$ rather than the variables $x_{i}, i \in I^{(k)} \cup I^{(\ell-k)}$ when $k \in\{1, \ldots, p-1\}$.
Consider

$$
\widehat{\mu}^{(0)}=\left(\left.\frac{\mu_{i}+i}{\ell} \right\rvert\, i \in I^{(0)}\right) \in \mathbb{Z}^{r_{0}} \text { and for } k>0, \widehat{\mu}^{(k)}=\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|+\operatorname{sign}(i) k}{\ell} \right\rvert\, i \in X^{(k)}\right) \in \mathbb{Z}^{r_{k}}
$$

where for any $i \in J_{n}, \operatorname{sign}(i)=1$ if $i>0$ and -1 otherwise. For any $i \in I$, we have $x_{i}^{-w(i)}=x_{\bar{i}}{ }^{-w(\bar{i})}$. Thus

$$
\prod_{i \in X^{(k)}} x_{i}^{-w(i)}=\prod_{i \in I^{(k)}} x_{i}^{-w(i)} \prod_{i \in I^{(\ell-k)}} x_{i}^{-w(i)} \text { and } x_{1}^{\left(\mu_{1}+1\right)} \cdots x_{n}^{\left(\mu_{n}+n\right)}=x_{I^{(0)}}^{\ell \ell^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell^{(k)}} \prod_{i \in X^{(k)}} x_{i}^{-k}
$$

by definition of the $\widehat{\mu}^{(k)}$ 's. Then (30) can be rewritten

$$
P_{\mu}=x_{I^{(0)}}^{\ell \ell^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell \hat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i)} \times \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-k}
$$

This gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=x_{I^{(0)}}^{\widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\hat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i)-k}\right)
$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if and only if

$$
\left\{\begin{array}{l}
i \in I^{(0)} \Longrightarrow w(i) \equiv 0 \bmod \ell  \tag{33}\\
i \in X^{(k)} \Longrightarrow w(i) \equiv-k \bmod \ell \text { for any } k=1, \ldots, p-1
\end{array}\right.
$$

Thus we must have $w\left(\bar{I}^{(0)} \cup I^{(0)}\right) \subset J^{(0)}$ and for any $k=1, \ldots, p-1, w\left(X^{(k)}\right) \subset J^{(\ell-k)}$. Recall that $\bar{J}^{(0)}=J^{(0)}$ and $\bar{J}^{(\ell-k)}=J^{(k)}$ for $k=1, \ldots, p-1$. Moreover

$$
I^{(0)} \cup \bar{I}^{(0)} \bigcup_{k=1}^{p-1} X^{(k)} \cup \bar{X}^{(k)}=J_{n} \text { and } J^{(0)} \cup \bigcup_{k=1}^{p-1} J^{(k)} \cup J^{(\ell-k)}=J_{n} .
$$

Since the sets appearing in the left hand side of these two equalities are pairwise disjoint, we must have $w\left(\bar{I}^{(0)} \cup I^{(0)}\right)=J^{(0)}$, and for $k=1, \ldots, p-1, w\left(X^{(k)}\right)=J^{(\ell-k)}$. In particular card $\left(J^{(0)}\right)=2 \operatorname{card}\left(I^{(0)}\right)=$ $2 r_{0}$ and $\operatorname{card}\left(J^{(\ell-k)}\right)=\operatorname{card}\left(X^{(k)}\right)=r_{k}$ for any $k=1, \ldots, p-1$. We have the equivalencies

$$
\varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i)+k}\right) \neq 0 \Longleftrightarrow\left\{\begin{array}{l}
\text { (i) }: w\left(I^{(0)} \cup \bar{I}^{(0)}\right)=J^{(0)}  \tag{34}\\
(\text { ii }): w\left(X^{(k)}\right)=J^{(\ell-k)} \text { for any } k=1, \ldots, p-1
\end{array}\right.
$$

Note that condition (ii) can be rewritten: $w\left(\bar{X}^{(k)}\right)=J^{(k)}$ for any $k=1, \ldots, p-1$.
We can set $J^{(0)}=\left(-r_{0} \ell, \ldots, r_{0} \ell\right)$ and for $k=1, \ldots, p-1$,

$$
J^{(\ell-k)}=\left(-k-\alpha_{k} \ell, \ldots,-k+\beta_{k} \ell\right), J^{(k)}=\left(k-\beta_{k} \ell, \ldots, k+\alpha_{k} \ell\right)
$$

with $\alpha_{k}+\beta_{k}+1=r_{k}$.
Consider $w_{0} \in W$ defined by

$$
\begin{align*}
& w_{0}\left(i_{a}^{(0)}\right)=a \ell \text { for } a \in\left\{1, \ldots, r_{0}\right\}  \tag{35}\\
& w_{0}\left(i_{a}^{(k)}\right)=-k-\alpha_{k} \ell+(a-1) \ell \text { for any } k=1, \ldots, p-1 .
\end{align*}
$$

Denote by $\mathcal{W}$ the set of signed permutations $w$ which verify (i) and (ii) in (34). We have $w_{0} \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w=w_{0} v$ where $v=\left(v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)}\right)$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$. Here $W_{I^{(0)}}$ is the group of signed permutations defined on $\bar{I}^{(0)} \cup I^{(0)}$ and for $k=1, \ldots, p-1, S_{X^{(k)}}$ is the group of signed permutations $\tau^{(k)}$ defined on $\bar{X}^{(k)} \cup X^{(k)}$ and verifying $\tau^{(k)}\left(X^{(k)}\right)=X^{(k)}$. Indeed if $\tau^{(k)}(x) \in \bar{X}^{(k)}$ and $x \in X^{(k)}$, we would have $w(x) \in J^{(k)}$ and $x \in X^{(k)}$ which contradicts (ii). This means that $S_{X^{(k)}}$ is in fact isomorphic to the symmetric group $S_{r_{k}}$. Since the sets $I^{(0)}$ and $X^{(k)}, k=1, \ldots, p-1$ are increasing subsequences of $J_{n}$, we have by Lemma 2.1.1 $(-1)^{l(w)}=(-1)^{l\left(w_{0}\right)}(-1)^{l\left(v^{(0)}\right)}(-1)^{l\left(\tau^{(1)}\right)} \times \cdots \times(-1)^{l\left(\tau^{(p-1)}\right)}$.

Set

$$
\begin{aligned}
P_{0} & =\sum_{v^{(0)} \in W_{I^{(0)}}}(-1)^{l\left(v^{(0)}\right)} \varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-w_{0} v^{(0)}(i)}\right) \text { and } \\
P_{k} & =\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \varphi_{\ell}\left(\prod_{i \in X^{(k)}} x_{i}^{-w_{0} \tau^{(k)}(i)-k}\right), k \in\{1, \ldots, p-1\} .
\end{aligned}
$$

We obtain

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\hat{\mu}^{(0)}} P_{0} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_{k} .
$$

From (23) and (35), we have $w_{0} v^{(0)}\left(i_{a}^{(0)}\right)=v^{(0)}(a) \ell$ for any $a=1, \ldots, r_{0}$ and

$$
w_{0} \tau^{(k)}\left(i_{a}^{(k)}\right)=-k-\alpha_{k} \ell+\left(\tau^{(k)}(a)-1\right) \ell \text { for any } a=1, \ldots, r_{k} .
$$

This yields

$$
\begin{aligned}
& P_{0}=\sum_{v^{(0)} \in W_{I^{(0)}}}(-1)^{l\left(v^{(0)}\right)} \prod_{a=1}^{r_{0}} x_{i_{a}^{(0)}}^{-v^{(0)}(a)}=x_{I^{(0)}}^{-\rho_{r_{0}}} \Delta_{I^{(0)}} \text { and } \\
& P_{k}=\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\tau^{(k)}(a)+\left(\alpha_{k}+1\right)}=x_{X^{(k)}}^{\eta_{\eta_{k}}} \Delta_{X^{(k)}}
\end{aligned}
$$

where for any $k=1, \ldots, p-1, \eta_{r_{k}}=-\rho_{r_{k}}+\left(\alpha_{k}+1, \ldots, \alpha_{k}+1\right) \in \mathbb{Z}^{r_{k}}$,

$$
\Delta_{I^{(0)}}=\prod_{i<j}\left(1-\frac{x_{j}}{x_{i}}\right) \prod_{r \leq s \in I^{(0)}}\left(1-x_{r} x_{s}\right)
$$

and $\Delta_{X^{(k)}}=\prod_{i<j}\left(1-\frac{x_{j}}{x_{i}}\right)$ for any $k=1, \ldots, p-1$.
Finally, this gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}-\rho_{r_{0}}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}-\eta_{r_{k}}} \Delta_{X^{(k)}}=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}
$$

where

$$
\begin{equation*}
\mu^{(0)}=\left(\left.\frac{\mu_{i}+i}{\ell} \right\rvert\, i \in I^{(0)}\right)-\left(1, \ldots, r_{0}\right) \in \mathbb{Z}^{r_{0}} \tag{36}
\end{equation*}
$$

and for any $k=1, \ldots, p-1$

$$
\begin{equation*}
\widehat{\mu}^{(k)}=\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|+\operatorname{sign}(i) k}{\ell} \right\rvert\, i \in X^{(k)}\right)-\left(1, \ldots, r_{k}\right)+\left(\alpha_{k}+1, \ldots ., \alpha_{k}+1\right) \in \mathbb{Z}^{r_{k}} \tag{37}
\end{equation*}
$$

Recall that the weights corresponding to the Levi subgroup $G_{\mathcal{I}}$ are written following the convention (8).

Theorem 3.2.3 Consider a partition $\mu$ of length $n$ and $\ell=2 p-1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (31). For any $k=1, \ldots, p-1$ define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (32).

- If $\operatorname{card}\left(I^{(0)}\right) \neq \frac{1}{2} \operatorname{card}\left(J^{(0)}\right)$ or if there exists $k \in\{1, \ldots, p-1\}$ such that $\operatorname{card}\left(X^{(k)}\right) \neq \operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, set $r_{0}=\operatorname{card}\left(I^{(0)}\right)$ and for any $k=1, \ldots, p-1, r_{k}=\operatorname{card}\left(X^{(k)}\right)$. Let $w_{0} \in W$ be as in (35). Consider $\binom{\mu}{\ell}=\left(\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(p-1)}\right)$ where the $\mu^{(k)}$ 's are defined by (36) and (37). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$with $\mathcal{I}=\left\{I^{(0)}, X^{(1)} \ldots, X^{(p-1)}\right\}$ and we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{T}} .
$$

Proof. The proof is essentially the same as in Theorem 3.2.1 We obtain

$$
\varphi_{\ell}\left(\Delta \times x^{\mu}\right)=(-1)^{l\left(w_{0}\right)} \Delta_{\mathcal{I}} \times x^{\left({ }_{\ell}^{\mu}\right)}
$$

where in the right hand side of the preceding equality $\binom{\mu}{\ell}$ is expressed on the basis $\left\{\varepsilon_{i} \mid i \in I_{n}\right\}$ (see (10)). This permits to write as in the case $G=G L_{n}$

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} \mathrm{H} \circ \varphi_{\ell}\left(\Delta \times x^{\mu}\right)=(-1)^{l\left(w_{0}\right)} \mathrm{H}\left(\Delta_{\mathcal{I}} \times x^{\binom{\mu}{\ell}}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}} .
$$

## Example 3.2.4

Consider $\mu=(1,2,3,4,4,4,6,6)$ and take $\ell=3$. We have $\mu+\rho_{8}=(2,4,6,8,9,10,13,14)$. Thus $I^{(0)}=$ $\{3,5\}, X^{(1)}=\{\overline{7}, \overline{6}, \overline{2}, 1,4,8\}$ and $J^{(0)}=\{\overline{6}, \overline{3}, 3,6\}, J^{(1)}=\{\overline{8}, \overline{5}, \overline{2}, 1,4,7\}, J^{(2)}=\{\overline{7}, \overline{4}, \overline{1}, 2,5,8\}$. In particular $\alpha_{1}=2$. Then $\mu^{(0)}=(1,1)$ and $\mu^{(1)}=$

$$
\begin{array}{r}
\left(-\frac{13-1}{3}-1+3,-\frac{10-1}{3}-2+3,-\frac{4-1}{3}-3+3, \frac{2+1}{3}-4+3, \frac{8+1}{3}-5+3, \frac{14+1}{3}-6+3\right) \\
=(-2,-2,-1,0,1,2)
\end{array}
$$

with the convention (8).
The even case $\ell=2 p$ With the same notation as in the odd case, (30) can be rewritten

$$
P_{\mu}=x_{I^{(0)}}^{\ell \hat{\mu}^{(0)}} x_{I^{(p)}}^{\ell \hat{\mu}^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\ell \hat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{i \in I^{(p)}} x_{i}^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i)-k} .
$$

where

$$
\widehat{\mu}^{(p)}=\left(\left.\frac{\mu_{i}+i+p}{\ell} \right\rvert\, i \in I^{(p)}\right) .
$$

This gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=x_{I^{(0)}}^{\widehat{\mu}^{(0)}}{\underset{I}{(p)}}_{\widehat{\mu}^{(p)}}^{\left.\prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(p)}} \times \sum_{w \in W}(-1)^{l(w)} \varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{i \in I^{(p)}} x_{i}^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i)-k}\right)\right), ~\left({ }^{p-1}\right)}
$$

The contribution of a fixed $w \in W$ in the above sum is nonzero if conditions (34) are verified and

$$
i \in I^{(p)} \Longrightarrow w(i) \in J^{(p)}
$$

Since $p \equiv-p \bmod \ell$ we have $J^{(p)}=\bar{J}^{(p)}=\left\{-p-\alpha_{p} \ell, \ldots,-p, p, \ldots, p+\alpha_{p} \ell\right\}$. This implies that $w\left(I^{(p)} \cup \bar{I}^{(p)}\right)=J^{(p)}$ and thus $\operatorname{card}\left(I^{(p)}\right)=\frac{1}{2} \operatorname{card}\left(J^{(p)}\right)$. We then define $w_{0}$ by requiring (35) and $w_{0}\left(i_{a}^{(p)}\right)=p+(a-1) \ell$ for $a \in\left\{1, \ldots, r_{p}\right\}$. By using similar arguments as in the odd case, we obtain that $w$ can be written $w=w_{0} v$ where $\tau=\left(v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)}, v^{(p)}\right.$ belongs to the direct product $\underline{W_{I^{(0)}}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(\ell-1)}} \times W_{I^{(p)}}$ where $W_{I^{(p)}}$ is the group of signed permutations defined on $\overline{I^{(p)}} \cup I^{(p)}$. Note that $W_{I^{(p)}}$ is a Weyl group of type $B_{r_{p}}$. By Lemma 2.1.1 we have also $(-1)^{l(w)}=$ $(-1)^{l\left(w_{0}\right)}(-1)^{l\left(v^{(0)}\right)}(-1)^{l\left(\tau^{(1)}\right)} \times \cdots \times(-1)^{l\left(\tau^{(p)}\right)} \times(-1)^{l\left(v^{(p)}\right)}$. We obtain
$\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}} P_{0} \times x_{I^{(p)}}^{\widehat{\mu}^{(p)}} P_{p} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_{k}$ where $P_{p}=\sum_{v^{(p)} \in W_{I^{(p)}}}(-1)^{l\left(v^{(p)}\right)} \varphi_{\ell}\left(\prod_{i \in I^{(p)}} x_{i}^{-w_{0} v^{(p)}(i)-p}\right)$.
The functions $P_{k}, k=0, \ldots, p-1$ can be computed as in the odd case. For $P_{p}$, observe that each $v^{(p)} \in W_{I^{(p)}}$ can be written $v^{(p)}=\zeta \sigma$ according to the decomposition of $W_{I^{(p)}}$ as the semidirect product $(\mathbb{Z} / 2 \mathbb{Z})^{r_{p}} \propto S_{I^{(p)}}$. We have then for any $a=1, \ldots, r_{p}, w_{0} v^{(p)}\left(i_{a}^{(p)}\right)=\xi(a)(p+(\sigma(a)-1) \ell)$. This yields

$$
P_{p}=\sum_{v^{(p)} \in W_{I^{(p)}}}(-1)^{l\left(v^{(p)}\right)} \varphi_{\ell}\left(\prod_{a=1}^{r_{p}} x_{i}^{-\xi(a)(p+(\sigma(a)-1) \ell)-p}\right)=\sum_{v^{(p)} \in W_{I^{(p)}}}(-1)^{l\left(v^{(p)}\right)} \prod_{a=1}^{r_{p}} x_{i_{a}^{(p)}}^{-\frac{1-\xi(a)}{2}-\xi \sigma(a)}
$$

Thus

$$
P_{p}=\prod_{i \in I^{(p)}} x_{i}^{-1 / 2} \sum_{v^{(p)} \in W_{I^{(p)}}}(-1)^{l\left(v^{(p)}\right)}\left(v^{(p)} \cdot \prod_{a=1}^{r_{p}} x_{i_{a}^{(p)}}^{-\left(a-\frac{1}{2}\right)}\right)=x_{I^{(p)}}^{-\rho_{r_{p}}} \Delta_{I^{(p)}, B_{r_{p}}}
$$

where

$$
\Delta_{I^{(p)}, B_{r_{p}}}=\prod_{i<j}\left(1-\frac{x_{j}}{x_{i}}\right) \prod_{r<j \in s \in I^{(p)}}\left(1-x_{r} x_{s}\right) \prod_{i \in I^{(p)}}\left(1-x_{i}\right) .
$$

Indeed the half sum of positive roots is equal to $\left(\frac{1}{2}, \ldots, r_{p}-\frac{1}{2}\right)$ in type $B_{r_{p}}$. This means that when $\ell$ is even

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} \times x_{I^{(p)}}^{\mu^{(p)}} \Delta_{I^{(p)}, B_{r_{p}}}
$$

where $\mu^{(p)}=\widehat{\mu}^{(p)}-\left(1, \ldots, r_{p}\right)$. In particular the computation of $\varphi_{\ell}\left(P_{\mu}\right)$ makes appear positive roots corresponding to a root system of type $B_{r_{p}}$. These roots do not belong to the root lattice associated to $S p_{2 n}$. Hence, there cannot exist an analogue of Theorem 3.2.3 when $\ell$ is even. With the previous notation, we only obtain:

Proposition 3.2.5 Suppose $G=S P_{2 n}$ and $\ell=2 p$.

- If $\operatorname{card}\left(I^{(0)}\right) \neq \frac{1}{2} \operatorname{card}\left(J^{(0)}\right), \operatorname{card}\left(I^{(p)}\right) \neq \frac{1}{2} \operatorname{card}\left(J^{(p)}\right)$ or there exists $k \in\{1, \ldots, p-1\}$ such that $\operatorname{card}\left(X^{(k)}\right) \neq \operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, the coefficients appearing in the decomposition of $\varphi_{\ell}\left(s_{\mu}\right)$ on the basis of Weyl characters cannot be interpreted as branching coefficients and have signs alternatively positive and negative.


### 3.2.3 $\quad$ For $G=S O_{2 n}$

As for $G=S p_{2 n}$, the coefficients appearing in the decomposition of $\varphi_{\ell}\left(s_{\mu}\right)$ with $\ell=2 p$ on the basis of Weyl characters cannot be interpreted as branching coefficients. Note that there is an additional difficulty in this case. Indeed, $\varphi_{\ell}\left(P_{\mu}\right)$ cannot be factorized as a product of polynomials $\left(1-x^{\beta}\right)$ where $\beta \in \mathbb{Z}^{n}$. For example, we have for $S O_{4}$

$$
\varphi_{2}\left(P_{(0,0)}\right)=\varphi_{2}\left(\left(1-\frac{x_{2}}{x_{1}}\right)\left(1-x_{1} x_{2}\right)\right)=1+x_{2}
$$

This is due to the incompatibility between the signatures defined on the Weyl groups of types $B$ and $D$ when they are realized as subgroups of the permutation group $S_{J_{n}}$.
So we will suppose $\ell=2 p-1$ in this paragraph. Recall that the elements of $W$ are the signed permutations $w$ defined on $J_{n}=\{\bar{n}, \ldots, \overline{1}, 1, \ldots, n\}$ such that $\operatorname{card}\left(\left\{i \in I_{n} \mid w(i)<0\right\}\right)$ is even. Set $K_{n}=\{\overline{n-1}, \ldots, \overline{1}, 0,1, \ldots, n-1\}$. Each $w \in W$ can be written $w=\zeta \sigma$ according to the decomposition of $W$ as the semidirect product $(\mathbb{Z} / 2 \mathbb{Z})^{n-1} \propto S_{n}$. For any $x \in J_{n}$, we have then $\xi(x)=1$ if $w(x)>0$ and $\xi(x)=-1$ otherwise. Given $w \in W$ we define $\widehat{w}: J_{n} \rightarrow K_{n}$ such that $\widehat{w}(x)=w(x)-\xi(x)$ for any $x \in J_{n}$. Then $\widehat{w}(\bar{x})=\widehat{\widehat{w}(x)}$.
For type $D_{n}$, we have $\rho=\rho_{n}^{\prime}=(0,1, \ldots, n-1)=\rho_{n}-(1, \ldots, 1)$. Hence

$$
w \cdot \rho_{n}^{\prime}=w \cdot \rho_{n}-\left(\xi(1), \ldots, \xi(n)=(\widehat{w}(1), \ldots, \widehat{w}(n))=\widehat{w} \cdot \rho_{n}\right.
$$

Then we obtain

$$
P_{\mu}=x_{1}^{\left(\mu_{1}+0\right)} \cdots x_{n}^{\left(\mu_{n}+n-1\right)} \sum_{w \in W}(-1)^{l(w)} x_{1}^{-\widehat{w}(1)} \cdots x_{n}^{-\widehat{w}(n)}
$$

For any $k=0, \ldots, \ell-1$ set

$$
\begin{equation*}
I^{(k)}=\left(i \in I_{n} \mid \mu_{i}+i-1 \equiv k \bmod \ell\right) \text { and } J^{(k)}=\left(x \in K_{n} \mid x \equiv k \bmod \ell\right) . \tag{38}
\end{equation*}
$$

We then proceed essentially as in 3.2.2 by using $\widehat{w}$ instead of $w$ and $\rho_{n}^{\prime}=(0,1, \ldots, n-1)$ instead of $\rho_{n}=(1, \ldots, n)$. We only sketch below the main steps of the computation.
Set $r_{0}=\operatorname{card}\left(I^{(0)}\right)$ and for any $k=1, \ldots, p-1, s_{k}=\operatorname{card}\left(I_{k}\right), r_{k}=\operatorname{card}\left(I_{k}\right)+\operatorname{card}\left(I_{\ell-k}\right)$. For $k=1, \ldots, p-1, X^{(k)}$ is defined as the increasing reordering of $\bar{I}^{(k)} \cup I^{(\ell-k)}$. Consider

$$
\begin{aligned}
& \widehat{\mu}^{(0)}=\left(\left.\frac{\mu_{i}+i-1}{\ell} \right\rvert\, i \in I^{(0)}\right) \in \mathbb{Z}^{r_{0}} \text { and for } k>0, \\
& \widehat{\mu}^{(k)}=\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|-1+\operatorname{sign}(i) k}{\ell} \right\rvert\, i \in X^{(k)}\right) \in \mathbb{Z}^{r_{k}} .
\end{aligned}
$$

We obtain

$$
\varphi_{\ell}\left(P_{\mu}\right)=x_{I^{(0)}}^{\widehat{\mu}^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(p)}} \times \sum_{w \in W}(-1)^{l(w)} \varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-\widehat{w}(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-\widehat{w}(i)-k}\right) .
$$

We also have the equivalencies

$$
\varphi_{\ell}\left(\prod_{i \in I^{(0)}} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}}^{s_{k}} x_{i}^{-\widehat{w}(i)+k}\right) \neq 0 \Longleftrightarrow\left\{\begin{array}{l}
\text { (i) : } \widehat{w}\left(I^{(0)} \cup \bar{I}^{(0)}\right)=J^{(0)}  \tag{39}\\
\text { (ii) : } \widehat{w}\left(X^{(k)}\right)=J^{(\ell-k)} \text { for any } k=1, \ldots, p-1
\end{array}\right.
$$

We can write $J^{(0)}=\left(-\left(r_{0}-1\right) \ell, \ldots, 0, \ldots,\left(r_{0}-1\right) \ell\right)$ and for $k=1, \ldots, p$,

$$
J^{(\ell-k)}=\left(-k-\alpha_{k} \ell, \ldots,-k+\beta_{k} \ell\right), J^{(k)}=\left(k-\beta_{k} \ell, \ldots, k+\alpha_{k} \ell\right)
$$

with $\alpha_{k}+\beta_{k}+1=r_{k}$. Consider $w_{0} \in W$ defined by

$$
\begin{align*}
& \widehat{w}_{0}\left(i_{a}^{(0)}\right)=(a-1) \ell \text { for } a \in\left\{1, \ldots, r_{0}\right\}  \tag{40}\\
& \widehat{w}_{0}\left(i_{a}^{(k)}\right)=-k-\alpha_{k} \ell+(a-1) \ell \text { for any } k=1, \ldots, p-1 .
\end{align*}
$$

Denote by $\mathcal{W}$ the set of signed permutations $w \in W$ which verify (i) and (ii) in (39). We have $w_{0} \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w=w_{0} v$ where $v=\left(v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)}\right)$ belongs to the direct product $W_{I^{(0)}} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$ where $W_{I^{(0)}}$ is the Weyl group of type $D_{r_{0}}$ defined on $\bar{I}^{(0)} \cup I^{(0)}$. We have by Lemma 2.1.1 $(-1)^{l(w)}=(-1)^{l\left(w_{0}\right)}(-1)^{l\left(v^{(0)}\right)}(-1)^{l\left(\tau^{(1)}\right)} \times \cdots \times(-1)^{l\left(\tau^{(p-1)}\right)}$.
Set

$$
\begin{aligned}
& P_{0}=\sum_{v^{(0)} \in W_{I^{(0)}}}(-1)^{l\left(v^{(0)}\right)} \varphi_{\ell}\left(\prod_{i \in X^{(0)}} x_{i}^{-\widehat{w}_{0} v^{(0)}(i)}\right) \\
& P_{k}=\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \varphi_{\ell}\left(\prod_{i \in X^{(k)}} x_{i}^{-\widehat{w}_{0} v^{(k)}(i)-k}\right) \text { for any } k \in\{1, \ldots, p-1\} .
\end{aligned}
$$

We obtain

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\hat{\mu}^{(0)}} P_{0} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_{k} .
$$

Given $v^{(0)} \in W_{I^{(0)}}$, we define $\widehat{v}^{(0)}=v^{(0)}-\xi_{v}$ where $\xi_{v}\left(i_{a}\right)=1$ if $v\left(i_{a}\right)>0$ and -1 otherwise. By (40), we have for any $a=1, \ldots, r_{0}, \widehat{w}_{0} v^{(0)}\left(i_{a}^{(0)}\right)=\widehat{v}^{(0)}(a) \ell$. Moreover since $\tau^{(k)} \in S_{X^{(k)}}$

$$
\widehat{w}_{0} \tau^{(k)}\left(i_{a}^{(k)}\right)=-k-\alpha_{k} \ell+\left(\tau^{(k)}(a)-1\right) \ell \text { for any } a=1, \ldots, r_{k}
$$

This yields

$$
\begin{aligned}
& P_{0}=\sum_{v^{(0)} \in W_{I^{(0)}}}(-1)^{l\left(v^{(0)}\right)} \prod_{a=1}^{r_{0}} x_{i_{a}^{(0)}}^{-\widehat{v}^{(0)}(a)}=x_{I^{(0)}}^{-\rho_{r_{0}^{\prime}}^{\prime}} \Delta_{I^{(0)}} \text { and } \\
& P_{k}=\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\tau^{(k)}(a)+\left(\alpha_{k}+1\right)}=x_{X^{(k)}}^{\eta_{\eta_{k}}} \Delta_{X^{(k)}}
\end{aligned}
$$

where for any $k=1, \ldots, p-1, \eta_{r_{k}}=-\rho_{r_{k}}^{\prime}+\left(\alpha_{k}, \ldots, \alpha_{k}\right)=-\rho_{r_{k}}+\left(\alpha_{k}+1, \ldots, \alpha_{k}+1\right) \in \mathbb{Z}^{r_{k}}$,

$$
\begin{aligned}
\Delta_{I^{(0)}} & =\prod_{i<j, i, j \in I^{(0)}}\left(1-\frac{x_{j}}{x_{i}}\right) \prod_{r<s, r, s \in I^{(0)}}\left(1-x_{r} x_{s}\right) \text { and } \\
\Delta_{X^{(k)}} & =\prod_{i<j, i, j \in X^{(k)}}\left(1-\frac{x_{j}}{x_{i}}\right) \text { for any } k=1, \ldots, p-1 .
\end{aligned}
$$

This gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\widehat{\mu}^{(0)}-\rho_{r_{0}}^{\prime}} \Delta_{I^{(0)}}^{p} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\widehat{\mu}^{(k)}-\eta_{r_{k}}} \Delta_{X^{(k)}}=(-1)^{l\left(w_{0}\right)} x_{I^{(0)}}^{\mu^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}
$$

where

$$
\begin{equation*}
\mu^{(0)}=\left(\left.\frac{\mu_{i}+i-1}{\ell} \right\rvert\, i \in I^{(0)}\right)-\left(0, \ldots, r_{0}-1\right) \in \mathbb{Z}^{r_{0}} \tag{41}
\end{equation*}
$$

and for any $k=1, \ldots, p-1$,

$$
\begin{equation*}
\widehat{\mu}^{(k)}=\left\lvert\, i \in X^{(k)}\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|-1+\operatorname{sign}(i) k}{\ell} \right\rvert\, i \in X^{(k)}\right)-\left(0, \ldots, r_{k}-1\right)+\left(\alpha_{k}, \ldots ., \alpha_{k}\right) \in \mathbb{Z}^{r_{k}}\right. \tag{42}
\end{equation*}
$$

Note that these formulas are essentially the same as for $G=S p_{2 n}$, except that we use $\rho_{n}^{\prime}=(0, \ldots, n-1)$ instead of $\rho_{n}=(1, \ldots, n)$ for the half sum of positive roots. This gives the following theorem whose proof is identical to that of Theorem 3.2.3

Theorem 3.2.6 Consider a partition $\mu$ of length $n$ and $\ell=2 p-1$ a positive integer. Let $I^{(0)}$ and $J^{(0)}$ be as in (38). For any $k=0, \ldots, p-1$ define the sets $X^{(k)}$ and $J^{(k)}$ by (31) and (38).

- If $\operatorname{card}\left(I^{(0)}\right) \neq \frac{1}{2}\left(\operatorname{card}\left(J^{(0)}\right)+1\right)$ or if there exists $k \in\{1, \ldots, p-1\}$ such that $\operatorname{card}\left(X^{(k)}\right) \neq$ $\operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, set $r_{0}=\operatorname{card}\left(I^{(0)}\right)$ and for any $k=0, \ldots, p-1, r_{k}=\operatorname{card}\left(X^{(k)}\right)$. Let $w_{0} \in W$ be as in 40). Consider $\binom{\mu}{\ell}=\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)$ where the $\mu^{(k)}$ 's are defined by 41) and 42). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$with $\mathcal{I}=\left\{I^{(0)}, X^{(1)} \ldots, X^{(p-1)}\right\}$ and we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}} .
$$

## Example 3.2.7

Consider $\mu=(1,2,3,4,4,4,6,6)$ and take $\ell=3$. We have $\mu+\rho_{8}^{\prime}=(1,3,5,7,8,9,12,13)$. Thus $I^{(0)}=$ $\{2,6,7\}, X^{(1)}=\{\overline{8}, \overline{4}, \overline{1}, 3,5\}$ and $J^{(0)}=\{\overline{6}, \overline{3}, 0,3,6\}, J^{(1)}=\{\overline{5}, \overline{2}, 1,4,7\}$ and $J^{(2)}=\{\overline{7}, \overline{4}, \overline{1}, 2,5\}$. In particular $\alpha_{1}=2$. Then $\mu^{(0)}=(1,2,2)$ and

$$
\begin{aligned}
\mu^{(1)}=\left(-\frac{13-1}{3}-1+3,-\frac{7-1}{3}-2+3,-\frac{1-1}{3}-3+3, \frac{5+1}{3}-4+3, \frac{8+1}{3}\right. & -5+3) \\
& =(-2,-1,0,1,1)
\end{aligned}
$$

### 3.2.4 $\quad$ For $G=S O_{2 n+1}$

Set $L_{n}=\{\overline{n-1}, \ldots, \overline{1}, 0,1, \ldots, n\}$. Each $w \in W$ can be written $w=\zeta \sigma$ according to the decomposition of $W$ as the semidirect product $(\mathbb{Z} / 2 \mathbb{Z})^{n} \propto S_{n}$. Given $w \in W$ we define $\widetilde{w}: J_{n} \rightarrow L_{n}$ such that $\widetilde{w}(x)=w(x)+\frac{1}{2}(1-\xi(x))$ for any $x \in J_{n}$. For any $y \in L_{n}$, set $y^{*}=\bar{y}+1$. We have then $\widetilde{w}(\bar{x})=$ $(w(x))^{*}=\overline{w(x)}+1$.
Observe that $\rho=\rho_{n}^{\prime \prime}=\left(\frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{1}{2}\right)=\rho_{n}-\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Thus

$$
w \cdot \rho_{n}^{\prime \prime}=w \cdot \rho_{n}-\frac{1}{2}\left(\xi(1), \ldots, \xi(n)=(\widetilde{w}(1), \ldots, \widetilde{w}(n))-\frac{1}{2}(1, \ldots, 1) .\right.
$$

This permits to write

$$
\begin{equation*}
P_{\mu}=x_{1}^{\left(\mu_{1}+1\right)} \cdots x_{n}^{\left(\mu_{n}+n\right)} \sum_{w \in W}(-1)^{l(w)} x_{1}^{-\widetilde{w}(1)} \cdots x_{n}^{-\widetilde{w}(n)} . \tag{43}
\end{equation*}
$$

For any $k=1, \ldots, \ell$ set

$$
\begin{equation*}
I^{(k)}=\left(i \in I_{n} \mid \mu_{i}+i \equiv k \bmod \ell\right) \text { and } J^{(k)}=\left(x \in L_{n} \mid x \equiv k \bmod \ell\right) . \tag{44}
\end{equation*}
$$

Note that $\left(J^{(k)}\right)^{*}=J^{(l-k+1)}$. We then proceed essentially as in 3.2.2 by using $\widetilde{w}$ instead of $w$. We are going to see that for $G=S O_{2 n+1}$, there exists an analogue of Theorem 3.2.3 whatever the parity of $\ell$.

The even case $\ell=2 p \quad$ For any $k=1, \ldots, p$, set $s_{k}=\operatorname{card}\left(I^{(k)}\right), r_{k}=\operatorname{card}\left(I^{(k)}\right)+\operatorname{card}\left(I^{(\ell-k+1)}\right)$ and define $X^{(k)}$ as the increasing reordering of $\bar{I}^{(k)} \cup I^{(\ell-k+1)}$. Set $I^{(0)}=\left\{i_{1}^{(0)}, \ldots, i_{r_{0}}^{(0)}\right\}$ and

$$
\begin{equation*}
X^{(k)}=\left(i_{1}^{(k)}, \ldots, i_{r_{k}}^{(k)}\right) . \tag{45}
\end{equation*}
$$

For $k=1, \ldots, p$ consider the $r_{k}$-tuple $\widehat{\mu}^{(k)}$ such that

$$
\widehat{\mu}^{(k)}=\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|+\operatorname{sign}(i) k-\frac{1+\operatorname{sign}(i)}{2}}{\ell} \right\rvert\, i \in X^{(k)}\right) \in \mathbb{Z}^{r_{k}} .
$$

For any $i \in I^{(k)}$ with $k=1, \ldots, p$, we have $x_{i}^{-\widetilde{w}(i)-1}=x_{\bar{i}}^{-\widetilde{w}(\bar{i})}$. Thus

$$
\prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)}=\prod_{i \in \bar{I}^{(k)}} x_{i}^{-\widetilde{w}(i)} \prod_{i \in I^{(\ell-k+1)}} x_{i}^{-\widetilde{w}(i)}=\prod_{i \in I^{(k)}} x_{i}^{-\widetilde{w}(i)-1} \prod_{i \in I^{(\ell-k+1)}} x_{i}^{-\widetilde{w}(i)}
$$

and by definition of the $\widehat{\mu}^{(k)}$ 's, (43) can be rewritten

$$
P_{\mu}=\prod_{k=1}^{p} x_{X^{(k)}}^{\ell \widehat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)} \times \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-k+1}
$$

We obtain

$$
\varphi_{\ell}\left(P_{\mu}\right)=\prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \varphi_{\ell}\left(\prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)-k+1}\right)
$$

We deduce the equivalencies

$$
\begin{equation*}
\varphi_{\ell}\left(\prod_{k=0}^{p} \prod_{i \in X^{(k)}}^{s_{k}} x_{i}^{-w(i)+k-1}\right) \neq 0 \Longleftrightarrow \widetilde{w}\left(X^{(k)}\right)=J^{(\ell-k+1)} \text { for any } k=1, \ldots, p \tag{46}
\end{equation*}
$$

In particular we must have $\operatorname{card}\left(J^{(\ell-k+1)}\right)=\operatorname{card}\left(J^{(k)}\right)=r_{k}$. We can write

$$
J^{(\ell-k+1)}=\left(-k+1-\alpha_{k} \ell, \ldots,-k+1+\beta_{k} \ell\right) \text { and } J^{(k)}=\left(k-\beta_{k} \ell, \ldots, k+\alpha_{k} \ell\right)
$$

with $\alpha_{k}+\beta_{k}+1=r_{k}$. Consider $w_{0} \in W$ defined by

$$
\begin{equation*}
\widetilde{w}_{0}\left(i_{a}^{(k)}\right)=-k+1-\alpha_{k} \ell+(a-1) \ell \text { for any } k=1, \ldots, p . \tag{47}
\end{equation*}
$$

Denote by $\mathcal{W}$ the set of signed permutations $w \in W$ which verify the right hand side of (46). We have $w_{0} \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w=w_{0} v$ where $\tau=\left(\tau^{(1)}, \ldots, \tau^{(p)}\right)$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}}$. We have also by Lemma 2.1.1 $(-1)^{l(w)}=(-1)^{l\left(w_{0}\right)}(-1)^{l\left(\tau^{(1)}\right)} \times \cdots \times(-1)^{l\left(\tau^{(p)}\right)}$. For any $k=1, \ldots, p$, set

$$
P_{k}=\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \varphi_{\ell}\left(\prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}_{0} \tau^{(k)}(i)-k+1}\right) .
$$

We obtain

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_{k} .
$$

By (47) we have

$$
w_{0} \tau^{(k)}\left(i_{a}^{(k)}\right)=-k+1-\alpha_{k} \ell+\left(\tau^{(k)}(a)-1\right) \ell \text { for any } a=1, \ldots, r_{k} .
$$

This yields

$$
P_{k}=\sum_{\tau^{(k)} \in S_{X^{(k)}}}(-1)^{l\left(\tau^{(k)}\right)} \prod_{a=1}^{r_{k}} x_{i_{a}^{(k)}}^{-\tau^{(k)}(a)+\left(\alpha_{k}+1\right)}=x_{X^{(k)}}^{\eta_{r_{k}}} \Delta_{X^{(k)}}
$$

where for any $k=1, \ldots, p, \eta_{r_{k}}=-\rho_{r_{k}}+\left(\alpha_{k}+1, \ldots, \alpha_{k}+1\right) \in \mathbb{Z}^{r_{k}}$ and

$$
\Delta_{X^{(k)}}=\prod_{i<j}\left(1-\frac{x_{j}}{x_{i}}\right) .
$$

Note that the computation only makes appear root systems of type $A$ in this case. This gives

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}-\eta_{r_{k}}} \Delta_{X^{(k)}}=(-1)^{l\left(w_{0}\right)} \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}}
$$

where for any $k=1, \ldots, p$,

$$
\begin{equation*}
\widehat{\mu}^{(k)}=\left(\left.\operatorname{sign}(i) \frac{\mu_{|i|}+|i|+\operatorname{sign}(i) k-\frac{1+\operatorname{sign}(i)}{2}}{\ell} \right\rvert\, i \in X^{(k)}\right)-\left(1, \ldots, r_{k}\right)+\left(\alpha_{k+1}, \ldots, \alpha_{k+1}\right) \in \mathbb{Z}^{r_{k}} \tag{48}
\end{equation*}
$$

Similarly to Theorem 3.2.3 we obtain:

Theorem 3.2.8 Consider a partition $\mu$ of length $n$ and $\ell=2 p$ a positive integer. For any $k=1, \ldots, p$ define the sets $X^{(k)}, J^{(k)}$ by 44) and 45).

- If there exists $k \in\{1, \ldots, p\}$ such that $\operatorname{card}\left(X^{(k)}\right) \neq \operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, for any $k=1, \ldots, p$, set $r_{k}=\operatorname{card}\left(X^{(k)}\right)$. Let $w_{0} \in W$ be as in 47). Consider $\binom{\mu}{\ell}=\left(\mu^{(1)}, \ldots, \mu^{(p)}\right)$ where the $\mu^{(k)}$ 's are defined by 48). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$ with $\mathcal{I}=\left\{X^{(1)} \ldots, X^{(p)}\right\}$ and we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}} .
$$

Example 3.2.9 Consider $\mu=(1,5,5,6,7,9)$ and take $\ell=2$. We have $\mu+\rho_{6}=(2,7,8,10,12,15)$. Thus $X^{(1)}=\{\overline{6}, \overline{2}, 1,3,4,5\}$ and $J^{(2)}=\{\overline{4}, \overline{2}, 0,2,4,6\}, J^{(1)}=\{\overline{5}, \overline{3}, \overline{1}, 1,3,5\}$. In particular $\alpha_{0}=$ 2. Then

$$
\begin{equation*}
\mu^{(1)}=\left(-\frac{15-1}{2}-1+3,-\frac{7-1}{2}-2+3, \frac{2}{2}-3+3, \frac{8}{2}-4+3, \frac{10}{2}-5+3, \frac{12}{2}-6+3\right)= \tag{-5,-2,1,3,3,3}
\end{equation*}
$$

The case $\ell=2 p+1$ In addition to the sets $X^{(k)}, k=1, \ldots, p$ defined in (45), we have also to consider $I^{(p+1)}=\left\{i_{1}^{(p+1)}, \ldots, i_{r_{p+1}}^{(p+1)}\right\}$. This yields to defined

$$
\widehat{\mu}^{(p+1)}=\left(\left.\frac{\mu_{i}+i+p}{\ell} \right\rvert\, i \in I^{(p+1)}\right)
$$

We have

$$
\varphi_{\ell}\left(P_{\mu}\right)=x_{I^{(p+1)}}^{\widehat{\mu}^{(p+1)}} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} \times \sum_{w \in W}(-1)^{l(w)} \varphi_{\ell}\left(\prod_{i \in I^{(p+1)}} x_{i}^{-\widetilde{w}(i)-p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-\widetilde{w}(i)-k-1}\right)
$$

and the equivalence

$$
\varphi_{\ell}\left(\prod_{i \in I^{(p+1)}} x_{i}^{-\widetilde{w}(i)-p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i}^{-w(i)+k-1}\right) \neq 0 \Longleftrightarrow\left\{\begin{array}{l}
\widetilde{w}\left(X^{(k)}\right)=J^{(\ell-k+1)} \text { for any } k=1, \ldots, p  \tag{49}\\
\widetilde{w}\left(I^{(p+1)} \cup \bar{I}^{(p+1)}\right)=J^{(\ell-p)}=J^{(p+1)}
\end{array} .\right.
$$

Indeed $\left(J^{(p+1)}\right)^{*}=J^{(p+1)}$. In particular we must have $\operatorname{card}\left(J^{(p+1)}\right)=2 \operatorname{card}\left(I^{(p+1)}\right)=2 r_{p+1}$. Thus we can set $J^{(p+1)}=\left(-p-\left(r_{p+1}-1\right) \ell, \ldots,-p+r_{p+1} \ell\right)$. Consider $w_{0} \in W$ defined by (47) and

$$
\begin{equation*}
\widetilde{w}_{0}\left(i_{a}^{(p+1)}\right)=-p+a \ell \text { for any } a=1, \ldots, r_{p+1} . \tag{50}
\end{equation*}
$$

Denote by $\mathcal{W}$ the set of signed permutations $w \in W$ which verify the right hand side of (49). We have $w_{0} \in \mathcal{W}$. Each $w \in \mathcal{W}$ can be written $w=w_{0} v$ where $v=\left(\tau^{(1)}, \ldots, \tau^{(p)}, v^{(p+1)}\right)$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}} \times W_{I^{(p+1)}}$. We have also $(-1)^{l(w)}=(-1)^{l\left(w_{0}\right)}(-1)^{l\left(\tau^{(1)}\right)} \times \cdots \times$ $(-1)^{l\left(\tau^{(p)}\right)}(-1)^{l\left(v^{(p+1)}\right)}$. This permits to write

$$
\begin{aligned}
\varphi_{\ell}\left(P_{\mu}\right) & =(-1)^{l\left(w_{0}\right)} x_{I^{(p+1)}}^{\widehat{\mu}^{(p+1)}} P_{p+1} \prod_{k=1}^{p} x_{X^{(k)}}^{\widehat{\mu}^{(k)}} P_{k} \text { where } \\
P_{p+1} & =\sum_{v^{(p+1)} \in W_{I^{(p+1)}}}(-1)^{l\left(v^{(p+1)}\right)} \varphi_{\ell}\left(\prod_{i \in I^{(p+1)}} x_{i}^{-\widetilde{w}_{0} v^{(p+1)}(i)-p}\right) .
\end{aligned}
$$

The functions $P_{k}, k=1, \ldots, p$ can be computed as in the even case. For $P_{p+1}$, observe that each $v^{(p+1)} \in W_{I^{(p+1)}}$ can be written $v^{(p+1)}=\zeta \sigma$ with $\sigma \in S_{I^{(p+1)}}$. According to this decomposition we have for any $a=1, \ldots, r_{p+1}, \widetilde{w}_{0} v^{(p+1)}\left(i_{a}^{(p+1)}\right)=\xi(a)(-p+\sigma(a) \ell)$.

$$
\begin{aligned}
& P_{p+1}= \sum_{v^{(p+1)} \in W_{I^{(p+1)}}}(-1)^{l\left(v^{(p+1)}\right)} \varphi_{\ell}\left(\prod_{a=1}^{r_{p+1}} x_{i_{a}^{(p+1)}}^{-\xi(a)(-p+\sigma(a) \ell)-p}\right)= \\
& \sum_{v^{(p+1)} \in W_{I^{(p+1)}}}(-1)^{l\left(v^{(p+1)}\right)} \prod_{a=1}^{r_{p+1}} x_{i_{a}^{(p+1)}}^{-\frac{1-\xi(a)}{2}}-\xi \sigma(a)
\end{aligned}
$$

Thus

$$
\begin{gathered}
P_{p+1}=\prod_{i \in I^{(p+1)}} x_{i}^{-1 / 2} \sum_{v^{(p+1)} \in W_{I^{(p+1)}}}(-1)^{l\left(v^{(p+1)}\right)}\left(\nu^{(p+1)} \cdot \prod_{a=1}^{r_{p+1}} x_{i_{a}^{(p+1)}}^{-\left(a-\frac{1}{2}\right)}\right)=x_{I^{(p+1)}}^{-\rho_{r_{p+1}}} \Delta_{I^{(p+1)}} \\
\Delta_{I^{(p+1)}}=\prod_{i<j}\left(1-\frac{x_{j}}{x_{i}}\right) \prod_{r<j \in I^{(p+1)}}\left(1-x_{r} x_{s}\right) \prod_{i \in I^{(p+1)}}\left(1-x_{i}\right) .
\end{gathered}
$$

This means that when $\ell$ is odd

$$
\varphi_{\ell}\left(P_{\mu}\right)=(-1)^{l\left(w_{0}\right)} \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} \times x_{I^{(p+1)}}^{\mu^{(p+1)}} \Delta_{I^{(p+1)}}
$$

where

$$
\begin{equation*}
\mu_{i_{a}}^{(p+1)}=\left(\left.\frac{\mu_{i}+i+p}{\ell} \right\rvert\, i \in I^{(p+1)}\right)-\left(1, \ldots, r_{p+1}\right) \in \mathbb{Z}^{r_{p+1}} \tag{51}
\end{equation*}
$$

This gives the following theorem:
Theorem 3.2.10 Consider a partition $\mu$ of length $n$ and $\ell=2 p+1$ a positive integer. Define $X^{(k)}, J^{(k)}$ $k=1, \ldots, p$ and $I^{(p+1)}, J^{(p+1)}$ by 44) and (45).

- If $\operatorname{card}\left(I^{(p+1)}\right) \neq \frac{1}{2} \operatorname{card}\left(J^{(p+1)}\right)$ or if there exists $k \in\{1, \ldots, p\}$ such that $\operatorname{card}\left(X^{(k)}\right) \neq \operatorname{card}\left(J^{(k)}\right)$ then $\varphi_{\ell}\left(s_{\mu}\right)=0$.
- Otherwise, set $r_{p+1}=\operatorname{card}\left(I^{(p+1)}\right)$ and for any $k=1, \ldots, p, r_{k}=\operatorname{card}\left(X^{(k)}\right)$. Let $w_{0} \in W$ verifying (47) and (50). Consider $\binom{\mu}{\ell}=\left(\mu^{(p+1)}, \mu^{(1)}, \ldots, \mu^{(p)}\right)$ where the $\mu^{(k)}$ 's are defined by (48) and (51). Then $\binom{\mu}{\ell}$ is a dominant weight of $P_{\mathcal{I}}^{+}$with $\mathcal{I}=\left\{I^{(p+1)}, X^{(1)}, \ldots, X^{(p)}\right\}$ and we have

$$
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{T}} .
$$

Example 3.2.11 Consider $\mu=(1,5,5,6,7,9)$ and take $\ell=3$. We have $\mu+\rho_{6}=(2,7,8,10,12,15)$. Thus $X^{(1)}=\{\overline{4}, \overline{2}, 5,6\}, I^{(2)}=\{1,3\}$ and $J^{(1)}=\{\overline{5}, \overline{2}, 1,4\}, J^{(2)}=\{\overline{4}, \overline{1}, 2,5\}, J^{(3)}=\{\overline{3}, 0,3,6\}$. In particular $\alpha_{0}=1$. Then

$$
\mu^{(1)}=\left(-\frac{10-1}{3}-1+1,-\frac{7-1}{3}-2+1, \frac{12}{3}-3+1, \frac{15}{3}-4+1\right)=(-2,-2,3,3)
$$

and $\mu^{(2)}=\left(\frac{2+1}{3}-1, \frac{8+1}{3}-2\right)=(0,1)$.

## 4 Parabolic Kazhdan-Lusztig polynomials

We recall briefly in this section, some basics on Affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials associated to classical root systems. The reader is referred to 11 and 13 for detailed expositions. Note that the definition of the Hecke algebra used in [1] coincides with that used in [8] and 13] (with generators $H_{w}$ ) up to the change $q \rightarrow q^{-1}$.

### 4.1 Extended affine Weyl group

Consider a root system of type $A_{n-1}, B_{n}, C_{n}$ or $D_{n}$. For any $\beta \in P$, we denote by $t_{\beta}$ the translation defined in $\mathfrak{h}_{\mathbb{R}}^{*}$ by $\gamma \longmapsto \gamma+\beta$. The extended affine Weyl group $\widehat{W}$ is the group

$$
\widehat{W}=\left\{w t_{\beta} \mid w \in W, \beta \in P\right\}
$$

with multiplication determined by the relations $t_{\beta} t_{\gamma}=t_{\beta+\gamma}$ and $w t_{\beta}=t_{w \cdot \beta} w$. The group $\widehat{W}$ is not a Coxeter group but contains the affine Weyl group $\widetilde{W}$ generated by reflections through the affine hyperplanes $H_{\alpha, k}=\left\{\beta \in \mathfrak{h}_{\mathbb{R}}^{*} \mid\left(\beta, \alpha^{\vee}\right)=k\right\}$. It makes sense to define a length function on $\widehat{W}$ verifying

$$
\begin{equation*}
l\left(w t_{\beta}\right)=\sum_{\alpha \in R_{+}}\left|\left(\beta, \alpha^{\vee}\right)+1_{R_{-}}(w \cdot \alpha)\right| \tag{52}
\end{equation*}
$$

where for any $w \in W, 1_{R_{-}}(w \cdot \alpha)=0$ if $w \cdot \alpha \in R_{+}$and $1_{R_{-}}(w \cdot \alpha)=1$ if $w \cdot \alpha \in-R_{+}=R_{-}$. Write $n_{\beta}$ for the element of maximal length in $W t_{\beta} W$. It follows from (52) that for any $\lambda \in P_{+}$we have $l\left(w t_{\lambda}\right)=l(w)+l\left(t_{\lambda}\right)$. This gives

$$
\begin{equation*}
n_{\lambda}=w_{0} t_{\lambda} \tag{53}
\end{equation*}
$$

where $w_{0}$ denotes the longest element of $W$. There exists a unique element $\eta \in R_{+}$such that the fundamental alcove

$$
\mathcal{A}=\left\{\beta \in \mathfrak{h}_{\mathbb{R}}^{*} \mid\left(\beta, \alpha^{\vee}\right) \geq 0 \forall \alpha \in R_{+} \text {and }\left(\beta, \eta^{\vee}\right)<1\right\}
$$

is a fundamental region for the action of $\widetilde{W}$ on $\mathfrak{h}_{\mathbb{R}}^{*}$. This means that, for any $\beta \in \mathfrak{h}_{\mathbb{R}}^{*}$, the orbit $\widetilde{W} \cdot \beta$ intersects $\mathcal{A}$ in a unique point. Each $w \in \widehat{W}$ can be written on the form $w=w_{\mathcal{A}} w_{\text {aff }}$ where $w_{\text {aff }} \in \widetilde{W}$ and $w_{\mathcal{A}}$ belongs to the stabilizer of $\mathcal{A}$ under the action of $\widehat{W}$. This implies that $\mathcal{A}$ is also a fundamental domain for the action of $\widehat{W}$ on $\mathfrak{h}_{\mathbb{R}}^{*}$. The Bruhat ordering on $\widehat{W}$ is defined by taking the transitive closure of the relations

$$
w<s w \text { whenever } l(w)<l(s w)
$$

for all $w \in \widehat{W}$ and all (affine) reflections $s \in \widetilde{W}$.
In fact the natural action of $\widehat{W}$ on the weight lattice $P$ obtained by considering $P$ as a sublattice of $\mathfrak{h}_{\mathbb{R}}^{*}$ is not that which is relevant for our purpose. For any integer $m \in \mathbb{Z}^{*}$ we obtain a faithful representation $\pi_{m}$ of $\widehat{W}$ on $P$ by setting for any $\beta, \gamma \in P, w \in W$

$$
\pi_{m}(w) \cdot \gamma=w \cdot \gamma \text { and } \pi_{m}\left(t_{\beta}\right) \cdot \gamma=\gamma+m \beta
$$

Warning: In the sequel, the extended affine Weyl group $\widehat{W}$ acts on the weight lattice $P$ via $\pi_{-\ell}$ where $\ell$ is a fixed nonnegative integer.

We write for simplicity $w t_{\beta} \cdot \gamma$ rather that $\pi_{-\ell}\left(w t_{\beta}\right) \cdot \gamma$. Hence for any $w \in W$ and any $\beta \in P$, we have $w t_{\beta} \cdot \gamma=w \cdot \gamma-\ell w \cdot \beta$. The fundamental region for this new action of $\widehat{W}$ on $P$ is the alcove $\mathcal{A}_{\ell}$ obtained by expanding $\mathcal{A}$ with the factor $-\ell$. This gives

$$
\mathcal{A}_{\ell}=\left\{\begin{array}{l}
\left\{\nu=\left(\nu_{1}, \ldots, \nu_{n}\right) \mid 0 \geq \nu_{1} \geq \cdots \geq \nu_{n}>-\ell\right\} \text { for types } A, B, C \\
\left\{\nu=\left(\nu_{1}, \ldots, \nu_{n}\right)\left|0 \geq-\left|\nu_{1}\right| \geq \nu_{2} \geq \cdots \geq \nu_{n}>-\ell\right\} \text { for type } D\right.
\end{array} .\right.
$$

Consider a weight $\beta \in P$. Then its orbit intersects $\mathcal{A}_{\ell}$ in a unique weight $\nu$. Then there is a unique $w(\beta) \in \widehat{W}$ of minimal length such that $w(\beta) \cdot \nu=\beta$. We denote by $W_{\nu}$ the stabilizer of $\nu \in \mathcal{A}_{\ell}$ in $\widehat{W}$. Since $\nu \in \mathcal{A}_{\ell}, W_{\nu}$ is in fact a subgroup of $W$.

Lemma 4.1.1 Consider $\lambda \in P^{+}$and suppose $\ell>n$. Then

1. $w(\ell \lambda+\rho)=n_{\lambda^{*}} \tau^{-n+1}$ with $\lambda^{*}=-w_{0}(\lambda)$ and $\tau=s_{1} s_{2} \cdots s_{r-1} t_{\varepsilon_{1}}$ for type $A$.
2. $w(\ell \lambda+\rho)=n_{\lambda}$ for types $B, C$ and $D$.

Proof. 1: See Lemma 2.3 in [8].
2 : Observe first that $w_{0} \cdot \rho=-\rho$ belongs to $\mathcal{A}_{\ell}$ for types $B, C, D$ since $\ell>n$. We have

$$
\ell \lambda+\rho=t_{-\lambda} \cdot \rho=t_{-\lambda} w_{0} \cdot\left(w_{0} \cdot \rho\right)=t_{-\lambda} w_{0} \cdot(-\rho)
$$

Moreover $W_{-\rho}=\{1\}$. Since $-\rho \in \mathcal{A}_{\ell}$ this means that $w(\ell \lambda+\rho)=t_{-\lambda} w_{0}=w_{0} t_{w_{0} \cdot(-\lambda)}=w_{0} t_{\lambda}=n_{\lambda}$ where the last equality follows from (53).

### 4.2 Affine Hecke algebra and K-L polynomials

The Hecke algebra associated to the root system $R$ of type $A_{n}, B_{n}, C_{n}$ or $D_{n}$ is the $\mathbb{Z}\left[q, q^{-1}\right]$-algebra defined by the generators $T_{w}, w \in \widehat{W}$ and relations

$$
\begin{aligned}
T_{w_{1}} T_{w_{2}} & =T_{w_{1}} T_{w_{2}} \text { if } l\left(w_{1} w_{2}\right)=l\left(w_{1}\right)+l\left(w_{2}\right) \\
T_{s_{i}} T_{w} & =\left(q^{-1}-q\right) T_{w}+T_{s_{i} w} \text { if } l\left(s_{i} w\right)<l(w) \text { and } i \in I_{n} .
\end{aligned}
$$

In particular we have $T_{i}^{2}=\left(q^{-1}-q\right) T_{i}+1$ for any $i \in I_{n}$. The bar involution on $\widehat{H}$ is the $\mathbb{Z}$-linear automorphism defined by

$$
\bar{q}=q^{-1} \text { and } \bar{T}_{w}=T_{w^{-1}}^{-1} \text { for any } w \in \widehat{W} .
$$

Kazhdan and Lusztig have proved that there exists a unique basis $\left\{C_{w}^{\prime} \mid w \in \widehat{W}\right\}$ of $\widehat{H}$ such that

$$
\bar{C}_{w}^{\prime}=C_{w}^{\prime} \text { and } C_{w}^{\prime}=\sum_{y \leq w} p_{y, w} T_{y}
$$

where $p_{w, w}=1$ and $p_{y, w} \in q \mathbb{Z}[q]$ for any $y<w$. We will refer to the polynomials $p_{y, w}(q)$ as KazhdanLusztig polynomials. They are renormalizations of the polynomials $P_{y, w}$ originally introduced by Kazhdan and Lusztig in [5. Namely we have $p_{y, w}=q^{l(w)-l(y)} P_{y, w}$.
Let us define the $q$-partition function $\mathcal{P}_{q}$ by

$$
\prod_{\alpha \in R_{+}} \frac{1}{1-q x^{\alpha}}=\sum_{\beta \in \mathbb{Z}^{n}} \mathcal{P}_{q}(\beta) x^{\beta} .
$$

Given $\lambda$ and $\mu$ in $P$, the Lusztig $q$-analogue $K_{\lambda, \mu}(q)$ is defined by

$$
K_{\lambda, \mu}(q)=\sum_{w \in W}(-1)^{l(w)} \mathcal{P}_{q}(w \circ \lambda-\mu) .
$$

Then one has the following theorem due to Lusztig
Theorem 4.2.1 Suppose $\lambda, \mu$ are dominant weights. Then $K_{\lambda, \mu}(q)=p_{n_{\mu}, n_{\mu}}(q)$.

One defines the action of the bar involution on the parabolic module $P_{\nu}=\widehat{H} \nu, \nu \in P$, by setting $\bar{q}=q^{-1}$ and $\overline{w \cdot \nu}=\bar{w} \cdot \nu$. Deodhar has proved that there exist two bases $\left\{C_{\lambda}^{+} \mid \lambda \in \widehat{W} \cdot \nu\right\}$ and $\left\{C_{\lambda}^{-} \mid \lambda \in \widehat{W} \cdot \nu\right\}$ of $P_{\nu}$ belonging respectively to

$$
L_{\nu}^{+}=\bigoplus_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}[q] \lambda \text { and } L_{\nu}^{-}=\bigoplus_{\lambda \in \widehat{W} \cdot \nu} \mathbb{Z}\left[q^{-1}\right] \lambda
$$

characterized by

$$
\left\{\begin{array} { l } 
{ \overline { C } _ { \lambda } ^ { + } = C _ { \lambda } ^ { + } } \\
{ C _ { \lambda } ^ { + } \equiv \lambda \operatorname { m o d } q L _ { \nu } ^ { + } }
\end{array} \quad \text { and } \left\{\begin{array}{l}
\bar{C}_{\lambda}^{-}=C_{\lambda}^{-} \\
C_{\lambda}^{-} \equiv \lambda \bmod q^{-1} L_{\nu}^{-}
\end{array}\right.\right.
$$

We will only need the basis $\left\{C_{\lambda}^{-} \mid \lambda \in \widehat{W} \cdot \nu\right\}$ in the sequel. The parabolic Kazhdan-Lusztig polynomials $P_{\lambda, \mu}^{-}$are then defined by the expansion

$$
C_{\lambda}^{-}=\sum_{\mu \in \widehat{W} \cdot \lambda}(-1)^{l(w(\lambda))+l(w(\mu))} P_{\mu, \lambda}^{-}\left(q^{-1}\right) \mu
$$

(see [13] Theorem 3.5). In particular they belong to $\mathbb{Z}[q]$. Their expansion in terms of the ordinary Kazhdan-Lusztig polynomials is given by the Following theorem due to Deodhar

Theorem 4.2.2 Consider $\nu \in P$ and $\lambda \in \widehat{W} \cdot \nu$. Then for any $\mu \in \widehat{W} \cdot \lambda$ we have

$$
P_{\lambda, \mu}^{-}(q)=\sum_{z \in W_{\nu}}(-q)^{l(z)} p_{w(\mu) z, w(\lambda)}(q)
$$

with the notation of 4.1 .
Remark: When $\nu$ is regular, that is $W_{\nu}=\{1\}$, we have $P_{\lambda, \mu}^{-}(q)=p_{w(\mu), w(\lambda)}(q)$.

## 5 Generalized Hall-Littlewood functions

### 5.1 Plethysm and parabolic K-L polynomials

Consider $\ell$ a nonnegative integer and $\zeta \in \mathbb{C}$ such that $\zeta^{2}$ is a primitive $\ell$-th root of 1 . We briefly recall in this paragraph the arguments of [8] which establish that the coefficients of the plethysm $\psi_{\ell}\left(s_{\lambda}\right)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q=1$.
For any $\lambda \in P_{+}$, denote by $V_{q}(\lambda)$ the finite dimensional $U_{q}(\mathfrak{g})$-module of highest weight $\lambda$. Its character is also the Weyl character $s_{\lambda}$. Let $U_{q, \mathbb{Z}(\mathfrak{g})}$ be the $\mathbb{Z}\left[q, q^{-1}\right]$-subalgebra of $U_{q}(\mathfrak{g})$ generated by the elements

$$
E_{i}^{(k)}=\frac{E_{i}^{(k)}}{[k]_{i}!}, F_{i}^{(k)}=\frac{F_{i}^{(k)}}{[k]_{i}!} \text { and } K_{i}^{ \pm 1}
$$

where $E_{i}, F_{i}, K_{i}^{ \pm 1}, i \in I_{n}$ are the generators of $U_{q}(\mathfrak{g})$. The indeterminate $q$ can be specialized at $\zeta$ in $U_{q, \mathbb{Z}}(\mathfrak{g})$. Thus it makes sense to define $U_{\zeta}(\mathfrak{g})=U_{q, \mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}\left[q, q^{-1}\right]} \mathbb{C}$ where $\mathbb{Z}\left[q, q^{-1}\right]$ acts on $\mathbb{C}$ by $q \mapsto \zeta$. Fix a highest weight vector $v_{\lambda}$ in $V_{q}(\lambda)$. We have $V_{q}(\lambda)=U_{q}(\mathfrak{g}) \cdot v_{\lambda}$. Similarly $V_{\zeta}(\lambda)=U_{\zeta}(\mathfrak{g}) \cdot v_{\lambda}$ is a $U_{\zeta}(\mathfrak{g})$ module called a Weyl module and one has $\operatorname{char}\left(V_{\zeta}(\lambda)\right)=s_{\lambda}$. The module $V_{\zeta}(\lambda)$ is not simple but admits a unique simple quotient denoted by $L(\lambda)$.
From results due to Kazhdan-Lusztig and Kashiwara-Tanisaki one obtains the following decomposition of $\operatorname{char}(L(\lambda))$ on the basis of Weyl characters:

Theorem 5.1.1 Consider $\lambda \in P_{+}$.

1. The character of $L(\lambda)$ decomposes on the form

$$
\begin{equation*}
\operatorname{char}(L(\lambda))=\sum_{\mu}(-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho, \lambda+\rho}^{-}(1) s_{\mu} \tag{54}
\end{equation*}
$$

where the sum runs over the dominant weights $\mu \in P_{+}$such that $\mu+\rho \in \widehat{W} \cdot(\lambda+\rho)$.
2. The parabolic Kazhdan-Lusztig polynomials $P_{\mu+\rho, \lambda+\rho}^{-}(q)$ have nonnegative integer coefficients.

Remark: The decomposition (54) has been conjectured by Kazhdan-Lusztig and proved by KashiwaraTanisaki. In [4, Kashiwara and Tanisaki have also obtained that the parabolic Kazhdan-Lusztig polynomials have nonnegative integer coefficients as soon as the Coxeter system considered corresponds to the Weyl group of a Kac-Moody Lie algebra, thus in the particular context of this paper.

Consider a nonnegative integer $\ell$. The Frobenius map $\mathrm{Fr}_{\ell}$ is the algebra homomorphism defined from $U_{\zeta}(\mathfrak{g})$ to $U(\mathfrak{g})$ by $\operatorname{Fr}_{\ell}\left(K_{i}\right)=1$ and

$$
\operatorname{Fr}_{\ell}\left(E_{i}^{(k)}\right)=\left\{\begin{array}{l}
e_{i}^{(k / \ell)} \text { if } \ell \text { divides } k \\
0 \text { otherwise }
\end{array} \quad \text { and } \operatorname{Fr}_{\ell}\left(F_{i}^{(k)}\right)=\left\{\begin{array}{l}
f_{i}^{(k / \ell)} \text { if } \ell \text { divides } k \\
0 \text { otherwise }
\end{array}\right.\right.
$$

where $e_{i}, f_{i}, i \in I_{n}$ are the Chevalley generators of the enveloping algebra $U(\mathfrak{g})$. This permits to endow each $U(\mathfrak{g})$-module $M$ with the structure of a $U_{\zeta}(\mathfrak{g})$-module $M^{\mathrm{Fr}_{\ell}}$. Then we have

$$
\operatorname{char}\left(M^{\mathrm{Fr} \ell}\right)=\psi_{\ell}(\operatorname{char}(M))
$$

in particular for any $\lambda \in P_{+}, \operatorname{char}\left(V(\lambda)^{\mathrm{Fr} \ell}\right)=\psi_{\ell}\left(s_{\lambda}\right)$.
Each dominant weight $\lambda \in P_{+}$, can be uniquely decomposed on the form $\lambda=\stackrel{r}{\lambda}+\ell \stackrel{q}{\lambda}$ where $\stackrel{r}{\lambda}, \stackrel{q}{\lambda} \in P_{+}$ and $\stackrel{r}{\lambda}=\left(\stackrel{r}{\lambda_{1}}, \ldots, \stackrel{r}{\lambda}{ }_{n}\right)$ verifies $0 \leq \stackrel{r}{\lambda_{i+1}}-\stackrel{r}{\lambda_{i}}<\ell$ for any $i \in I_{n}$.
Theorem 5.1.2 (Lusztig) The simple $U_{\zeta}(\mathfrak{g})$-module $L(\lambda)$ is isomorphic to the tensor product

$$
L(\lambda) \simeq L(\stackrel{r}{\lambda}) \otimes V(\stackrel{q}{\lambda})^{\mathrm{Fr} \ell} .
$$

By replacing $\lambda$ by $\ell \lambda$ in the previous theorem, we have $\stackrel{r}{\lambda}=0$ and $\stackrel{q}{\lambda}=\lambda$. Thus $L(\ell \lambda) \simeq V(\lambda)^{\mathrm{Fr} \ell}$. Then one deduces from (54) the equality

$$
\psi_{\ell}\left(s_{\lambda}\right)=\operatorname{char}(L(\ell \lambda))=\sum_{\mu+\rho \in \widehat{W} \cdot(\ell \lambda+\rho)}(-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho, \ell \lambda+\rho}^{-}(1) s_{\mu}
$$

which shows that the coefficients of the expansion of $\psi_{\ell}\left(s_{\lambda}\right)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q=1$. This gives

$$
\left|<\psi_{\ell}\left(s_{\lambda}\right), s_{\mu}>\left|=\left|<s_{\lambda}, \varphi\left(s_{\mu}\right)>\right|=P_{\mu+\rho, \ell \lambda+\rho}^{-}(1) .\right.\right.
$$

By definition of the action of $\widehat{W}$ on $P$ we have $\widehat{W} \cdot(\ell \lambda+\rho)=\widehat{W} \cdot \rho$. This implies the
Corollary 5.1.3 (of Theorems 4.2.2 and 5.1.2). For any nonnegative integer $\ell$

$$
\psi_{\ell}\left(s_{\lambda}\right)=\sum_{\mu+\rho \in \widehat{W} \cdot \rho}(-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P_{\mu+\rho, \ell \lambda+\rho}^{-}(1) s_{\mu} .
$$

In particular $\varphi\left(s_{\mu}\right) \neq 0$ if and only if $\mu+\rho \in \widehat{W} \cdot \rho$, that is $\mu+\rho=w \cdot \rho-\ell \beta$ with $w \in W$ and $\beta \in P$.
Remark: The equivalence $\varphi\left(s_{\mu}\right) \neq 0 \Longleftrightarrow \mu+\rho \in \widehat{W} \cdot \rho$ can also be obtained more elementary from algorithms described in 3.2

### 5.2 Parabolic K-L polynomials and branching coefficients

Warning: In the sequel of the paper we will suppose that $\ell$ is odd when the Lie groups considered are of type $C$ or $D$.
Under this hypothesis we have for any $\mu \in \mathcal{P}_{n} \varphi_{\ell}\left(s_{\mu}\right)=0$ or

$$
\begin{equation*}
\varphi_{\ell}\left(s_{\mu}\right)=(-1)^{l\left(w_{0}\right)} S_{\binom{\mu}{\ell}, \mathcal{I}} \tag{55}
\end{equation*}
$$

according to the results of 3.2
Remark: Following the algorithms described in 3.2, when $\varphi_{\ell}\left(s_{\mu}\right) \neq 0$, the cardinalities of the sets $I^{(k)}$ or $X^{(k)}$ contained in $\mathcal{I}$ are determined by those of the sets $J^{(k)}$. In particular they depend only on $n$ and $\ell$ and not on the partition $\mu$ considered. Thus in (55), the underlying Levi subgroup $G_{\mathcal{I}}$ is, up to isomorphism, independent on $\mu$.

By using Proposition 2.5 .2 and Theorems 3.2.1 3.2.3, 3.2.6, 3.2.8 3.2.10 we deduce from Corollary 5.1.3 the

Theorem 5.2.1 For any $\lambda, \mu \in \mathcal{P}_{n}$ such that $\mu+\rho \in \widehat{W} \cdot \rho$

$$
P_{\mu+\rho, \ell \lambda+\rho}^{-}(1)=\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]
$$

where $\binom{\mu}{\ell}$ and $\mathcal{I}$ are obtained from $\mu$ and $\ell$ by applying the algorithms described in 3.2.

### 5.3 The functions $H_{\mu}^{\ell}$

For any $\mu \in \mathcal{P}_{n}$, we define the function $G_{\mu}^{\ell}$ by setting

$$
\begin{equation*}
G_{\mu}^{\ell}=\sum_{\lambda \in \mathcal{P}_{n}}\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q} s_{\lambda} \tag{56}
\end{equation*}
$$

where for any $\lambda \in \mathcal{P}_{n},\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q}=P_{\mu+\rho, \ell \lambda+\rho}^{-}(q)$. We also consider the function $H_{\mu}^{\ell}$ such that

$$
\begin{equation*}
H_{\mu}^{\ell}=G_{\ell \mu}^{\ell} \tag{57}
\end{equation*}
$$

Theorem 5.3.1 Consider a partition $\mu \in \mathcal{P}_{n}$.

1. The coefficients of $H_{\mu}^{\ell}$ on the basis of Weyl characters are polynomials in $q$ with nonnegative integer coefficients.
2. We have $H_{\mu}^{1}=s_{\mu}$
3. For $\ell$ sufficiently large $H_{\mu}^{\ell}=Q_{\mu}^{\prime}$, that is $H_{\mu}^{\ell}$ coincide with the Hall-Littlewood function associated to $\mu$.

To prove our theorem we need the following Lemma:
Lemma 5.3.2 Consider $\beta \in P$.

- In type $A_{n-1}$, suppose $\ell>n$. Then the weight $\ell \beta+\rho$ is regular.
- In type $B_{n}, C_{n}$ or $D_{n}$, suppose $\ell>2 n$. Then the weight $\ell \beta+\rho$ is regular.

Proof. Consider $w \in W$ and $t_{\gamma}$ such that $t_{\gamma} w \cdot(\ell \beta+\rho)=\ell \beta+\rho$. Then $\delta=\ell \beta+\rho-w \cdot(\ell \beta+\rho) \in(\ell \mathbb{Z})^{\ell}$. Set $\beta=\left(\beta_{1}, \ldots, \beta_{n}\right)$. For any $i=1, \ldots, n$, the $i$-th coordinate of $\delta$ is $\delta_{i}=\ell \beta_{i}+\rho_{i}-\ell \beta_{w(i)}-\rho_{w(i)}$. Since $\delta_{i} \in \ell \mathbb{Z}$, we must have $\left|\rho_{i}-\rho_{w(i)}\right| \in \ell \mathbb{Z}$. One verifies easily that for type $A_{n-1},\left|\rho_{i}-\rho_{w(i)}\right| \leq$ $n-1$ and for types $B_{n}, C_{n}, D_{n}\left|\rho_{i}-\rho_{w(i)}\right| \leq 2 n$. Hence when the conditions of the lemma are verified, $\left|\rho_{i}-\rho_{w(i)}\right|=0$ for any $i=1, \ldots, n$. This gives $w=1$. The equality $t_{\gamma} w \cdot(\ell \beta+\rho)=\ell \beta+\rho$ implies then that $\gamma=0$. Thus the stabilizer of $\ell \beta+\rho$ is reduced to $\{1\}$, that is $\ell \beta+\rho$ is regular.

Proof. (of Theorem 5.3.1)
1 : Follows from Theorem [5.1.1 and (56).
2: When $\ell=1$, we have seen that $G=G_{\mathcal{I}}$ and $\binom{\mu}{\ell}=\mu$. Thus $\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q} \neq 0$ only if $\lambda=\mu$. In this case $H_{\mu}^{1}=s_{\mu}$ for $\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q}=[V(\lambda): V(\lambda)]_{q}=1$.
3 : Suppose $\ell$ as in the previous lemma. We have $\left[V(\lambda): V_{\mathcal{I}}\binom{\ell \mu}{\ell}\right]_{q}=P_{\ell \mu+\rho, \ell \lambda+\rho}^{-}(q)$. Since $\ell \lambda+\rho$ is regular for the action of $\widehat{W}$, we obtain by Theorem4.2.2, $P_{\ell \mu+\rho, \ell \lambda+\rho}^{-}(q)=p_{w(\ell \mu+\rho), w(\ell \lambda+\rho)}(q)$. By using Lemma 4.1.1] we deduce $P_{\ell \mu+\rho, \ell \lambda+\rho}^{-}(q)=p_{n_{\mu}, n_{\lambda}}(q)$. Now by Theorem4.2.1 this gives $P_{\ell \mu+\rho, \ell \lambda+\rho}^{-}(q)=K_{\lambda, \mu}(q)$. Finally

$$
\begin{equation*}
H_{\mu}^{\ell}=\sum_{\lambda \in \mathcal{P}_{n}}\left[V(\lambda): V_{\mathcal{I}}\binom{\ell \mu}{\ell}\right]_{q} s_{\lambda}=\sum_{\lambda \in \mathcal{P}_{n}} K_{\lambda, \mu}(q) s_{\lambda}=Q_{\mu}^{\prime} \tag{58}
\end{equation*}
$$

## Remarks:

(i): By the previous theorem the functions $H_{\mu}^{\ell}$ interpolate between the Weyl characters and the Hall-Littlewood functions.
(ii) : When $\ell$ is even for types $C$ and $D$, one can also define the functions $G_{\mu}^{\ell}$ and $H_{\mu}^{\ell}$ by setting $G_{\mu}^{\ell}=\sum_{\lambda \in \mathcal{P}_{n}} P_{\mu+\rho, \ell \lambda+\rho}^{-}(q) s_{\lambda}$ and $H_{\mu}^{\ell}=G_{\ell \mu}^{\ell}$, respectively. When $\ell>2 n$ we have yet $H_{\mu}^{\ell}=Q_{\mu}^{\prime}$, but the polynomials $P_{\mu+\rho, \ell \lambda+\rho}^{-}(q)$ cannot be interpreted as quantizations of branching coefficients.
(iii) : The conditions $\ell>n$ for type $A_{n-1}$ and $\ell>2 n$ for types $B_{n}, C_{n}, D_{n}$ appear also naturally in the algorithms of 3.2 When they are fulfilled, one has $\varphi_{\ell}\left(s_{\ell \mu}\right)=0$, or $J_{k}=I_{k}$ for any $k=1, \ldots, n$ and $J_{k}=I_{k}=\emptyset$ for $k \notin\{1, \ldots, n\}$. Then $[(\ell \mu) / \ell]=\mu$ and $G_{\mathcal{I}}=H$. Hence $\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]=K_{\lambda, \mu}$ for any $\lambda \in \mathcal{P}_{n}$. This yields equality (58) specialized at $q=1$.

## 6 Further remarks

### 6.1 Quantization of tensor product coefficients

Consider $\mu \in \mathcal{P}_{n}$ and set $\mu=\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)$ as in Theorem 3.2.1 For $G=G L_{n}$, the duality $c_{\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)}^{\lambda}=\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]$ yields a $q$-analogue of the Littlewood-Richardson coefficient $c_{\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)}^{\lambda}$ defined by setting

$$
\begin{equation*}
c_{\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)}^{\lambda}(q)=\left[V(\lambda): V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q}=P_{\mu+\rho, \ell \lambda+\rho}^{-}(q) . \tag{59}
\end{equation*}
$$

By Theorem 5.1.1] $c_{\left(\mu^{(0)}, \ldots, \mu^{(\ell-1)}\right)}^{\lambda}(q)$ have then nonnegative integer coefficients.
In [9, we have shown that there also exists a duality between tensor product coefficients for types $B, C, D$ defined as the analogues of the Littlewood-Richardson coefficients by counting the multiplicities of the isomorphic irreducible components in a tensor product of irreducible representations and branching coefficients. These branching coefficients correspond to the restriction of $\mathrm{SO}_{2 n}$ to subgroups of the form $S O_{2 r_{0}} \times \cdots S O_{2 r_{p}}$ where the $r_{i}$ 's are positive integers summing $n$. These subgroups are not Levi subgroups, thus the Littlewood-Richardson coefficients for types $B, C, D$ cannot be quantified as in (59) by using parabolic Kazhdan-Lusztig polynomials.

For $G=S O_{2 n+1}, S p_{2 n}$ or $S O_{2 n}$ and $\lambda \in \mathcal{P}_{n}$, denote by $\mathfrak{V}(\lambda)$ the restriction of the irreducible finite dimensional $G L_{N}$-module of highest weight $\lambda$ to $G$. Consider a $p$-tuple ( $\mu^{(0)}, \ldots, \mu^{(p-1)}$ ) of partitions such that $\mu^{(k)} \in \mathcal{P}_{r_{k}}$ for any $k=0, \ldots, p-1$. One can define the coefficients $\mathfrak{D}_{\mu^{(0)}, \ldots, \mu^{(p-1)}}^{\lambda}$ as the multiplicity of $V(\lambda)$ in $\mathfrak{V}\left(\mu^{(0)}\right) \otimes \cdots \otimes \mathfrak{V}\left(\mu^{(p-1)}\right)$, that is such that

$$
\mathfrak{V}\left(\mu^{(0)}\right) \otimes \cdots \otimes \mathfrak{V}\left(\mu^{(p-1)}\right) \simeq \bigoplus_{\lambda \in \mathcal{P}_{n}} V(\lambda)^{\oplus \mathfrak{D}_{\mu}^{\mathcal{X}}(0), \ldots, \mu^{(p-1)}}
$$

We have also obtained in [9] a duality result between the coefficients $\mathfrak{D}_{\mu^{(0)}, \ldots, \mu^{(p-1)}}^{\lambda}$ and branching coefficients corresponding to the restriction of $G$ to the Levi subgroup $G L_{r_{0}} \times \cdots \times G L_{r_{p-1}}$. The coefficients $\mathfrak{D}_{\mu^{(0)}, \ldots, \mu^{(p-1)}}^{\lambda}$ can be expressed by using a partition function similarly to Proposition 2.5.1, By quantifying this partition function, one shows that they admit nonnegative $q$-analogues. It is conjectured that stable one-dimensional sums defined in [3] from affine crystals obtained by considering the affinizations of the classical root systems are special cases of the $q$-analogues obtained in this way. Recall that the Levi subgroups $G_{\mathcal{I}}$ obtained in the theorems of 3.2 are, up to isomorphism, determined only by $G$ and $\ell$. This implies that there exist Levi subgroups $L$ in $G$ which are not isomorphic to a subgroup $G_{\mathcal{I}}$. This is in particular the case when $G=S p_{2 n}$ for the Levi subgroups $G_{\mathcal{I}} \simeq$ $G L_{r_{0}} \times \cdots \times G L_{r_{p-1}}$ such that $r_{k}>1$ for any $k=0, \ldots, p-1$. Indeed, by Theorem 3.2.3 when $r_{0}=\operatorname{card}\left(I^{(0)}\right)>1, G_{\mathcal{I}}$ is isomorphic to

$$
S p_{2 r_{0}} \times G L_{r_{1}} \times \cdots \times G L_{r_{p-1}} .
$$

This implies that one cannot obtain in general a quantization of the tensor product coefficients $\mathfrak{D}_{\mu^{(0)}, \ldots, \mu^{(p-1)}}^{\lambda}$ by using parabolic Kazhdan-Lusztig polynomials as in (59).

### 6.2 Combinatorial description of the functions $G_{\mu}^{\ell}$

When $G=G L_{n}$, the functions $G_{\mu}^{\ell}$ defined in (56]) admit the following combinatorial description

$$
G_{\mu}^{\ell}=\sum_{T \in \operatorname{Tab}_{\ell}(\mu)} q^{s(T)} x^{T}
$$

where $\operatorname{Tab}_{\ell}(\mu)$ is the set of $\ell$-ribbon tableaux of shape $\mu$ on $I_{n}$ and $s$ the spin statistic defined on ribbon tableaux (see [7] page 1057). Recently, Haglund, Haiman and Loehr have obtained the expansion of the Macdonald polynomials in terms of simple renormalizations of the LLT polynomials $G_{\mu}^{\ell}$. This expansion yields a combinatorial formula for the Macdonald polynomials [2].
This suggests to investigate the following combinatorial problem:
Problem 6.2.1 Find a combinatorial description of the polynomials $G_{\mu}^{\ell}$ and the $q$-analogues $[V(\lambda)$ : $\left.V_{\mathcal{I}}\binom{\mu}{\ell}\right]_{q}$ related to the roots systems of type $B, C$ or $D$.

### 6.3 Exceptional root systems

It is also possible to define the plethysm $\psi_{\ell}$ and the dual plethysm $\varphi_{\ell}$ for exceptional root systems. Consider such an exceptional root system $R$ and $\mu$ a dominant weight for $R$. Denote also by $s_{\mu}$ the Weyl character of the irreducible finite dimensional module of highest weight $\lambda$. When $\ell$ is sufficiently large (the bound depends on $R$ ), we have $\varphi_{\ell}\left(s_{\mu}\right)=s_{\mu}$. For the other values of $\ell$, one shows that the polynomial $\varphi_{\ell}\left(e^{\mu} \prod_{\alpha \in R_{+}}\left(1-e^{\alpha}\right)\right)$ do not factorize in general as a product of factors $\left(1-x^{\beta}\right)$ where $\beta$ is a positive root. This implies that one cannot define generalized Hall-Littlewood functions for exceptional types by proceeding as in (57).

## References

[1] G. Goodman, N. R Wallach, Representation theory and invariants of the classical groups, Cambridge University Press.
[2] J. Haglund, M. Haiman, N. Loehr, A combinatorial formula for Macdonald polynomials, J. Amer. Math. Soc. 18, n ${ }^{\circ}$ 3, 735-761 (2005).
[3] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on fermionic formula, in N. Jing and K. C. Misra, eds. Recent Developments in Quantum Affine Algebras and Related Topics, Contemporary Mathematics 248, AMS, Providence, 243-291, (1999).
[4] G. Kashiwara, A. Tanisaki, Parabolic Kazhdan-Lusztig polynomials and Schubert varieties, J. Algebra, 249, 306-325 (2002).
[5] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Inventiones 53, 191-213 (1979).
[6] K. Koike, I. Terada, Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank, Advances in Mathematics, 79, 104-135 (1990).
[7] A. Lascoux, B. Leclerc, J. Y. Thibon, Ribbon tableaux, Hall Littelwood functions, quantum affine algebras, J. Math. Phys. 38, 1041-1068 (1996).
[8] B. Leclerc, J. Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials, Advance Studies in Pure Mathematics 28, Combinatorial Methods in representation Theory, 155-220 (2000).
[9] C. Lecouvey, Quantization of branching coefficients for classical Lie groups, submitted.
[10] I-G. Macdonald, Symmetric functions and Hall polynomials, Second edition, Oxford Mathematical Monograph, Oxford University Press, New York, (1995).
[11] K. Nelsen, A. Ram, Kostka-Foulkes polynomials and Macdonald spherical functions, Surveys in Combinatorics 2003, C. Wensley ed., London Math. Soc. Lect. Notes 307 , Cambridge University Press, 325-370 (2003).
[12] A. Ram, Weyl group, symmetric functions and the representation theory of Lie algebras, Proceedings of the 4th conference "Formal Power Series and Algebraic Combinatorics", Publ. LACIM 11, 327-342 (1992).
[13] W. Soergel, Kazhdan-Lusztig polynomials and a combinatorics for tilting modules, Represent. Theory 1, 83-114 (1997).

