FROM LINE-SYSTEMS TO SPHERE-SYSTEMS — SCHLÄFLI'S DOUBLE SIX, LIE'S LINE-SPHERE TRANSFORMATION, AND GRACE'S THEOREM

HIROSHI MAEHARA AND NORIHIDE TOKUSHIGE

Abstract. If each four spheres in a set of five unit spheres in $\mathbb{R}^3$ have nonempty intersection, then all five spheres have nonempty intersection. This result is proved using Grace's theorem: the circumsphere of a tetrahedron encloses none of its escribed spheres. This paper provides self-contained proofs of these results; including Schläfli's double six theorem and modified version of Lie's line-sphere transformation. Some related problems are also posed.

1. Introduction

Let us start with the following result concerning a family of identical circles, which can be proved easily, see [6], [17].

Theorem 1.1. *If three identical circles intersect at a point, then the remaining three intersections determine a circle of the same radius. (see Figure 1).*

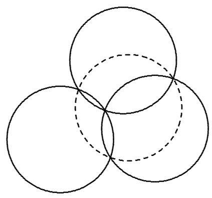

Figure 1. Four identical circles

Thus there is a family of four unit circles in the plane such that each three of them have nonempty intersection, but the intersection of all four circles is empty. Is there a similar family of unit spheres in higher dimensions?

Problem 1.2. *Is there a family of $d+2$ unit spheres in $\mathbb{R}^d$ such that each $d+1$ of them have nonempty intersection, but the intersection of all $d+2$ spheres is empty?*

If we allow the sphere radii to differ, then, for each $d \geq 1$, there exists a family of $d+2$ spheres in $\mathbb{R}^d$ such that each $d+1$ of them have nonempty intersection, and yet the intersection of all $d+2$ spheres is empty. On the other hand, it is known [14] that, if a family of $n \geq d+3$ spheres in $\mathbb{R}^d$ satisfies the condition that each $d+1$ spheres in the family have nonempty intersection, then the intersection of all n spheres is also nonempty.

Date: November 11, 2008, 04:07pm.

Now, in the unit sphere case of Problem 1.2, the answer depends on the dimension d. There is no such family for $d = 1$, but there are many such families for $d = 2$ by Theorem 1.1. And, the answer to the problem is also affirmative for all $d \geq 4$ [3, 15]. Then, what is the answer for $d = 3$? Is there a family of five unit spheres in $\mathbb{R}^3$ such that each four of them have nonempty intersection, and the intersection of all five spheres is empty? We were convinced that there is no such family, and proved in [15] that this assertion follows from the following rather obvious looking conjecture: "the circumsphere of a tetrahedron never encloses any escribed sphere." We tried to prove this conjecture, but did not succeed.

Meanwhile, Professor Margaret M. Bayer informed us that our conjecture is a theorem proved by John Hilton Grace (1873–1958) nearly a century ago [7, 8]. His proof was based on an ingenious idea to convert Schläfli's double six of lines into "double six of spheres" by using Lie's line-sphere transformation. Nevertheless, the proof itself is elementary in the sense that it is accessible for undergraduates if appropriate preliminaries are provided. It is a good example that a combination of several elementary facts bring an unexpected result.

The original proof given by Grace is, however, rather difficult to follow. In fact, he only gave an outline of the proof. In this paper, we reconstruct his proof in detail, and prove the nonexistence of a family of five unit spheres mentioned above in a self-contained way. We provide sections on Schläfli's double six theorem [1, 10, 11, 19, 20] from the viewpoint of quadratic surfaces, Plücker coordinate of lines [18, 5, 21], and Lie's line-sphere transformation [4, 9, 13, 16]. Lie's transformation is a bijection from the set of lines in $\mathbb{P}^3(\mathbb{C})$ to the set of all oriented spheres in $\mathbb{P}^3(\mathbb{C})$. We present a variant of the transformation, namely, a bijection from a special family Λ of lines in $\mathbb{P}^3(\mathbb{C})$ to the set Θ of all oriented spheres in $\mathbb{R}^3 \cup \{\infty\}$. This version fits sphere-systems in $\mathbb{R}^3$. In the last section, we pose some related problems together with examples.

2. Quadrics and Schläfli's double six theorem

In this section, we consider lines and quadrics in a projective space on the base field $k = \mathbb{R}$ or $\mathbb{C}$. Each point of $\mathbb{P}^3$ has a homogeneous coordinate $[x, y, z, t]$, namely,

$$\mathbb{P}^3 = \{[x, y, z, t] : x, y, z, t \in k,\ (x, y, z, t) \neq (0, 0, 0, 0)\}.$$

Let $Q = Q(x, y, z, t)$ be a quadratic form of variables x, y, z, t, that is, a homogeneous quadratic polynomial of x, y, z, t. We use the same symbol Q to represent the corresponding quadratic surface (simply **quadric**), i.e., the set $\{[x, y, z, t] \in \mathbb{P}^3 : Q(x, y, z, t) = 0\}$. We list below some basic properties of quadrics as a theorem without proof, where, and in what follows, "skew lines" mean mutually non-intersecting lines.

Theorem 2.1. *The following holds for quadrics in $\mathbb{P}^3$.*

(i) *If a quadric Q contains three points on a line, then Q contains the line.*
(ii) *If a quadric Q contains three lines in a plane, then Q contains the plane.*
(iii) *For any three skew lines, there is a unique quadric that contains the three lines.*

(iv) *Let Q be a quadric containing three skew lines. Then for each point of Q, Q contains exactly two lines that passes through the point. Moreover, all lines lying on Q are divided into two classes so that lines from different classes always intersect, while lines in the same class never intersect.*

(v) *If a quadric Q contains a plane, then Q is either a double plane or the union of two planes.*

To illustrate how to use the above properties, we consider the following specific situation of two quadrics, which we will meet soon.

Lemma 2.2. *Let Q_1 and Q_2 be quadrics containing no plane. Suppose that the intersection $Q_1 \cap Q_2$ contains two intersecting lines ℓ_1 and ℓ_2, and $Q_1 \cap Q_2 \neq \ell_1 \cup \ell_2$. Then the points in $(Q_1 \cap Q_2) \setminus (\ell_1 \cup \ell_2)$ lie on a plane π with $Q_1 \cap \pi = Q_2 \cap \pi$.*

Proof. Let π' be the plane containing ℓ_1 and ℓ_2. By (i) and (ii), we have $(Q_1 \cup Q_2) \cap \pi' = \ell_1 \cup \ell_2$. Let P' be any point in $\pi' \setminus (\ell_1 \cup \ell_2)$. Then there is a nonzero scalar λ such that $\lambda Q_1(P') + Q_2(P') = 0$. Again by (i) and (ii), the quadric $Q := \lambda Q_1 + Q_2$ contains the plane π'. Then, by (v), Q is the union of two distinct planes π' and π. Moreover, we have $Q_1 \cap \pi = Q_2 \cap \pi$. In fact, if $P \in Q_1 \cap \pi$, then $Q_2(P) = Q(P) - \lambda Q_1(P) = 0$ and $P \in Q_2 \cap \pi$. ☐

We say that a line h is a **transversal** of lines $g_1, g_2, \ldots$ if h intersects all $g_1, g_2, \ldots$. Now, for given four skew lines a, b, c, d, how many transversals of the four lines are there? Three skew lines a, b, c determine a quadric Q by (ii). The fourth line d would meet Q at two points, say, p, q, unless d is in a special position such as d is entirely contained in Q, or d is tangent to Q. Applying (iv) to the surface Q, the 'class' of lines on Q that intersect a, b, c contains two lines passing through p, q respectively. This means that there are exactly two transversals of the four given lines if they are in general position in some sense. Note also that the two transversals do not meet, because they belong to the same class of lines on Q.

Definition 2.3. *We say that four skew lines are in* **regular position** *if there exist exactly two transversals of the four lines. We say that n (≥ 4) skew lines are in regular position if each four of the n lines are in regular position.*

Definition 2.4. *A* **Schläfli's double six** *is a pair of six lines $a_1, \ldots, a_6$; $b_1, \ldots, b_6$ in $\mathbb{P}^3$ such that a_i and b_j intersect iff $i \neq j$, and no other two lines intersect. A* **line table** *is an array of lines (symbols) written in two rows such that two lines intersect iff they are written in different rows and in different columns.*

For example, the following is a line table for a double six:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \tag{1}$$

The next theorem gives a necessary and sufficient condition for the line table

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & \\ & & & & & b_6 \end{matrix} \tag{2}$$

to be extendable to a line table (1).

Theorem 2.5 (Double six theorem [20])**.** *Let $a_1, a_2, \ldots, a_5$ be five skew lines in $\mathbb{P}^3$, and let b_6 be a transversal of the five lines. Then these six lines can be extended to a double six if and only if $a_1, a_2, \ldots, a_5$ are in regular position. Moreover if they are extendable to a double six, then the extension is unique.*

Proof. First we show the "only if" part. If the six given lines can be extended to a double six, then, with consulting the line table (1), each four (say, a_1, a_2, a_3, a_4) of the lines among $a_1, a_2, \ldots, a_5$ have two transversals (say b_5, b_6), and there is a line (say, b_4) that intersects exactly three (say, a_1, a_2, a_3) lines of the four. Hence, the following lemma shows that $a_1, \ldots, a_5$ are in regular position.

Lemma 2.6. *Suppose that four skew lines g_1, g_2, g_3, g_4 (referred as the four lines) in $\mathbb{P}^3$ have two transversals h_5, h_6, and there is a transversal h_4 of exactly three lines out of the four lines. Then the four lines are in regular position.*

Proof. Suppose that there is a transversal h of the four lines other than h_5, h_6. Then h_5, h_6, h are skew. Let Q be the quadric determined by h_5, h_6, h. Then the four lines lie on Q. Since h_4 intersects three lines among the four lines, the line h_4 is entirely contained in Q, and thus h_4 meets all of the four lines, which is a contradiction. $\square$

Next we show the "if" part. For $i = 1, 2, \ldots, 5$, let b_i be the transversal of $\{a_1, a_2, a_3, a_4, a_5\} \setminus \{a_i\}$ besides b_6 (see the line table (3) at the end of this section). First suppose that there is a transversal a_6 of the five lines b_1, b_2, b_3, b_4, b_5. (We will show the existence of a_6 shortly.) The six lines $b_1, b_2, \ldots, b_6$ are skew, and in regular position by Lemma 2.6. The transversals of b_3, b_4, b_5, b_6 are exactly a_1 and a_2, and thus a_6 and b_6 do not intersect (otherwise a_6 is another transversal of b_3, b_4, b_5, b_6). This means that $a_1, \ldots, a_6; b_1, \ldots, b_6$ is a double six. Moreover $b_1, b_2, \ldots, b_5$ and a_6 are uniquely determined, and the extension to the double six is unique. We are going to show the existence of a transversal a_6. Our proof is essentially the same as the ones in [1] pp. 159–160, and [12]. For $i \neq j$, let P_{ij} be the intersection of a_i and b_j.

Lemma 2.7. *There exist a transversal a of four lines b_1, b_2, b_3, b_4, passing through a point P, where P is the intersection of b_1 and the plane determined by three points P_{23}, P_{32}, P_{16}. (Actually, we will see that $a = a_6$ and $P = P_{61}$ later).*

Proof. Let Q_1 and Q_2 be the quadrics containing $\{a_2, a_3, a_5\}$ and $\{b_2, b_3, b_4\}$ respectively. Since a_5 meets b_2, b_3, b_4, we have $a_5 \subset Q_1 \cap Q_2$ by Theorem 2.1 (i). Similarly, b_4 meets a_2, a_3, a_5, and $b_4 \subset Q_1 \cap Q_2$. Thus $Q_1 \cap Q_2$ contains two intersecting lines a_5 and b_4.

Let us find points in $(Q_1 \cap Q_2) \setminus (a_5 \cup b_4)$. Clearly, we have P_{23}, P_{32}. Also P_{16} is such a point, for a_1 meets b_2, b_3, b_4, and $a_1 \subset Q_2 \setminus a_5$ by (i), similarly, b_6 meets a_2, a_3, a_5, and $b_6 \subset Q_1 \setminus b_4$. Now we show that the three points P_{23}, P_{32}, P_{16} determine a plane π. Suppose for a contradiction that the three points lie on a line ℓ. Then $\ell \subset Q_1$, and ℓ meets a_2 at P_{23}. Thus, by (iv), ℓ meets a_3 and a_5, too. Moreover ℓ meets a_1 at P_{16}. Then ℓ is a transversal of four lines a_1, a_2, a_3, a_5 in regular position, and thus $\ell = b_4$ or b_6. But this is impossible, because ℓ meets b_3 at P_{23}.

By Lemma 2.2 the points in $(Q_1 \cap Q_2) \setminus (a_5 \cup b_4)$ lie in the plane π and $Q_1 \cap \pi = Q_2 \cap \pi$. To see $b_1 \not\subset \pi$, suppose for a contradiction that $b_1 \subset \pi$. Using $b_1 \subset Q_1$, we have $b_1 \subset Q_1 \cap \pi = Q_2 \cap \pi \subset Q_2$. By (iv) this is impossible, because $a_1, b_2 \subset Q_2$.

Let P be the intersection of b_1 and π. Since $b_1 \subset Q_1$ and $P \in Q_1 \cap \pi = Q_2 \cap \pi$, we also have $P \in Q_2$. By (iv), there are two lines in Q_2 passing through P, and one of them is the desired transversal a. ☐

Similarly there is a transversal a' of b_1, b_2, b_3, b_5, passing through the same point P. To see this, just replace a_5 and b_4 with a_4 and b_5, respectively, in Lemma 2.7. In this replacement, the plane π remains the same, and so does the point P. Since both a and a' pass through the point $P \in b_1$ and intersect two skew lines b_2, b_3 with $P \notin b_2 \cup b_3$, they must coincide with each other. This is the transversal of five line b_1, b_2, b_3, b_4, b_5, namely, $a = a' = a_6$. This completes the proof of Theorem 2.5. ☐

We have shown that if five skew lines a_1, a_2, a_3, a_4, a_5 are in regular position, then the line table (2) can be extended uniquely to the following line table:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \tag{3}$$

and then extended uniquely to the line table (1). Let us state this fact as a corollary, which we will use to prove Theorem 6.3, an analogous result for spheres.

Corollary 2.8. *If a set $\{a_1, \dots, a_5, b_1, \dots, b_6\}$ of eleven lines constitute the line table (3), then there is a unique line a_6 that intersects only the five lines b_1, b_2, b_3, b_4, b_5 among the eleven lines.*

Let us briefly explain the relation between a double six and the 27 lines in a cubic surface. Let $\{a_1, \dots, a_6, b_1, \dots, b_6\}$ be a double six in $\mathbb{P}^3$. Then there is a unique nonsingular cubic surface containing these 12 lines. This surface contains exactly 27 lines, and the remaining 15 lines are given by (ij), $1 \le i < j \le 6$, where (ij) is the line of intersection between the plane containing a_i, b_j and the plane containing a_j, b_i. See [10], [11] §25, [19] §7 for more details. We will not use the facts mentioned in this paragraph to prove our results.

3. Plücker coordinates of lines

Plücker coordinate [18] is a homogeneous coordinate which represents a line in $\mathbb{P}^3$ as a point in $\mathbb{P}^5$. The set of Plücker coordinates of all lines in $\mathbb{P}^3$ form a quadric M in $\mathbb{P}^5$. Moreover, for a pair of points on M corresponding to two mutually intersecting lines in $\mathbb{P}^3$, the line in $\mathbb{P}^5$ determined by the two points on M is entirely contained in M. This property gives a condition for the Plücker coordinates of two intersecting lines. In this section we derive these things, cf. [5] Chap. 8 §6, [21] Chap. 1 §7.

Fix a base field $\mathbb{R}$ or $\mathbb{C}$. We use bold lower-case characters $\boldsymbol{x}$ etc. for points in $\mathbb{P}^3$, and $[x_1, x_2, x_3, x_4]$ etc. for the corresponding homogeneous coordinates. For two distinct points $\boldsymbol{a}, \boldsymbol{b}$ in $\mathbb{P}^3$, let $\ell(\boldsymbol{a}, \boldsymbol{b})$ denote the line determined by $\boldsymbol{a}, \boldsymbol{b}$, that is,

$$\ell(\boldsymbol{a}, \boldsymbol{b}) = \{s\boldsymbol{a} + t\boldsymbol{b} : [s, t] \in \mathbb{P}^1\}. \tag{4}$$

For $1 \le i, j \le 4$, $i \ne j$, set

$$p_{ij} = p_{ij}(\boldsymbol{a}, \boldsymbol{b}) = a_i b_j - a_j b_i. \tag{5}$$

Then a straightforward calculation shows

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{6}$$

If $\boldsymbol{a} \neq \boldsymbol{b}$, then the two vectors (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are linearly independent, and the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

has rank 2. Since p_{ij} is a determinant of a minor matrix of this matrix, some p_{ij} is nonzero. The **Plücker coordinate** $f(\ell) \in \mathbb{P}^5$ of a line $\ell = \ell(\boldsymbol{a}, \boldsymbol{b})$ is defined by

$$f(\ell) = [p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}] = [a_1b_2 - a_2b_1, \ldots, a_2b_3 - a_3b_2]. \tag{7}$$

We have to check that (7) is independent of a choice of two points that determine the line ℓ. Choose two distinct points $\boldsymbol{a}', \boldsymbol{b}' \in \ell(\boldsymbol{a}, \boldsymbol{b})$. Then we can write $\boldsymbol{a}' = s\boldsymbol{a} + t\boldsymbol{b}$, $\boldsymbol{b}' = u\boldsymbol{a} + v\boldsymbol{b}$ with some $[s, t], [u, v] \in \mathbb{P}^1$ by (4). Using (5) and (7) with $sv - tu \neq 0$, we get $f(\ell(\boldsymbol{a}', \boldsymbol{b}')) = (sv - tu)f(\ell(\boldsymbol{a}, \boldsymbol{b})) = f(\ell(\boldsymbol{a}, \boldsymbol{b}))$.

Lemma 3.1. *Let L be the set of all lines in $\mathbb{P}^3$ and let M be a quadric in $\mathbb{P}^5$ defined by*

$$M = \{[\alpha, \beta, \gamma, \xi, \eta, \zeta] \in \mathbb{P}^5 : \alpha\xi + \beta\eta + \gamma\zeta = 0\}.$$

Then, for each line $\ell \in L$, the Plücker coordinate $f(\ell)$ is a point of M by (6), and $f : L \to M$ is a bijection.

Proof. First we show that f is injective. Suppose that two lines $\ell, \ell' \in L$ satisfy $f(\ell) = f(\ell')$. Choose two points $\boldsymbol{a}, \boldsymbol{b}$ on ℓ. Then $f(\ell)$ is defined by (7), and we may assume that $p_{12} \neq 0$ by changing the coordinate system if necessary. By (4), we have the following two distinct points on ℓ:

$$\boldsymbol{c} = b_1\boldsymbol{a} - a_1\boldsymbol{b} = [0, p_{12}, p_{13}, p_{14}], \quad \boldsymbol{d} = b_2\boldsymbol{a} - a_2\boldsymbol{b} = [p_{12}, 0, -p_{23}, p_{42}]. \tag{8}$$

Let $f(\ell') = [p'_{12}, p'_{13}, p'_{14}, p'_{34}, p'_{42}, p'_{23}]$. Since $f(\ell) = f(\ell')$ there is some $\lambda \neq 0$ such that $p_{ij} = \lambda p'_{ij}$ for all i, j. This together with (8) gives $\boldsymbol{c} = [0, p_{12}, p_{13}, p_{14}] = [0, p'_{12}, p'_{13}, p'_{14}] = b'_1\boldsymbol{a}' - a'_1\boldsymbol{b}'$ for any distinct $\boldsymbol{a}', \boldsymbol{b}' \in \ell'$, and thus $\boldsymbol{c} \in \ell'$. Similarly we get $\boldsymbol{d} \in \ell'$. Since $\boldsymbol{c}$ and $\boldsymbol{d}$ are two distinct points on ℓ and ℓ', the two lines ℓ and ℓ' must coincide.

Next we show that f is surjective. Let $P = [\alpha, \beta, \gamma, \xi, \eta, \zeta]$ be an arbitrary point in M. We may assume that $\alpha \neq 0$. For two distinct points $\boldsymbol{c} = [0, \alpha, \beta, \gamma]$ and $\boldsymbol{d} = [\alpha, 0, -\zeta, \eta]$ in $\mathbb{P}^3$, let $\ell = \ell(\boldsymbol{c}, \boldsymbol{d}) \in L$. Then by (7) we have

$$f(\ell) = [\alpha^2, \alpha\beta, \alpha\gamma, -\beta\eta - \gamma\zeta, \alpha\eta, \alpha\zeta]. \tag{9}$$

Since $P \in M$, we have $\alpha\xi = -\beta\eta - \gamma\zeta$, and the fourth coordinate of (9) is equal to $\alpha\xi$. Then, dividing the RHS of (9) by $\alpha \neq 0$, we have $P = f(\ell)$. $\square$

Lemma 3.2. *Let ℓ and ℓ' be two lines in $\mathbb{P}^3$, and let $f(\ell) = [\alpha, \beta, \gamma, \xi, \eta, \zeta]$ and $f(\ell') = [\alpha', \beta', \gamma', \xi', \eta', \zeta']$ be their Plücker coordinates in M. Then the following (i), (ii), and (iii) are equivalent.*

(i) *Two lines ℓ and ℓ' intersect.*

(ii) *The line in $\mathbb{P}^5$ determined by two points $f(\ell)$ and $f(\ell')$ is entirely contained in M.*

(iii) $\alpha\xi' + \alpha'\xi + \beta\eta' + \beta'\eta + \gamma\zeta' + \gamma'\zeta = 0.$

Proof. (i)⇒(ii). Suppose that the line ℓ contains two points $\boldsymbol{a}$ and $\boldsymbol{b}$, and ℓ' contains $\boldsymbol{a}$ and $\boldsymbol{c}$. Then we have

$$f(\ell) = [a_1b_2 - a_2b_1, \dots, a_2b_3 - a_3b_2], \; f(\ell') = [a_1c_2 - a_2c_1, \dots, a_2c_3 - a_3c_2]. \tag{10}$$

Let $[\alpha'', \beta'', \gamma'', \xi'', \eta'', \zeta''] \in \mathbb{P}^5$ be a point on the line determined by the two points in (10). Then we have $[\alpha'', \dots, \zeta''] = sf(\ell) + tf(\ell')$ with some $[s, t] \in \mathbb{P}^1$, and $\alpha'' = s(a_1b_2 - a_2b_1) + t(a_1c_2 - a_2c_1)$ etc. A simple computation shows $\alpha''\xi'' + \beta''\eta'' + \gamma''\zeta'' = 0$, which means $[\alpha'', \dots, \zeta''] \in M$.

(ii)⇒(iii). Let $P = f(\ell) + f(\ell') = [\alpha, \dots, \zeta] + [\alpha', \dots, \zeta']$. Then P is a point on the line determined by $f(\ell)$ and $f(\ell')$, and $P \in M$ by our assumption. Then (6) gives

$$\begin{aligned} 0 &= (\alpha + \alpha')(\xi + \xi') + (\beta + \beta')(\eta + \eta') + (\gamma + \gamma')(\zeta + \zeta') \\ &= (\alpha\xi + \beta\eta + \gamma\zeta) + (\alpha'\xi' + \beta'\eta' + \gamma'\zeta') + (\alpha\xi' + \alpha'\xi + \beta\eta' + \beta'\eta + \gamma\zeta' + \gamma'\zeta). \end{aligned}$$

Again (6) gives (iii), because $f(\ell), f(\ell') \in M$.

(iii)⇒(i). Suppose that ℓ contains $\boldsymbol{a}, \boldsymbol{b}$, and ℓ' contains $\boldsymbol{c}, \boldsymbol{d}$. Then, by (5), we have $\alpha = a_1b_2 - a_2b_1, \xi' = c_3d_4 - c_4d_3$, etc, and the LHS of (iii) is nothing but the Laplace expansion of

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

By the assumption of (iii), the above determinant vanishes. This means four points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are coplanar in $\mathbb{P}^3$, and thus two lines ℓ and ℓ' intersect. □

4. A family of lines in $\mathbb{P}^3(\mathbb{C})$

In this section, we identify a line in $\mathbb{P}^3$ and its Plücker coordinate. Thus a line in $\mathbb{P}^3$ is sometimes regarded as a point in $\mathbb{P}^5$. Let us write the Plücker coordinate of a line in $\mathbb{P}^3(\mathbb{C})$ in the form

$$[a, x + y\sqrt{-1}, b, c, x - y\sqrt{-1}, d], \tag{11}$$

where $a, b, c, d, x, y \in \mathbb{C}$. We denote this coordinates by $\lambda(a, b, c, d, x, y)$. Then (6) becomes

$$ac + bd + x^2 + y^2 = 0. \tag{12}$$

Definition 4.1. *Let Λ be the set of lines in $\mathbb{P}^3(\mathbb{C})$ whose Plücker coordinate can be written in the form of (11) with reals a, b, c, d, x, y, that is,*

$$\Lambda = \{\lambda(a, b, c, d, x, y) \in \mathbb{P}^5(\mathbb{C}) : a, b, c, d, x, y \text{ are reals satisfying (12)}\}.$$

Lemma 4.2. *Let $g_1, g_2, g_3, g_4 \in \Lambda$ be four skew lines. Suppose that these four lines have exactly two transversals in $\mathbb{P}^3(\mathbb{C})$. If one of the transversals belongs to Λ, then so does the other.*

Proof. For two vectors $u = (a, b, c, d, x, y)$ and $u' = (a', b', c', d', x', y')$, we write

$$\langle u, u' \rangle = ac' + a'c + bd' + b'd + 2xx' + 2yy'.$$

Let $g_j = \lambda(u_j)$ $(j = 1, 2, 3, 4)$ be four skew lines such that $u_j \in \mathbb{R}^6$. Then $\lambda(v)$ $(v \in \mathbb{C}^6)$ is a transversal of all g_j's if and only if v satisfies, by Lemma 3.2 (iii), the linear system of equations

$$\langle u_j, v\rangle = 0 \quad (j = 1, 2, 3, 4), \tag{13}$$

as well as, by (12), the quadratic equation

$$\langle v, v\rangle = 0. \tag{14}$$

Since (13) has real coefficients as equations for v, we have linearly independent real solutions $v_k \in \mathbb{R}^6$ $(k = 1, 2, \dots)$ of (13), and each transversal of g_1, g_2, g_3, g_4 are represented by a linear combination (with coefficient in $\mathbb{C}$) of these real solutions. (Notice that $\lambda(v_k)$ is not a line unless v_k satisfies (14).) Suppose that one of the two transversals belongs to Λ, say, v_1 satisfies (14) and $\lambda(v_1) \in \Lambda$.

Let $\lambda(u)$ $(u \in \mathbb{C}^6)$ be the other transversal of g_1, g_2, g_3, g_4. If some v_k $(k \geq 2)$ satisfies (14), then we have $\lambda(u) = \lambda(v_k)$ and $\lambda(v_k) \in \Lambda$, as desired. So, suppose that $\langle v_k, v_k\rangle \neq 0$ for all $k \geq 2$. In this case u is a linear combination of v_k, that is, $u = \sum_{k\geq 1} \alpha_k v_k$ $(\alpha_k \in \mathbb{C})$. Since v_1 satisfies (14) we have $\langle u, v_1\rangle = \langle \sum_{k\geq 1} \alpha_k v_k, v_1\rangle = \sum_{k\geq 2} \alpha_k \langle v_k, v_1\rangle$. On the other hand, $\lambda(u)$ and $\lambda(v_1)$ do not meet (see the comment just before Definition 2.3), and it follows from Lemma 3.2 that $\langle u, v_1\rangle \neq 0$. Thus there is some $j \geq 2$ such that $\langle v_j, v_1\rangle \neq 0$, say, $\langle v_2, v_1\rangle \neq 0$.

Let $\mu = 2\langle v_2, v_1\rangle/\langle v_2, v_2\rangle \in \mathbb{R}$, and let $v' = v_1 - \mu v_2 \in \mathbb{R}^6$. Then v' is a real solution of the equations (13) and (14). In fact, we have $\langle u_j, v'\rangle = \langle u_j, v_1\rangle - \mu\langle u_j, v_2\rangle = 0$ (because $\langle u_j, v_k\rangle = 0$ for all j, k), and $\langle v', v'\rangle = \langle v_1, v_1\rangle - 2\mu\langle v_2, v_1\rangle + \mu^2\langle v_2, v_2\rangle = 0$ (due to the definition of μ). So, $\lambda(v') \in \Lambda$ is a transversal of g_1, g_2, g_3, g_4. Moreover, we have $\lambda(v') \neq \lambda(v_1)$, because $v' = v_1 - \mu v_2$ $(\mu \neq 0)$ is linearly independent from v_1. Finally, $\lambda(v')$ is the other transversal and $\lambda(v') \in \Lambda$. □

The set Λ is closed under the extensions of the line tables from (2) to (1). Namely, we have the following.

Theorem 4.3 (Double six in Λ)**.** *If a double six $a_1, a_2, \dots, a_6; b_1, b_2, \dots, b_6$ in $\mathbb{P}^3(\mathbb{C})$ satisfies $a_1, a_2, a_3, a_4, a_5, b_6 \in \Lambda$, then the remaining six lines also belong to Λ.*

Proof. By Lemma 4.2, all of $b_1, \dots, b_5$ and a_6 belong to Λ. □

5. Oriented spheres in $\mathbb{R}^3 \cup \{\infty\}$

By adding a point ∞ at infinity to $\mathbb{R}^3$, we get a compact space $\mathbb{R}^3 \cup \{\infty\}$. An **oriented sphere** in $\mathbb{R}^3 \cup \{\infty\}$ is one of the following:

- **oriented true sphere:** a usual sphere in $\mathbb{R}^3$ in which one side (inside or outside) is distinguished as its positive side.
- **oriented plane:** a plane in $\mathbb{R}^3$ (with ∞ at infinity) in which one side is distinguished as its positive side.
- **null sphere:** a singleton in $\mathbb{R}^3 \cup \{\infty\}$ (whose 'positive side' is defined to be the empty set $\emptyset$).

Let us present a homogeneous coordinate of an oriented sphere in $\mathbb{R}^3 \cup \{\infty\}$. For reals a, b, c, d, e (not all 0), consider a geometric figure defined by

$$a(x^2 + y^2 + z^2) + b - 2cx - 2dy - 2ez = 0. \tag{15}$$

When a, b, c, d, e are replaced with $\lambda a, \lambda b, \lambda c, \lambda d, \lambda e$ ($\lambda \neq 0$), the equation (15) still defines the same figure. So, we may use the homogeneous coordinate $[a, b, c, d, e] \in \mathbb{P}^4(\mathbb{R})$ as the parameter for the figure.

If $a \neq 0$, we can rewrite (15) as $(x - c/a)^2 + (y - d/a)^2 + (z - e/a)^2 = (f/a)^2$, where

$$f^2 = c^2 + d^2 + e^2 - ab. \tag{16}$$

If $f \neq 0$, then (15) represents a true sphere in $\mathbb{R}^3$ with center $(c/a, d/a, e/a)$ and radius $|f/a|$. If $f = 0$, then (15) represents a singleton in $\mathbb{R}^3$. Notice that a, b, c, d, e have to be chosen so that the RHS of (16) is nonnegative. If $f \neq 0$ there are two choices for f (either positive or negative). Hence, we can designate the positive side of the sphere by the sign of f/a. Namely, we can define the positive side of the sphere to be the inside if $f/a > 0$, and to be the outside if $f/a < 0$. Thus, an oriented true sphere or a null sphere can be presented by a homogeneous coordinate $[a, b, c, d, e, f]$ satisfying (16) and $a \neq 0$.

If $a = 0$ and $c^2 + d^2 + e^2 > 0$ then (15) represents a plane, and the unit normal vector of this plane is given by $(c/f, d/f, e/f)$. Therefore, we can define the positive side of the plane by this normal vector. Thus, an oriented plane is also presented by a homogeneous coordinate $[0, b, c, d, e, f]$ satisfying (16) and $f \neq 0$.

Finally, in the case $b \neq 0$, $a = c = d = e = 0$, the equation (15) gives a void statement. So, let us assign $[0, b, 0, 0, 0, 0]$ with $b \neq 0$ to the null sphere $\{\infty\}$ with empty positive side.

Thus, every oriented sphere in $\mathbb{R}^3 \cup \{\infty\}$ can be represented uniquely by a homogeneous coordinate $[a, b, c, d, e, f]$ satisfying (16). For example, the null sphere $\{P\}$ with $P = (c, d, e) \in \mathbb{R}^3$ has coordinate $[1, c^2 + d^2 + e^2, c, d, e, 0]$, and the null sphere $\{\infty\}$ has coordinate $[0, 1, 0, 0, 0, 0]$. An oriented true sphere with center (x_0, y_0, z_0) and radius r has coordinate $[1, x_0^2 + y_0^2 + z_0^2 - r^2, x_0, y_0, z_0, \pm r]$.

Definition 5.1. *Let Θ be the set of all oriented spheres in $\mathbb{R}^3 \cup \{\infty\}$, that is,*

$$\Theta = \{[a, b, c, d, e, f] \in \mathbb{P}^5(\mathbb{R}) : -ab + c^2 + d^2 + e^2 - f^2 = 0\}.$$

For an oriented sphere $\sigma \in \Theta$, let us denote the positive side of σ by $[\sigma]$. For a true sphere S in $\mathbb{R}^3$, the oriented true sphere S^+ and S^- are defined by setting $[S^+] =$ (the inside) and $[S^-] =$ (the outside). Then, every oriented true sphere can be represented as S^+ or S^- using some true sphere S in $\mathbb{R}^3$. The sign $+$ or $-$ appearing in the shoulder is called the **sign** of the oriented true sphere. For an oriented true sphere with coordinate $[a, b, c, d, e, f]$, its sign coincides with the sign of f/a.

Two distinct oriented spheres $\sigma, \tau \in \Theta$ are said to be in **oriented contact** if (i) $\sigma \cap \tau = \{\text{one point}\}$, and (ii) $[\sigma] \subset [\tau]$ or $[\tau] \subset [\sigma]$. Then, by definition,

$$\begin{array}{ll} & \text{two true spheres } S \text{ and } T \text{ touch externally (resp. internally)} \\ \text{iff} & S^+ \text{ and } T^- \text{ (resp. } T^+\text{) are in oriented contact (see Figure 2).} \end{array} \tag{17}$$

Since the positive side of a null sphere $\{P\}$ is empty, $\{P\}$ and an oriented sphere σ are in oriented contact if and only if P lies on the oriented sphere σ. Since ∞ lies on every oriented plane, the null sphere $\{\infty\}$ and an oriented plane are always in oriented contact.

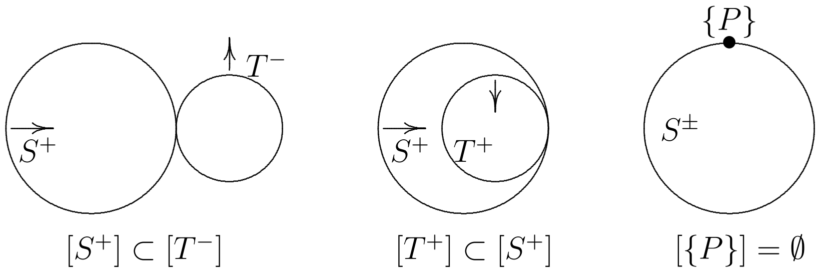

FIGURE 2. Oriented spheres in oriented contact

Lemma 5.2. *Two oriented spheres* $[a, b, c, d, e, f]$ *and* $[a', b', c', d', e', f']$ *are in oriented contact if and only if*

$$-ab' - ba' + 2cc' + 2dd' + 2ee' - 2ff' = 0.$$

Proof. We prove only the case of two oriented true spheres. We may assume that $a = a' = 1$, and the two spheres have centers (c, d, e) and (c', d', e'), and radii $|f|$ and $|f'|$. Then these spheres are in oriented contact if and only if

$$\begin{aligned}
&(c-c')^2 + (d-d')^2 + (e-e')^2 = (f-f')^2 \\
\Leftrightarrow\quad & -2cc' - 2dd' - 2ee' + 2ff' + (c^2 + d^2 + e^2 - f^2) + (c'^2 + d'^2 + e'^2 - f'^2) = 0 \\
\Leftrightarrow\quad & -2cc' - 2dd' - 2ee' + 2ff' + b + b' = 0 \qquad (\because (16)) \\
\Leftrightarrow\quad & -ab' - a'b + 2cc' + 2dd' + 2ee' - 2ff' = 0. \qquad (\because a = a' = 1)
\end{aligned}$$

One can show other cases similarly. ☐

6. LIE'S LINE-SPHERE TRANSFORMATION

We can assign a line with Plücker coordinate $[\alpha, \beta, \gamma, \xi, \eta, \zeta]$ to an oriented sphere with coordinate $[a, b, c, d, e, f]$ by setting

$$\begin{array}{lll} \alpha = a, & \beta = c + d\sqrt{-1}, & \gamma = e + f, \\ \xi = -b, & \eta = c - d\sqrt{-1}, & \zeta = e - f. \end{array} \tag{18}$$

In fact, we have $\alpha\xi + \beta\eta + \gamma\zeta = -ab + c^2 + d^2 + e^2 - f^2 = 0$ by (18), (6) and (16). Moreover, (18) gives a bijection from Θ to Λ, and the inverse is as follows:

$$\begin{array}{lll} a = \alpha, & c = (\beta + \eta)/2, & e = (\gamma + \zeta)/2, \\ b = -\xi, & d = (\beta - \eta)/(2\sqrt{-1}), & f = (\gamma - \zeta)/2. \end{array} \tag{19}$$

Definition 6.1. *The bijection* $\varphi : \Lambda \to \Theta$, $[\alpha, \beta, \gamma, \xi, \eta, \zeta] \mapsto [a, b, c, d, e, f]$ *given by (19) is called* **Lie's line-sphere transformation** [4, 9, 13, 16].

Lemma 6.2. *Let $\varphi : \Lambda \to \Theta$ be the line-sphere transformation. Then two lines a and b intersect if and only if the corresponding two oriented spheres $\varphi(a)$ and $\varphi(b)$ are in oriented contact.*

Proof. Let $[\alpha, \beta, \gamma, \xi, \eta, \zeta]$, $[\alpha', \beta', \dots, \zeta']$ be lines, and $[a, b, c, d, e, f]$, $[a', b', \dots, f']$ be the corresponding oriented spheres. Then (18) (or (19)) gives

$$\alpha\xi' + \alpha'\xi + \beta\eta' + \beta'\eta + \gamma\zeta' + \gamma'\zeta = 0 \Leftrightarrow -ab' - ba' + 2cc' + 2dd' + 2ee' - 2ff' = 0.$$

Now the lemma follows from Lemma 3.2 and Lemma 5.2. □

A **sphere table** is just a sphere version of a line table, and it is defined similarly as in Definition 2.4 by replacing "line" with "oriented sphere," and "intersect" with "are in oriented contact." The next theorem guarantees the following extension of sphere tables:

$$\begin{array}{cccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 & \tau_6 \end{array} \Rightarrow \begin{array}{cccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 & \tau_6 \end{array}$$

This is a sphere version of Corollary 2.8. The twelve riented spheres $\sigma_1, \dots, \sigma_6$; $\tau_1, \dots, \tau_6$ in the table (above right) are called a **double six of spheres**. See Example 9.1 for an example of a double six of spheres in $\mathbb{R}^3$.

Theorem 6.3 (Double six of spheres in Θ)**.** *Let $\{\sigma_1, \dots, \sigma_5, \tau_1, \dots, \tau_6\}$ be a set of eleven oriented spheres in Θ such that σ_i and τ_j are in oriented contact if and only if $i \neq j$, and no other pair of spheres are in oriented contact. Then there exists a unique oriented sphere $\sigma_6 \in \Theta$, which is in oriented contact with only $\tau_1, \dots, \tau_5$ among the eleven spheres.*

Proof. Let $a_i = \varphi^{-1}(\sigma_i) \in \Lambda$, $b_j = \varphi^{-1}(\tau_j) \in \Lambda$. Then it follows from Lemma 6.2 that $a_1, \dots, a_5; b_1, \dots, b_6$ constitute the line table (3). By Corollary 2.8, there is a unique line a_6 in $\mathbb{P}^3(\mathbb{C})$ that intersects only the five lines $b_1, \dots, b_5$. Then Theorem 4.3 implies that $a_6 \in \Lambda$. Set $\sigma_6 = \varphi(a_6)$. Now the theorem follows from Lemma 6.2. □

7. Grace's theorem on a tetrahedron

Throughout this section, T denotes a tetrahedron in $\mathbb{R}^3$ ($\subset \mathbb{R}^3 \cup \{\infty\}$) with vertices A, B, C, D. By a face of a tetrahedron, we mean an extended face, that is, a plane determined by three vertices of the tetrahedron. Let a, b, c, d be the faces of T opposite to the vertices A, B, C, D, respectively. Let a^+ be the oriented plane determined by a whose positive side contains A, and let a^- be the oriented plane such that $A \notin [a^-]$. Define $b^+, b^-, c^+, c^-, d^+, d^-$ similarly. Then $[a^+] \cap [a^-] = a$, $[a^+] \cup [a^-] = \mathbb{R}^3 \cup \{\infty\}$ and $T \subset [a^+] \cap [b^+] \cap [c^+] \cap [d^+]$.

A **circumsphere** of T is the sphere passing through the vertices A, B, C, D. A **tangent sphere** of T is a sphere that is tangent to all the faces a, b, c, d. Tangent spheres of a tetrahedron are classified in the following way, see [2] p.296.

- **inscribed sphere:** a unique tangent sphere lying inside the tetrahedron.
- **escribed sphere:** a sphere that is tangent to three faces from the inside, and tangent to the remaining face from the outside. There are precisely four such spheres.

roof sphere: a sphere that is tangent to two faces from the inside, and tangent to the other two faces from the outside. (This is a tangent sphere inscribed in a roof such as $[a^-] \cap [b^-] \cap [c^+] \cap [d^+]$, or $XYZWCD$ in Figure 3.)

Not every roof has a roof sphere, in fact, the number of roof spheres varies from 0 to 3 depending on the shape of the tetrahedron. If there is a roof sphere lying in a roof then no roof sphere lies in the opposite roof. We note that

$$\text{the circumsphere never enclose any roof sphere,} \tag{20}$$

because, if the roof $XYZWCD$ contains a tangent sphere, then the contact point of the sphere and the face d lies in the angular region XCY in the plane d, which is apparently exterior to the circumsphere.

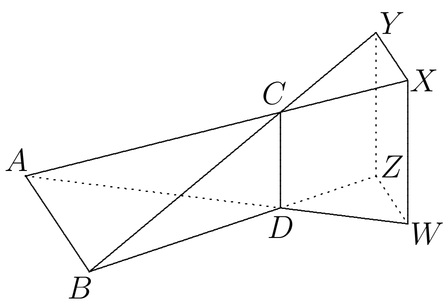

FIGURE 3. A roof of a tetrahedron

Theorem 7.1 (Grace [8] §13). *Among the tangent spheres of a tetrahedron, only the inscribed sphere is enclosed by the circumsphere. Every escribed sphere is cut by the circumsphere, while every roof sphere is entirely exterior to the circumsphere.*

This theorem follows from the next lemma.

Lemma 7.2 (Grace's sphere [7] §24–25). *For a tetrahedron T in $\mathbb{R}^3$ with vertices A, B, C, D, let σ_I, σ_C, σ_E be the inscribed sphere, the circumsphere, and the escribed sphere in $[a^-]$, respectively. Let σ_R be the roof sphere inscribed in $[a^-] \cap [b^-] \cap [c^+] \cap [d^+]$. Then we have the following:*

(i) *There is a true sphere τ_1 passing through the vertices B, C, D, to which both σ_I and σ_E are internally tangent, and σ_C cuts σ_E.*

(ii) *There is a true sphere τ_2 passing through the vertices A, C, D, and externally tangent to both σ_E and σ_R, and σ_R is entirely exterior to σ_C.*

Proof. Let $\sigma_I^+, \sigma_E^+, \sigma_R^+$ be oriented true spheres obtained from $\sigma_I, \sigma_E, \sigma_R$, respectively. Then we have the following sphere table:

$$\begin{array}{cccccc} a^- & a^+ & b^+ & c^+ & d^+ & \\ \sigma_I^+ & \sigma_E^+ & \{B\} & \{C\} & \{D\} & \{\infty\} \end{array}$$

By Theorem 6.3 there is an oriented sphere τ_1 that is in oriented contact exactly with $\sigma_I^+, \sigma_E^+, \{B\}, \{C\}, \{D\}$. The sphere τ_1 passes through B, C, D, and $\tau_1 \neq a^+, a^-$. Thus, τ_1 is a true sphere tangent to both σ_I and σ_E. We notice that τ_1 encloses the triangle BCD, and the face a touches two spheres σ_I and σ_E in this triangle. This implies that σ_I and σ_E are both internally tangent to τ_1.

If τ_1 encloses the cap $[a^+] \cap \sigma_C$ (Figure 4 left), then τ_1 encloses T (and thus σ_I), and this contradicts the fact that τ_1 and σ_I are tangent. So, τ_1 must enclose the cap $[a^-] \cap \sigma_C$ (Figure 4 right). Then, σ_E touches the face a, and σ_E also touches τ_1 at a point in $[a^-] \cap \tau_1$, namely, σ_C cuts σ_E. This proves (i).

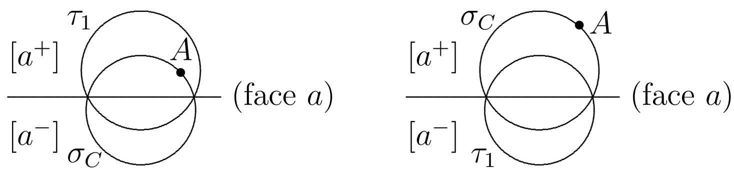

FIGURE 4. τ_1 and σ_C

Next we show (ii). We have the following sphere table:

$$\begin{array}{cccccc} a^- & b^- & b^+ & c^+ & d^+ & \\ \{A\} & \sigma_E^+ & \sigma_R^+ & \{C\} & \{D\} & \{\infty\} \end{array}$$

Then, by Theorem 6.3, there is a true sphere τ_2 passing through A, C, D. Recall from (17) that τ_2 and σ_E touch internally (resp. externally) iff τ_2 and σ_R touch internally (resp. externally).

If τ_2 encloses the cap $[b^+] \cap \sigma_C$ (Figure 5 left), then τ_2 encloses T (and also the contact point of T and σ_E). Thus, σ_E is internally tangent to τ_2, and so is σ_R. Then, σ_C encloses σ_R, contradicting (20). So, τ_2 must enclose the cap $[b^-] \cap \sigma_C$ (Figure 5 right). If σ_E is internally tangent to τ_2 in $[b^+] \cap \tau_2$, then σ_C encloses σ_E, which contradicts (i). Consequently, τ_2 is externally tangent to both σ_E and σ_R. Since τ_2 encloses the cap $[b^-] \cap \sigma_C$, we can conclude that σ_R is external to σ_C. $\square$

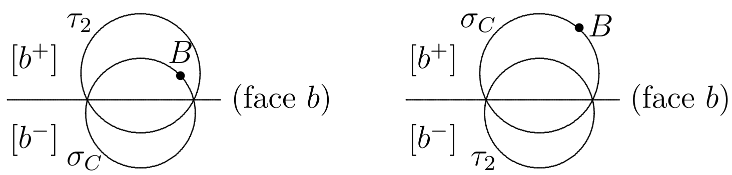

FIGURE 5. τ_2 and σ_C

8. On families of five unit spheres in $\mathbb{R}^3$

We are ready to prove our main theorem.

Theorem 8.1. *In a family of five unit spheres in $\mathbb{R}^3$, if each four spheres have nonempty intersection, then all five spheres have nonempty intersection.*

We recall here what an inversion of $\mathbb{R}^d \cup \{\infty\}$ is. An inversion with respect to a reference sphere of center $p \in \mathbb{R}^d$ and radius $r > 0$ is a transformation of $\mathbb{R}^d \cup \{\infty\}$ that sends each point x ($\neq p, \infty$) to a point x' on the ray $\overrightarrow{px}$ such that $|x - p| \cdot |x' - p| = r^2$, and that switches p and ∞. By an **inversion at** p, denoted by ψ_p, we mean the inversion with respect to a unit sphere centered at $p \in \mathbb{R}^d$. This is an involution (i.e., $\psi_p \circ \psi_p = id$), which switches the inside and the outside of the reference sphere, and maps a sphere to another sphere. More precisely, the following holds.

Lemma 8.2. *Let ψ_p be the inversion at $p \in \mathbb{R}^d$.*

(i) *ψ_p maps a sphere not passing through p to a sphere not passing through p.*
(ii) *ψ_p maps a sphere passing through p to a hyperplane not passing through p.*
(iii) *ψ_p maps a hyperplane passing through p to the same hyperplane.*
(iv) *If p lies inside a sphere S_1 and S_1 lies inside another sphere S_2, then p lies inside $\psi_p(S_2)$ and $\psi_p(S_2)$ lies inside $\psi_p(S_1)$. If p lies outside S_1, then p lies outside $\psi(S_1)$.*

Lemma 8.3. *Let γ, Γ be circles and Δ be a triangle in the plane such that γ lies inside Δ and Δ lies inside Γ. Let ψ be the inversion at the center of the circle γ. Then the diameter of $\psi(\Gamma)$ is at most the radius of $\psi(\gamma)$, and they are equal only when γ is the circumscribed circle of Δ, and γ is the inscribed circle of Δ.*

Proof. By replacing Δ with a larger triangle if necessary, we may assume that Δ is inscribed in Γ. Since Δ encloses γ, there is a triangle Δ' homothetic to Δ and circumscribed to γ, see Figure 6. The center of the homothety is contained in Δ', and the homothety sends Γ to the circumscribed circle Γ' of Δ'. Thus Γ' is contained in Γ, and by Lemma 8.2 (iv), $\psi(\Gamma)$ is contained in $\psi(\Gamma')$. So, the diameter of $\psi(\Gamma)$ is at most the diameter of $\psi(\Gamma')$.

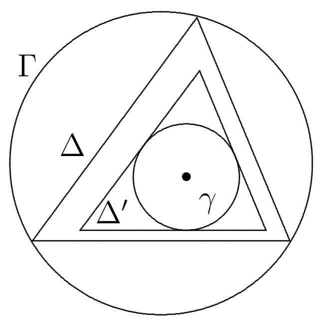

Figure 6. Circle–triangle–circle

Let O be the center of the circle γ, and r be its radius. Then $\psi(\gamma)$ has radius $1/r$. Let a, b, c be the three lines determined by the sides of the triangle Δ'. Then, $\psi(a), \psi(b), \psi(c)$ are three circles passing through O with diameter $1/r$ (and internally tangent to $\psi(\gamma)$). These three circles determine three intersections other than O, which determine the circle $\psi(\Gamma')$. By Theorem 1.1, the diameter of $\psi(\Gamma')$ coincides with the diameter of $\psi(a)$ $(= 1/r)$, which equals to the radius of $\psi(\gamma)$. So, the diameter of $\psi(\Gamma)$ is at most $1/r$, and it is equal to $1/r$ only when $\Gamma = \Gamma'$. □

Lemma 8.4. *For a tetrahedron T in $\mathbb{R}^3$, let σ be the inscribed sphere, and Σ be the circumsphere. Let ψ be the inversion at the center of σ. Then the diameter of $\psi(\Sigma)$ is smaller than the radius of $\psi(\sigma)$.*

Proof. Let O be the center of σ, and let p be the center of Σ. Let π be a plane determined by O, p and a vertex of T. Cutting Σ, T, σ by the plane π, we get a circle Γ, a triangle Δ contained in Γ, and a circle γ contained in Δ as plane figures on the plane π. Then $\psi(\Gamma)$ and $\psi(\Sigma)$ have the same diameter, and $\psi(\gamma)$ and $\psi(\sigma)$ have the same radius. Since Γ is not the circumcircle of Δ, it follows from Lemma 8.3 that the diameter of $\psi(\Sigma)$ is smaller than the radius of $\psi(\sigma)$. □

Proof of Theorem 8.1. Suppose for a contradiction that there are five unit spheres $S_0, S_1, \dots, S_4$ such that each four spheres have nonempty intersection, but the intersection of the five spheres is empty. Let us call the intersection of each four spheres a *junction*. Note that there are five junctions, and they are all different. Let q_i be the junction of the four spheres other than S_i. Then q_0 is the junction of S_1, S_2, S_3, S_4, and the other four junctions q_1, q_2, q_3, q_4 lie on the sphere S_0.

We show that there is some i such that q_i lies inside S_i. Let ψ be the inversion at q_0, and let $\Sigma_i := \psi(S_i)$, $p_i := \psi(q_i)$. Then Σ_0 is a sphere, and $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ are planes, because $q_0 \notin S_0$ and $q_0 \in S_j$ for $j = 1, 2, 3, 4$ (see Lemma 8.2 (i) (ii)). These four planes span a tetrahedron T with vertices p_1, p_2, p_3, p_4. Then Σ_i is the face opposite to p_i, and Σ_0 is the circumsphere of T. Now, suppose that q_0 lies outside S_0. Then, q_0 also lies outside Σ_0 by Lemma 8.2 (iv), and hence lies outside the tetrahedron T. By (ii), the four planes spanning T do not pass through q_0. So we can choose one of the planes closest to q_0, say, Σ_i. Then the line segment connecting q_0 and p_i intersect Σ_i, and hence q_i lies inside $\psi(\Sigma_i) = S_i$.

By changing indices if necessary, we may assume that q_0 lies inside S_0. Let S be the sphere of radius 2 centered at q_0. We claim that $\sigma := \psi(S)$ is a tangent sphere of T and it is contained in the circumsphere Σ_0. In fact, since all four spheres S_1, S_2, S_3, S_4 (with diameter 2) pass through q_0, they are internally tangent to S. Also it follows from (iv) that Σ_0 $(= \psi(S_0))$ encloses σ $(= \psi(S))$, because S_0 lies inside S

Here we come to the point where we invoke Theorem 7.1 by Grace. The sphere σ is the inscribed sphere of T. By Lemma 8.4, the diameter of $S_0 = \psi(\Sigma_0)$ is smaller than the radius of $\psi(\sigma)$, namely, diameter of S_0 is less than 2. This contradicts to the fact that S_0 is a unit sphere. □

9. Examples and problems on sphere-systems

In this section, a sphere implies a *true* sphere.

A **double k of spheres** is a sphere-system formed by a set of $2k$ distinct oriented true spheres $A_1, \dots, A_k; B_1, \dots, B_k$ in $\mathbb{R}^d$ such that if $i \neq j$ then A_i and B_j are in oriented contact and no other pair of spheres are in oriented contact. Thus, if the signs of A_i, B_j are the same, then they are internally tangent, otherwise they are externally tangent. Notice that A_i and B_i may be in contact (non-orientedly).

Example 9.1. The following table presents a double six of spheres in $\mathbb{R}^3$.

sphere	center	radius	sign.	sphere	center	radius	sign.
A_1	$(2,0,0)$	1	$+$	B_1	$(-2,0,0)$	3	$+$
A_2	$(-1,\sqrt{3},0)$	1	$+$	B_2	$(1,-\sqrt{3},0)$	3	$+$
A_3	$(-1,-\sqrt{3},0)$	1	$+$	B_3	$(1,\sqrt{3},0)$	3	$+$
A_4	$(0,0,3/2)$	$1/2$	$+$	B_4	$(0,0,3/2)$	$7/2$	$+$
A_5	$(0,0,-3/2)$	$1/2$	$+$	B_5	$(0,0,-3/2)$	$7/2$	$+$
A_6	$(0,0,0)$	5	$+$	B_6	$(0,0,0)$	1	$-$

In this example, B_6 and A_i are externally tangent for all $i \neq 6$, and other pair of spheres A_i and B_j $(i \neq j)$ are internally tangent. (Also A_i and B_i for $i = 1,2,3$ are externally tangent, but not in oriented contact.) Among the twelve lines in Λ corresponding to these oriented spheres by line-sphere transformation, the lines corresponding to A_2, A_3, B_2, B_3 are not contained in $\mathbb{R}^3$.

Example 9.2. One can find an example of double six of lines in $\mathbb{R}^3$ with picture in [11] Chap. 8 §25. The corresponding double six of spheres (obtained by the line-sphere transformation) are not contained in $\mathbb{R}^3$ (but in $\mathbb{C}^3$, of course). In general, there is no double six of lines in $\mathbb{R}^3$ whose corresponding double six of spheres are also in $\mathbb{R}^3$.

Theorem 9.3. *Let $d \geq 1$. If there is a double k of spheres in $\mathbb{R}^d$, then $k \leq d+3$.*

Proof. Let $A_1, \dots, A_k; B_1, \dots, B_k$ be a double k of true spheres. Let A_i have center $(a_{i1}, a_{i2}, \dots, a_{id})$, radius $r_i > 0$, and sign ϵ_i. Let B_j have center $(b_{j1}, b_{j2}, \dots, b_{jd})$, radius $r'_j > 0$, and sign ϵ'_j. Define a polynomial f_{A_i} associated to the sphere A_i by

$$f_{A_i}(x_1, x_2, \dots, x_d, r) = (x_1 - a_{i1})^2 + (x_2 - a_{i2})^2 + \cdots + (x_d - a_{id})^2 + (r - \epsilon_i r_i)^2.$$

Then we have

$$f_{A_i}(B_j) = f_{A_i}(b_{j1}, b_{j2}, \dots, b_{jd}, \epsilon'_j r'_j) \begin{cases} \neq 0 & \text{if} \quad i = j \\ = 0 & \text{if} \quad i \neq j, \end{cases}$$

and $\{f_{A_i}\}$ are linearly independent. The vector space spanned by $\{f_{A_i}\}$ has a basis

$$x_1^2 + x_2^2 + \cdots + x_d^2 + r^2, x_1, x_2, \dots, x_d, r, 1.$$

Thus the dimension of the space is at most $d+3$, which means the number of linearly independent $\{f_{A_i}\}$ is at most $d+3$, that is, $k \leq d+3$. □

Problem 9.4. *Determine all possible combinations of signs for a double six of spheres in $\mathbb{R}^3$. Is it possible that A_i and B_j are externally tangent for all $i \neq j$?*

Example 9.5. There is a double $(d+2)$ of spheres in $\mathbb{R}^d$ in which tangent spheres are always externally tangent. The following shows a construction. Fix a regular d-simplex with vertices $v_1, \ldots, v_{d+1}$ in $\mathbb{R}^d$. Take spheres $A_1, \ldots, A_{d+2}$ of sufficiently small radius, and put them at each vertex v_i, and put A_{d+2} at the barycenter. Then, for each $i = 1, \ldots, d+2$, we can choose a sphere B_i so that it is externally tangent to all the spheres of $\{A_1, \ldots, A_{d+2}\} \setminus \{A_i\}$. Assign positive sign to all A_i, and negative sign to all B_i.

Example 9.6. There is no double four of spheres in $\mathbb{R}^1$. To see this, suppose, on the contrary, that there is a double four of spheres $A_1, \ldots, A_4; B_1, \ldots, B_4$. Each sphere consists of two points. Without loss of generality, we may assume that one of the points in A_1 is contained in B_2, B_3. In this case, since A_1 and B_2, and also A_1 and B_3 are in oriented contact, B_2 and B_3 must become in oriented contact, a contradiction.

Problem 9.7. *Let $\mathcal{D}$ be the set of d such that there is a double $(d+3)$ of spheres in $\mathbb{R}^d$. We know that $1 \notin \mathcal{D}$ and $3 \in \mathcal{D}$. Is there a double five of circles in the plane? Determine $\mathcal{D}$.*

Example 9.8. There are ten circles $A_1, \ldots, A_5; B_1, \ldots, B_5$ in the plane such that if $i \neq j$ then A_i and B_j are tangent, and no other pair of circles are tangent to each other. They are given in Figure 7, where small circles and big circles are corresponding to A_i and B_j respectively. It is impossible to assign signs to these circles so that they become a double five of circles.

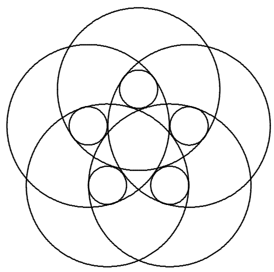

FIGURE 7. A family of ten circles

ACKNOWLEDGMENT

The authors would like to thank Professor Margaret M. Bayer for informing them about the articles of Grace [7, 8].

REFERENCES

[1] H. F. Baker. *Principles of Geometry. Vol. III. Solid Geometry.* The Cambridge Press, 1923.
[2] M. Berger. Geometry I, Springer-Verlag, New York 1987.
[3] K. Bezdek, Z. Lángi, M. Naszódi, P. Papez. Ball polyhedra, *Discrete Comput. Geom.*, 38 (2007) 201–230.
[4] J. L. Coolidge. *A treatise on the circle and the sphere*, Chelsea, New York 1916.
[5] D. A. Cox, J. B. Little, D. O'Shea. *Ideals, Varieties, And Algorithms: An Introduction to Computational Algebraic Geometry And Commutative Algebra*, Springer 1992.

[6] P. J. Davis, R. Hersh. *The Mathematical Experience*, Boston 1982, Birkhäuser.
[7] J. H. Grace. Circles, spheres, and linear complexes, *Trans. Cambridge Philosophical Society*, 14 (1898) 153–190.
[8] J. H. Grace. Tetrahedra in relation to spheres and quadrics, *Proc. London Math. Soc.* S2-17 (1918) 259–271.
[9] S. Helgason. Sophus Lie, the mathematician, *Proc. Sophus Lie Memorial Conference, Oslo*, 1994 3–21.
[10] A. Henderson. *The twenty-seven lines upon the cubic surface*, Hafner Publishing Company, 1911, New York.
[11] D. Hilbert, S. Cohn-Vossen, *Anschuliche geometrie,*Springer Verlarg, 1932, Berlin.
[12] A. Ichida. A simple poof of double six theorem, *Tohoku Math. J.* 32 (1929) 52–53.
[13] S. Lie. Marius Sophus Lie's doctoral thesis "Over en Classe geometriske Transformationer" ["On a class of geometrical transformations"]. Translated from the Norwegian by Erik Trell. Commentary by Trell and Ruggero Maria Santilli. Algebras Groups Geom. 15 (1998), no. 4, 395–445.
[14] H. Maehara. Helly-type theorems for spheres, *Discrete Comput Geom.*, 4 (1989) 279–285.
[15] H. Maehara, N. Tokushige. On a special arrangement of spheres, *Ryukyu Math. J.* 19 (2006) 15–24.
[16] R. Milson. An overview of Lie's line–sphere transformation, *Contemporary Mathematics* 285 (2001) 1–10.
[17] G. Pólya. Mathematical Discovery, New York 1981, Wiley.
[18] J. Plücker. *System der Geometrie des Raumes in neuer analytischer Behandlungsweise*, Crelle's Journal 1846.
[19] M. Reid. *Undergraduate algebraic geometry*, Cambridge Univ. Press 1988.
[20] L. Schläfli. An attempt to determine the twenty-seven lines upon a surface of the third order, and to derive such surfaces in species, in reference to the reality of the lines upon the surface, *Quart. J. of pure and applied Math.*, 2 (1858) 55–65, 110–120.
[21] B.L. van der Waerden. *Einführung in die algebraische Geometrie*, Springer Verlag, 1939.

College of Education, Ryukyu University
E-mail address: `hmaehara@edu.u-ryukyu.ac.jp`

College of Education, Ryukyu University
E-mail address: `hide@edu.u-ryukyu.ac.jp`