# ON TREE-PARTITION-WIDTH 

DAVID R. WOOD


#### Abstract

A tree-partition of a graph $G$ is a proper partition of its vertex set into 'bags', such that identifying the vertices in each bag produces a forest. The tree-partition-width of $G$ is the minimum number of vertices in a bag in a tree-partition of $G$. An anonymous referee of the paper by Ding and Oporowski [J. Graph Theory, 1995] proved that every graph with tree-width $k \geq 3$ and maximum degree $\Delta \geq 1$ has tree-partition-width at most $24 k \Delta$. We prove that this bound is within a constant factor of optimal. In particular, for all $k \geq 3$ and for all sufficiently large $\Delta$, we construct a graph with tree-width $k$, maximum degree $\Delta$, and tree-partition-width at least $\left(\frac{1}{8}-\epsilon\right) k \Delta$. Moreover, we slightly improve the upper bound to $\frac{5}{2}(k+1)\left(\frac{7}{2} \Delta-1\right)$ without the restriction that $k \geq 3$.


## 1. Introduction

A graph $H$ is a partition of a graph $G$ if:

- each vertex of $H$ is a set of vertices of $G$ (called a $b a g)$,
- every vertex of $G$ is in exactly one bag of $H$, and
- distinct bags $A$ and $B$ are adjacent in $H$ if and only if some edge of $G$ has one endpoint in $A$ and the other endpoint in $B$.

The width of a partition is the maximum number of vertices in a bag. Informally speaking, the graph $H$ is obtained from a proper partition of $V(G)$ by identifying the vertices in each part, deleting loops, and replacing parallel edges by a single edge.

If a forest $T$ is a partition of a graph $G$, then $T$ is a tree-partition of $G$. The tree-partition-width ${ }^{2}$ of $G$, denoted by $\operatorname{tpw}(G)$, is the minimum width of a tree-partition of $G$. Tree-partitions were independently introduced by Seese [23] and Halin [19], and have since been widely investigated [6, 7, 12, 13, 17, 24]. Applications of tree-partitions include graph drawing [9, 14, 15, 25], graph colouring [2], partitioning graphs into subgraphs with only small components [1], monadic second-order logic [20], and network emulations [3, 4, 8, 18]. Planar-partitions and other more general structures have also recently been studied [11, 25].

[^0]What bounds can be proved on the tree-partition-width of a graph? Let $\operatorname{tw}(G)$ denote the tree-width ${ }^{3}$ of a graph $G$. Seese [23] proved the lower bound,

$$
2 \operatorname{tpw}(G) \geq \operatorname{tw}(G)+1
$$

In general, tree-partition-width is not bounded from above by any function solely of treewidth. For example, wheel graphs have bounded tree-width and unbounded tree-partitionwidth [7]. However, tree-partition-width is bounded for graphs of bounded tree-width and bounded degree [12, 13]. The best known upper bound is due to an anonymous referee of the paper by Ding and Oporowski 12], who proved that

$$
\operatorname{tpw}(G) \leq 24 \operatorname{tw}(G) \Delta(G)
$$

whenever $\operatorname{tw}(G) \geq 3$ and $\Delta(G) \geq 1$. Using a similar proof, we make the following improvement to this bound without the restriction that $\operatorname{tw}(G) \geq 3$.

Theorem 1. Every graph $G$ with tree-width $\operatorname{tw}(G) \geq 1$ and maximum degree $\Delta(G) \geq 1$ has tree-partition-width

$$
\operatorname{tpw}(G)<\frac{5}{2}(\operatorname{tw}(G)+1)\left(\frac{7}{2} \Delta(G)-1\right)
$$

Theorem 1 is proved in Section 2. Note that Theorem 1 can be improved in the case of chordal graphs. In particular, a simple extension of a result by Dujmović et al. [14] implies that

$$
\operatorname{tpw}(G) \leq \operatorname{tw}(G)(\Delta(G)-1)
$$

for every chordal graph $G$ with $\Delta(G) \geq 2$; see [24] for a simple proof. Nevertheless, the following theorem proves that $\mathcal{O}(\operatorname{tw}(G) \Delta(G))$ is the best possible upper bound, even for chordal graphs.

Theorem 2. For every $\epsilon>0$ and integer $k \geq 3$, for every sufficiently large integer $\Delta \geq$ $\Delta(k, \epsilon)$, for infinitely many values of $N$, there is a chordal graph $G$ with $N$ vertices, tree-width $\mathrm{tw}(G) \leq k$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width

$$
\operatorname{tpw}(G) \geq\left(\frac{1}{8}-\epsilon\right) \operatorname{tw}(G) \Delta(G)
$$

Theorem [2 is proved in Section 3. Note that Theorem 2 is for $k \geq 3$. For $k=1$, every tree is a tree-partition of itself with width 1 . For $k=2$, we prove that the upper bound $\mathcal{O}(\Delta(G))$ is again best possible; see Section 4.

## 2. Upper Bound

In this section we prove Theorem 1 The proof relies on the following separator lemma by Robertson and Seymour [22].

Lemma 1 ([22]). For every graph $G$ with tree-width at most $k$, for every set $S \subseteq V(G)$, there are edge-disjoint subgraphs $G_{1}$ and $G_{2}$ of $G$ such that $G_{1} \cup G_{2}=G$, $\left|V\left(G_{1}\right) \cap V\left(G_{2}\right)\right| \leq k+1$, and $\left|S-V\left(G_{i}\right)\right| \leq \frac{2}{3}\left|S-\left(V\left(G_{1}\right) \cap V\left(G_{2}\right)\right)\right|$ for each $i \in\{1,2\}$.

[^1]Theorem 1 is a corollary of the following stronger result.
Lemma 2. Let $\alpha:=1+1 / \sqrt{2}$ and $\gamma:=1+\sqrt{2}$. Let $G$ be a graph with tree-width at most $k \geq 1$ and maximum degree at most $\Delta \geq 1$. Then $G$ has tree-partition-width

$$
\operatorname{tpw}(G) \leq \gamma(k+1)(3 \gamma \Delta-1)
$$

Moreover, for each set $S \subseteq V(G)$ such that

$$
(\gamma+1)(k+1) \leq|S| \leq 3(\gamma+1)(k+1) \Delta
$$

there is a tree-partition of $G$ with width at most

$$
\gamma(k+1)(3 \gamma \Delta-1),
$$

such that $S$ is contained in a single bag containing at most $\alpha|S|-\gamma(k+1)$ vertices.
Proof. We proceed by induction on $|V(G)|$.
Case 1. $|V(G)|<(\gamma+1)(k+1)$ : Then no set $S$ is specified, and the tree-partition in which all the vertices are in a single bag satisfies the lemma. Now assume that $|V(G)| \geq$ $(\gamma+1)(k+1)$, and without loss of generality, $S$ is specified.

Case 2. $|V(G)-S|<(\gamma+1)(k+1)$ : Then the tree-partition in which $S$ is one bag and $V(G)-S$ is another bag satisfies the lemma. Now assume that $|V(G)-S| \geq(\gamma+1)(k+1)$.

Case 3. $|S| \leq 3(\gamma+1)(k+1)$ : Let $N$ be the set of vertices in $G$ that are adjacent to some vertex in $S$ but are not in $S$. Then $|N| \leq \Delta|S| \leq 3(\gamma+1)(k+1) \Delta$. If $|N|<(\gamma+1)(k+1)$ then add arbitrary vertices from $V(G)-(S \cup N)$ to $N$ until $|N| \geq(\gamma+1)(k+1)$. This is possible since $|V(G)-S| \geq(\gamma+1)(k+1)$.

By induction, there is a tree-partition of $G-S$ with width at most $\gamma(k+1)(3 \gamma \Delta-1)$, such that $N$ is contained in a single bag. Create a new bag only containing $S$. Since all the neighbours of $S$ are in a single bag, we obtain a tree-partition of $G$. ( $S$ corresponds to a leaf in the pattern.) Since $|S| \geq(\gamma+1)(k+1)$, it follows that $|S| \leq \alpha|S|-\gamma(k+1)$ as desired. Now $|S| \leq 3(\gamma+1)(k+1)<\gamma(k+1)(3 \gamma \Delta-1)$. Since the other bags do not change we have the desired tree-partition of $G$.

Case 4. $|S| \geq 3(\gamma+1)(k+1)$ : By Lemma 1 , there are edge-disjoint subgraphs $G_{1}$ and $G_{2}$ of $G$ such that $G_{1} \cup G_{2}=G,\left|V\left(G_{1}\right) \cap V\left(G_{2}\right)\right| \leq k+1$, and $\left|S-V\left(G_{i}\right)\right| \leq \frac{2}{3}\left|S-\left(V\left(G_{1}\right) \cap V\left(G_{2}\right)\right)\right|$ for each $i \in\{1,2\}$. Let $Y:=V\left(G_{1}\right) \cap V\left(G_{2}\right)$. Let $a:=|S \cap Y|$ and $b:=|Y-S|$. Thus $a+b \leq$ $k+1$. Let $p_{i}:=\left|\left(S \cap V\left(G_{i}\right)\right)-Y\right|$. Then $p_{1} \leq 2 p_{2}$ and $p_{2} \leq 2 p_{1}$. Let $S_{i}:=\left(S \cap V\left(G_{i}\right)\right) \cup Y$. Note that $\left|S_{i}\right|=p_{i}+a+b$.

Now $p_{1}+p_{2}+a=|S| \geq 3(\gamma+1)(k+1)$. Thus $3 p_{i}+a \geq 3(\gamma+1)(k+1)$ and $3 p_{i}+3 a+3 b \geq$ $3(\gamma+1)(k+1)$. That is, $\left|S_{i}\right| \geq(\gamma+1)(k+1)$ for each $i \in\{1,2\}$.

Now $p_{1}+p_{2}+a \leq 3(\gamma+1)(k+1) \Delta$. Thus $\frac{3}{2} p_{i}+a \leq 3(\gamma+1)(k+1) \Delta$ and $p_{i} \leq 2(\gamma+1)(k+1) \Delta$. Thus $p_{i}+a+b \leq 2(\gamma+1)(k+1) \Delta+(k+1)$. Hence $\left|S_{i}\right|=p_{i}+a+b<3(\gamma+1)(k+1) \Delta$.

Thus we can apply induction to the set $S_{i}$ in the graph $G_{i}$ for each $i \in\{1,2\}$. We obtain a tree-partition of $G_{i}$ with width at most $\gamma(k+1)(3 \gamma \Delta-1)$, such that $S_{i}$ is contained in a single bag $T_{i}$ containing at most $\alpha\left|S_{i}\right|-\gamma(k+1)$ vertices.

Construct a partition of $G$ by uniting $T_{1}$ and $T_{2}$. Each vertex of $G$ is in exactly one bag since $V\left(G_{1}\right) \cap V\left(G_{2}\right)=Y \subseteq S_{i} \subseteq T_{i}$. Since $G_{1}$ and $G_{2}$ are edge-disjoint, the pattern of


Figure 1. Illustration of Case 4.
this partition of $G$ is obtained by identifying one vertex of the pattern of the tree-partition of $G_{1}$ with one vertex of the pattern of the tree-partition of $G_{2}$. Since the patterns of the tree-partitions of $G_{1}$ and $G_{2}$ are forests, the pattern of the partition of $G$ is a forest, and we have a tree-partition of $G$.

Moreover, $S$ is contained in a single bag $T_{1} \cup T_{2}$ and

$$
\begin{aligned}
\left|T_{1} \cup T_{2}\right| & =\left|T_{1}\right|+\left|T_{2}\right|-|Y| \\
& \leq \alpha\left|S_{1}\right|-\gamma(k+1)+\alpha\left|S_{2}\right|-\gamma(k+1)-(a+b) \\
& =\alpha\left(p_{1}+a+b\right)-\gamma(k+1)+\alpha\left(p_{2}+a+b\right)-\gamma(k+1)-(a+b) \\
& =\alpha\left(p_{1}+p_{2}+a\right)-2 \gamma(k+1)+(\alpha-1) a+(2 \alpha-1) b \\
& \leq \alpha|S|-2 \gamma(k+1)+(2 \alpha-1)(a+b) \\
& \leq \alpha|S|-2 \gamma(k+1)+(2 \alpha-1)(k+1) \\
& =\alpha|S|-\gamma(k+1)
\end{aligned}
$$

Thus $\left|T_{1} \cup T_{2}\right| \leq \alpha \cdot 3(\gamma+1)(k+1) \Delta-\gamma(k+1)=\gamma(k+1)(3 \gamma \Delta-1)$. Since the other bags do not change we have the desired tree-partition of $G$.

## 3. General Lower Bound

The remainder of the paper studies lower bounds on the tree-partition-width. The graphs employed are chordal. We first show that tree-partitions of chordal graphs can be assumed to have certain useful properties.

Lemma 3. Every chordal graph $G$ has a tree-partition $T$ with width $\operatorname{tpw}(G)$, such that for every independent set $S$ of simplicia ${ }^{4}$ vertices of $G$, and for every bag $B$ of $T$, either $B=\{v\}$ for some vertex $v \in S$, or the induced subgraph $G[B-S]$ is connected.

[^2]Proof. Let $T_{0}$ be a tree-partition of a chordal graph $G$ with width $\operatorname{tpw}(G)$. Let $T$ be the partition of $G$ obtained from $T_{0}$ by replacing each bag $B$ of $T_{0}$ by bags corresponding to the connected components of $G[B]$. Then $T$ has width at most $\operatorname{tpw}(G)$.

To prove that $T$ is a forest, suppose on the contrary that $T$ contains an induced cycle $C$. Since each bag in $C$ induces a connected subgraph of $G, G$ contains an induced cycle $D$ with at least one vertex from each bag in $C$. Since $G$ is chordal, $D$ is a triangle. Thus $C$ is a triangle, implying that the vertices in $D$ were in distinct bags in $T_{0}$ (since the bags of $T$ that replaced each bag of $T_{0}$ form an independent set). Hence the bags of $T_{0}$ that contain $D$ induce a triangle in $T_{0}$, which is the desired contradiction since $T_{0}$ is a forest. Hence $T$ is a forest.

Let $S$ be an independent set of simplicial vertices of $G$. Consider a bag $B$ of $T$. By construction, $G[B]$ is connected. First suppose that $B \subseteq S$. Since $S$ is an independent set and $G[B]$ is connected, $B=\{v\}$ for some vertex $v \in S$.

Now assume that $B-S \neq \emptyset$. Suppose on the contrary that $G[B-S]$ is disconnected. Thus $B \cap S$ is a cut-set in $G[B]$. Let $v$ and $w$ be vertices in distinct components of $G[B-S]$ such that the distance between $v$ and $w$ in $G[B]$ is minimised. (This is well-defined since $G[B]$ is connected.) Since $S$ is an independent set, every shortest path between $v$ and $w$ in $G[B]$ has only two edges. That is, $v$ and $w$ have a common neighbour $x$ in $B \cap S$. Since $x$ is simplicial, $v$ and $w$ are adjacent. This contradiction proves that $G[B-S]$ is connected.

The next lemma is the key component of the proof of Theorem 2. For integers $a<b$, let $[a, b]:=\{a, a+1, \ldots, b\}$ and $[b]:=[1, b]$.

Lemma 4. For all integers $k \geq 2$ and $\Delta \geq 3 k+1$, for infinitely many values of $N$ there is a chordal graph $G$ with $N$ vertices, tree-width $\operatorname{tw}(G)=2 k-1$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width $\operatorname{tpw}(G)>\frac{1}{4} k(\Delta-3 k)$.

Proof. Let $n$ be an integer with $n>\max \left\{\frac{1}{2} k(\Delta-3 k), 2\right\}$. Let $H$ be the graph with vertex set $\{(x, y): x \in[n], y \in[k]\}$, where distinct vertices $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ are adjacent if and only if $\left|x_{1}-x_{2}\right| \leq 1$. The set of vertices $\{(x, y): y \in[k]\}$ is the $x$-column. The set of vertices $\{(x, y): x \in[n]\}$ is the $y$-row. Observe that each column induces a $k$-vertex clique, and each row induces an $n$-vertex path.

Let $C$ be an induced cycle in $H$. If $(x, y)$ is a vertex in $C$ with $x$ minimum then the two neighbours of $(x, y)$ in $C$ are adjacent. Thus $C$ is a triangle. Hence $H$ is chordal. Observe that each pair of consecutive columns form a maximum clique of $2 k$ vertices in $H$. Thus $H$ has tree-width $2 k-1$. Also note that $H$ has maximum degree $3 k-1$.

An edge of $H$ between vertices $(x, y)$ and $(x+1, y)$ is horizontal. As illustrated in Figure 2, construct a graph $G$ from $H$ as follows. For each horizontal edge $v w$ of $H$, add $\left\lceil\frac{1}{2}(\Delta-3 k)\right\rceil$ new vertices, each adjacent to $v$ and $w$. Since $H$ is chordal and each new vertex is simplicial, $G$ is chordal. The addition of degree- 2 vertices to $H$ does not increase the maximum clique size (since $k \geq 2$ ). Thus $G$ has clique number $2 k$ and tree-width $2 k-1$. Since each vertex of $H$ is incident to at most two horizontal edges, $G$ has maximum degree $3 k-1+2\left\lceil\frac{1}{2}(\Delta-3 k)\right\rceil \leq \Delta$.

Observe that $V(G)-V(H)$ is an independent set of simplicial vertices in $G$. By Lemma 3, $G$ has a tree-partition $T$ with width $\operatorname{tpw}(G)$, such that for every bag $B$ of $T$, either $B=\{v\}$


Figure 2. The graph $G$ with $k=4, \Delta=15$, and $n=9$.
for some vertex $v$ of $G-H$, or the induced subgraph $H[B]$ is connected. Since $G$ is connected, $T$ is a (connected) tree. Let $U$ be the tree-partition of $H$ induced by $T$. That is, to obtain $U$ from $T$ delete the vertices of $G-H$ from each bag, and delete empty bags. Since $H$ is connected, $U$ is a (connected) tree. By Lemma 3, each bag of $U$ induces a connected subgraph of $H$.

Suppose that $U$ only has two bags $B$ and $C$. Then one of $B$ and $C$ contains at least $\frac{1}{2} n k$ vertices. Since $k \geq 2$, we have $\operatorname{tpw}(G) \geq \frac{1}{2} n k>\frac{1}{4} k(\Delta-3 k)$, as desired. Now assume that $U$ has at least three bags.

Consider a bag $B$ of $U$. Let $\ell(B)$ be the minimum integer such that some vertex in $B$ is in the $\ell(B)$-column, and let $r(B)$ be the maximum integer such that some vertex in $B$ is in the $r(B)$-column. Since $H[B]$ is connected, there is a path in $B$ from the $\ell(B)$ column to the $r(B)$-column. By the definition of $H$, for each $x \in[\ell(B), r(B)]$, the $x$-column contains a vertex in $B$. Let $I(B)$ be the closed real interval from $\ell(B)-\frac{1}{2}$ to $r(B)+\frac{1}{2}$. Observe that two bags $B$ and $C$ of $U$ are adjacent if and only if $I(B) \cap I(C) \neq \emptyset$. Thus $\{I(B): B$ is a bag of $U\}$ is an interval representation of the tree $U$. Every tree that is an interval graph is a caterpillar5; see [16] for example. Thus $U$ is a caterpillar.

Let $\preceq$ be the relation on the set of non-leaf bags of $U$ defined by $A \preceq B$ if and only if $\ell(A) \leq \ell(B)$ and $r(A) \leq r(B)$. We claim that $\preceq$ is a total order. It is immediate that $\preceq$ is reflexive and transitive. To prove that $\preceq$ is antisymmetric, suppose on the contrary that $A \preceq B$ and $B \preceq A$ for distinct non-leaf bags $A$ and $B$. Thus $\ell(A)=\ell(B)$ and $r(A)=r(B)$.

[^3]Since $U$ has at least three bags, there is a third bag $C$ that contains a vertex in the $(\ell(A)-1)$ column or in the $(r(A)+1)$-column. Thus $\{A, B, C\}$ induce a triangle in $U$, which is the desired contradiction. Hence $\preceq$ is antisymmetric. To prove that $\preceq$ is total, suppose on the contrary that $A \npreceq B$ and $B \npreceq A$ for distinct non-leaf bags $A$ and $B$. Now $A \npreceq B$ implies that $\ell(A)>\ell(B)$ or $r(A)>r(B)$. Without loss of generality, $\ell(A)>\ell(B)$. Thus $B \npreceq A$ implies that $r(B)>r(A)$. Hence the interval $[\ell(A), r(A)]$ is strictly within the interval $[\ell(B), r(B)]$ at both ends. For each $x \in[\ell(A), r(A)]$, every vertex in the $x$-column is in $A \cup B$, as otherwise $U$ would contain a triangle (since each column is a clique in $H$ ). Moreover, every vertex in the $(\ell(A)-1)$-column or in the $(r(A)+1)$-column is in $B$, as otherwise $U$ would contain a triangle (since the union of consecutive columns is a clique in $H$ ). Thus every neighbour of every vertex in $A$ is in $B$. That is, $A$ is a leaf in $U$. This contradiction proves that $\preceq$ is a total order on the set of non-leaf bags of $U$.

Suppose that $U$ has a 4 -vertex path $(A, B, C, D)$ as a subgraph.
Thus $B$ and $C$ are non-leaf bags. Without loss of generality, $B \prec C$. If every column contains vertices in both $B$ and $C$, then $B$ and $C$ and any other bag would induce a triangle in $U$ (since each column induces a clique in $H$ ). Thus some column contains a vertex in $B$ but no vertex in $C$, and some column contains a vertex in $C$ but no vertex in $B$. Let $p$ be the maximum integer such that some vertex in $B$ is in the $p$-column, but no vertex in $C$ is in the $p$-column. Let $q$ be the minimum integer such that some vertex in $C$ is in the $q$-column, but no vertex in $B$ is in the $q$-column. Now $p<q$ since $B \prec C$.

We claim that the $(p+1)$-column contains a vertex in $C$. If not, then the $(p+1)$-column contains no vertex in $B$ by the definition of $p$. Thus $r(B)=p$ since $H[B]$ is connected. Since $B$ is adjacent to $C$ in $U, \ell(C) \leq r(B)+1=p+1$. In particular, the $(p+1)$-column contains a vertex in $C$. Since $H[C]$ is connected, for $x \in[p+1, q]$, each $x$-column contains a vertex in $C$. In fact, $\ell(C)=p+1$ since the $p$-column contains no vertex in $C$. By symmetry, for $x \in[p, q-1]$, each $x$-column contains a vertex in $B$, and $r(C)=q-1$.

The union of the $p$-column and the $(p+1)$-column only contains vertices in $B \cup C$, as otherwise $U$ would contain a triangle (since the union of two consecutive columns is a clique in $H)$. By the definition of $p$, no vertex in the $p$-column is in $C$. Thus every vertex in the $p$-column is in $B$. By symmetry, every vertex in the $q$-column is in $C$. Now for each $y \in[k]$, the vertices $(p, y),(p+1, y), \ldots,(q, y)$ are all in $B \cup C$, the first vertex $(p, y)$ is in $B$, and the last vertex $(q, y)$ is in $C$. Thus $(x, y) \in B$ and $(x+1, y) \in C$ for some $x \in[p, q-1]$. That is, in every row of $H$ there is a horizontal edge with one endpoint in $B$ and the other in $C$.

Thus there are at least $k$ horizontal edges with one endpoint in $B$ and the other in $C$ (now considered to be bags of $T$ ). For each such horizontal edge $v w$, each vertex of $G-H$ adjacent to $v$ and $w$ is in $B \cup C$, as otherwise $T$ would contain a triangle. There are $\left\lceil\frac{1}{2}(\Delta-3 k)\right\rceil$ such vertices of $G-H$ for each of the $k$ horizontal edges between $B$ and $C$. Thus $|B \cup C| \geq \frac{1}{2} k(\Delta-3 k)$. Thus one of $B$ and $C$ has at least $\frac{1}{4} k(\Delta-3 k)$ vertices. Hence $\operatorname{tpw}(G) \geq \frac{1}{4} k(\Delta-3 k)$ as desired.

Now assume that $U$ has no 4 -vertex path as a subgraph.
A tree is a star if and only if it has no 4 -vertex path as a subgraph. Hence $U$ is a star. Let $R$ be the root bag of $U$. If $R$ contains a vertex in every column then $|R| \geq n$, implying
$\operatorname{tpw}(G) \geq n \geq \frac{1}{4} k(\Delta-3 k)$, as desired. Now assume that for some $x \in[n]$, the $x$-column of $H$ contains no vertex in $R$. Let $B$ be a bag containing some vertex in the $x$-column. The $x$-column induces a clique in $H$, the only bag in $U$ that is adjacent to $B$ is $R$, and $R$ contains no vertex in the $x$-column. Thus every vertex in the $x$-column is in $B$. Since $R$ is the only bag in $U$ adjacent to $B$, there are at least $k$ horizontal edges with one endpoint in $B$ and the other endpoint in $R$. As in the case when $U$ contained a 4-vertex path, we conclude that $\operatorname{tpw}(G) \geq \frac{1}{4} k(\Delta-3 k)$ as desired.

Proof of Theorem [2. Let $\ell:=\left\lceil\frac{k}{2}\right\rceil$. Thus $\ell \geq 2$. By Lemma 4, for each integer $\Delta \geq \Delta(k, \epsilon):=$ $\max \left\{3 \ell+1, \frac{3 \ell}{8 \epsilon}\right\}$, there are infinitely many values of $N$ for which there is a chordal graph $G$ with $N$ vertices, tree-width $\operatorname{tw}(G)=2 \ell-1 \leq k$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width $\operatorname{tpw}(G)>\frac{1}{4} \ell(\Delta-3 \ell)$, which is at least $\left(\frac{1}{8}-\epsilon\right) k \Delta$ since $\Delta \geq \frac{3 \ell}{8 \epsilon}$.

A domino tree decomposition $\sqrt[6]{ }$ is a tree decomposition in which each vertex appears in at most two bags. The domino tree-width of a graph $G$, denoted by $\operatorname{dtw}(G)$, is the minimum width of a domino tree decomposition of $G$. Domino tree-width behaves like tree-partitionwidth in the sense that $\operatorname{dtw}(G) \geq \operatorname{tw}(G)$, and $\operatorname{dtw}(G)$ is bounded for graphs of bounded tree-width and bounded degree [7]. The best upper bound is

$$
\operatorname{dtw}(G) \leq(9 \operatorname{tw}(G)+7) \Delta(G)(\Delta(G)+1)-1
$$

which is due to Bodlaender [6], who also constructed a graph $G$ with

$$
\operatorname{dtw}(G) \geq \frac{1}{12} \operatorname{tw}(G) \Delta(G)-2
$$

Tree-partition-width and domino tree-width are related in that every graph $G$ satisfies

$$
\operatorname{dtw}(G) \geq \operatorname{tpw}(G)-1
$$

as observed by Bodlaender and Engelfriet [7]. Thus Theorem 2 provides examples of graphs $G$ with

$$
\operatorname{dtw}(G) \geq\left(\frac{1}{8}-\epsilon\right) \operatorname{tw}(G) \Delta(G)
$$

This represents a small constant-factor improvement over the above lower bound by Bodlaender 6].

## 4. Lower Bound for Tree-width 2

We now prove a lower bound on the tree-partition-width of graphs with tree-width 2 .
Theorem 3. For all odd $\Delta \geq 11$ there is a chordal graph $G$ with tree-width 2, maximum degree $\Delta$, and tree-partition-width $\operatorname{tpw}(G) \geq \frac{2}{3}(\Delta-1)$.

Proof. As illustrated in Figure 3, let $G$ be the graph with

$$
V(G):=\{r\} \cup\left\{v_{i}: i \in[\Delta]\right\} \cup\left\{w_{i, \ell}: i \in[\Delta-1], \ell \in\left[\frac{1}{2}(\Delta-3)\right]\right\}
$$

and
$E(G):=\left\{r v_{i}: i \in[\Delta]\right\} \cup\left\{v_{i} v_{i+1}: i \in[\Delta-1]\right\} \cup\left\{v_{i} w_{i, \ell}, v_{i+1} w_{i, \ell}: i \in[\Delta-1], \ell \in\left[\frac{1}{2}(\Delta-3)\right]\right\}$.

[^4]Observe that $G$ has maximum degree $\Delta$. Clearly every induced cycle of $G$ is a triangle. Thus $G$ is chordal. Observe that $G$ has no 4 -vertex clique. Thus $G$ has tree-width 2.


Figure 3. Illustration for Theorem 3 with $\Delta=13$.

Let $T$ be the tree-partition of $G$ from Lemma 3. Then $T$ has width $\operatorname{tpw}(G)$, and every bag induces a connected subgraph of $G$. Let $R$ be the bag containing $r$. Let $B_{1}, \ldots, B_{d}$ be the bags, not including $R$, that contain some vertex $v_{i}$. Thus $R$ is adjacent to each $B_{j}$ (since $r$ is adjacent to each $v_{i}$ ). Since $\left\{w_{i, \ell}: i \in[\Delta-1], \ell \in\left[\frac{1}{2}(\Delta-3)\right]\right\}$ is an independent set of simplicial vertices, by Lemma 3, for each $j \in[d]$, the vertices $\left\{v_{1}, v_{2}, \ldots, v_{\Delta}\right\} \cap B_{j}$ induce a (connected) subpath of $G$.

First suppose that $d=0$. Then the $\Delta+1$ vertices $\left\{r, v_{1}, \ldots, v_{\Delta}\right\}$ are contained in one bag $R$. Thus $\operatorname{tpw}(G) \geq \Delta+1 \geq \frac{2}{3}(\Delta-1)$.

Now suppose that $d=1$. Thus $\left\{r, v_{1}, \ldots, v_{\Delta}\right\} \subseteq R \cup B_{1}$. In addition, at least one edge $v_{i} v_{i+1}$ has one endpoint in $R$ and the other endpoint in $B_{1}$. Thus $w_{i, \ell} \in R \cup B_{1}$ for each $\left.\ell \in\left[\frac{1}{2}(\Delta-3)\right\}\right]$. Hence $1+\Delta+\frac{1}{2}(\Delta-3)$ vertices are contained in two bags. Thus one bag contains at least $\frac{1}{4}(3 \Delta-1)$ vertices, and $\operatorname{tpw}(G) \geq \frac{1}{4}(3 \Delta-1) \geq \frac{2}{3}(\Delta-1)$.

Finally suppose that $d \geq 2$. Since $\left\{v_{1}, v_{2}, \ldots, v_{\Delta}\right\} \cap B_{j}$ induce a subpath in each bag $B_{j}$, we can assume that $\left\{v_{1}, v_{2}, \ldots, v_{\Delta}\right\} \cap B_{j}=\left\{v_{i}: i \in[f(j), g(j)]\right\}$, where

$$
1 \leq f(1) \leq g(1)<f(2) \leq g(2)<\cdots<f(d) \leq g(d) \leq \Delta
$$

Distinct $B_{j}$ bags are not adjacent (since $T$ is a tree). Thus $v_{f(j)-1} \in R$ for each $j \in[2, d]$. Similarly, $v_{g(j)+1} \in R$ for each $j \in[d-1]$. Thus $w_{f(j)-1, \ell} \in R \cup B_{j}$ for each $j \in[2, d]$ and $\left.\ell \in\left[\frac{1}{2}(\Delta-3)\right\}\right]$. Similarly, $w_{g(j), \ell} \in R \cup B_{j}$ for each $j \in[d-1]$ and $\left.\ell \in\left[\frac{1}{2}(\Delta-3)\right\}\right]$.

Hence the bags $R, B_{1}, \ldots, B_{d}$ contain at least

$$
1+\Delta+2(d-1) \cdot \frac{1}{2}(\Delta-3)
$$

vertices. Therefore one of these bags has at least

$$
(1+\Delta+(d-1)(\Delta-3)) /(d+1)
$$

vertices, which is at least $\frac{2}{3}(\Delta-1)$. Hence $\operatorname{tpw}(G) \geq \frac{2}{3}(\Delta-1)$.

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Figure 4. Illustration for Theorem 3 with $\Delta=19$ and $d=4$.
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Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Barcelona, Spain
E-mail address: david.wood@upc.es


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    ${ }^{1}$ All graphs considered are undirected, simple, and finite. Let $V(G)$ and $E(G)$ respectively be the vertex set and edge set of a graph $G$. Let $\Delta(G)$ be the maximum degree of $G$.
    ${ }^{2}$ Tree-partition-width has also been called strong tree-width [7, 23].

[^1]:    ${ }^{3}$ A graph is chordal if every induced cycle is a triangle. The tree-width of a graph $G$ can be defined to be the minimum integer $k$ such that $G$ is a subgraph of a chordal graph with no clique on $k+2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see [5, 21] for surveys.

[^2]:    ${ }^{4}$ A vertex is simplicial if its neighbourhood is a clique.

[^3]:    ${ }^{5}$ A caterpillar is a tree such that deleting the leaves gives a path.

[^4]:    ${ }^{6}$ See 10 for an introduction to tree decompositions.

