On arithmetic partitions of \mathbb{Z}_n

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Abstract. Generalizing a classical problem in enumerative combinatorics, Mansour and Sun counted the number of subsets of \mathbb{Z}_n without certain separations. Chen, Wang, and Zhang then studied the problem of partitioning \mathbb{Z}_n into arithmetical progressions of a given type under some technical conditions. In this paper, we improve on their main theorems by applying a convolution formula for cyclic multinomial coefficients due to Raney-Mohanty.

Keywords: cycle dissection, m-AP-partition, cyclic multinomial coefficient, Raney-Mohanty's identity

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1. Introduction

In his solution of problème des ménages Kaplansky [10] showed that the number of ways of selecting k elements, no two consecutive, from n objects arrayed on a cycle is $\frac{n}{n-k} \binom{n-k}{k}$. Let $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ be the set of congruence classes modulo n with usual arithmetic. Then Yamamoto [22] (see also [19, p. 222]) proved that if $n \ge pk + 1$ the number of ways of selecting k elements from \mathbb{Z}_n , no two consecutive, is

$$\frac{n}{n-pk} \binom{n-pk}{k},\tag{1.1}$$

when $i \pm 1, \dots, i \pm p$ are regarded as consecutive to i.

In the last three decades a lot of generalizations and variations of Kaplansky's problem have been studied by several authors (see, for example, [4, 7-9, 11, 12, 15-17, 20]). In particular, Konvalina [12] considered the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of \mathbb{Z}_n such that $x_i - x_j \neq 2$ for all $1 \leq i, j \leq k$, and found that the answer is $\frac{n}{n-k} \binom{n-k}{k}$ if $n \geq 2k+1$. Hwang [8] then generalized Konvalina's result to the case $x_i - x_j \neq m$ and deduced that the desired number is given by the same formula if $n \geq mk+1$. Recently, Mansour and Sun [13] gave the following unification of Yamamoto's and Hwang's formulas.

Theorem 1.1 (Mansour-Sun). Let m, n, p, k be positive integers such that $n \ge mpk + 1$. Then the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of \mathbb{Z}_n such that

$$x_i - x_j \notin \{m, 2m, \dots, pm\} \quad (1 \leqslant i, j \leqslant k), \tag{1.2}$$

is also given by (1.1).

A short proof of Theorem 1.1 was given by Guo [5] by using Rothe's identity. In order to generalize Mansour-Sun's result, Chen, Wang, and Zhang [3] defined an m-AP-block of length k to be a sequence (x_1, x_2, \ldots, x_k) of distinct elements in \mathbb{Z}_n such that $x_{i+1} - x_i = m$ for $1 \leq i \leq k-1$ and studied the problem of partitioning \mathbb{Z}_n into m-AP-blocks. The type of such a partition is defined to be the type of the multiset of the lengths of the blocks. For example, the following is a 3-AP-partition of \mathbb{Z}_{20} of type $1^4 2^3 3^2 4^1$:

$$(2), (4,7), (5,8), (6), (9,12,15), (10), (11), (13,16,19), (14,17,0,3), (18,1).$$

We need to emphasize that (x, x + m, ..., x + (n-1)m) and (x + m, x + 2m, ..., x + (n-1)m, x) and so on are deemed as different m-AP-blocks in \mathbb{Z}_{mn} . For example, all the 2-AP-partitions of \mathbb{Z}_6 of type 3^2 are

$$\{(i, i+2, i+4), (j+1, j+3, j+5)\}_{i,j=0,2,4}$$

Chen, Wang, and Zhang [3] constructed a bijection between m-AP-partitions and m'-AP-partitions of \mathbb{Z}_n under some technical conditions, and established the following theorem.

Theorem 1.2 (Chen-Wang-Zhang). Let $m, n, k_1, k_2, \ldots, k_r$ and i_2, \ldots, i_r be positive integers such that $1 < i_2 < \cdots < i_r$ and

$$k_1 > (k_2 + \dots + k_r) ((m-1)(i_r - 1) - 1).$$
 (1.3)

Then the number of partitions of \mathbb{Z}_n into m-AP-blocks of type $1^{k_1}i_2^{k_2}\cdots i_r^{k_r}$ does not depend on m, and is given by the cyclic multinomial coefficient

$$\frac{n}{k_1 + \dots + k_r} \binom{k_1 + \dots + k_r}{k_1, \dots, k_r}.$$
(1.4)

If we specialize the type to $1^{n-(p+1)k}(p+1)^k$, then the condition (1.3) becomes $n \ge mpk+1$. Furthermore, if $(x_1,x_1+m,\ldots,x_1+pm),\ldots,(x_k,x_k+m,\ldots,x_k+pm)$ are the k blocks of length p+1 in an m-AP-partition of \mathbb{Z}_n of type $1^{n-(p+1)k}(p+1)^k$, then the set $\{x_1,\ldots,x_k\}$ satisfies (1.2), and vice versa. Therefore Theorem 1.2 implies Theorem 1.1.

In this paper we shall improve and complete Theorem 1.2 by establishing the following two theorems.

Theorem 1.3. Let m, n be positive integers, and let k_1, k_2, \ldots, k_r be nonnegative integers such that $n = k_1 + 2k_2 + \cdots + rk_r$. Let $d = \gcd(m, n)$. If

$$\Delta := n - d(n - k_1 - \dots - k_r) > 0, \tag{1.5}$$

then the number of partitions of \mathbb{Z}_n into m-AP-blocks of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ is given by (1.4).

It is not hard to see that the condition (1.5) is weaker than (1.3), i.e., the condition (1.3) implies that (1.5). In other words, for fixed n and a given type, there are in general many more m's satisfying (1.5) than satisfying (1.3). For example, by Theorem 1.3, the numbers of m-AP-partitions of \mathbb{Z}_{120} of type $1^{89}2^33^25^17^2$ are all equal for

$$m = 1, 2, 3, 4, 5, 7, 9, 11, 13, 14, 17, 19, 21, 22, 23, 25, 26, 27, 28, 29, 31, 33, 34, 35, 37, 38, 39, 41, 43, 44, 46, 47, 49, 51, 52, 53, 55, 57, 58, 59,$$

i.e., for d = 1, 2, 3, 4, 5. However, Theorem 1.2 only asserts that these numbers for m = 1, 2, 3 are equal.

Theorem 1.4. Let $k_1, k_2, \ldots, k_r, m, n, d$ and Δ be given as in Theorem 1.3. Then the number of partitions of \mathbb{Z}_n into m-AP-blocks of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ is given by

$$\begin{cases} \frac{n}{k_1 + \dots + k_r} \binom{k_1 + \dots + k_r}{k_1, \dots, k_r} + \frac{n(-1)^{k_2 + \dots + k_r}}{k_2 + \dots + k_r} \binom{k_2 + \dots + k_r}{k_2, \dots, k_r}, & \text{if } \Delta = 0, \\ \frac{n}{k_1 + \dots + k_r} \binom{k_1 + \dots + k_r}{k_1, \dots, k_r} + \clubsuit (-1)^{k_2 + \dots + k_r} \binom{k_2 + \dots + k_r}{k_2, \dots, k_r}, & \text{if } \Delta = -d, \end{cases}$$

where

When the type in Theorem 1.4 is $1^{n-(p+1)k}(p+1)^k$ again, then $\Delta = n - mpk$. To assure that there is an m-AP-block of length p+1 in \mathbb{Z}_n , we need to assume that n > pm, which is equivalent to $k \geq 2$ if $\Delta = 0$ and pk > p+1 if $\Delta = -m$. As mentioned after Theorem 1.2, each family of k m-AP-blocks in \mathbb{Z}_n is in one-to-one correspondence with a k-subset of \mathbb{Z}_n satisfying (1.2), we derive the following two results, which can be viewed as complements to Theorem 1.1.

Corollary 1.5. Let $m, p \ge 1$, $k \ge 2$ and n = mpk. Then the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of \mathbb{Z}_n such that $x_i - x_j \notin \{m, 2m, \ldots, pm\}$ for all $1 \le i, j \le k$, is given by

$$\frac{n}{n-pk} \binom{n-pk}{k} + (-1)^k \frac{n}{k}.$$
 (1.6)

Actually the above formula is deduced for $m \ge 2$, i.e., $n \ge (p+1)k$, but it also holds for m = 1 if we take the convention

$$\lim_{x \to 0} \frac{n}{x} \binom{x}{k} = (-1)^{k-1} \frac{n}{k},$$

and so (1.6) is equal to 0 in this case. Here is an example for Corollary 1.5. For m=p=k=2, the number of 2-subsets $\{x_1,x_2\}$ of \mathbb{Z}_8 such that $x_1-x_2,x_2-x_1\notin\{2,4\}$ is equal to

$$\frac{8}{4} \binom{4}{2} + 4 = 16,$$

and the corresponding subsets are $\{i, i+1\}$ and $\{i, i+3\}$, where $i \in \mathbb{Z}_8$.

Corollary 1.6. Let $m, p, k \ge 1$ with pk > p+1 and let n = mpk-m. Then the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of \mathbb{Z}_n such that $x_i - x_j \notin \{m, 2m, \ldots, pm\}$ for all $1 \le i, j \le k$, is given by

$$\begin{cases} \frac{n}{n-k} \binom{n-k}{k} + (-1)^{k-1} n(m-2), & \text{if } p = 1, \\ \frac{n}{n-pk} \binom{n-pk}{k} + (-1)^k n, & \text{if } p \geqslant 2. \end{cases}$$

Similarly, although the above formula is deduced for $mpk - m \ge (p+1)k$, it also holds without this condition. The details are left to the interested reader.

Remark. For 0 < m < n, let $g_m(n,k)$ denote the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of \mathbb{Z}_n such that $x_i - x_j \neq m$ for all $1 \leq i, j \leq k$. Hwang [8, Corollary 2] obtained

$$g_m(n,k) = \sum_{j=0}^{\lfloor d/2 \rfloor} (-1)^{nj/d} {d \choose j} \frac{n - 2nj/d}{n - k - nj/d} {n - k - nj/d \choose k - nj/d},$$
(1.7)

where $d = \gcd(m, n)$. Letting n = mk or n = mk - m in (1.7), we are led to the p = 1 case of Corollaries 1.5 or 1.6. However, since there are two cases in Corollary 1.6, it seems impossible to give a formula like (1.7) to unify Corollaries 1.5 and 1.6 for general p.

We recall and establish some necessary lemmas in Section 2 and prove Theorems 1.3 and 1.4 in Sections 3 and 4, respectively. Our main idea is the following: Lemma 2.4 permits us to reduce the general m-AP-partition problem of \mathbb{Z}_n to the case where m divides n. For the latter we may write the partition number as a multiple sum, which can be computed by applying Raney-Mohanty's identity.

2. Some lemmas

A dissection of an n-cycle is a 1-AP-partition of \mathbb{Z}_n , which can be depicted by inserting a bar between any two consecutive blocks on an n-cycle. For example, Figure 1 illustrates a 20-cycle dissection of type $1^42^33^24^1$. It is easy to see that the number of dissections of \mathbb{Z}_n is given by (1.4). Indeed, deleting the segment containing 0 in any dissection of n-cycle of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ yields a dissection of a (n-i)-line of type $1^{k_1}\ldots i^{k_i-1}\ldots r^{k_r}$ if the segment containing 0 is of length i ($1 \leq i \leq n$). So the number of such dissections of n-cycle is equal to

$$i \binom{k_1 + \dots + (k_i - 1) + \dots + k_r}{k_1, \dots, k_i - 1, \dots, k_r}.$$
 (2.1)

Summing (2.1) over all i yields the following known result (see [2, Lemma 3.1]).

Lemma 2.1 (Chen-Lih-Yeh). For an n-cycle, the number of dissections of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ is given by the cyclic multinomial coefficient (1.4).

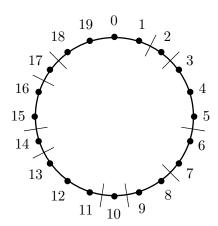


Figure 1: A 20-cycle dissection of type $1^42^33^24^1$.

For any variable x and nonnegative integers k_1, \ldots, k_r define the multinomial coefficient

$$\binom{x}{k_1, k_2, \dots, k_r} := \frac{x(x-1)\cdots(x-k_1-\dots-k_r+1)}{k_1!k_2!\cdots k_r!}.$$

Note that when $x = k_1 + k_2 + \ldots + k_r$ the above definition coincides with the classical definition of multinomial coefficient and

$$\binom{k_1+\cdots+k_r}{k_1,\ldots,k_r} = \binom{k_1+\cdots+k_r}{k_2,\ldots,k_r}.$$

The following convolution formula for multinomial coefficients is due to Raney-Mohanty [14,18]. For other proofs of (2.2), we refer the reader to [6,21,23].

Lemma 2.2 (Raney-Mohanty's identity). For any variables x, y, z_1, \ldots, z_m and nonnegative integers N_1, \ldots, N_m , there holds

$$\sum_{\substack{0 \leqslant t_i \leqslant N_i \\ i=1,\dots,m}} \frac{x}{x - t_1 z_1 - \dots - t_m z_m} {x - t_1 z_1 - \dots - t_m z_m \choose t_1,\dots,t_m} \\
\times \frac{y}{y - (N_1 - t_1) z_1 - \dots - (N_m - t_m) z_m} {y - (N_1 - t_1) z_1 - \dots - (N_m - t_m) z_m \choose N_1 - t_1,\dots,N_m - t_m} \\
= \frac{x + y}{x + y - N_1 z_1 - \dots - N_m z_m} {x + y - N_1 z_1 - \dots - N_m z_m \choose N_1,\dots,N_m}.$$
(2.2)

We also need the following elementary arithmetical result (see [1, Theorem 5.32 and Exercise 16 on page 127] or [5]).

Lemma 2.3. Let m, n be positive integers. If gcd(m, n) = d, then there exists an integer a such that gcd(a, n) = 1 and $am \equiv d \pmod{n}$.

The following is our key lemma.

Lemma 2.4. If $m, n \ge 1$ and gcd(m, n) = d, then there is a bijection from the set of m-AP-partitions of \mathbb{Z}_n to the set of d-AP-partitions of \mathbb{Z}_n . Moreover this bijection keeps the type of partitions.

Proof. By Lemma 2.3, there exists an inversible element $a \in \mathbb{Z}_n$ such that am = d. Let a^{-1} be the inverse of a. For any subset B of \mathbb{Z}_n and $x \in \mathbb{Z}_n$, let $xB = \{xb : b \in B\}$. If $\{B_1, B_2, \ldots, B_s\}$ is an m-AP-partition of \mathbb{Z}_n , then $\{aB_1, aB_2, \ldots, aB_s\}$ is a d-AP-partition of \mathbb{Z}_n . Conversely, if $\{C_1, C_2, \ldots, C_s\}$ is a d-AP-partition of \mathbb{Z}_n , then $\{a^{-1}C_1, a^{-1}C_2, \ldots, a^{-1}C_s\}$ is an m-AP-partition of \mathbb{Z}_n . Obviously, this correspondence keeps the type of partitions. This proves the lemma.

It follows from Lemma 2.4 that if there exists an m-AP-partition of \mathbb{Z}_n of a given type $1^{k_1}2^{k_2}\cdots r^{k_r}$ $(k_r>0)$ then $\gcd(m,n)r\leqslant n$.

3. Proof of Theorem 1.3

By Lemma 2.4, it suffices to consider the case where m divides n, i.e., d = m. Let $n = mn_1$ and divide \mathbb{Z}_n into m subsets of the same cardinality n_1 :

$$\mathbb{Z}_{n,j} = \{ mi + j : i = 0, \dots, n_1 - 1 \}, \quad 0 \le j \le m - 1.$$

Hence $\mathbb{Z}_n = \biguplus_{j=0}^{m-1} \mathbb{Z}_{n,j}$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_s\}$ be an m-AP-partition of \mathbb{Z}_n of type $1^{k_1} 2^{k_2} \cdots r^{k_r}$ $(r \leqslant n_1)$. Then $B_{i,j} = \mathbb{Z}_{n,j} \cap B_i$ is equal to \emptyset or B_i for $1 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant m-1$. Furthermore, since the transformation $x \mapsto (x-j)/m$ maps each m-AP-block $B_{i,j}$ of $\mathbb{Z}_{n,j}$ $(0 \leqslant j \leqslant m-1)$ to a 1-AP-block $B'_{i,j}$ of \mathbb{Z}_{n_1} , each m-AP-partition $\mathcal{B}_j = \{B_{1,j}, \ldots, B_{s,j}\}$ corresponds bijectively to a 1-AP-partition \mathcal{B}'_j of \mathbb{Z}_{n_1} with the same type. Thus, we have established a bijection between the set of m-AP-partitions of \mathbb{Z}_n and the set of m-tuples of 1-AP-partitions of \mathbb{Z}_{n_1} : $\mathcal{B} \leftrightarrow (\mathcal{B}'_0, \ldots, \mathcal{B}'_{m-1})$.

Now assume that the m-AP-partition \mathcal{B} is of type $1^{k_1}2^{k_2}\cdots r^{k_r}$ $(r\leqslant n_1)$, and the corresponding 1-AP-partition \mathcal{B}'_j is of type $1^{k_{1,j}}2^{k_{2,j}}\cdots r^{k_{r,j}}$ $(0\leqslant j\leqslant m-1)$. Clearly,

$$\begin{cases}
k_{2,0} + k_{2,1} + \dots + k_{2,m-1} = k_2, \\
k_{3,0} + k_{3,1} + \dots + k_{3,m-1} = k_3, \\
\dots \\
k_{r,0} + k_{r,1} + \dots + k_{r,m-1} = k_r.
\end{cases}$$
(3.1)

By Lemma 2.1 and noticing that $n_1 = k_{1,j} + 2k_{2,j} + \cdots + rk_{r,j}$, the number of 1-AP-partitions of \mathbb{Z}_{n_1} of type $1^{k_{1,j}}2^{k_{2,j}}\cdots r^{k_{r,j}}$ is equal to

$$\frac{n_1}{k_{1,j} + k_{2,j} + \dots + k_{r,j}} \binom{k_{1,j} + k_{2,j} + \dots + k_{r,j}}{k_{1,j}, k_{2,j}, \dots, k_{r,j}} \\
= \frac{n_1}{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}} \binom{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}}{k_{2,j}, \dots, k_{r,j}}.$$

For $m, n \ge 1$ let $f_{m,n}(k_1, \ldots, k_r)$ be the number of partitions of \mathbb{Z}_n into m-AP-blocks of type $1^{k_1}2^{k_2}\cdots r^{k_r}$. Then

$$f_{m,n}(k_1,\ldots,k_r) = \sum_{(k_{i,j})} \prod_{j=0}^{m-1} \frac{n_1}{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}} \binom{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}}{k_{2,j},\ldots,k_{r,j}},$$
(3.2)

where the summation is over all matrices $(k_{i,j})_{\substack{2 \leqslant i \leqslant r \\ 0 \leqslant j \leqslant m-1}}$ of nonnegative integral coefficients $k_{i,j}$ satisfying (3.1) and

$$\begin{cases}
n_{1} - k_{2,0} - \dots - (r-1)k_{r,0} > 0, \\
n_{1} - k_{2,1} - \dots - (r-1)k_{r,1} > 0, \\
\dots \\
n_{1} - k_{2,m-1} - \dots - (r-1)k_{r,m-1} > 0.
\end{cases}$$
(3.3)

Recall that

$$\Delta = n - m(n - k_1 - \dots - k_r) = mn_1 - m(k_2 + \dots + (r-1)k_r).$$

If $\Delta > 0$, then we have $n_1 > k_2 + \cdots + (r-1)k_r$, and thus all nonnegative integral solutions to (3.1) also satisfy (3.3) as $k_i \ge k_{i,j}$ ($2 \le i \le r$, $0 \le j \le m-1$).

It remains to prove that the right-hand side of (3.2) is equal to (1.4), namely

$$\frac{mn_1}{mn_1 - k_2 - \dots - (r-1)k_r} \binom{mn_1 - k_2 - \dots - (r-1)k_r}{k_2, \dots, k_r}.$$
 (3.4)

We proceed by induction on $m \ge 1$. This is equivalent to repeatedly applying Raney-Mohanty's identity (2.2). The case m = 1 is obviously true. Suppose that the formula is true for m - 1 with $m \ge 2$ and let $k_{i,0} + k_{i,1} + \cdots + k_{i,m-2} = k'_i$ be fixed for $i = 2, \ldots, r$. Then

$$\sum_{\substack{k_{i,0},\dots,k_{i,m-2}\\i=2,\dots,r}} \prod_{j=0}^{m-2} \frac{n_1}{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}} \binom{n_1 - k_{2,j} - \dots - (r-1)k_{r,j}}{k_{2,j},\dots,k_{r,j}}$$

$$= \frac{(m-1)n_1}{(m-1)n_1 - k'_2 - \dots - (r-1)k'_r} \binom{(m-1)n_1 - k'_2 - \dots - (r-1)k'_r}{k'_2,\dots,k'_r}.$$

Plugging this into (3.2) yields

$$f_{m,n}(k_1, \dots, k_r) = \sum_{\substack{k'_i + k_{i,m-1} = k_i \\ i = 2, \dots, r}} \frac{(m-1)n_1}{(m-1)n_1 - k'_2 - \dots - (r-1)k'_r} \binom{(m-1)n_1 - k'_2 - \dots - (r-1)k'_r}{k'_2, \dots, k'_r} \times \frac{n_1}{n_1 - k_{2,m-1} - \dots - (r-1)k_{r,m-1}} \binom{n_1 - k_{2,m-1} - \dots - (r-1)k_{r,m-1}}{k_{2,m-1}, \dots, k_{r,m-1}}$$

which is (3.4) by applying Raney-Mohanty's identity (2.2).

4. Proof of Theorem 1.4

For the case $\Delta = 0$ or $\Delta = -m$, the number $f_{m,n}(k_1, \ldots, k_r)$ is again given by (3.2). However, we will meet with $n_1 - k_{2,j} - \cdots - (r-1)k_{r,j} \leq 0$ for some $0 \leq j \leq m-1$ in some nonnegative integral solutions $(k_{i,j})$ to (3.1). It is convenient here to consider a more general form of (3.2) as follows. For any variable x, let $f_{m,n}(x; k_1, \ldots, k_r)$ be the following expression

$$\sum_{\substack{(k_{i,j})}} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \dots - (r-1)k_{r,j}} {x \choose k_{2,j}, \dots, k_{r,j}},$$

where $(k_{i,j})$ ranges over the same integral matrices as (3.2).

Let M be the set of all nonnegative integral matrices $(k_{i,j})_{\substack{2 \leqslant i \leqslant r \\ 0 \leqslant j \leqslant m-1}}$ satisfying (3.1), and let S be the set of all $(k_{i,j})$ in M such that (3.3) does not hold. Then

$$f_{m,n}(x; k_1, \dots, k_r) = \sum_{(k_{i,j}) \in M} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \dots - (r-1)k_{r,j}} \binom{x - k_{2,j} - \dots - (r-1)k_{r,j}}{k_{2,j}, \dots, k_{r,j}} - \sum_{(k_{i,j}) \in S} \prod_{j=0}^{m-1} \frac{x}{x - k_{2,j} - \dots - (r-1)k_{r,j}} \binom{x - k_{2,j} - \dots - (r-1)k_{r,j}}{k_{2,j}, \dots, k_{r,j}}.$$

$$(4.1)$$

When $\Delta = 0$, we have $n_1 = k_2 + 2k_3 + \cdots + (r-1)k_r$, and S reduces to

 $S_1 := \{(k_{i,j}): \text{ for some } j_0 \text{ and all } i, \text{ we have } k_{i,j_0} = k_i \text{ and } k_{i,j} = 0 \text{ if } j \neq j_0\}$.

So the second summation on the right-hand side of (4.1) becomes

$$\frac{mx}{x-n_1} \binom{x-n_1}{k_2,\ldots,k_r},$$

while the first summation can be summed by using Raney-Mohanty's identity. It follows that

$$f_{m,n}(x; k_1, \dots, k_r) = \frac{mx}{mx - k_2 - \dots - (r-1)k_r} {mx - k_2 - \dots - (r-1)k_r \choose k_2, \dots, k_r} - \frac{mx}{x - n_1} {x - n_1 \choose k_2, \dots, k_r}.$$
(4.2)

Letting $x = n_1$ in (4.2) and noticing the following fact

$$\lim_{z \to 0} \frac{1}{z} \binom{z}{a_1, \dots, a_s} = \frac{(-1)^{a_1 + \dots + a_s - 1}}{a_1 + \dots + a_s} \binom{a_1 + \dots + a_s}{a_1, \dots, a_s}, \tag{4.3}$$

one obtains the first formula in Theorem 1.4.

When $\Delta = -m$, we have $n_1 = k_2 + 2k_3 + \cdots + (r-1)k_r - 1$. If $k_2 = 0$, then $S = S_1$, while if $k_2 > 0$, then

$$S = S_1 \cup \{(k_{i,j}): \text{ for some } j_0 \neq j_1, \text{ we have } k_{2,j_0} = k_2 - 1, k_{2,j_1} = 1,$$

 $k_{i,j_0} = k_i \ (2 < i \le r) \text{ and } k_{i,j} = 0 \text{ otherwise}\}.$

It follows that

$$f_{m,n}(x; k_1, \dots, k_r) = \frac{mx}{mx - k_2 - \dots - (r-1)k_r} {mx - k_2 - \dots - (r-1)k_r \choose k_2, \dots, k_r} - \frac{mx}{x - n_1 - 1} {x - n_1 - 1 \choose k_2, \dots, k_r} - \chi(k_2 > 0) \frac{m(m-1)x^2}{x - n_1} {x - n_1 \choose k_2 - 1, k_3, \dots, k_r}.$$
(4.4)

Letting $x = n_1$ in (4.4) and using (4.3) and

$$\binom{-1}{a_1,\ldots,a_s} = (-1)^{a_1+\cdots+a_s} \binom{a_1+\cdots+a_s}{a_1,\ldots,a_s},$$

we obtain the second formula in Theorem 1.4.

Remark. It is also possible to compute $f_{m,n}(k_1,\ldots,k_r)$ for the case $\Delta=-2\gcd(m,n)$ or $\Delta=-3\gcd(m,n)$. But the result is more complicated and is omitted here.

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