# On arithmetic partitions of $\mathbb{Z}_{n}$ 

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#### Abstract

Generalizing a classical problem in enumerative combinatorics, Mansour and Sun counted the number of subsets of $\mathbb{Z}_{n}$ without certain separations. Chen, Wang, and Zhang then studied the problem of partitioning $\mathbb{Z}_{n}$ into arithmetical progressions of a given type under some technical conditions. In this paper, we improve on their main theorems by applying a convolution formula for cyclic multinomial coefficients due to Raney-Mohanty.


Keywords: cycle dissection, m-AP-partition, cyclic multinomial coefficient, Raney-Mohanty's identity

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## 1. Introduction

In his solution of problème des ménages Kaplansky [10] showed that the number of ways of selecting $k$ elements, no two consecutive, from $n$ objects arrayed on a cycle is $\frac{n}{n-k}\binom{n-k}{k}$. Let $\mathbb{Z}_{n}:=\{0,1, \ldots, n-1\}$ be the set of congruence classes modulo $n$ with usual arithmetic. Then Yamamoto [22] (see also [19, p. 222]) proved that if $n \geqslant p k+1$ the number of ways of selecting $k$ elements from $\mathbb{Z}_{n}$, no two consecutive, is

$$
\begin{equation*}
\frac{n}{n-p k}\binom{n-p k}{k} \tag{1.1}
\end{equation*}
$$

when $i \pm 1, \ldots, i \pm p$ are regarded as consecutive to $i$.
In the last three decades a lot of generalizations and variations of Kaplansky's problem have been studied by several authors (see, for example, $[4,7-9,11,12,15-17,20]$ ). In particular, Konvalina [12] considered the number of $k$-subsets $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ of $\mathbb{Z}_{n}$ such that $x_{i}-x_{j} \neq 2$ for all $1 \leqslant i, j \leqslant k$, and found that the answer is $\frac{n}{n-k}\binom{n-k}{k}$ if $n \geqslant 2 k+1$. Hwang [8] then generalized Konvalina's result to the case $x_{i}-x_{j} \neq m$ and deduced that the desired number is given by the same formula if $n \geqslant m k+1$. Recently, Mansour and Sun [13] gave the following unification of Yamamoto's and Hwang's formulas.

Theorem 1.1 (Mansour-Sun). Let $m, n, p, k$ be positive integers such that $n \geqslant m p k+1$. Then the number of $k$-subsets $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ of $\mathbb{Z}_{n}$ such that

$$
\begin{equation*}
x_{i}-x_{j} \notin\{m, 2 m, \ldots, p m\} \quad(1 \leqslant i, j \leqslant k), \tag{1.2}
\end{equation*}
$$

is also given by (1.1).
A short proof of Theorem 1.1 was given by Guo [5] by using Rothe's identity. In order to generalize Mansour-Sun's result, Chen, Wang, and Zhang [3] defined an $m$-AP-block of length $k$ to be a sequence $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ of distinct elements in $\mathbb{Z}_{n}$ such that $x_{i+1}-x_{i}=m$ for $1 \leqslant i \leqslant k-1$ and studied the problem of partitioning $\mathbb{Z}_{n}$ into $m$-AP-blocks. The type of such a partition is defined to be the type of the multiset of the lengths of the blocks. For example, the following is a 3-AP-partition of $\mathbb{Z}_{20}$ of type $1^{4} 2^{3} 3^{2} 4^{1}$ :

$$
(2),(4,7),(5,8),(6),(9,12,15),(10),(11),(13,16,19),(14,17,0,3),(18,1)
$$

We need to emphasize that $(x, x+m, \ldots, x+(n-1) m)$ and $(x+m, x+2 m, \ldots, x+$ $(n-1) m, x)$ and so on are deemed as different $m$-AP-blocks in $\mathbb{Z}_{m n}$. For example, all the 2-AP-partitions of $\mathbb{Z}_{6}$ of type $3^{2}$ are

$$
\{(i, i+2, i+4),(j+1, j+3, j+5)\}_{i, j=0,2,4} .
$$

Chen, Wang, and Zhang [3] constructed a bijection between $m$-AP-partitions and $m^{\prime}$-APpartitions of $\mathbb{Z}_{n}$ under some technical conditions, and established the following theorem.

Theorem 1.2 (Chen-Wang-Zhang). Let $m, n, k_{1}, k_{2}, \ldots, k_{r}$ and $i_{2}, \ldots, i_{r}$ be positive integers such that $1<i_{2}<\cdots<i_{r}$ and

$$
\begin{equation*}
k_{1}>\left(k_{2}+\cdots+k_{r}\right)\left((m-1)\left(i_{r}-1\right)-1\right) \tag{1.3}
\end{equation*}
$$

Then the number of partitions of $\mathbb{Z}_{n}$ into $m$-AP-blocks of type $1^{k_{1}} i_{2}^{k_{2}} \cdots i_{r}^{k_{r}}$ does not depend on $m$, and is given by the cyclic multinomial coefficient

$$
\begin{equation*}
\frac{n}{k_{1}+\cdots+k_{r}}\binom{k_{1}+\cdots+k_{r}}{k_{1}, \ldots, k_{r}} . \tag{1.4}
\end{equation*}
$$

If we specialize the type to $1^{n-(p+1) k}(p+1)^{k}$, then the condition (1.3) becomes $n \geqslant$ $m p k+1$. Furthermore, if $\left(x_{1}, x_{1}+m, \ldots, x_{1}+p m\right), \ldots,\left(x_{k}, x_{k}+m, \ldots, x_{k}+p m\right)$ are the $k$ blocks of length $p+1$ in an $m$-AP-partition of $\mathbb{Z}_{n}$ of type $1^{n-(p+1) k}(p+1)^{k}$, then the set $\left\{x_{1}, \ldots, x_{k}\right\}$ satisfies (1.2), and vice versa. Therefore Theorem 1.2 implies Theorem 1.1.

In this paper we shall improve and complete Theorem 1.2 by establishing the following two theorems.

Theorem 1.3. Let $m, n$ be positive integers, and let $k_{1}, k_{2}, \ldots, k_{r}$ be nonnegative integers such that $n=k_{1}+2 k_{2}+\cdots+r k_{r}$. Let $d=\operatorname{gcd}(m, n)$. If

$$
\begin{equation*}
\Delta:=n-d\left(n-k_{1}-\cdots-k_{r}\right)>0 \tag{1.5}
\end{equation*}
$$

then the number of partitions of $\mathbb{Z}_{n}$ into $m$-AP-blocks of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}$ is given by (1.4).

It is not hard to see that the condition (1.5) is weaker than (1.3), i.e., the condition (1.3) implies that (1.5). In other words, for fixed $n$ and a given type, there are in general many more $m$ 's satisfying (1.5) than satisfying (1.3). For example, by Theorem 1.3, the numbers of $m$-AP-partitions of $\mathbb{Z}_{120}$ of type $1^{89} 2^{3} 3^{2} 5^{1} 7^{2}$ are all equal for

$$
\begin{aligned}
m= & 1,2,3,4,5,7,9,11,13,14,17,19,21,22,23,25,26,27,28,29,31, \\
& 33,34,35,37,38,39,41,43,44,46,47,49,51,52,53,55,57,58,59,
\end{aligned}
$$

i.e., for $d=1,2,3,4,5$. However, Theorem 1.2 only asserts that these numbers for $m=$ $1,2,3$ are equal.
Theorem 1.4. Let $k_{1}, k_{2}, \ldots, k_{r}, m, n, d$ and $\Delta$ be given as in Theorem 1.3. Then the number of partitions of $\mathbb{Z}_{n}$ into $m$-AP-blocks of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}$ is given by

$$
\left\{\begin{array}{c}
\frac{n}{k_{1}+\cdots+k_{r}}\binom{k_{1}+\cdots+k_{r}}{k_{1}, \ldots, k_{r}}+\frac{n(-1)^{k_{2}+\cdots+k_{r}}}{k_{2}+\cdots+k_{r}}\binom{k_{2}+\cdots+k_{r}}{k_{2}, \ldots, k_{r}}, \quad \text { if } \Delta=0, \\
\frac{n}{k_{1}+\cdots+k_{r}}\binom{k_{1}+\cdots+k_{r}}{k_{1}, \ldots, k_{r}}+\boldsymbol{\mu}(-1)^{k_{2}+\cdots+k_{r}}\binom{k_{2}+\cdots+k_{r}}{k_{2}, \ldots, k_{r}}, \quad \text { if } \Delta=-d,
\end{array}\right.
$$

where

$$
\dot{\&}= \begin{cases}n, & \text { if } k_{2}=0, \\ n\left(1-\frac{n\left(1-d^{-1}\right) k_{2}}{\left(k_{2}+\cdots+k_{r}\right)\left(k_{2}+\cdots+k_{r}-1\right)}\right), & \text { if } k_{2}>0 .\end{cases}
$$

When the type in Theorem 1.4 is $1^{n-(p+1) k}(p+1)^{k}$ again, then $\Delta=n-m p k$. To assure that there is an $m$-AP-block of length $p+1$ in $\mathbb{Z}_{n}$, we need to assume that $n>p m$, which is equivalent to $k \geqslant 2$ if $\Delta=0$ and $p k>p+1$ if $\Delta=-m$. As mentioned after Theorem 1.2, each family of $k m$-AP-blocks in $\mathbb{Z}_{n}$ is in one-to-one correspondence with a $k$-subset of $\mathbb{Z}_{n}$ satisfying (1.2), we derive the following two results, which can be viewed as complements to Theorem 1.1.

Corollary 1.5. Let $m, p \geqslant 1, k \geqslant 2$ and $n=m p k$. Then the number of $k$-subsets $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ of $\mathbb{Z}_{n}$ such that $x_{i}-x_{j} \notin\{m, 2 m, \ldots, p m\}$ for all $1 \leqslant i, j \leqslant k$, is given by

$$
\begin{equation*}
\frac{n}{n-p k}\binom{n-p k}{k}+(-1)^{k} \frac{n}{k} . \tag{1.6}
\end{equation*}
$$

Actually the above formula is deduced for $m \geqslant 2$, i.e., $n \geqslant(p+1) k$, but it also holds for $m=1$ if we take the convention

$$
\lim _{x \rightarrow 0} \frac{n}{x}\binom{x}{k}=(-1)^{k-1} \frac{n}{k},
$$

and so (1.6) is equal to 0 in this case. Here is an example for Corollary 1.5. For $m=p=$ $k=2$, the number of 2 -subsets $\left\{x_{1}, x_{2}\right\}$ of $\mathbb{Z}_{8}$ such that $x_{1}-x_{2}, x_{2}-x_{1} \notin\{2,4\}$ is equal to

$$
\frac{8}{4}\binom{4}{2}+4=16
$$

and the corresponding subsets are $\{i, i+1\}$ and $\{i, i+3\}$, where $i \in \mathbb{Z}_{8}$.
Corollary 1.6. Let $m, p, k \geqslant 1$ with $p k>p+1$ and let $n=m p k-m$. Then the number of $k$-subsets $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ of $\mathbb{Z}_{n}$ such that $x_{i}-x_{j} \notin\{m, 2 m, \ldots, p m\}$ for all $1 \leqslant i, j \leqslant k$, is given by

$$
\begin{cases}\frac{n}{n-k}\binom{n-k}{k}+(-1)^{k-1} n(m-2), & \text { if } p=1 \\ \frac{n}{n-p k}\binom{n-p k}{k}+(-1)^{k} n, & \text { if } p \geqslant 2\end{cases}
$$

Similarly, although the above formula is deduced for $m p k-m \geqslant(p+1) k$, it also holds without this condition. The details are left to the interested reader.
Remark. For $0<m<n$, let $g_{m}(n, k)$ denote the number of $k$-subsets $\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$ of $\mathbb{Z}_{n}$ such that $x_{i}-x_{j} \neq m$ for all $1 \leqslant i, j \leqslant k$. Hwang [8, Corollary 2] obtained

$$
\begin{equation*}
g_{m}(n, k)=\sum_{j=0}^{\lfloor d / 2\rfloor}(-1)^{n j / d}\binom{d}{j} \frac{n-2 n j / d}{n-k-n j / d}\binom{n-k-n j / d}{k-n j / d}, \tag{1.7}
\end{equation*}
$$

where $d=\operatorname{gcd}(m, n)$. Letting $n=m k$ or $n=m k-m$ in (1.7), we are led to the $p=1$ case of Corollaries 1.5 or 1.6. However, since there are two cases in Corollary 1.6, it seems impossible to give a formula like (1.7) to unify Corollaries 1.5 and 1.6 for general $p$.

We recall and establish some necessary lemmas in Section 2 and prove Theorems 1.3 and 1.4 in Sections 3 and 4, respectively. Our main idea is the following: Lemma 2.4 permits us to reduce the general $m$-AP-partition problem of $\mathbb{Z}_{n}$ to the case where $m$ divides $n$. For the latter we may write the partition number as a multiple sum, which can be computed by applying Raney-Mohanty's identity.

## 2. Some lemmas

A dissection of an n-cycle is a 1-AP-partition of $\mathbb{Z}_{n}$, which can be depicted by inserting a bar between any two consecutive blocks on an $n$-cycle. For example, Figure 1 illustrates a 20 -cycle dissection of type $1^{4} 2^{3} 3^{2} 4^{1}$. It is easy to see that the number of dissections of $\mathbb{Z}_{n}$ is given by (1.4). Indeed, deleting the segment containing 0 in any dissection of $n$-cycle of type $1^{k_{1}} 2^{k_{2}} \ldots r^{k_{r}}$ yields a dissection of a $(n-i)$-line of type $1^{k_{1}} \ldots i^{k_{i}-1} \ldots r^{k_{r}}$ if the segment containing 0 is of length $i(1 \leqslant i \leqslant n)$. So the number of such dissections of $n$-cycle is equal to

$$
\begin{equation*}
i\binom{k_{1}+\cdots+\left(k_{i}-1\right)+\cdots+k_{r}}{k_{1}, \ldots, k_{i}-1, \ldots, k_{r}} . \tag{2.1}
\end{equation*}
$$

Summing (2.1) over all $i$ yields the following known result (see [2, Lemma 3.1]).
Lemma 2.1 (Chen-Lih-Yeh). For an $n$-cycle, the number of dissections of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}$ is given by the cyclic multinomial coefficient (1.4).


Figure 1: A 20-cycle dissection of type $1^{4} 2^{3} 3^{2} 4^{1}$.

For any variable $x$ and nonnegative integers $k_{1}, \ldots, k_{r}$ define the multinomial coefficient

$$
\binom{x}{k_{1}, k_{2}, \ldots, k_{r}}:=\frac{x(x-1) \cdots\left(x-k_{1}-\cdots-k_{r}+1\right)}{k_{1}!k_{2}!\cdots k_{r}!} .
$$

Note that when $x=k_{1}+k_{2}+\ldots+k_{r}$ the above definition coincides with the classical definition of multinomial coefficient and

$$
\binom{k_{1}+\cdots+k_{r}}{k_{1}, \ldots, k_{r}}=\binom{k_{1}+\cdots+k_{r}}{k_{2}, \ldots, k_{r}} .
$$

The following convolution formula for multinomial coefficients is due to Raney-Mohanty $[14,18]$. For other proofs of (2.2), we refer the reader to $[6,21,23]$.

Lemma 2.2 (Raney-Mohanty's identity). For any variables $x, y, z_{1}, \ldots, z_{m}$ and nonnegative integers $N_{1}, \ldots, N_{m}$, there holds

$$
\begin{align*}
& \sum_{\substack{0 \leqslant t_{i} \leqslant N_{i} \\
i=1, \ldots, m}} \frac{x}{x-t_{1} z_{1}-\cdots-t_{m} z_{m}}\binom{x-t_{1} z_{1}-\cdots-t_{m} z_{m}}{t_{1}, \ldots, t_{m}} \\
& \quad \times \frac{y}{y-\left(N_{1}-t_{1}\right) z_{1}-\cdots-\left(N_{m}-t_{m}\right) z_{m}}\binom{y-\left(N_{1}-t_{1}\right) z_{1}-\cdots-\left(N_{m}-t_{m}\right) z_{m}}{N_{1}-t_{1}, \ldots, N_{m}-t_{m}} \\
& =\frac{x+y}{x+y-N_{1} z_{1}-\cdots-N_{m} z_{m}}\binom{x+y-N_{1} z_{1}-\cdots-N_{m} z_{m}}{N_{1}, \ldots, N_{m}} . \tag{2.2}
\end{align*}
$$

We also need the following elementary arithmetical result (see [1, Theorem 5.32 and Exercise 16 on page 127] or [5]).

Lemma 2.3. Let $m, n$ be positive integers. If $\operatorname{gcd}(m, n)=d$, then there exists an integer a such that $\operatorname{gcd}(a, n)=1$ and $a m \equiv d(\bmod n)$.

The following is our key lemma.

Lemma 2.4. If $m, n \geqslant 1$ and $\operatorname{gcd}(m, n)=d$, then there is a bijection from the set of $m$-AP-partitions of $\mathbb{Z}_{n}$ to the set of $d$ - $A P$-partitions of $\mathbb{Z}_{n}$. Moreover this bijection keeps the type of partitions.

Proof. By Lemma 2.3, there exists an inversible element $a \in \mathbb{Z}_{n}$ such that $a m=d$. Let $a^{-1}$ be the inverse of $a$. For any subset $B$ of $\mathbb{Z}_{n}$ and $x \in \mathbb{Z}_{n}$, let $x B=\{x b: b \in$ $B\}$. If $\left\{B_{1}, B_{2}, \ldots, B_{s}\right\}$ is an $m$-AP-partition of $\mathbb{Z}_{n}$, then $\left\{a B_{1}, a B_{2}, \ldots, a B_{s}\right\}$ is a $d$ -AP-partition of $\mathbb{Z}_{n}$. Conversely, if $\left\{C_{1}, C_{2}, \ldots, C_{s}\right\}$ is a d-AP-partition of $\mathbb{Z}_{n}$, then $\left\{a^{-1} C_{1}, a^{-1} C_{2}, \ldots, a^{-1} C_{s}\right\}$ is an $m$-AP-partition of $\mathbb{Z}_{n}$. Obviously, this correspondence keeps the type of partitions. This proves the lemma.

It follows from Lemma 2.4 that if there exists an $m$-AP-partition of $\mathbb{Z}_{n}$ of a given type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}\left(k_{r}>0\right)$ then $\operatorname{gcd}(m, n) r \leqslant n$.

## 3. Proof of Theorem 1.3

By Lemma 2.4, it suffices to consider the case where $m$ divides $n$, i.e., $d=m$. Let $n=m n_{1}$ and divide $\mathbb{Z}_{n}$ into $m$ subsets of the same cardinality $n_{1}$ :

$$
\mathbb{Z}_{n, j}=\left\{m i+j: i=0, \ldots, n_{1}-1\right\}, \quad 0 \leqslant j \leqslant m-1 .
$$

Hence $\mathbb{Z}_{n}=\biguplus_{j=0}^{m-1} \mathbb{Z}_{n, j}$. Let $\mathcal{B}=\left\{B_{1}, B_{2} \ldots, B_{s}\right\}$ be an $m$-AP-partition of $\mathbb{Z}_{n}$ of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}\left(r \leqslant n_{1}\right)$. Then $B_{i, j}=\mathbb{Z}_{n, j} \cap B_{i}$ is equal to $\emptyset$ or $B_{i}$ for $1 \leqslant i \leqslant s$ and $0 \leqslant j \leqslant m-1$. Furthermore, since the transformation $x \mapsto(x-j) / m$ maps each $m$ -AP-block $B_{i, j}$ of $\mathbb{Z}_{n, j}(0 \leqslant j \leqslant m-1)$ to a 1 -AP-block $B_{i, j}^{\prime}$ of $\mathbb{Z}_{n_{1}}$, each $m$-AP-partition $\mathcal{B}_{j}=\left\{B_{1, j}, \ldots, B_{s, j}\right\}$ corresponds bijectively to a 1-AP-partition $\mathcal{B}_{j}^{\prime}$ of $\mathbb{Z}_{n_{1}}$ with the same type. Thus, we have established a bijection between the set of $m$-AP-partitions of $\mathbb{Z}_{n}$ and the set of $m$-tuples of 1-AP-partitions of $\mathbb{Z}_{n_{1}}: \mathcal{B} \leftrightarrow\left(\mathcal{B}_{0}^{\prime}, \ldots, \mathcal{B}_{m-1}^{\prime}\right)$.

Now assume that the $m$-AP-partition $\mathcal{B}$ is of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}\left(r \leqslant n_{1}\right)$, and the corresponding 1-AP-partition $\mathcal{B}_{j}^{\prime}$ is of type $1^{k_{1, j}} 2^{k_{2, j}} \cdots r^{k_{r, j}}(0 \leqslant j \leqslant m-1)$. Clearly,

$$
\left\{\begin{array}{l}
k_{2,0}+k_{2,1}+\cdots+k_{2, m-1}=k_{2},  \tag{3.1}\\
k_{3,0}+k_{3,1}+\cdots+k_{3, m-1}=k_{3}, \\
\cdots \\
k_{r, 0}+k_{r, 1}+\cdots+k_{r, m-1}=k_{r} .
\end{array}\right.
$$

By Lemma 2.1 and noticing that $n_{1}=k_{1, j}+2 k_{2, j}+\cdots+r k_{r, j}$, the number of 1-APpartitions of $\mathbb{Z}_{n_{1}}$ of type $1^{k_{1, j}} 2^{k_{2, j}} \cdots r^{k_{r, j}}$ is equal to

$$
\begin{aligned}
& \frac{n_{1}}{k_{1, j}+k_{2, j}+\cdots+k_{r, j}}\binom{k_{1, j}+k_{2, j}+\cdots+k_{r, j}}{k_{1, j}, k_{2, j}, \ldots, k_{r, j}} \\
& =\frac{n_{1}}{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}} .
\end{aligned}
$$

For $m, n \geqslant 1$ let $f_{m, n}\left(k_{1}, \ldots, k_{r}\right)$ be the number of partitions of $\mathbb{Z}_{n}$ into $m$-AP-blocks of type $1^{k_{1}} 2^{k_{2}} \cdots r^{k_{r}}$. Then

$$
\begin{equation*}
f_{m, n}\left(k_{1}, \ldots, k_{r}\right)=\sum_{\left(k_{i, j}\right)} \prod_{j=0}^{m-1} \frac{n_{1}}{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}}, \tag{3.2}
\end{equation*}
$$

where the summation is over all matrices $\left(k_{i, j}\right)_{\substack{2 \leqslant i \leqslant r \\ 0 \leqslant j \leqslant m-1}}$ of nonnegative integral coefficients $k_{i, j}$ satisfying (3.1) and

$$
\left\{\begin{array}{l}
n_{1}-k_{2,0}-\cdots-(r-1) k_{r, 0}>0  \tag{3.3}\\
n_{1}-k_{2,1}-\cdots-(r-1) k_{r, 1}>0 \\
\cdots \\
n_{1}-k_{2, m-1}-\cdots-(r-1) k_{r, m-1}>0
\end{array}\right.
$$

Recall that

$$
\Delta=n-m\left(n-k_{1}-\cdots-k_{r}\right)=m n_{1}-m\left(k_{2}+\cdots+(r-1) k_{r}\right) .
$$

If $\Delta>0$, then we have $n_{1}>k_{2}+\cdots+(r-1) k_{r}$, and thus all nonnegative integral solutions to (3.1) also satisfy (3.3) as $k_{i} \geqslant k_{i, j}(2 \leqslant i \leqslant r, 0 \leqslant j \leqslant m-1)$.

It remains to prove that the right-hand side of (3.2) is equal to (1.4), namely

$$
\begin{equation*}
\frac{m n_{1}}{m n_{1}-k_{2}-\cdots-(r-1) k_{r}}\binom{m n_{1}-k_{2}-\cdots-(r-1) k_{r}}{k_{2}, \ldots, k_{r}} . \tag{3.4}
\end{equation*}
$$

We proceed by induction on $m \geqslant 1$. This is equivalent to repeatedly applying RaneyMohanty's identity (2.2). The case $m=1$ is obviously true. Suppose that the formula is true for $m-1$ with $m \geqslant 2$ and let $k_{i, 0}+k_{i, 1}+\cdots+k_{i, m-2}=k_{i}^{\prime}$ be fixed for $i=2, \ldots, r$. Then

$$
\begin{aligned}
& \sum_{\substack{k_{i, 0}, \ldots, k_{i, m-2} \\
i=2, \ldots, r}} \prod_{j=0}^{m-2} \frac{n_{1}}{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}} \\
= & \frac{(m-1) n_{1}}{(m-1) n_{1}-k_{2}^{\prime}-\cdots-(r-1) k_{r}^{\prime}}\binom{(m-1) n_{1}-k_{2}^{\prime}-\cdots-(r-1) k_{r}^{\prime}}{k_{2}^{\prime}, \ldots, k_{r}^{\prime}} .
\end{aligned}
$$

Plugging this into (3.2) yields

$$
\begin{aligned}
& f_{m, n}\left(k_{1}, \ldots, k_{r}\right) \\
& =\sum_{\substack{k_{i}^{\prime}+k_{i, m-1}=k_{i} \\
i=2, \ldots, r}} \frac{(m-1) n_{1}}{(m-1) n_{1}-k_{2}^{\prime}-\cdots-(r-1) k_{r}^{\prime}}\binom{(m-1) n_{1}-k_{2}^{\prime}-\cdots-(r-1) k_{r}^{\prime}}{k_{2}^{\prime}, \ldots, k_{r}^{\prime}} \\
& \quad \times \frac{n_{1}}{n_{1}-k_{2, m-1}-\cdots-(r-1) k_{r, m-1}}\binom{n_{1}-k_{2, m-1}-\cdots-(r-1) k_{r, m-1}}{k_{2, m-1}, \ldots, k_{r, m-1}}
\end{aligned}
$$

which is (3.4) by applying Raney-Mohanty's identity (2.2).

## 4. Proof of Theorem 1.4

For the case $\Delta=0$ or $\Delta=-m$, the number $f_{m, n}\left(k_{1}, \ldots, k_{r}\right)$ is again given by (3.2). However, we will meet with $n_{1}-k_{2, j}-\cdots-(r-1) k_{r, j} \leqslant 0$ for some $0 \leqslant j \leqslant m-1$ in some nonnegative integral solutions $\left(k_{i, j}\right)$ to (3.1). It is convenient here to consider a more general form of (3.2) as follows. For any variable $x$, let $f_{m, n}\left(x ; k_{1}, \ldots, k_{r}\right)$ be the following expression

$$
\sum_{\left(k_{i, j}\right)} \prod_{j=0}^{m-1} \frac{x}{x-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{x-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}}
$$

where $\left(k_{i, j}\right)$ ranges over the same integral matrices as (3.2).
Let $M$ be the set of all nonnegative integral matrices $\left(k_{i, j}\right) \substack{2 \leqslant i \leqslant r \\ 0 \leqslant j \leqslant m-1} ~$ satisfying (3.1), and let $S$ be the set of all $\left(k_{i, j}\right)$ in $M$ such that (3.3) does not hold. Then

$$
\begin{align*}
& f_{m, n}\left(x ; k_{1}, \ldots, k_{r}\right) \\
& =\sum_{\left(k_{i, j}\right) \in M} \prod_{j=0}^{m-1} \frac{x}{x-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{x-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}} \\
& \quad-\sum_{\left(k_{i, j}\right) \in S} \prod_{j=0}^{m-1} \frac{x}{x-k_{2, j}-\cdots-(r-1) k_{r, j}}\binom{x-k_{2, j}-\cdots-(r-1) k_{r, j}}{k_{2, j}, \ldots, k_{r, j}} . \tag{4.1}
\end{align*}
$$

When $\Delta=0$, we have $n_{1}=k_{2}+2 k_{3}+\cdots+(r-1) k_{r}$, and $S$ reduces to

$$
S_{1}:=\left\{\left(k_{i, j}\right): \text { for some } j_{0} \text { and all } i, \text { we have } k_{i, j_{0}}=k_{i} \text { and } k_{i, j}=0 \text { if } j \neq j_{0}\right\}
$$

So the second summation on the right-hand side of (4.1) becomes

$$
\frac{m x}{x-n_{1}}\binom{x-n_{1}}{k_{2}, \ldots, k_{r}},
$$

while the first summation can be summed by using Raney-Mohanty's identity. It follows that

$$
\begin{align*}
& f_{m, n}\left(x ; k_{1}, \ldots, k_{r}\right) \\
& \quad=\frac{m x}{m x-k_{2}-\cdots-(r-1) k_{r}}\binom{m x-k_{2}-\cdots-(r-1) k_{r}}{k_{2}, \ldots, k_{r}}-\frac{m x}{x-n_{1}}\binom{x-n_{1}}{k_{2}, \ldots, k_{r}} . \tag{4.2}
\end{align*}
$$

Letting $x=n_{1}$ in (4.2) and noticing the following fact

$$
\begin{equation*}
\lim _{z \rightarrow 0} \frac{1}{z}\binom{z}{a_{1}, \ldots, a_{s}}=\frac{(-1)^{a_{1}+\cdots+a_{s}-1}}{a_{1}+\cdots+a_{s}}\binom{a_{1}+\cdots+a_{s}}{a_{1}, \ldots, a_{s}}, \tag{4.3}
\end{equation*}
$$

one obtains the first formula in Theorem 1.4.

When $\Delta=-m$, we have $n_{1}=k_{2}+2 k_{3}+\cdots+(r-1) k_{r}-1$. If $k_{2}=0$, then $S=S_{1}$, while if $k_{2}>0$, then

$$
\begin{array}{r}
S=S_{1} \cup\left\{\left(k_{i, j}\right): \text { for some } j_{0} \neq j_{1}, \text { we have } k_{2, j_{0}}=k_{2}-1, k_{2, j_{1}}=1\right. \\
\left.k_{i, j_{0}}=k_{i}(2<i \leqslant r) \text { and } k_{i, j}=0 \text { otherwise }\right\}
\end{array}
$$

It follows that

$$
\begin{align*}
& f_{m, n}\left(x ; k_{1}, \ldots, k_{r}\right) \\
& \quad=\frac{m x}{m x-k_{2}-\cdots-(r-1) k_{r}}\binom{m x-k_{2}-\cdots-(r-1) k_{r}}{k_{2}, \ldots, k_{r}} \\
& \quad-\frac{m x}{x-n_{1}-1}\binom{x-n_{1}-1}{k_{2}, \ldots, k_{r}}-\chi\left(k_{2}>0\right) \frac{m(m-1) x^{2}}{x-n_{1}}\binom{x-n_{1}}{k_{2}-1, k_{3}, \ldots, k_{r}} . \tag{4.4}
\end{align*}
$$

Letting $x=n_{1}$ in (4.4) and using (4.3) and

$$
\binom{-1}{a_{1}, \ldots, a_{s}}=(-1)^{a_{1}+\cdots+a_{s}}\binom{a_{1}+\cdots+a_{s}}{a_{1}, \ldots, a_{s}}
$$

we obtain the second formula in Theorem 1.4.
Remark. It is also possible to compute $f_{m, n}\left(k_{1}, \ldots, k_{r}\right)$ for the case $\Delta=-2 \operatorname{gcd}(m, n)$ or $\Delta=-3 \operatorname{gcd}(m, n)$. But the result is more complicated and is omitted here.
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