Distance-regular graphs and the q-tetrahedron algebra

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In honor of Eiichi Bannai on his 60th Birthday

Abstract

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) and $b \neq 1$, $\alpha = b - 1$. The condition on α implies that Γ is formally self-dual. For $b = q^2$ we use the adjacency matrix and dual adjacency matrix to obtain an action of the q-tetrahedron algebra $\boxtimes_q$ on the standard module of Γ. We describe four algebra homomorphisms into $\boxtimes_q$ from the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$; using these we pull back the above $\boxtimes_q$-action to obtain four actions of $U_q(\widehat{\mathfrak{sl}}_2)$ on the standard module of Γ.

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1 Introduction

In [25] B. Hartwig and the second author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\boxtimes$ by generators and relations, and displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. The algebra $\boxtimes$ has essentially six generators, and it is natural to identify these with the six edges of a tetrahedron. For each face of the tetrahedron the three surrounding edges form a basis for a subalgebra of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$ [25, Corollary 12.4]. Any five of the six edges of the tetrahedron generate a subalgebra of $\boxtimes$ that is isomorphic to the $\mathfrak{sl}_2$ loop algebra [25, Corollary 12.6]. Each pair of opposite edges of the tetrahedron generate a subalgebra of $\boxtimes$ that is isomorphic to the Onsager algebra [25, Corollary 12.5]. Let us call these Onsager subalgebras. Then $\boxtimes$ is the direct sum of its three Onsager subalgebras [25, Theorem 11.6]. In [20] Elduque found an attractive decomposition of $\boxtimes$ into a direct sum of three abelian subalgebras, and he showed how these subalgebras are related to the Onsager

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subalgebras. In [35] Pascasio and the second author give an action of $\boxtimes$ on the standard module of a Hamming graph. In [4] Bremner obtained the universal central extension of the three-point $\mathfrak{sl}_2$ loop algebra. By modifying the defining relations for $\boxtimes$, Benkart and the second author obtained a presentation for this extension by generators and relations [2]. In [24] Hartwig obtained the irreducible finite-dimensional $\boxtimes$-modules over an algebraically closed field with characteristic 0.

In [30] we introduced a quantum analog of $\boxtimes$ which we call $\boxtimes_q$. We defined $\boxtimes_q$ using generators and relations. We showed how $\boxtimes_q$ is related to the quantum group $U_q(\mathfrak{sl}_2)$ in roughly the same way that $\boxtimes$ is related to $\mathfrak{sl}_2$ [30, Proposition 7.4]. We showed how $\boxtimes_q$ is related to the $U_q(\mathfrak{sl}_2)$ loop algebra in roughly the same way that $\boxtimes$ is related to the $\mathfrak{sl}_2$ loop algebra [30, Proposition 8.3]. In [28] we considered an algebra $\mathcal{A}_q$ on two generators subject to the cubic q-Serre relations. $\mathcal{A}_q$ is often called the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$. We showed how $\boxtimes_q$ is related to $\mathcal{A}_q$ in roughly the same way that $\boxtimes$ is related to the Onsager algebra [30, Proposition 9.4]. In [30] and [31] we described the finite-dimensional irreducible $\boxtimes_q$-modules under the assumption that q is not a root of 1, and the underlying field is algebraically closed.

In the present paper we consider a distance-regular graph Γ that has classical parameters (D, b, α, β) and $b \neq 1$, $\alpha = b - 1$. The condition on α implies that Γ is formally self-dual [5, p. 71]. For $b = q^2$ we use the adjacency matrix and dual adjacency matrix to construct an action of $\boxtimes_q$ on the standard module of Γ. We describe four algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ to $\boxtimes_q$; using these homomorphisms we pull back the above $\boxtimes_q$-action to obtain four actions of $U_q(\widehat{\mathfrak{sl}}_2)$ on the standard module of Γ. Several well-known families of distance-regular graphs satisfy the above parameter restriction; for instance the bilinear forms graph [5, p. 280], the alternating forms graph [5, p. 282], the Hermitean forms graph [5, p. 285], the quadratic forms graph [5, p. 290], the affine E_6 graph [5, p. 340], and the extended ternary Golay code graph [5, p. 359].

All of the original results in this paper are about distance-regular graphs. However, in order to motivate things and develop some machinery, we will initially discuss $\boxtimes_q$ and its relationship to certain quantum groups. The paper is organized as follows. In Section 2 we define $\boxtimes_q$ and mention a few of its properties. In Section 3 we recall how $\boxtimes_q$ is related to $U_q(\mathfrak{sl}_2)$. In Section 4 we discuss how $\boxtimes_q$ is related to $U_q(\widehat{\mathfrak{sl}}_2)$. In Section 5 we recall how $\boxtimes_q$ is related to $\mathcal{A}_q$. In Section 6 we discuss the finite-dimensional irreducible $\boxtimes_q$-modules. In Section 7 we consider a distance-regular graph Γ and discuss its basic properties. In Section 8 we impose a parameter restriction on Γ needed to construct our $\boxtimes_q$-module. In Sections 9, 10 we define some matrices that will be used to construct our $\boxtimes_q$-module. In Section 11 we display an action of $\boxtimes_q$ on the standard module of Γ; Theorem 11.1 is the main result of the paper. In Section 12 we discuss how the above $\boxtimes_q$-action is related to the subconstituent algebra of Γ. In Section 13 we give some suggestions for further research.

Throughout the paper $\mathbb{C}$ denotes the field of complex numbers.

2 The q-tetrahedron algebra $\boxtimes_q$

In this section we recall the q-tetrahedron algebra. We fix a nonzero scalar $q \in \mathbb{C}$ such that $q^2 \neq 1$ and define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, 2, \ldots$$

We let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 2.1 [30, Definition 6.1] Let $\boxtimes_q$ denote the unital associative $\mathbb{C}$-algebra that has generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4,\ j - i = 1 \text{ or } j - i = 2\}$$

and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij} x_{ji} = 1.$$

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1), (1, 2), (2, 1)$,

$$\frac{q x_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1.$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0. \qquad (1)$$

We call $\boxtimes_q$ the *q-tetrahedron algebra* or "q-tet" for short.

Note 2.2 The equations (1) are the cubic q-Serre relations [33, p. 10].

We make some observations.

Lemma 2.3 [30, Lemma 6.3] *There exists a $\mathbb{C}$-algebra automorphism ϱ of $\boxtimes_q$ that sends each generator x_{ij} to $x_{i+1,j+1}$. Moreover $\varrho^4 = 1$.*

Lemma 2.4 [30, Lemma 6.5] *There exists a $\mathbb{C}$-algebra automorphism of $\boxtimes_q$ that sends each generator x_{ij} to $-x_{ij}$.*

3 The algebra $U_q(\mathfrak{sl}_2)$

In this section we recall how the algebra $\boxtimes_q$ is related to $U_q(\mathfrak{sl}_2)$. We start with a definition.

Definition 3.1 [32, p. 122] Let $U_q(\mathfrak{sl}_2)$ denote the unital associative $\mathbb{C}$-algebra with generators $K^{\pm 1}$, $e^{\pm}$ and the following relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ Ke^{\pm}K^{-1} &= q^{\pm 2}e^{\pm}, \\ [e^+, e^-] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

The following presentation of $U_q(\mathfrak{sl}_2)$ will be useful.

Lemma 3.2 [29, Theorem 2.1] *The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative $\mathbb{C}$-algebra with generators $x^{\pm 1}$, y, z and the following relations:*

$$\begin{aligned} xx^{-1} = x^{-1}x &= 1, \\ \frac{qxy - q^{-1}yx}{q - q^{-1}} &= 1, \\ \frac{qyz - q^{-1}zy}{q - q^{-1}} &= 1, \\ \frac{qzx - q^{-1}xz}{q - q^{-1}} &= 1. \end{aligned}$$

An isomorphism with the presentation in Definition 3.1 is given by:

$$\begin{aligned} x^{\pm 1} &\mapsto K^{\pm 1}, \\ y &\mapsto K^{-1} + e^-, \\ z &\mapsto K^{-1} - K^{-1}e^+q(q - q^{-1})^2. \end{aligned}$$

The inverse of this isomorphism is given by:

$$\begin{aligned} K^{\pm 1} &\mapsto x^{\pm 1}, \\ e^- &\mapsto y - x^{-1}, \\ e^+ &\mapsto (1 - xz)q^{-1}(q - q^{-1})^{-2}. \end{aligned}$$

Proposition 3.3 [30, Proposition 7.4] *For $i \in \mathbb{Z}_4$ there exists a $\mathbb{C}$-algebra homomorphism from $U_q(\mathfrak{sl}_2)$ to $\boxtimes_q$ that sends*

$$x \mapsto x_{i,i+2}, \quad x^{-1} \mapsto x_{i+2,i}, \quad y \mapsto x_{i+2,i+3}, \quad z \mapsto x_{i+3,i}.$$

4 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this section we consider how $\boxtimes_q$ is related to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. We start with a definition.

Definition 4.1 [9, p. 262] The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the unital associative $\mathbb{C}$-algebra with generators $K_i^{\pm 1}$, $e_i^{\pm}$, $i \in \{0, 1\}$ and the following relations:

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\
K_0 K_1 &= K_1 K_0, \\
K_i e_i^{\pm} K_i^{-1} &= q^{\pm 2} e_i^{\pm}, \\
K_i e_j^{\pm} K_i^{-1} &= q^{\mp 2} e_j^{\pm}, \qquad i \neq j, \\
[e_i^+, e_i^-] &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
[e_0^{\pm}, e_1^{\mp}] &= 0,
\end{aligned}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \qquad i \neq j.$$

The following presentation of $U_q(\widehat{\mathfrak{sl}}_2)$ will be useful.

Theorem 4.2 ([27, Theorem 2.1], [42]) *The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the unital associative $\mathbb{C}$-algebra with generators $x_i^{\pm 1}$, y_i, z_i, $i \in \{0, 1\}$ and the following relations:*

$$\begin{aligned}
x_i x_i^{-1} = x_i^{-1} x_i &= 1, \\
x_0 x_1 \ \ &\textit{is central}, \\
\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} &= 1, \\
\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} &= 1, \\
\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} &= 1, \\
\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} &= x_0^{-1} x_1^{-1}, \qquad i \neq j,
\end{aligned}$$

$$\begin{aligned}
y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \qquad i \neq j, \\
z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0, \qquad i \neq j.
\end{aligned}$$

An isomorphism with the presentation in Definition 4.1 is given by:

$$\begin{aligned}
x_i^{\pm 1} &\mapsto K_i^{\pm 1}, \\
y_i &\mapsto K_i^{-1} + e_i^-, \\
z_i &\mapsto K_i^{-1} - K_i^{-1} e_i^+ q (q - q^{-1})^2.
\end{aligned}$$

The inverse of this isomorphism is given by:

$$\begin{aligned} K_i^{\pm 1} &\mapsto x_i^{\pm 1}, \\ e_i^- &\mapsto y_i - x_i^{-1}, \\ e_i^+ &\mapsto (1 - x_i z_i) q^{-1} (q - q^{-1})^{-2}. \end{aligned}$$

Proposition 4.3 *For $i \in \mathbb{Z}_4$ there exists a $\mathbb{C}$-algebra homomorphism from $U_q(\widehat{\mathfrak{sl}}_2)$ to $\boxtimes_q$ that sends*

$$\begin{array}{llll} x_1 \mapsto x_{i,i+2}, & x_1^{-1} \mapsto x_{i+2,i}, & y_1 \mapsto x_{i+2,i+3}, & z_1 \mapsto x_{i+3,i}, \\ x_0 \mapsto x_{i+2,i}, & x_0^{-1} \mapsto x_{i,i+2}, & y_0 \mapsto x_{i,i+1}, & z_0 \mapsto x_{i+1,i+2}. \end{array}$$

Proof: Compare the defining relations for $U_q(\widehat{\mathfrak{sl}}_2)$ given in Theorem 4.2 with the relations in Definition 2.1. □

5 The algebra $\mathcal{A}_q$

In this section we recall how $\boxtimes_q$ is related to the algebra $\mathcal{A}_q$. We start with a definition.

Definition 5.1 Let $\mathcal{A}_q$ denote the unital associative $\mathbb{C}$-algebra defined by generators x, y and relations

$$\begin{aligned} x^3 y - [3]_q x^2 y x + [3]_q x y x^2 - y x^3 &= 0, \\ y^3 x - [3]_q y^2 x y + [3]_q y x y^2 - x y^3 &= 0. \end{aligned}$$

Definition 5.2 Referring to Definition 5.1, we call x, y the *standard generators* for $\mathcal{A}_q$.

Note 5.3 [33, Corollary 3.2.6] The algebra $\mathcal{A}_q$ is often called the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$.

Proposition 5.4 [30, Proposition 9.4] *For $i \in \mathbb{Z}_4$ there exists a homomorphism of $\mathbb{C}$-algebras from $\mathcal{A}_q$ to $\boxtimes_q$ that sends the standard generators x, y to $x_{i,i+1}, x_{i+2,i+3}$ respectively.*

6 The finite-dimensional irreducible $\boxtimes_q$-modules

In this section we recall how the finite-dimensional irreducible modules for $\boxtimes_q$ and $\mathcal{A}_q$ are related. We start with some comments. Let V denote a finite-dimensional vector space over $\mathbb{C}$. A linear transformation $A : V \to V$ is said to be *nilpotent* whenever there exists a positive integer n such that $A^n = 0$. Let V denote a finite-dimensional irreducible $\mathcal{A}_q$-module. This module is called *NonNil* whenever the standard generators x, y are not nilpotent on V [28, Definition 1.3]. Assume V is NonNil. Then by [28, Corollary 2.8] the standard generators x, y are semisimple on V. Moreover there exist an integer $d \geq 0$ and nonzero scalars $\alpha, \alpha^* \in \mathbb{C}$ such that the set of distinct eigenvalues of x (resp. y) on V is $\{\alpha q^d, \alpha q^{d-2}, \ldots, \alpha q^{-d}\}$ (resp. $\{\alpha^* q^d, \alpha^* q^{d-2}, \ldots, \alpha^* q^{-d}\}$). We call the ordered pair (α, α^*) the *type* of V. Replacing x, y by $x/\alpha, y/\alpha^*$ the type becomes $(1, 1)$. Now let V denote a finite-dimensional irreducible

$\boxtimes_q$-module. By [30, Theorem 12.3] each generator x_{ij} is semisimple on V. Moreover there exist an integer $d \geq 0$ and a scalar $\varepsilon \in \{1, -1\}$ such that for each generator x_{ij} the set of distinct eigenvalues on V is $\{\varepsilon q^d, \varepsilon q^{d-2}, \ldots, \varepsilon q^{-d}\}$. We call ε the *type* of V. Replacing each generator x_{ij} by εx_{ij} the type becomes 1. The finite-dimensional irreducible modules for $\boxtimes_q$ and $\mathcal{A}_q$ are related according to the following two theorems and subsequent remark.

Theorem 6.1 [30, Theorem 10.3] *Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then there exists a unique $\mathcal{A}_q$-module structure on V such that the standard generators x and y act as x_{01} and x_{23} respectively. This $\mathcal{A}_q$-module is irreducible, NonNil, and type* $(1, 1)$.

Theorem 6.2 [30, Theorem 10.4] *Let V denote a NonNil finite-dimensional irreducible $\mathcal{A}_q$-module of type* $(1, 1)$*. Then there exists a unique $\boxtimes_q$-module structure on V such that the standard generators x and y act as x_{01} and x_{23} respectively. This $\boxtimes_q$-module structure is irreducible and type* 1.

Remark 6.3 [30, Remark 10.5] Combining Theorem 6.1 and Theorem 6.2 we obtain a bijection between the following two sets:

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules of type 1;

(ii) the isomorphism classes of NonNil finite-dimensional irreducible $\mathcal{A}_q$-modules of type $(1, 1)$.

7 Distance-regular graphs; preliminaries

We now turn our attention to distance-regular graphs. After a brief review of their basic properties we consider a special type said to be formally self-dual with classical parameters. From such a distance-regular graph we will obtain a $\boxtimes_q$-module.

We now review some definitions and basic concepts concerning distance-regular graphs. For more information we refer the reader to [1, 5, 23, 38].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitean inner product $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where t denotes transpose and $^-$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For an integer $k \geq 0$ we say that Γ is *regular with valency k* whenever each vertex of Γ is adjacent to exactly k distinct vertices of Γ. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i = p^i_{1,i-1}$ $(1 \leq i \leq D)$, $b_i = p^i_{1,i+1}$ $(0 \leq i \leq D-1)$, $a_i = p^i_{1i}$ $(0 \leq i \leq D)$.

For the rest of this paper we assume Γ is distance-regular; to avoid trivialities we always assume $D \geq 3$. Note that Γ is regular with valency $k = b_0$. Moreover $k = c_i + a_i + b_i$ for $0 \leq i \leq D$, where $c_0 = 0$ and $b_D = 0$.

We mention a fact for later use. By the triangle inequality, for $0 \leq h, i, j \leq D$ we have $p^h_{ij} = 0$ (resp. $p^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two.

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. The matrix A_1 is often called the *adjacency matrix* of Γ. We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^D A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (iv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (v) $A_i A_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \leq i, j \leq D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, called the *Bose-Mesner algebra* of Γ. It turns out that A_1 generates M [1, p. 190]. By [5, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1} J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$; (iv) $E_i^t = E_i$ $(0 \leq i \leq D)$; (v) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A_1 = \sum_{i=0}^D \theta_i E_i$. Observe $A_1 E_i = E_i A_1 = \theta_i E_i$ for $0 \leq i \leq D$. By [1, p. 197] the scalars $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct since A_1 generates M. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \leq i \leq D)$. Observe

$$V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum)}.$$

For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of A_1 associated with θ_i.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so M is closed under $\circ$. Thus there exist complex scalars q^h_{ij} $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q^h_{ij} E_h \qquad (0 \leq i, j \leq D).$$

By [3, p. 170], q^h_{ij} is real and nonnegative for $0 \leq h, i, j \leq D$. The q^h_{ij} are called the *Krein parameters* of Γ. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [5, p. 235]. See [6, 7, 8, 12, 13, 16, 17, 34] for background information on the Q-polynomial property. For the rest of this section we assume Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We recall the dual Bose-Mesner algebra of Γ. For the rest of this paper we fix a vertex $x \in X$. We view x as a "base vertex." For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E^*_i)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X). \tag{2}$$

We call E^*_i the ith *dual idempotent* of Γ with respect to x [38, p. 378]. We observe (i) $\sum_{i=0}^D E^*_i = I$; (ii) $\overline{E^*_i} = E^*_i$ $(0 \leq i \leq D)$; (iii) $E^{*t}_i = E^*_i$ $(0 \leq i \leq D)$; (iv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \leq i, j \leq D)$. By these facts $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [38, p. 378]. For $0 \leq i \leq D$ let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry $(A^*_i)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A^*_0, A^*_1, \ldots, A^*_D$ is a basis for M^* [38, p. 379]. Moreover (i) $A^*_0 = I$; (ii) $\overline{A^*_i} = A^*_i$ $(0 \leq i \leq D)$; (iii) $A^{*t}_i = A^*_i$ $(0 \leq i \leq D)$; (iv) $A^*_i A^*_j = \sum_{h=0}^D q^h_{ij} A^*_h$ $(0 \leq i, j \leq D)$ [38, p. 379]. We call $A^*_0, A^*_1, \ldots, A^*_D$ the *dual distance matrices* of Γ with respect to x. The matrix A^*_1 is often called the *dual adjacency matrix* of Γ with respect to x. The matrix A^*_1 generates M^* [38, Lemma 3.11].

We recall the dual eigenvalues of Γ. Since $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for M^* there exist complex scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ such that $A^*_1 = \sum_{i=0}^D \theta^*_i E^*_i$. Observe $A^*_1 E^*_i = E^*_i A^*_1 = \theta^*_i E^*_i$ for $0 \leq i \leq D$. By [38, Lemma 3.11] the scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are in $\mathbb{R}$. The scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are mutually distinct since A^*_1 generates M^*. We call θ^*_i the *dual eigenvalue* of Γ associated with E^*_i $(0 \leq i \leq D)$.

We recall the subconstituents of Γ. From (2) we find

$$E^*_i V = \mathrm{span}\{\hat{y} \mid y \in X, \quad \partial(x, y) = i\} \qquad (0 \leq i \leq D). \tag{3}$$

By (3) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \qquad \text{(orthogonal direct sum)}.$$

For $0 \leq i \leq D$ the space $E^*_i V$ is the eigenspace of A^*_1 associated with θ^*_i. We call $E^*_i V$ the ith *subconstituent* of Γ with respect to x.

We recall the subconstituent algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [38, Definition 3.3]. Observe that T has finite dimension. Moreover T is semisimple since it is closed under the conjugate transpose map [15, p. 157]. By [38, Lemma 3.2] the following are relations in T:

$$E^*_h A_i E^*_j = 0 \quad \text{iff} \quad p^h_{ij} = 0, \qquad (0 \leq h, i, j \leq D), \tag{4}$$

$$E_h A^*_i E_j = 0 \quad \text{iff} \quad q^h_{ij} = 0, \qquad (0 \leq h, i, j \leq D). \tag{5}$$

See [10, 11, 14, 19, 21, 22, 26, 36, 38, 39, 40] for more information on the subconstituent algebra.

We recall the T-modules. By a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module [22, p. 802]. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. In particular V is an orthogonal direct sum of irreducible T-modules.

Let W denote an irreducible T-module. Observe that W is the direct sum of the nonzero spaces among $E^*_0W, \ldots, E^*_DW$. Similarly W is the direct sum of the nonzero spaces among $E_0W, \ldots, E_DW$. By the *endpoint* of W we mean $\min\{i|0 \leq i \leq D,\ E^*_iW \neq 0\}$. By the *diameter* of W we mean $|\{i|0 \leq i \leq D,\ E^*_iW \neq 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i|0 \leq i \leq D,\ E_iW \neq 0\}$. By the *dual diameter* of W we mean $|\{i|0 \leq i \leq D,\ E_iW \neq 0\}| - 1$. It turns out that the diameter of W is equal to the dual diameter of W [34, Corollary 3.3]. We finish this section with a comment.

Lemma 7.1 [38, Lemma 3.4, Lemma 3.9, Lemma 3.12] *Let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then ρ, τ, d are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)–(iv) hold.*

(i) $E^*_iW \neq 0$ *if and only if* $\rho \leq i \leq \rho + d, \quad (0 \leq i \leq D)$.

(ii) $W = \sum_{h=0}^{d} E^*_{\rho+h}W \qquad$ (*orthogonal direct sum*).

(iii) $E_iW \neq 0$ *if and only if* $\tau \leq i \leq \tau + d, \quad (0 \leq i \leq D)$.

(iv) $W = \sum_{h=0}^{d} E_{\tau+h}W \qquad$ (*orthogonal direct sum*).

8 A restriction on the intersection numbers

From now on we impose the following restriction on the intersection numbers of Γ.

Assumption 8.1 We fix $b, \beta \in \mathbb{C}$ such that $b \neq 1$, and assume Γ has classical parameters (D, b, α, β) with $\alpha = b - 1$. This means that the intersection numbers of Γ satisfy

$$\begin{aligned} c_i &= b^{i-1}\frac{b^i - 1}{b - 1}, \\ b_i &= (\beta + 1 - b^i)\frac{b^D - b^i}{b - 1} \end{aligned}$$

for $0 \leq i \leq D$ [5, p. 193]. We remark that b is an integer and $b \neq 0$, $b \neq -1$ [5, Proposition 6.2.1]. For notational convenience we fix $q \in \mathbb{C}$ such that

$$b = q^2.$$

We note that q is nonzero and not a root of unity.

Remark 8.2 Referring to Assumption 8.1, the restriction $\alpha = b-1$ implies that Γ is formally self-dual [5, Corollary 8.4.4]. Consequently there exists an ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents of Γ, with respect to which the Krein parameter q^h_{ij} is equal to the intersection number p^h_{ij} for $0 \leq h, i, j \leq D$. In particular Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. We fix this ordering of the primitive idempotents for the rest of the paper.

Remark 8.3 In the notation of Bannai and Ito [1, p. 263], the Q-polynomial structure from Remark 8.2 is type I with $s = 0, s^* = 0$.

Example 8.4 The following distance-regular graphs satisfy Assumption 8.1: the bilinear forms graph [5, p. 280], the alternating forms graph [5, p. 282], the Hermitean forms graph [5, p. 285], the quadratic forms graph [5, p. 290], the affine E_6 graph [5, p. 340], and the extended ternary Golay code graph [5, p. 359].

With reference to Assumption 8.1 we will display an action of $\boxtimes_q$ on the standard module of Γ. To describe this action we define eight matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A, \quad A^*, \quad B, \quad B^*, \quad K, \quad K^*, \quad \Phi, \quad \Psi. \tag{6}$$

These matrices will be defined in the next two sections.

9 The matrices A and A^*

In this section we define the matrices A, A^* and discuss their properties. We start with a comment.

Lemma 9.1 [5, Corollary 8.4.4] *With reference to Assumption 8.1, there exist $\alpha_0, \alpha_1 \in \mathbb{C}$ such that each of θ_i, θ^*_i is $\alpha_0 + \alpha_1 q^{D-2i}$ for $0 \leq i \leq D$. Moreover $\alpha_1 \neq 0$.*

Definition 9.2 With reference to Assumption 8.1 we define $A, A^* \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$\begin{aligned} A_1 &= \alpha_0 I + \alpha_1 A, \\ A^*_1 &= \alpha_0 I + \alpha_1 A^*, \end{aligned}$$

where α_0, α_1 are from Lemma 9.1. Thus for $0 \leq i \leq D$ the space E_iV (resp. E^*_iV) is an eigenspace of A (resp. A^*) with eigenvalue q^{D-2i}.

Lemma 9.3 *With reference to Assumption 8.1 and Definition 9.2, the following (i), (ii) hold for all $0 \leq i, j \leq D$ such that $|i - j| > 1$:*

(i) $E^*_i A E^*_j = 0$,

(ii) $E_i A^* E_j = 0$.

Proof: (i) We have $p^i_{1j} = 0$ since $|i - j| > 1$, so $E^*_i A_1 E^*_j = 0$ in view of (4). The result now follows using the first equation of Definition 9.2.
(ii) Similar to the proof of (i) above. □

The following is essentially a special case of [40, Lemma 5.4].

Lemma 9.4 [40, Lemma 5.4] *With reference to Assumption 8.1 and Definition 9.2 the matrices A, A^* satisfy the q-Serre relations*

$$A^3A^* - [3]_qA^2A^*A + [3]_qAA^*A^2 - A^*A^3 = 0, \qquad (7)$$
$$A^{*3}A - [3]_qA^{*2}AA^* + [3]_qA^*AA^{*2} - AA^{*3} = 0. \qquad (8)$$

Proof: We first show (7). By the last sentence in Definition 9.2, for $0 \leq i \leq D$ we have $AE_i = E_iA = \sigma_iE_i$ where $\sigma_i = q^{D-2i}$. Let C denote the expression on the left in (7). We show $C = 0$. Since $I = E_0 + \cdots + E_D$ it suffices to show $E_iCE_j = 0$ for $0 \leq i, j \leq D$. Let i, j be given. By our preliminary comment and the definition of C we find $E_iCE_j = E_iA^*E_j\alpha_{ij}$ where

$$\begin{aligned}\alpha_{ij} &= \sigma_i^3 - [3]_q\sigma_i^2\sigma_j + [3]_q\sigma_i\sigma_j^2 - \sigma_j^3 \\ &= (\sigma_i - \sigma_jq^2)(\sigma_i - \sigma_j)(\sigma_i - \sigma_jq^{-2}). \qquad (9)\end{aligned}$$

If $|i - j| > 1$ then $E_iA^*E_j = 0$ by Lemma 9.3(ii). If $|i - j| \leq 1$ then $\alpha_{ij} = 0$ by (9) and the definition of $\sigma_0, \ldots, \sigma_D$. In either case $E_iCE_j = 0$ as desired. It follows that $C = 0$ and line (7) is proved. The proof of (8) is similar to the proof of (7). $\Box$

We finish this section with a comment.

Lemma 9.5 *With reference to Assumption 8.1 and Definition 9.2 the matrices A, A^* together generate T.*

Proof: By definition T is generated by M and M^*. The algebra M (resp. M^*) is generated by A_1 (resp. A_1^*) and hence by A (resp. A^*) in view of Definition 9.2. The result follows. $\Box$

10 The matrices B, B^*, K, K^*, Φ, Ψ

In the previous section we defined the matrices A, A^*. In this section we define the remaining matrices from the list (6).

Definition 10.1 With reference to Assumption 8.1, for $-1 \leq i, j \leq D$ we define

$$\begin{aligned}V_{i,j}^{\downarrow\downarrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV), \\ V_{i,j}^{\uparrow\downarrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV), \\ V_{i,j}^{\downarrow\uparrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V), \\ V_{i,j}^{\uparrow\uparrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).\end{aligned}$$

In each of the above four equations we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

Definition 10.2 With reference to Assumption 8.1 and Definition 10.1, for $\eta, \mu \in \{\downarrow, \uparrow\}$ and $0 \le i, j \le D$ we have $V^{\eta\mu}_{i-1,j} \subseteq V^{\eta\mu}_{i,j}$ and $V^{\eta\mu}_{i,j-1} \subseteq V^{\eta\mu}_{i,j}$. Therefore

$$V^{\eta\mu}_{i-1,j} + V^{\eta\mu}_{i,j-1} \subseteq V^{\eta\mu}_{i,j}.$$

Referring to the above inclusion, we define $\tilde{V}^{\eta\mu}_{i,j}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is

$$\tilde{V}^{\eta\mu}_{i,j} = (V^{\eta\mu}_{i-1,j} + V^{\eta\mu}_{i,j-1})^{\perp} \cap V^{\eta\mu}_{i,j}.$$

The following result is a mild generalization of [41, Corollary 5.8].

Lemma 10.3 *With reference to Assumption 8.1 and Definition 10.2 the following holds for $\eta, \mu \in \{\downarrow, \uparrow\}$:*

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}^{\eta\mu}_{i,j} \qquad \textit{(direct sum)}.$$

Proof: For $\eta = \downarrow$, $\mu = \downarrow$ this is just [41, Corollary 5.8]. For general values of η, μ, in the proof of [41, Corollary 5.8] replace the sequence $E^*_0, \ldots, E^*_D$ (resp. $E_0, \ldots, E_D$) by $E^*_D, \ldots, E^*_0$ (resp. $E_D, \ldots, E_0$) if $\eta = \uparrow$ (resp. $\mu = \uparrow$). $\square$

Definition 10.4 With reference to Assumption 8.1 and Definition 10.2, we define B, B^*, K, K^*, Φ, Ψ to be the unique matrices in $\mathrm{Mat}_X(\mathbb{C})$ that satisfy the requirements of the following table for $0 \le i, j \le D$.

The matrix	is 0 on
$B - q^{i-j}I$	$\tilde{V}^{\downarrow\uparrow}_{i,j}$
$B^* - q^{j-i}I$	$\tilde{V}^{\uparrow\downarrow}_{i,j}$
$K - q^{i-j}I$	$\tilde{V}^{\downarrow\downarrow}_{i,j}$
$K^* - q^{i-j}I$	$\tilde{V}^{\uparrow\uparrow}_{i,j}$
$\Phi - q^{i+j-D}I$	$\tilde{V}^{\downarrow\downarrow}_{i,j}$
$\Psi - q^{i+j-D}I$	$\tilde{V}^{\downarrow\uparrow}_{i,j}$

11 An action of $\boxtimes_q$ on the standard module of Γ

We now state our main result, in which we display an action of $\boxtimes_q$ on the standard module V of Γ.

Theorem 11.1 *With reference to Assumption 8.1, there exists a $\boxtimes_q$-module structure on V such that the generators x_{ij} act as follows:*

generator	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}
action on V	$A\Phi\Psi^{-1}$	$B\Phi^{-1}$	$A^*\Phi\Psi$	$B^*\Phi^{-1}$	$K\Psi^{-1}$	$K^*\Psi$

The proof of Theorem 11.1 is given at the end of this section. First we need some lemmas.

Lemma 11.2 *With reference to Assumption 8.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then there exists a unique $\boxtimes_q$-module structure on W such that the generators x_{01}, x_{23} act as $Aq^{d-D+2\tau}$, $A^*q^{d-D+2\rho}$ respectively. This $\boxtimes_q$-module structure is irreducible and type* 1.

Proof: The matrices A, A^* satisfy the q-Serre relations (7), (8). These relations are homogeneous so they still hold if A, A^* are replaced by $Aq^{d-D+2\tau}$, $A^*q^{d-D+2\rho}$ respectively. Therefore there exists an $\mathcal{A}_q$-module structure on W such that the standard generators act as $Aq^{d-D+2\tau}$ and $A^*q^{d-D+2\rho}$. The $\mathcal{A}_q$-module W is irreducible since A, A^* generate T and since the T-module W is irreducible. By Lemma 7.1(iii),(iv) the action of A on W is semisimple with eigenvalues $q^{D-2\tau-2i}$ $(0 \le i \le d)$. Therefore the action of $Aq^{d-D+2\tau}$ on W is semisimple with eigenvalues q^{d-2i} $(0 \le i \le d)$. By Lemma 7.1(i),(ii) the action of A^* on W is semisimple with eigenvalues $q^{D-2\rho-2i}$ $(0 \le i \le d)$. Therefore the action of $A^*q^{d-D+2\rho}$ on W is semisimple with eigenvalues q^{d-2i} $(0 \le i \le d)$. By these comments and the first paragraph of Section 6 the $\mathcal{A}_q$-module W is NonNil and type $(1,1)$. So far we have shown that the $\mathcal{A}_q$-module W is irreducible, NonNil, and type $(1,1)$. Combining this with Theorem 6.2 we obtain the result. $\square$

Lemma 11.3 *With reference to Assumption 8.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Consider the $\boxtimes_q$-module structure on W from Lemma 11.2. For each generator x_{rs} of $\boxtimes_q$ and for $0 \le i \le d$, the eigenspace of x_{rs} on W associated with the eigenvalue q^{d-2i} is given in the following table.*

r	s	eigenspace of x_{rs} for the eigenvalue q^{d-2i}
0	1	$E_{\tau+i}W$
1	2	$(E^*_\rho W + \cdots + E^*_{\rho+d-i}W) \cap (E_{\tau+d-i}W + \cdots + E_{\tau+d}W)$
2	3	$E^*_{\rho+i}W$
3	0	$(E^*_{\rho+d-i}W + \cdots + E^*_{\rho+d}W) \cap (E_\tau W + \cdots + E_{\tau+d-i}W)$
0	2	$(E^*_\rho W + \cdots + E^*_{\rho+d-i}W) \cap (E_\tau W + \cdots + E_{\tau+i}W)$
1	3	$(E^*_{\rho+i}W + \cdots + E^*_{\rho+d}W) \cap (E_{\tau+d-i}W + \cdots + E_{\tau+d}W)$

Proof: Referring to the table, we first verify row $(r,s) = (0,1)$. By Lemma 11.2 the generator x_{01} acts on W as $Aq^{d-D+2\tau}$. By Lemma 7.1(iii),(iv) the space $E_{\tau+i}W$ is the eigenspace of A on W for the eigenvalue $q^{D-2\tau-2i}$. By these comments $E_{\tau+i}W$ is the eigenspace of x_{01} on W for the eigenvalue q^{d-2i}. We have now verified row $(r,s) = (0,1)$. Next we verify row $(r,s) = (2,3)$. By Lemma 11.2 the generator x_{23} acts on W as $A^*q^{d-D+2\rho}$. By Lemma 7.1(i),(ii) the space $E^*_{\rho+i}W$ is the eigenspace of A^* on W for the eigenvalue $q^{D-2\rho-2i}$. By these comments $E^*_{\rho+i}W$ is the eigenspace of x_{23} on W for the eigenvalue q^{d-2i}. We have now verified row $(r,s) = (2,3)$. The remaining rows are valid by [30, Theorem 16.4]. $\square$

The following result is a mild generalization of [41, Lemma 6.1].

Lemma 11.4 *With reference to Assumption 8.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then the following (i)–(iv) hold for $0 \le i \le d$.*

(i) The space

$$(E^*_\rho W + \cdots + E^*_{\rho+d-i} W) \cap (E_{\tau+d-i} W + \cdots + E_{\tau+d} W)$$

is contained in $\tilde{V}^{\downarrow\uparrow}_{\rho+d-i,D-d-\tau+i}$.

(ii) The space

$$(E^*_{\rho+d-i} W + \cdots + E^*_{\rho+d} W) \cap (E_\tau W + \cdots + E_{\tau+d-i} W)$$

is contained in $\tilde{V}^{\uparrow\downarrow}_{D-d-\rho+i,\tau+d-i}$.

(iii) The space

$$(E^*_\rho W + \cdots + E^*_{\rho+d-i} W) \cap (E_\tau W + \cdots + E_{\tau+i} W)$$

is contained in $\tilde{V}^{\downarrow\downarrow}_{\rho+d-i,\tau+i}$.

(iv) The space

$$(E^*_{\rho+i} W + \cdots + E^*_{\rho+d} W) \cap (E_{\tau+d-i} W + \cdots + E_{\tau+d} W)$$

is contained in $\tilde{V}^{\uparrow\uparrow}_{D-\rho-i,D-d-\tau+i}$.

Proof: Assertion (iii) is just [41, Lemma 6.1]. To get (i), in the proof of [41, Lemma 6.1] replace the sequence $E_0, \ldots, E_D$ by $E_D, \ldots, E_0$. To get (ii), in the proof of [41, Lemma 6.1] replace $E^*_0, \ldots, E^*_D$ by $E^*_D, \ldots, E^*_0$. To get (iv), in the proof of [41, Lemma 6.1] replace $E^*_0, \ldots, E^*_D$ (resp. $E_0, \ldots, E_D$) by $E^*_D, \ldots, E^*_0$ (resp. $E_D, \ldots, E_0$). $\square$

Lemma 11.5 *With reference to Assumption 8.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Consider the $\boxtimes_q$-module structure on W from Lemma 11.2. In the table below, each row contains a matrix in $Mat_X(\mathbb{C})$ and an element of $\boxtimes_q$. The action of these two objects on W coincide.*

matrix	element of $\boxtimes_q$
A	$q^{D-d-2\tau} x_{01}$
B	$q^{d-D+\rho+\tau} x_{12}$
A^*	$q^{D-d-2\rho} x_{23}$
B^*	$q^{d-D+\rho+\tau} x_{30}$
K	$q^{\rho-\tau} x_{02}$
K^*	$q^{\tau-\rho} x_{13}$
Φ	$q^{d-D+\rho+\tau} 1$
Ψ	$q^{\rho-\tau} 1$

Proof: By Lemma 11.2 the expressions $A - q^{D-d-2\tau}x_{01}$ and $A^* - q^{D-d-2\rho}x_{23}$ are each 0 on W. Next we show that $B - q^{d-D+\rho+\tau}x_{12}$ is 0 on W. To this end we pick $w \in W$ and show $Bw = q^{d-D+\rho+\tau}x_{12}w$. Recall that x_{12} is semisimple on W with eigenvalues q^{d-2i} $(0 \le i \le d)$. Therefore without loss of generality we may assume that there exists an integer i $(0 \le i \le d)$ such that $x_{12}w = q^{d-2i}w$. By row $(r,s) = (1,2)$ in the table of Lemma 11.3 and by Lemma 11.4(i), we find $w \in \tilde{V}^{\downarrow\uparrow}_{\rho+d-i,D-d-\tau+i}$. By this and the first row in the table of Definition 10.4 we find $Bw = q^{2d-D+\rho+\tau-2i}w$. From these comments we find $Bw = q^{d-D+\rho+\tau}x_{12}w$ as desired. We have now shown that $B - q^{d-D+\rho+\tau}x_{12}$ is 0 on W. Similarly one shows that each of $B^* - q^{d-D+\rho+\tau}x_{30}$, $K - q^{\rho-\tau}x_{02}$, $K^* - q^{\tau-\rho}x_{13}$ is 0 on W. We now show that $\Phi - q^{d-D+\rho+\tau}I$ is 0 on W. To this end we pick $v \in W$ and show $\Phi v = q^{d-D+\rho+\tau}v$. Recall that x_{02} is semisimple on W with eigenvalues q^{d-2i} $(0 \le i \le d)$. Therefore without loss of generality we may assume that there exists an integer i $(0 \le i \le d)$ such that $x_{02}v = q^{d-2i}v$. By row $(r,s) = (0,2)$ in the table of Lemma 11.3 and by Lemma 11.4(iii), we find $v \in \tilde{V}^{\downarrow\downarrow}_{\rho+d-i,\tau+i}$. By this and the second to the last row in the table of Definition 10.4 we find $\Phi v = q^{d-D+\rho+\tau}v$ as desired. We have now shown that $\Phi - q^{d-D+\rho+\tau}I$ is 0 on W. Similarly one shows that $\Psi - q^{\rho-\tau}I$ is 0 on W. □

Corollary 11.6 *With reference to Assumption 8.1, let W denote an irreducible T-module and consider the $\boxtimes_q$-action on W from Lemma 11.2. In the table below, each column contains a generator for $\boxtimes_q$ and a matrix in $Mat_X(\mathbb{C})$. The action of these two objects on W coincide.*

generator	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{13}
matrix	$A\Phi\Psi^{-1}$	$B\Phi^{-1}$	$A^*\Phi\Psi$	$B^*\Phi^{-1}$	$K\Psi^{-1}$	$K^*\Psi$

Proof: Immediate from Lemma 11.5. □

It is now a simple matter to prove Theorem 11.1.

Proof of Theorem 11.1: The standard module V decomposes into a direct sum of irreducible T-modules. Each irreducible T-module in this decomposition supports a $\boxtimes_q$-module structure from Lemma 11.2. Combining these $\boxtimes_q$-modules we get a $\boxtimes_q$-module structure on V. It remains to show that this $\boxtimes_q$-module satisfies the requirements of Theorem 11.1. This is the case since by Corollary 11.6, for each column in the table of Theorem 11.1 the given $\boxtimes_q$ generator and the matrix beneath it coincide on each of the irreducible T-modules in the above decomposition and hence on V. □

Remark 11.7 In Theorem 11.1 we displayed an action of $\boxtimes_q$ on the standard module V of Γ. In Proposition 4.3 we displayed four $\mathbb{C}$-algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ to $\boxtimes_q$. Using these homomorphisms to pull back the $\boxtimes_q$-action we obtain four $U_q(\widehat{\mathfrak{sl}}_2)$-module structures on V.

12 How $\boxtimes_q$ is related to T

In Theorem 11.1 we displayed an action of $\boxtimes_q$ on the standard module of Γ; observe that this action induces a $\mathbb{C}$-algebra homomorphism $\boxtimes_q \to \mathrm{Mat}_X(\mathbb{C})$ which we will denote by ϑ.

In this section we clarify how the image $\vartheta(\boxtimes_q)$ is related to the subconstituent algebra T.

Lemma 12.1 *With reference to Assumption 8.1, the following (i), (ii) hold.*

(i) Each of the matrices from the list (6) is contained in T.

(ii) Each of Φ, Ψ is contained in the center $Z(T)$.

Proof: (i) By Lemma 11.5 each matrix in the list (6) leaves invariant every irreducible T-module. Let T' denote the set of matrices in $\mathrm{Mat}_X(\mathbb{C})$ that leave invariant every irreducible T-module. We observe that T' is a subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ that contains T as well as each matrix in the list (6). We show that $T = T'$. To this end we first show that T' is semisimple. By the construction each irreducible T-module is an irreducible T'-module. We mentioned in Section 7 that the standard module V is a direct sum of irreducible T-modules. Therefore V is a direct sum of irreducible T'-modules, so T' is semisimple. Next, let W_1, W_2 denote irreducible T-modules. We claim that any isomorphism of T-modules $\gamma : W_1 \to W_2$ is an isomorphism of T'-modules. This is readily checked using the fact that $\{w + \gamma(w) \mid w \in W_1\}$ is an irreducible T-module and therefore invariant under T'. By our above comments the vector spaces T and T' have the same dimension; this dimension is $\sum_\lambda d_\lambda^2$ where the sum is over all isomorphism classes λ of irreducible T-modules and d_λ denotes the dimension of an irreducible T-module in the isomorphism class λ. Since T' contains T and they have the same dimension we find $T = T'$. The result follows.
(ii) By Lemma 11.5 each of Φ, Ψ acts as a scalar multiple of the identity on every irreducible T-module. $\square$

Theorem 12.2 *With reference to Assumption 8.1 the following (i), (ii) hold.*

(i) The image $\vartheta(\boxtimes_q)$ is contained in T.

(ii) T is generated by $\vartheta(\boxtimes_q)$ together with Φ, Ψ.

Proof: Combine Lemma 9.5, Theorem 11.1, and Lemma 12.1. $\square$

13 Directions for further research

In this section we give some suggestions for further research.

Problem 13.1 For the spaces in Definition 10.1, find a combinatorial interpretation and an attractive basis.

Problem 13.2 With reference to Assumption 8.1, the matrices Φ, Ψ commute by Lemma 12.1(ii) and they are semisimple by Definition 10.4. Therefore the standard module of Γ decomposes into a direct sum of their common eigenspaces. For these common eigenspaces find a combinatorial interpretation and an attractive basis.

Problem 13.3 With reference to Assumption 8.1, for $y, z \in X$ and for each of B, B^*, K, K^*, Φ, Ψ find the (y,z)-entry in terms of the distances $\partial(x,y)$, $\partial(y,z)$, $\partial(z,x)$ ($x =$ base vertex from Section 7) and other combinatorial parameters as needed. When is this entry 0?

Problem 13.4 Find all the distance-regular graphs that have classical parameters (D, b, α, β) and $b \neq 1$, $\alpha = b - 1$. Some examples are given in Example 8.4.

Problem 13.5 The finite-dimensional irreducible $U_q(\widehat{\mathfrak{sl}}_2)$-modules are classified by V. Chari and A. Pressley [9]; see also [18], [37]. Use this and Remark 11.7 to describe the irreducible T-modules for each of the graphs in Example 8.4.

Conjecture 13.6 *With reference to Assumption 8.1, for $0 \leq i, j \leq D$ the spaces $\tilde{V}_{ij}^{\downarrow\downarrow}$ and $\tilde{V}_{rs}^{\uparrow\uparrow}$ are orthogonal unless $i + r = D$ and $j + s = D$. Moreover $\tilde{V}_{ij}^{\downarrow\uparrow}$ and $\tilde{V}_{rs}^{\uparrow\downarrow}$ are orthogonal unless $i + r = D$ and $j + s = D$.*

Problem 13.7 With reference to Assumption 8.1, note by Lemma 11.5 that the following are equivalent: (i) for each irreducible T-module the endpoint and dual endpoint coincide; (ii) $\Psi = I$. For which of the graphs in Example 8.4 do these equivalent conditions hold?

Conjecture 13.8 *With reference to Assumption 8.1, each of Φ, Ψ is symmetric and*

$$B^t = B^*, \qquad\qquad K^t = K^{*-1}.$$

Under Assumption 8.1 we displayed an action of $\boxtimes_q$ on the standard module of Γ. For the moment replace Assumption 8.1 by the weaker assumption that Γ is Q-polynomial. We suspect that there is still a natural action of $\boxtimes_q$ (or $U_q(\widehat{\mathfrak{sl}}_2)$, $U_q(\mathfrak{sl}_2)$, $\widehat{\mathfrak{sl}}_2$, $\mathfrak{sl}_2, \ldots$ in degenerate cases) on the standard module of Γ. It is premature for us to guess how this action behaves in every case, but the general idea is conveyed in the following two conjectures.

Conjecture 13.9 *Assume Γ has classical parameters (D, b, α, β) with $b \neq 1$. In order to avoid degenerate situations, assume that Γ is not a dual polar graph [5, p. 274]. Then for $b = q^2$ there exists a $\boxtimes_q$-action on the standard module of Γ for which the adjacency matrix acts as a $Z(T)$-linear combination of $1, x_{01}, x_{12}$ and the dual adjacency matrix acts as a $Z(T)$-linear combination of $1, x_{23}$. We recall that $Z(T)$ denotes the center of T.*

Conjecture 13.10 *Assume Γ is Q-polynomial, with eigenvalues θ_i and dual eigenvalues θ_i^*. Recall that the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad\qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq D - 1$ [1, p. 263]. Denote this common value by $b + b^{-1} + 1$ and assume that b is not a root of unity. Further assume that, in the notation of Bannai and Ito [1, p. 263], the given Q-polynomial structure is type I with $s \neq 0$ and $s^* \neq 0$. Then for $b = q^2$ there exists a $\boxtimes_q$-action on the standard module of Γ for which the adjacency matrix acts as a $Z(T)$-linear combination of $1, x_{01}, x_{12}$ and the dual adjacency matrix acts as a $Z(T)$-linear combination of $1, x_{23}, x_{30}$.

Problem 13.11 A uniform poset [43] is ranked and has an algebraic structure similar to that of a Q-polynomial distance-regular graph. In [43, p. 200] 11 infinite families of uniform posets are given. For some uniform posets P it might be possible to adapt the method of the present paper to get an action of $\boxtimes_q$ on the standard module of P.

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References

[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1984.

[2] G. Benkart and P. Terwilliger. The universal central extension of the three-point $\mathfrak{sl}_2$ loop algebra. *Proc. Amer. Math. Soc.*, accepted. `arXiv:math.RA/0512422`.

[3] N. Biggs *Algebraic Graph Theory. Second edition.* Cambridge University Press, Cambridge, 1993.

[4] M. Bremner. Generalized affine Kac-Moody Lie algebras over localizations of the polynomial ring in one variable. *Canad. Math. Bull.* **37** (1994), 21–28.

[5] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin, 1989.

[6] A. E. Brouwer, C. D. Godsil, J. H. Koolen, W. J. Martin. Width and dual width of subsets in polynomial association schemes. *J. Combin. Theory Ser. A* **102** (2003), 255-271.

[7] J. S. Caughman IV. Spectra of bipartite P- and Q-polynomial association schemes. *Graphs Combin.* **14** (1998), 321–343.

[8] J. S. Caughman IV. The Terwilliger algebras of bipartite P- and Q-polynomial association schemes. *Discrete Math.* **196** (1999), 65–95.

[9] V. Chari and A. Pressley. Quantum affine algebras. *Comm. Math. Phys.* **142** (1991), 261–283.

[10] B. Curtin. Bipartite distance-regular graphs I. *Graphs Combin.* **15** (1999), 143–158.

[11] B. Curtin. Bipartite distance-regular graphs II. *Graphs Combin.* **15** (1999), 377–391.

[12] B. Curtin. 2-homogeneous bipartite distance-regular graphs. *Discrete Math.* **187** (1998), 39–70.

[13] B. Curtin. Distance-regular graphs which support a spin model are thin. *16th British Combinatorial Conference (London, 1997).* Discrete Math. **197/198** (1999), 205–216.

[14] B. Curtin and K. Nomura. Distance-regular graphs related to the quantum enveloping algebra of sl(2). *J. Algebraic Combin.* **12** (2000), 25–36.

[15] C. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* Interscience, New York, 1962.

[16] G. Dickie. Twice Q-polynomial distance-regular graphs are thin. *European J. Combin.* **16** (1995), 555–560.

[17] G. Dickie and P. Terwilliger. A note on thin P-polynomial and dual-thin Q-polynomial symmetric association schemes. *J. Algebraic Combin.* **7** (1998), 5–15.

[18] J. Ding and I. B. Frenkel. Isomorphism of two realizations of quantum affine algebra $U_q(gl(n))$. *Comm. Math. Phys.* **156** (1993), 277–300.

[19] E. Egge. A generalization of the Terwilliger algebra. *J. Algebra* **233** (2000), 213–252.

[20] A. Elduque. The S_4-action on the tetrahedron algebra. Preprint. `arXiv:math.RA/0604218`

[21] J. T. Go. The Terwilliger algebra of the hypercube. *European J. Combin.* **23** (2002), 399–429.

[22] J. T. Go and P. Terwilliger. Tight distance-regular graphs and the subconstituent algebra. *European J. Combin.* **23** (2002), 793–816.

[23] C. D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, Inc., New York, 1993.

[24] B. Hartwig. The Tetrahedron algebra and its finite-dimensional irreducible modules. *Linear Algebra Appl.*, submitted. `arXiv:math.RT/0606197`

[25] B. Hartwig and P. Terwilliger. The Tetrahedron algebra, the Onsager algebra, and the $\mathfrak{sl}_2$ loop algebra. *J. Algebra*, submitted. `arXiv:math-ph/0511004`.

[26] S. A. Hobart and T. Ito. The structure of nonthin irreducible T-modules: ladder bases and classical parameters. *J. Algebraic Combin.* **7** (1998), 53–75.

[27] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$. *Ramanujan J.*, to appear; `arXiv:math.QA/0310042`.

[28] T. Ito and P. Terwilliger. Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations. *J. Algebra Appl.*, submitted. `arXiv:math.QA/0508398`.

[29] T. Ito, P. Terwilliger, and C. W. Weng. The quantum algebra $U_q(\mathfrak{sl}_2)$ and its equitable presentation. *J. Algebra* **298** (2006), 284–301. `arXiv:math.QA/0507477`.

[30] T. Ito and P. Terwilliger. The q-tetrahedron algebra and its finite-dimensional irreducible modules. *Comm. Algebra*; accepted. `arXiv:math.QA/0602199`.

[31] T. Ito and P. Terwilliger. q-Inverting pairs of linear transformations and the q-tetrahedron algebra. *Linear Algebra Appl.*; submitted. `arXiv:math.RT/0606237`.

[32] C. Kassel. *Quantum Groups*, Springer-Verlag, New York, 1995.

[33] G. Lusztig. *Introduction to Quantum Groups*, Birkhauser, Boston, 1990.

[34] A. A. Pascasio. On the multiplicities of the primitive idempotents of a Q-polynomial distance-regular graph. *European J. Combin.* **23** (2002), 1073–1078.

[35] A. A. Pascasio and P. Terwilliger. The Tetrahedron algebra and the Hamming graphs. In preparation.

[36] K. Tanabe. The irreducible modules of the Terwilliger algebras of Doob schemes. *J. Algebraic Combin.* **6** (1997), 173–195.

[37] V. O. Tarasov. Irreducible monodromy matrices for the R-matrix of the XXZ-model, and lattice local quantum Hamiltonians. *Theor. Math. Phys.* **63** (1985), 440–454.

[38] P. Terwilliger. The subconstituent algebra of an association scheme I. *J. Algebraic Combin.* **1** (1992), 363–388.

[39] P. Terwilliger. The subconstituent algebra of an association scheme II. *J. Algebraic Combin.* **2** (1993), 73–103.

[40] P. Terwilliger. The subconstituent algebra of an association scheme III. *J. Algebraic Combin.* **2** (1993), 177–210.

[41] P. Terwilliger. The displacement and split decompositions for a Q-polynomial distance-regular graph. *Graphs Combin.* **21** (2005), 263–276. `arXiv:math.CO/0306142`.

[42] P. Terwilliger. The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. *J. Algebra* **298** (2006), 302–319. `arXiv:math.QA/0507478`.

[43] P. Terwilliger. The incidence algebra of a uniform poset. *Coding theory and design theory, Part I*, 193–212, IMA Vol. Math. Appl., **20**, Springer, New York, 1990.

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