THE MORPHOLOGY OF INFINITE TOURNAMENTS. APPLICATION TO THE GROWTH OF THEIR PROFILE

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Dedicated to Michel Deza at the occasion of his 65^{th} birthday.

Abstract. A tournament is *acyclically indecomposable* if no acyclic autonomous set of vertices has more than one element. We identify twelve infinite acyclically indecomposable tournaments and prove that every infinite acyclically indecomposable tournament contains a subtournament isomorphic to one of these tournaments. The *profile* of a tournament T is the function φ_T which counts for each integer n the number $\varphi_T(n)$ of tournaments induced by T on the n-element subsets of T, isomorphic tournaments being identified. As a corollary of the result above we deduce that the growth of φ_T is either polynomial, in which case $\varphi_T(n) \simeq an^k$, for some positive real a, some non-negative integer k, or as fast as some exponential.

1. Introduction and presentation of the results

An important chapter of the theory of graphs is about the decompositions of graphs into simpler subgraphs. A wealth of results has been obtained along the lines pioneered by T. Gallai [13], [15] with the notion of indecomposable graph (see [9] for an example). This paper is about tournaments. We study acyclically indecomposable tournaments, objects introduced by Culus and Jouve in 2005 [7]. A consequence of our study is the existence of a gap in the growth rate of the profile of tournaments.

1.1. **Lexicographical sums of acyclic tournaments and acyclically indecomposable tournaments.** A tournament is *acyclically indecomposable* if no acyclic autonomous set of vertices has more than one element. Our first result is quite elementary:

Theorem 1. *Every tournament T decomposes into a lexicographical sum of acyclic tournaments indexed by an acyclically indecomposable tournament. The decomposition is unique and up to an isomorphism, this acyclically indecomposable tournament is unique.*

The blocks of the decomposition are the *acyclic components* of T. We denote by $\check{T}$ the acyclically indecomposable tournament indexing them.

The next one is more involved:

Date: November 16, 2021.
2000 *Mathematics Subject Classification.* Combinatorics (05A16), (05C20).
Key words and phrases. Tournaments, acyclic tournaments, asymptotic enumeration, polynomial growth, profile.
This work was done under the auspices of the Intas programme Universal algebra theory and supported by CMCU Franco-Tunisian "Outils mathématiques pour l'informatique".

Theorem 2. *There are twelve infinite acyclically indecomposable tournaments such that every infinite acyclically indecomposable tournament contains a subtournament isomorphic to one of these tournaments.*

The twelve tournaments mentionned in Theorem 2 are described in Section 5. At this point, we mention that they do not embed in each other, each one is countable and is the union of two acyclic tournaments. We also indicate that to an acyclic tournament C we associate a set $\mathfrak{B}_C$ consisting of (at most) six tournaments denoted respectively $C_{3[C]}, V_{[C]}, T_{[C]}, U_{[C]}, H_{[C]}$ and $K_{[C]}$. Let $\mathfrak{B} := \mathfrak{B}_\omega \cup \mathfrak{B}_{\omega *}$ where ω is the tournament made of $\mathbb{N}$ and the natural (strict) order and ω^* is the dual of ω. It turns out that each members of $\mathfrak{B}$, except $K_{[\omega]}$, is acyclically prime and that $\check{K}_{[\omega]}$ is obtained from $K_{[\omega]}$ by identifying two vertices. The twelve tournaments mentionned in Theorem 2 are obtained by replacing $K_{[\omega]}$ by $\check{K}_{[\omega]}$ in $\mathfrak{B}$. Indeed, as we will prove, every infinite acyclically indecomposable tournament contains a member of $\mathfrak{B}$. The proof is based on a separation lemma (Lemma 9) and Ramsey Theorem.

Theorem 2 has a finitary version. Denote by $\underline{n}$ the acyclic tournament made of $\{0, \dots, n-1\}$ with the natural (strict) order and set $\check{\mathfrak{B}}_{\underline{n}} := \{\check{T} : T \in \mathfrak{B}_{\underline{n}}\}$ for each non negative integer n.

Theorem 3. *For every non-negative integer n there is an integer $a(n)$ such that every finite tournament of size at least $a(n)$ which is acyclically indecomposable contains a subtournament isomorphic to a member of $\check{\mathfrak{B}}_{\underline{n}}$.*

An upper bound for $a(n)$ can be deduced from a careful analysis of the proof of Theorem 2. An existence proof is readily obtained by means of the compactness theorem of first order logic.

Indeed, suppose that the conclusion of Theorem 3 is false. Let n be such that for every integer m there is an acyclically indecomposable tournament $T(m)$ of size at least m which contains no subtournament isomorphic to a member of $\check{\mathfrak{B}}_{\underline{n}}$. With the terminology of *embeddability*, we simply say that no member of $\check{\mathfrak{B}}_{\underline{n}}$ is embeddable in T. The compactness theorem of first order logic (or an ultraproduct) yields a tournament T such that for every first order-sentence φ of the language of tournaments, φ holds in T whenever it holds in all of the $T(m)$, but finitely many. The fact that a given finite tournament is embeddable in a tournament can be expressed by the satisfaction of a first order sentence (in fact an existential one), thus no member of $\check{\mathfrak{B}}_{\underline{n}}$ is embeddable into T. Our separation lemma ensures that the fact that a tournament is acyclically indecomposable can be expressed by the satisfaction of a first order formula (Lemma 12). Hence T is a acyclically indecomposable. Since the size of the $T(m)$'s is unbounded, T is infinite, thus, from Theorem 2, some $\check{X}_{[\alpha]}$ with $X_{[\alpha]} \in \mathfrak{B}$ is embeddable in T. This tournament is an increasing union of $\check{X}_{[\underline{m'}]}$, for $m' \in \mathbb{N}$ (Corollary 9). Hence $\check{X}_{[\underline{n}]} \in \check{\mathfrak{B}}_{\underline{n}}$ is embeddable in T, a contradiction.

Let $\mathfrak{A}$ be the collection of tournaments T which can be written as a finite lexicographical sum of acyclic tournaments. Tournaments not in $\mathfrak{A}$ are *obstructions* to $\mathfrak{A}$. Clearly, no member of $\mathfrak{A}$ contains a subtournament isomorphic to an obstruction. From Theorem 2 (and the fact that $K_{[\omega]}$ can be embedded into $\check{K}_{[\omega]}$) $\mathfrak{B}$ is a set of obstructions characterizing $\mathfrak{A}$. And since the members of $\mathfrak{B}$ do not embed in each other, $\mathfrak{B}$ is a minimum sized set of obstructions. As a consequence:

Corollary 1. *Let T be an infinite tournament, then:*

- *Either T is a lexicographical sum of acyclic tournaments indexed by a finite tournament.*
- *Or T contains as a subtournament a tournament isomorphic to a member of $\mathfrak{B}$.*

1.2. **Application to the profile of tournaments.** The *profile* of a tournament T is the function φ_T which counts for each integer n the number $\varphi_T(n)$ of tournaments induced by T on the n-element subsets of T, isomorphic tournaments being identified. The *age* of T is the set $\mathcal{A}(T)$ of isomorphic types of subtournaments induced on the finite subsets of $V(T)$. Clearly, the profile of T depends only upon the age of T. We prove (see Section 5 and Section 6):

Lemma 1. *The ages of members of $\mathfrak{B}$ are six sets pairwise incomparable w.r.t. inclusion. For each $T \in \mathfrak{B}$ the growth of φ_T is at least exponential, that is $\varphi_T(n) \geq ac^n$ for some reals $a > 0$ and $c > 1$.*

It is easy to see that if T is a lexicographical sum of acyclic tournaments indexed by a finite tournament, say D, then φ_T is bounded from above by a polynomial (of degree at most $|D| - 1$). From Corollary 1 and Lemma 1 we deduce:

Theorem 4. *The profile of a tournament T is either bounded from above by a polynomial, in which case T is a lexicographical sum of acyclic tournaments indexed by a finite tournament, or it growth is at least exponential.*

We give a precise description of the profile of a lexicographical sum of acyclic tournaments indexed by a finite tournament.

Theorem 5. *If a tournament T is a lexicographic sum of acyclic tournaments indexed by a finite tournament then the generating series of the profile*

$$H_{\varphi_T} := \sum_{n=0}^{\infty} \varphi_T(n) x^n$$

is a rational fraction of the form:

$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$$

with $P(x) \in \mathbb{Z}[x]$ and $\varphi_T(n) \simeq an^{k-1}$ for some non-negative real a, the integer k being the number of infinite acyclic components of T.

The first part of Theorem 5 is a consequence of a more general result about relational structures recently obtained by N.Thiéry and the second author [20] that we record in Section 2.2.

There are acyclically indecomposable tournaments of size k for every integer $k \geq 3$, hence, according Theorem 5, there are tournaments of arbitrarily large polynomial growth.

An other consequence of Theorem 5 is this:

Corollary 2. *The growth of the profile of an infinite indecomposable tournament is at least exponential.*

This research leaves open the following:

Problem 1. *Find a result, similar to Theorem 2, for indecomposable tournaments and, possibly, a finitary version.*

The notion of acyclically indecomposable tournament was studied by J.F.Culus and B.Jouve in [7], [8], [14]. The notion of profile was introduced in 1971 by the second author (see [11], [12]) and developped in [17],[18], [19]; for a survey see [22]. The study of the orbital profile of permutation groups is intensively studied by P.J.Cameron and his school [2], [3], [4]. The survey [22] includes a presentation of Theorems 2, 4 and 5 with an application to the structure of the age algebra of Cameron.

This work was presented at the CGCS 2007(Luminy, France, May 2-4 2007) in honor of Michel Deza. We are pleased to thanks Y.Manoussakis and the other organisers of this conference for offering this opportunity to us. We thanks J.Nesetril for his comments and bibliographical references.

This paper is organized as follows. Section 2 contains the material needed about relational structures and tournaments. Section 3 contains the main properties of acyclic decompositions of tournaments, particularly the proof of Theorem 1 and of our separation lemma. Section 4 contains the proof of Theorem 5, Section 5 the description of $\mathfrak{B}_{\underline{n}}$ and $\mathfrak{B}$ with their main properties. Section section:computprofile contains the description of the profiles of members of $\mathfrak{B}$ and Section 7 the proof of Theorem 2.

2. Perequisite

We use standard set-theoretical notations. If E is a set, $|E|$ denotes its cardinality. If n is an integer, $[E]^n$ denotes the set of n-element subsets of E; whereas E^n denotes the set of n-tuples of elements of E.

2.1. Invariant structures and skew products.

A relation ρ on a set E is a map from a finite power E^n of E into the two element set $2 := \{0, 1\}$; the integer n is the *arity* of ρ, denoted $a(\rho)$ and ρ is said n-ary. If $n = 2$ we say that ρ is a *binary relation* and we denote $x\rho y$ the fact that $\rho(x, y) = 1$. A *relational structure* is a pair $R := (E, (\rho_i)_{i\in I})$ where each ρ_i is a relation on E. We denote by $R_{\restriction A}$ the relational structure induced by R on A. A *local automorphism* of R is any isomorphism h from an induced substructure of R onto an other one. A pair (E, ρ) where ρ is a binary relation is a *directed graph*. A *chain* is a pair $C := (A, \leq)$ where $\leq$ is a linear order on A. In this case a local automorphism of C is every map h from a subset F of C onto an other subset F' of C such that

$$x \leq y \Longleftrightarrow h(x) \leq h(y) \tag{1}$$

for every $x, y \in F$. For each integer n, let $[C]^n$ be the set of n-tuples $(c_1, \dots, c_n)$ of members of A such that $c_1 < \dots < c_n$. These n-tuples will be identified with the n-element subsets of A. If h is local automorphism of C, F is its domain, n is an integer and $\vec{c} := (c_1, \dots, c_n) \in [F]^n$, we will set $\overline{h}(\vec{c}) := (h(c_1), \dots, h(c_n))$.

Let $\mathfrak{L} :=< C, R, \Phi >$ be a triple made of a chain $C := (A, \leq)$, a relational structure $R := (E, (\rho_i)_{i\in I})$ and a set Φ of maps, each one being a map φ from $[C]^{a(\varphi)}$ into E, where $a(\varphi)$ is an integer, the *arity* of φ.

We say that $\mathfrak{L}$ is *invariant* if

$$\rho_i(\varphi_1(\vec{a_1}), \ldots, \varphi_m(\vec{a_m})) = \rho_i(\varphi_1(\overline{h}(\vec{a_1})), \ldots, \varphi_m(\overline{h}(\vec{a_m}))) \tag{2}$$

for every $i \in I$, $m := a(\rho_i)$, $\varphi_1, \ldots, \varphi_m \in \Phi$, $\vec{a_1} \in [C]^{a(\varphi_1)}, \ldots, \vec{a_m} \in [C]^{a(\varphi_m)}$ and every local automorphism h of C whose domain contains the union of $\vec{a_1}, \ldots, \vec{a_m}$.

This technical condition expresses the fact that each ρ_i is invariant under the transformations of the $a(\rho_i)$-tuples of E which are induced on E by the local isomorphisms of C. In the case of a single binary relation ρ and one n-ary function φ, it says that $\varphi(\vec{a})\rho\varphi(\vec{b})$ depends only upon the relative positions of the components $\vec{a}$ and $\vec{b}$ on the chain C.

If $\mathfrak{L} :=< C, R, \Phi >$ and B is a subset of A, $\Phi_{\restriction B} := \{\varphi_{\restriction [B]^{a(\varphi)}} : \varphi \in \Phi\}$ and $\mathfrak{L}_{\restriction B} :=< C_{\restriction B}, R, \Phi_{\restriction B} >$ is the *restriction* of $\mathfrak{L}$ to B.

We will use the following straightforward consequence of Ramsey's theorem.

Theorem 6. *Let $\mathfrak{L} :=< C, R, \Phi >$ be a structure such that the domain A of C is infinite, R consists of finitely many relations, and Φ is finite. Then there is an infinite subset A' of A such that the structure $\mathfrak{L}_{\restriction A'}$ is invariant.*

Let $S := (V, (\rho_i)_{i \in I})$ be a relational structure and $C := (A, \leq)$ be a chain. A relational structure R is a *skew product* of S and C, denoted by $S \bigotimes C$ if

(1) the domain is $A \times V$
(2) for every $x \in A$, the map $v \to (x, v)$ is an isomorphism from S into R
(3) for each local automorphism h of C, the map $(h, 1_V)$ defined by $(h, 1_V)(x, v) = (h(x), v)$ is a local automorphism of R.

Let $\mathfrak{L} :=< C, R, \Phi >$ where $\Phi := \{\varphi_v : v \in V\}$ and φ_v is the map from A into $A \times V$ defined by $\varphi_v(x) := (x, v)$. Condition (2) expresses that $\mathfrak{L}$ is invariant.

Theorem 6 yields:

Lemma 2. *Let R be a relational structure made of finitely many relations and defined on a product $A \times V$. If V is finite and A is infinite, then for every linear order $\leq$ on A there is some infinite subset A' of A such that $R_{\restriction A' \times V}$ is a skew product of $R_{\restriction \{a\} \times V}$ and $C_{\restriction A'}$, for some $a \in A$ and $C := (A, \leq)$.*

If R is a skew product of a finite relational structure S by a chain then φ_R is bounded from above by some exponential function. Indeed, if S is such that all its one-element restrictions are non-isomorphic, $\varphi(n) = \sum_{k=0}^{v} \varphi(n-k)\binom{n}{k}$ where v is the size of the domain of S, hence the generating series $\mathcal{H}_{\varphi_R}$ is a rational fraction and the result follows. If S is arbitrary, its profile is dominated tems by terms by the previous one. It is not known if the generating series of a skew product R of a finite relational structure S by a chain is a rational fraction. It is not even known if the profile of R is either polynomial or exponential.

In this paper, we will consider skew product of a two-element tournament by a chain. For those which are not acyclic, their profile is asymptotically bounded from above by $\frac{1}{2}(1+\sqrt{2})^n$. As we will see in Section 6, their profile is bounded from below by some exponential.

The notion of invariant structure appeared in [5], Theorem 6 was an handy tool for using Ramsey's theorem. The notion of a skew product of a relation has appeared (under other names) in various papers of the second author (see [22]). For some recent applications, see [21] and [23].

2.2. **Monomorphic decomposition of a relational structure.** Let R be a relational structure on E. A subset B of E is a *monomorphic part* of R if for every integer n and every pair A, A' of n-element subsets of E the induced structures on A and A' are isomorphic whenever $A \setminus B = A' \setminus B$. This notion has been introduced by N.Thiéry and the second author [20]. We present the results we need. The following lemma gathers the main properties of monomorphic parts.

Lemma 3. *(i) The empty set and the one element subsets of E are monomorphic parts of R;*
(ii) If B is a monomorphic part of R then every subset of B too;
(iii) Let B and B' be two monomorphic parts of R; if B and B' intersect, then $B \cup B'$ is a monomorphic part of R;
(iv) Let $\mathcal{B}$ be a family of monomorphic parts of R; if $\mathcal{B}$ is up-directed (that is the union of two members of $\mathcal{B}$ is contained into a third one), then their union $B := \bigcup \mathcal{B}$ is a monomorphic part of R.

Corollary 3. *For every $x \in E$, the set-union $R(x)$ of all the monomorphic parts of R containing x is a monomorphic part, the largest monomorphic part containing x.*

Proof. By (i) of Lemma 3, the set $R(x)$ contains x and by (iii) and (iv) this is a monomorphic part, thus the largest monomorphic part of R containing x. □

We call the set $R(x)$ a *monomorphic component of R*.

A *monomorphic decomposition* of a relational structure R is a partition $\mathcal{P}$ of E into blocks such that for every integer n, the induced structures on two n-elements subsets A and A' of E are isomorphic whenever the intersections $A \cap B$ and $A' \cap B$ over each block B of $\mathcal{P}$ have the same size.

Proposition 1. *The monomorphic components of R form a monomorphic decomposition of R of which every monomorphic decomposition of R is a refinement.*

We will call *canonical* the decomposition of R into monomorphic components.

Recently, N.Thiéry and the second author proved this:

Theorem 7. *Let R be an infinite relational structure admitting a monomorphic decomposition into finitely many blocks and let k be the number of infinite blocks of the canonical decomposition of R, then:*

1. *The generating series H_{φ_R} is a rational fraction of the form:*
$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$$
where $P \in \mathbb{Z}[x]$.
2. *$\varphi_R(n) \simeq an^{k-1}$ for some positive real a.*

The proof of the first part and the fact that $\varphi_R(n) \simeq an^{k'}$ for some $k' \leq k-1$ is in [20]. The proof that $k' = k-1$ was obtained afterward. We will give below the proof of the second part for the special case of tournaments.

2.3. **Basic terminology and notations for tournaments.** A *tournament* T is a pair $(V, \mathcal{E})$, where $\mathcal{E}$ is a binary relation on V which is irreflexive, antisymmetric and complete. Members of V are the *vertices* of T, pairs (x, y) of vertices such that $(x, y) \in \mathcal{E}$ are the *edges* of T. Given a pair $u := (x, y)$, resp. a set $\mathcal{F}$ of pairs, we set $u^{-1} := (y, x)$, resp. $\mathcal{F}^{-1} := \{u^{-1} : u \in \mathcal{F}\}$. The tournament $T^* := (V, \mathcal{E}^{-1})$ is the *dual* of T. If $A \subseteq V$, $T_{\restriction A} := (A, \mathcal{E} \cap A \times A)$ is the tournament *induced on* A or the *restriction of* T *to* A. A *subtournament* of T is any restriction of T to a subset of V. As usual, $V(T)$, resp. $E(T)$, stands for the set of vertices, resp. edges, of the tournament T. We also write $T(x, y) = 1$ for $(x, y) \in E(T)$ and $T(x, y) = 0$ for $(x, y) \notin E(T)$. An *isomorphism* from a tournament T onto a tournament T' is a bijective map $f : V(T) \to V(T')$ such that $T(x, y) = T'(f(x), f(y))$ for all $(x, y) \in V(T) \times V(T)$. If T' is a tournament, a tournament T is *isomorphic to* T', resp. *is embeddable into* T' if there is an isomorphism from T onto T', resp. onto a subtournament of T'. A tournament is *self-dual* if it is isomorphic to its dual. A 3-*element cycle*, or briefly a 3-*cycle*, of a tournament T is a 3-element subset $\{a, b, c\}$ of $V(T)$ such that $T(a, b) = T(b, c) = T(c, a)$. The tournament induced on a 3-cycle is also called a 3-*cycle*. As a tournament, we will denote it by C_3. A tournament T is *acyclic* if no subtournament is a 3-element cycle; this amounts to say that the relation $E(T) \cup \{(x, x) : x \in V(T)\}$ is a linear order. Up to reflexivity, acyclic tournaments and chains (alias totally ordered sets) being the same objects, we will use standard notions and notations used for chains. For example, we will say that the tournament $(\mathbb{N}, <)$, where $<$ is the strict order on $\mathbb{N}$, has type ω; its dual is isomorphic to the tournament made of the set of negative integers equipped with the strict order, we will say that it has type ω^*. Note that according to the theorem of Ramsey, every infinite tournament contains a subtournament which is isomorphic to ω or to ω^*.

Let D be a tournament and $(T_i)_{i \in V(D)}$ be a family of tournaments. The *lexicographical sum of the tournaments* T_i *indexed by the tournament* D is the tournament, denoted $\sum_{i \in D} T_i$, and defined as follows. The vertex set is the disjoint union of the family $(V(T_i))_{i \in V(D)}$, that is $\bigcup\{V(T_i) : i \in V(D)\}$ if the $V(T_i)$'s are pairwise disjoint and $\bigcup\{V(T_i) \times \{i\} : i \in V(D)\}$ otherwise. Members of this disjoint union being denoted by pairs (x, i) with $x \in V(T_i)$, a pair $((x, i), (y, j))$ of vertices is an edge if either $i = j$ and $(x, y) \in E(T_i)$ or $(i, j) \in E(D)$. If D has type ω, resp. ω^*, the lexicographical sum is an ω-sum, resp. an ω^*-sum. If $T_i = T$ for all $i \in V(D)$, this sum is a *lexicographical product* of T and D denoted $T.D$.

A subset $A \subseteq V(T)$ of a tournament T is *autonomous* if for every $x, x' \in A, y \notin A$, $(x, y) \in E(T)$ if and only if $(x', y) \in E(T)$. The empty set, the one-element subsets and the whole vertex set are autonomous and are said *trivial*. If T has no other autonomous subset, T is *indecomposable* (an other denomination is *simple*, see [10], [16]). If T is acyclic, autonomous subsets coincide with intervals of the linear order, hence if $|V(T)| \geq 3$, T is not indecomposable. We also recall that if T is a lexicographical sum $\sum_{i \in D} T_i$, the subsets of the form $V(T_i)$ are autonomous. Conversely, if the vertices of a tournament T are partionned into autonomous subsets, then T is the lexicographical sum of the blocks of the partition.

3. Acyclic decompositions of tournaments

3.1. **Proof of Theorem 1.** We recall the following result (which holds for arbitrary binary relations).

Lemma 4. *Let T be a tournament.*

1. *The union of two autonomous subsets of T with a non empty intersection is autonomous.*
2. *The union of a family $\mathcal{F}$ of automous subsets of T which is closed under finite union is autonomous.*

Lemma 5. *The union of two acyclic autonomous subsets of a tournament is acyclic.*

The proof is immediate.

Applying Lemma 4 and Lemma 5, we get:

Lemma 6. *Let T be a tournament and $x \in V(T)$. Then the set-union $Ac(T)(x)$ of all the acyclic autonomous subsets of T containing x is the largest acyclic autonomous subset containing x.*

An *acyclic component* of T is any subset of $V(T)$ of the form $Ac(T)(x)$.

From Lemma 6, we also have immediately:

Lemma 7. *Let T be a tournament. Then:*

1. *Every acyclic autonomous subset is contained into an acyclic component.*
2. *The acyclic components of T form a partition of $V(T)$ into autonomous subsets .*
3. *Every partition of $V(T)$ into acyclic autonomous subsets is a refinement of the partition into acyclic components.*

As a corollary of Item (3) we get:

Proposition 2. *A tournament is a lexicographical sum of acyclic tournaments indexed by a finite tournament if and only if it has only finitely many acyclic components.*

Let $Ac(T)$ be the set of acyclic components of a tournament T; set $ac(T) := \{|A| : A \in Ac(T)\}$ and $\overline{ac}(T)$ be the sequence of the elements of $ac(T)$ sorted in a decreasing order. As a corollary of the existence of acyclic components, we get:

Corollary 4. *If T and T' are two isomorphic finite tournaments, $\overline{ac}(T) = \overline{ac}(T')$.*

Let T be a tournament and let $p : V(T) \rightarrow Ac(T)$ defined by setting $p(x) := Ac(T)(x)$. Let $\mathcal{E} := \{(Ac(T)(x), Ac(T(y)) : Ac(T)(x) \neq Ac(T(y) \text{ and } T(x,y) = 1\}$ and let $\check{T} := (Ac(T), \mathcal{E})$.

Lemma 8. *Let T be a tournament. Then $\check{T}$ is acyclically indecomposable and T is the lexicographical sum of its acyclic components indexed by $\check{T}$.*

Proof. According to Item (2) of Lemma 7 above, T is the lexicographical sum of its acyclic components indexed by $\check{T}$. Let us prove that $\check{T}$ is acyclically indecomposable. Let A be an acyclic autonomous subset of $V(\check{T}) := Ac(T)$. Then $\bigcup A$ is an autonomous subset of T. Since $T_{\restriction \bigcup A}$ is a lexicographic sum of acyclic tournaments indexed by the acyclic tournament $\check{T}_{\restriction A}$, $\bigcup A$ is also acyclic. Consequently, $\bigcup A$ reduces to a single acyclic component and A is a singleton. This proves our assertion. $\square$

With this lemma the proof of Theorem 1 follows.

3.2. **A separation lemma.** A *diamond*, resp. a *double diamond*, is a tournament obtained by replacing a vertex of a 2-element tournament, resp. a 3-element acyclic tournament, by a 3-element cycle. A double diamond is self-dual if and only if the middle element of the 3-element acyclic tournament is replaced by a 3-element cycle.

We have the following separation lemma which generalizes Lemma 5:

Lemma 9. *Two vertices x, y of a tournament T do not belong to an acyclic autonomous subset of T if and only if x and y are distinct and either:*

(1) x and y belong to some 3-element cycle, or
(2) x and y belong to some diamond, or
(3) x and y belong to some self-dual double diamond.

Proof. Let x, y be two distinct vertices of T and A be an autonomous acyclic subset of $V(T)$ containing x, y. If x, y belong to some 3-element cycle C, then, since A is acyclic, the third element, say z, does not belong to A. Since C is a cycle, $T(z, x) = T(z, y)$ whereas, since A is autonomous, $T(z, x) = T(z, y)$. A contradiction. If x, y belong to some diamond δ, then from the previous case, they cannot belong to the 3-cycle of the diamond. With no loss of generality we may suppose that δ is a *positive* diamond, that is $\delta = \delta^{+} := (\{a, b, c, d\}, \{(a, b), (b, c), (c, a), (a, d), (b, d), (c, d)\})$, with $x := a, y := d$. Since A is autonomous, $a, d \in A$ and $T(b, a) \neq T(b, d)$, we get $b \in A$. Hence, A contains two vertices of the 3-cycle $\{a, b, c\}$. From the previous case, this contradicts the fact that A is acyclic. If x, y belong to a self-dual double diamond $D := \{a, b, c, d, d'\}$, then either they belong to one of the two diamonds included in D or they coincide with the end-points d, d' of D. As seen above, the first case contradicts the fact that A is acyclic. In the second case, since A is autonomous and $T(z, d) \neq T(z, d')$ for every element z of the 3-cycle, A must contain each element of the 3-element cycle, hence it cannot be acyclic.

Conversely, suppose that x and y does not belong to an acyclic autonomous subset of T, in particular $x \neq y$. Let $Z := \{x, y\}$, $Z_i := \{z \in V(T) \setminus Z : T(z, x) = T(z, y) = i\}$ for $i \in \{0, 1\}$ and $Z_{\frac{1}{2}} := \{z \in V(T) \setminus Z : T(z, x) \neq T(z, y)\}$. With no loss of generality, we may suppose $T(x, y) = 1$.

Claim 1. *(a) $Z_{\frac{1}{2}} \neq \emptyset$.*

(b) If x, y does not belong to a diamond or a 3-cycle then $Z_{\frac{1}{2}}$ is an autonomous subset of $Z \cup Z_{\frac{1}{2}}$ and $Z \cup Z_{\frac{1}{2}}$ is an autonomous subset of T.

(c) Furthermore, $Z_{\frac{1}{2}}$ contains a 3-cycle and x, y belong to a double diamond.

Proof of Claim 1.

(a) Since Z is acyclic, it cannot be autonomous. Hence, there is $z \notin Z$ such that $T(z, x) \neq T(z, y)$. We have $z \in Z_{\frac{1}{2}}$, hence $Z_{\frac{1}{2}} \neq \emptyset$.

(b) Suppose that x and y does not belong to a 3-cycle. Let $z \in Z_{\frac{1}{2}}$. Since $T(z, x) \neq T(z, y)$ we have $T(x, z) = T(z, y)$. Since $\{x, y, z\}$ is not a 3-cycle, we have

$T(x,z)=T(x,y)=T(z,y)$. Since the values $T(x,z)$ and $T(z,y)$ do not depend upon our choice of z, $Z_{\frac{1}{2}}$ is an autonomous subset of $Z\cup Z_{\frac{1}{2}}$. Let $i\in\{0,1\}$ and $z_i\in Z_i$. If $T(z_i,z)\neq i$ for some $z\in Z_{\frac{1}{2}}$, then $\{x,y,z,z_i\}$ is a diamond. Hence, supposing that x,y does not belong to a diamond, we have $T(z_i,z)=i$ for all $z\in Z\cup Z_{\frac{1}{2}}$, proving that $Z\cup Z_{\frac{1}{2}}$ is autonomous.

(c) From our hypothesis, $Z\cup Z_{\frac{1}{2}}$ cannot be acyclic. Since $Z_{\frac{1}{2}}$ is an autonomous subset of $Z\cup Z_{\frac{1}{2}}$, $Z_{\frac{1}{2}}$ and no 3-cycle contains $\{x,y\}$, $Z_{\frac{1}{2}}$ cannot be acyclic. Let C be a 3-cycle included into $Z_{\frac{1}{2}}$. Then $C\cup Z$ is double diamond containing x and y, as claimed. □

With this claim, the proof is complete. □

Lemma 10. *Let T be a tournament , A be a subset of $V(T)$ and κ the number of acyclic components of T which meet A. Then there is a subset A' of $V(T)$ containing A such that $|A'\setminus A|\leq 3.\binom{\kappa}{2}$ and the acyclic components of $T_{\restriction A'}$ are the traces on A' of the acyclic components of T.*

Proof. Let $U:=\{X\in Ac(T): X\cap A\neq\emptyset\}$. For each $X\in U$, select $a_X\in X\cap X$. According to Lemma 9, for each pair of distinct elements a_X, a_Y, we may select a subset $F_{X,Y}$ containing a_X and a_Y such that $T_{\restriction F_{X,Y}}$ is either a 3-element cycle, a diamond, or a self-dual double diamond. Set $A':=A\cup\bigcup\{F_{X,Y}:\{X,Y\}\in[U]^2\}$ and $T':=T_{\restriction A'}$. The traces on A' of the acyclic decomposition of T form a partition of A' into acyclic autonomous subsets. □

Corollary 5. *If T is an acyclically indecomposable tournament, every subset A of $V(T)$ extends to a subset A' such that $T_{\restriction A'}$ is acyclically indecomposable and $|A'\setminus A|\leq 3.\binom{|A|}{2}$.*

Corollary 6. *Let T be a tournament and A be a subset of $V(T)$. If A meet each acyclic component of T, then the acyclic components of $T_{\restriction A}$ are the traces on A of the acyclic components of T.*

3.3. **Relation with logic formulas.** Let L be the first order language with equality for which the only non logical symbol is a binary predicate denoted $<$. In this language, tournaments are models of a universal sentence, namely the sentence $\theta:=\forall x\forall y(((x=y)\vee(y<x)\vee(x<y))\wedge((\neg y<x)\vee(\neg x<y)))$

Lemma 11. *There is a two-variables first-order formula $\phi(x,y)$ of the language of tournaments such that for every tournament T and every $(a,b)\in V(T)\times V(T)$, the pair $\{a,b\}$ is no included into an acyclic autonomous subset of T if and only if T satisfies $\phi(a,b)$.*

Proof. Set $\phi(x,y):=\phi_1(x,y)\vee\phi_2(x,y)\vee\phi_3(x,y)$ such that the satisfaction of $\phi_1(a,b)$, resp. $\phi_2(a,b)$, resp. $\phi_3(a,b)$, in a tournament T, expresses that a and b are two vertices of a 3-cycle, resp. a diamond, resp. are the end-vertices of a self-dual double diamond. For an example, $\phi_1(x,y):=\theta(x,y)\vee\theta(x,y)$, where $\theta(x,y):=x<y\wedge(\exists z(y<z\wedge z<x))$. This extends easily to $\phi_2(x,y)$ and $\phi_3(x,y)$. In particular, $\phi(x,y)$ is a universal sentence. □

Lemma 12. *There is a first-order sentence ϕ of the language of tournaments such that a tournament T is acyclically indecomposable if and only if it satisfies ϕ.*

Proof. Set $\phi := \forall x \forall y \phi(x, y)$. □

3.4. **Acyclic components and monomorphic parts.** Let a, b be two distinct vertices of a tournament T. Let $C(a, b)$ be the set of vertices x such that $\{a, b, x\}$ is a 3-cycle of T (that is $C(a, b) := \{x : T(x, a) = T(a, b) = T(b, x)\}$.

Lemma 13. *Let T be a tournament and A be a subset of $V(T)$.*

(1) If A is acyclic and autonomous then A is a monomorphic part of T.
(2) If A is a monomorphic part and no pair of distinct vertices of A belongs to a 3-cycle of T then A is included into an acyclic component of T.
(3) If A is a monomorphic part, there is a 3-cycle which contains two vertices of A if and only if A is included into some 3-cycle of T and
- either A is an autonomous 3-cycle,
- or $A = \{a, b\}$, $C(a, b)$ is acyclic and $\{a, b\} \cup C(a, b)$ is autonomous in T .
(4) If A is a monomorphic component then either A is an autonomous 3-cycle or $A = \{x, y\}$ and $A \cup C(\{a, b\})$ is autonomous, or A is an acyclic component of T.

Proof. Assertion (1) is obvious.

Assertion (2). If A was no included into an acyclic component of T then, according to Lemma 9, two distinct vertices x and y of A would belong either to a 3-cycle of T, or to some diamond, or to some self-dual double diamond. The first case is excluded by our hypothesis. The two other cases cannot happen. Indeed, if x and y belong to some diamond δ, then they cannot belong to the 3-cycle of the diamond. With no loss of generality we may suppose that δ is a positive diamond, eg $\delta = \delta^+ := (\{a, b, c, d\}, \{(a, b), (b, c), (c, a), (a, d), (b, d), (c, d)\})$, with $x := a, y := d$. Since A is a monomorphic part, the tournaments $T_{\restriction \{a,b,c\}}$ and $T_{\restriction \{d,b,c\}}$ must be isomorphic, which is impossible since the first one is a 3-cycle and the other is acyclic. Thus this case cannot happen. If x and y belong to some double diamond $D := \{ab, c, d, d'\}$, then since the previous cases cannot happen, x and y are the end-poinds d, d' of the double diamond. The two tournament $T_{\restriction \{a,b,c,d\}}$ and $T_{\restriction \{a,b,c,d'\}}$ are opposite, hence cannot be isomorphic. Whereas, since A is a monomorphic part, they must be isomorphic. A contradiction.

Assertion (3). Let A be a monomorphic part of T. Let C be a 3-cycle containing at least two vertices $\{a, b\}$ of A and let c be the third vertex of C.

Claim 2. *$A \subseteq C$.*

Indeed, otherwise let $y \in A \setminus C$. Since C contains, at least, two elements of C, there is one, say a, such that $T(c, a) = T(c, y)$. Hence $D := \{a, c, y\}$ is an acyclic tournament. Since $D \setminus A = C \setminus A$ and A is a monomorphic part, D and C must be isomorphic, which is not the case. This proves our claim.

Case 1. A has three elements. From Claim 2 above, $C = A$. Let $y \notin A$. Since $|A| = 3$, there are two element $a, b \in A$ such that $T(a, y) = T(b, y)$. We claim that for the remaining element c of A, $T(c, y) = T(a, y)$. Otherwise, $T(c, y) = T(y, a)$ and since from the claim above, $\{a, c, y\}$ cannot be a 3-cycle, $T(y, a) = T(c, a)$. By the same token, $T(c,b) = T(y,b)$, hence $T(c,a) = T(c,b)$, contradicting the fact that A forms a 3-cycle. This proves our claim. It follows that A is autonomous.

Case 2. A has two elements. First $C(a,b)$ is acyclic. Otherwise, if D be a 3-cycle included into $C(a,b)$ then $\{a\} \cup D$ and $\{b\} \cup D$ are two opposite diamonds, hence cdannot be isomorphic, contradicting the fact that A is a monomorphic part. Next, $X := A \cup C(a,b)$ is autonomous. Let $x \in V(T) \setminus X$ and $z \in C(a,b)$. We claim that the tournament $T_{\restriction \{a,b,x,z\}}$ is a diamond. Indeed, as a 4-vertices tournaments, it contains at most two 3-cycles. Since it contains the 3-cycle $\{a,b,z\}$, the restrictions $T_{\restriction \{a,z,x\}}$ and $T_{\restriction \{b,x,z\}}$- which are isomorphic since A is a monomorphic part- must be acyclic. Since $x \notin C(a,b)$, $\{a,b,x\}$ cannot be a cycle. Since $T_{\restriction \{a,b,x,z\}}$ contains just one 3-cycle, this is a diamond. hence $T(x,a) = T(x,b) = T(x,z)$, proving that X is autonomous.

Conversely, one can easily check that if A satisfies the stated conditions then it is a monomorphic part.

Assertion (4) follows immediately from Assertion (2).

□

From Assertion (1) and Assertion (4) we get:

Corollary 7. *Let T be a tournament and A be a subset of $V(T)$. If $|A| \geq 4$ then A is an acyclic component of T if and only if A is a monomorphic component of T.*

4. Proof of Theorem 5

Let T be a tournament which is a lexicographical sum of finitely many acyclic tournaments. Let p be the number of acyclic components, k be the number of the infinite components. According to Lemma 13, each acyclic component is a monomorphic part of T, and according to Corollary 7, T has exactly k infinite monomorphic components, hence from part 1 of Theorem 5, the generating series H_{φ_T} is a rational fraction of the form:

$$\frac{P(x)}{(1-x)(1-x^2)\cdots(1-x^k)}$$

where $P \in \mathbb{Z}[x]$. Furthermore, $\varphi_R(n) \simeq an^{k'}$ for some $k' \leq k-1$. To complete the proof of Theorem 5, it remains to prove that $k' = k-1$. This is a consequence of Proposition 3 below.

Let n be a positive integer. A *partition of* n is a finite decreasing sequence $x_1 \geq \cdots \geq x_k$ of positive integers such that $x_1 + \cdots + x_k = n$. The integers in this sequence are the *parts of the partition.* Set $\mathfrak{p}_k(n)$ for the number of partitions of the integer n into at most k parts, and set $\mathfrak{p}_k(0) := 1$. As it is well-known, the generating series $\mathcal{H}_{\mathfrak{p}_k} := \sum_{n=0}^{\infty} \mathfrak{p}_k(n)x^n$ is the rational fraction $\frac{1}{(1-x)\cdots(1-x^k)}$ and $\mathfrak{p}_k(n) \simeq \frac{n^{k-1}}{(k-1)!k!}$. We also recall that the partition function $\mathfrak{p}$ counts the number $\mathfrak{p}(n)$ of partitions of the integer n. A famous result of Hardy and Ramanujan, 1918, asserts that $\mathfrak{p}(n) \simeq \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2n}{3}}}$.

Proposition 3. *If a tournament T is a finite lexicographical sum of acyclic tournaments, then*

$$\varphi_T(n) \geq \mathfrak{p}_k(n-p) \tag{3}$$

where p is the number of acyclic components of T, k is the number of the infinite one and $n \geq p$. In particular the growth of φ_T is at least a polynomial with degree $k-1$.

Proof. Let $Ac(T) := \{A_1, \cdots, A_p\}$ be the set of acyclic components of T, enumerated in such a way that $A_1, \ldots, A_k$ are infinite. Let $n \geq p$. To a decreasing sequence $\vec{x} := x_1 \geq \cdots \geq x_{k'}$ of positive integers such that $x_1 + \cdots + x_{k'} = n - p$ and $k' \leq k$ associate the p-element sequence $1 + \vec{x} = (1 + x_1, \ldots, 1 + x_k, 1, \ldots, 1)$ and a subset $A_{\vec{x}}$ of $V(T)$ such that $|A_{\vec{x}} \cap A_i| = x_i + 1$ if $i \leq k'$ and $|A_{\vec{x}} \cap A_i| = 1$ otherwise. Set $T_{\vec{x}} := T_{\restriction A_{\vec{x}}}$.

Claim If $\vec{x} \neq \vec{x'}$ then $T_{\vec{x}}$ and $T_{\vec{x'}}$ are not isomorphic.

Since T contains at least an element of each acyclic component of T, the acyclic decomposition of $T_{\vec{x}}$ is induced by the acyclic decomposition of T (Corollary 6). Hence, $\overline{ac}(T_{\vec{x}}) = 1 + \vec{x}$. If $T_{\vec{x}}$ and $T_{\vec{x'}}$ are isomorphic, $ac(T_{\vec{x}}) = ac(T_{\vec{x'}})$ (Corollary 4), thus $\overline{ac}(T_{\vec{x}}) = \overline{ac}(T_{\vec{x'}})$, that is $1 + \vec{x} = 1 + \vec{x'}$ which yields $\vec{x} = \vec{x'}$.

Inequality (3) follows immediately. □

5. Twelve tournaments

Let $C := (A, <)$ be an acyclic tournament. We define six tournaments, denoted respectively by $C_{3[C]}$, $V_{[C]}$, $T_{[C]}$, $U_{[C]}$, $H_{[C]}$ and $K_{[C]}$.

- The tournament $C_{3[C]}$ is the lexicographical product $C_3.C$ of C_3 by C.
- The vertex set of $V_{[C]}$ is $A \times 2 \cup \{a\}$, where $2 := \{0, 1\}$ and $a \notin A \times 2$. A pair (e, e') of vertices is an edge of $V_{[C]}$ in the following cases:

(i) $e = (x, i)$, $e' = (x', i')$ and either $x < x'$ or $x = x'$ and $i < i'$;
(ii) $e = a$, $e' = (x', 0)$;
(iii) $e = (x, 1)$ and $e' = a$.

- The four remaining tournaments have the same vertex set, namely $A \times 2$. In order to define their edge sets, let $i \in 2$. Set $h_i := ((0, i), (1, i))$, $v_i := ((i, 0), (i, 1))$, $d_i := ((0, i), (1, i+1))$ (where $i+1 = 1$ if $i = 0$ and 0 otherwise). Let $X \subseteq 2 \times 2 \setminus \{v_1, v_1^{-1}\}$. Let $\Delta(C, X)$ be the directed graph whose vertex set is $A \times 2$ and edge set the union of the following three sets:

(a) $\{((x, i), (x, j)) : ((0, i), (0, j)) \in X\}$;
(b) $\{((x, i), (y, j)) : (x, y) \in E(C) \text{ and} ((0, i), (1, j)) \in X\}$;
(c) $\{((x, i), (y, j)) : (y, x) \in E(C) \text{ and} ((1, i), (0, j)) \in X\}$.

Set $Y := \{h_0, v_0\}$. If $X := \{d_0^{-1}, d_1^{-1}, h_1\} \cup Y$, resp. $X := \{d_0^1, d_1, h_1^{-1}\} \cup Y$, resp. $X := \{d_0^{-1}, d_1, h_1\} \cup Y$, resp. $X := \{d_0^{-1}, d_1, h_1^{-1}\} \cup Y$ then $\Delta(C, X)$ is a tournament denoted by $T_{[C]}$, resp. $U_{[C]}$, resp. $H_{[C]}$, resp. $K_{[C]}$.

Conditions $(a), (b), (c)$ above simply mean that $\Delta(C, X)$ is a skew product of a binary relation on $\{0, 1\}$ by the chain $(A, \leq)$. We choose X in such a way that $\Delta(C, X)$ is a tournament and in fact a skew product of the tournament $\underline{2}$ (for which $0 < 1$) by $(A, \leq)$. Deciding furthermore that this tournament will contains all pairs $((x, 0), (y, 0))$ such that $x < y$, we have only eight possible choices for the three remaining pairs belonging to X. It turns out that three choices yield acyclic tournaments. On the remaining five choices, two tournaments are dual of each other, namely $U_{[C]}$ and $U'_{[C]} := \Delta(C, X)$ where $X := \{d_0^{-1}, d_1^{-1}, h_1^{-1}\} \cup Y$. We do not need to add to our list tournaments of the form $U'_{[C]}$. Indeed, our aim is to obtain a minimal list of unavoidable infinite acyclically indecomposable tournaments. And it follows from Item (14) of Lemma 14 below, that $U_{[\omega^*]}$ is embeddable $U'_{[\omega]}$ and $U_{[\omega]}$ is embeddable $U'[\omega^*]$.

Lemma 14. *Let C be an acyclic tournament, then: (i) $(C_3.C)^*$ is isomorphic to $C_3.(C^*)$; (ii) $(V_{[C]})^*$ is isomorphic to $V_{[C^*]}$; (iii) $(T_{[C]})^*$ is isomorphic to $T_{[C^*]}$; (iv) If $\underline{2}.C$ is embeddable in C then $(U_{[C]})^*$ and $U_{[C^*]}$ are embeddable in each other; (v) $(H_{[C]})^*$ is isomorphic to $H_{[C^*]}$; (vi) $K_{[C]}$ is self dual.*

Proof. We only check Assertion (iv). We claim that $(U_{[C]})^*$ is embeddable in $U_{[C^*]}$. Indeed, let $\varphi : A\times 2 \to (A\times 2)\times 2$ defined by $\varphi((x,i)) := ((x,i),i)$. Then, as it is easy to check, φ is an embedding from $(U_{[C]})^*$ into $U_{[(2.C)^*]}$. From our hypothesis $(\underline{2}.C)^*$ is embeddable in C^*, thus $U_{[(2.C)^*]}$ is embeddable in $U_{[C^*]}$. This proves our claim. Applying this claim to C^* we get that $U_{[C^*]}$ is embeddable in $(U_{[C]})^*$ as required. □

We denote respectively by $\mathfrak{C}_3$, $\mathfrak{V}$, $\mathfrak{T}$, $\mathfrak{U}$, $\mathfrak{H}$, and $\mathfrak{K}$ the collections of tournaments $C_{3[C]}$, $V_{[C]}$, $T_{[C]}$, $U_{[C]}$, $H_{[C]}$ and $K_{[C]}$ when C describe all possible acyclic tournaments. We denote by $\mathfrak{C}_{3,<\omega}$, resp. $\mathfrak{V}_{<\omega}$, $\mathfrak{T}_{<\omega}$, $\mathfrak{U}_{<\omega}$, $\mathfrak{H}_{<\omega}$, $\mathfrak{K}_{<\omega}$, the collection of finite tournaments which are embeddable into some member of the corresponding collection.

Some members of $\mathfrak{V}_{<\omega}$, $\mathfrak{T}_{<\omega}$ and $\mathfrak{U}_{<\omega}$ and have been considered previously. We will refer to some known properties of these tournaments. We use the presentation given in [1]. Let $h \geq 2$ be an integer, denote by T_{2h+1}, U_{2h+1} and V_{2h+1} the tournaments defined on $\{0,\dots,2h\}$ as follows.
(i)$T_{2h+1\restriction\{0,\dots,h\}} = U_{2h+1\restriction\{0,\dots,h\}} = 0 < \dots < h$, $T_{2h+1\restriction\{h+1,\dots,2h\}} = (U_{2h+1})^*_{\restriction\{h+1,\dots,2h\}} = h+1 < \dots < 2h$.
(ii) For every $i \in \{0,\dots,h-1\}$, if $j \in \{i+1,\dots,h\}$ and if $k \in \{0,\dots,i\}$, then $(j, i+h+1)$ and $(i+h+1, k)$ belong to $E(T_{2h+1})$ and $E(U_{2h+1})$.
(iii)$V_{2h+1\restriction\{0,\dots,2h-1\}} = 0 < \dots < 2h-1$ and for $i \in \{0,\dots,h-1\}$, $(2i+1, 2h)$ and $(2h, 2i)$ belong to $E(V_{2h+1})$.

According to Schmerl and Trotter [25], these tournaments are indecomposable and moreover a finite tournament T on at least five vertex is *critically indecomposable* (in the sense that T is indecomposable and for every $x \in V(T)$, the subtournament $T_{\restriction V(T)\setminus\{x\}}$ is not indecomposable) if and only if it is isomorphic to one of these tournaments.

We will need the following result [1].

Lemma 15. *Given three integers h_1, h_2, $h_3 \geq 2$, the tournaments V_{2h_3+1}, T_{2h_1+1} and U_{2h_2+1} are incomparable with respect to embeddability.*

These tournaments belong to $\mathfrak{V}_{<\omega}$, $\mathfrak{T}_{<\omega}$ and $\mathfrak{U}_{<\omega}$. Indeed:

Fact 1. *V_{2h+1} is isomorphic to $V_{\underline{h}}$, T_{2h+1} is isomorphic to $T_{[\underline{h+1}]}$ minus the vertex $(0,1)$ and U_{2h+1} is isomorphic to $U_{[\underline{h+1}]}$ minus the vertex $(0,0)$.*

Lemma 16. *(i) If $C := (A,<)$ is an non-empty acyclic tournament, $C_3[C]$ is acyclically indecomposable and not indecomposable except if $|A| = 1$. In fact, no indecomposable subset of $C_3[C]$ has more than three elements. (ii)$V[C]$ is indecomposable,* *hence acyclically indecomposable. (iii)$T_{[C]}$ is indecomposable, except if $|A| \geq 2$ and C has a least and largest element. In this latter case $\{(m,0),(M,1)\}$ (where m and M are the least and largest element of C) is an acyclic component, $T_{[C]}$ minus the vertex $(m,0)$ is isomorphic to $\check{T}_{[C]}$ and is indecomposable. (iv) $U_{[C]}$ is acyclically indecomposable for $|A| \geq 2$. If moreover C has a least element $U_{[C]}$ is not indecomposable, $U_{[C]}$ minus the vertex $(m,0)$ (where m is the least element of C) is isomorphic to $\check{U}_{[C]}$ and is indecomposable. (v) $H_{[C]}$ is indecomposable except for $|A| = 2$. (vi) $K_{[C]}$ is never indecomposable; in fact its indecomposable subsets have at most three elements. It is acyclically indecomposable except if C has a least element; in this latter case $\{(m,0),(m,1)\}$ (here m is the least element de C) is an acyclic component, and $K_{[C]}$ minus the vertex $(m,0)$ is isomorphic to $\check{K}_{[C]}$.*

Proof. Assertions (i). Every pair of distinct vertices of $C_3.C$ is included in a 3-cycle or a diamond. Thus from Lemma 9, $C_3.C$ is acyclically indecomposable. The second part of the sentence is obvious. Assertions (ii), (iii) and (iv) follow directly from Fact1 and the fact that V_{2h+1}, T_{2h+1} and U_{2h+1} are indecomposable. Assertion (v) follows by inspection. Note that every pair of distinct vertices of $H_{[C]}$ is included in a 3-cycle. Assertion (vi). The first part follows from the fact that $A' \times 2$ is an automous subset of $K_{[C]}$ for every initial interval A' of C. The second part follows from the fact that every pair of distinct vertices of $K_{[C]}$ or of $K_{[C]}$ minus the vertex $(m,0)$ if C has a least element m, is included in a 3-cycle. □

From this, we deduce first:

Corollary 8. *Each member of $\mathfrak{B}$ except $K_{[\omega]}$ is acyclically indecomposable. The tournament $\check{K}_{[\omega]}$ is isomorphic to $K_{[\omega]}$ minus the vertex $(0,0)$. In particular, it contains an isomorphic copy of $K_{[\omega]}$.*

From our definitions, we have immediately this:

Fact 2. *The age of V_ω and V_{ω^*} is $\mathfrak{V}_{<\omega}$. The age of $C_3.\omega$ and $C_3.\omega^*$ is $\mathfrak{C}_{3,<\omega}$.The age of T_ω and T_{ω^*} is $\mathfrak{T}_{<\omega}$. The age of U_ω and U_{ω^*} is $\mathfrak{U}_{<\omega}$. The age of H_ω and H_{ω^*} is $\mathfrak{H}_{<\omega}$. The age of K_ω and K_{ω^*} is $\mathfrak{K}_{<\omega}$.*

With the help of Lemma 16 we obtain:

Corollary 9. *For every $X_{[\alpha]} \in \mathfrak{B}$, $\check{X}_{[\alpha]}$ is an increasing union of $\check{X}_{[\underline{n}]}$ for $n \in \mathbb{N}$. In particular, the age of $\check{X}_{[\alpha]}$ is the collection of finite tournaments which are embeddable in some $\check{X}_{[\underline{n}]}$ for some integer n.*

Lemma 17. *The six ages $\mathfrak{C}_{3,<\omega}$, $\mathfrak{V}_{<\omega}$, $\mathfrak{T}_{<\omega}$, $\mathfrak{U}_{<\omega}$, $\mathfrak{H}_{<\omega}$, $\mathfrak{K}_{<\omega}$ are incomparable with respect to inclusion.*

Proof. Let $\mathcal{A} := \{\mathfrak{C}_{3,<\omega}, \mathfrak{V}_{<\omega}, \mathfrak{T}_{<\omega}, \mathfrak{U}_{<\omega}, \mathfrak{H}_{<\omega}, \mathfrak{K}_{<\omega}\}$. Denote by $\neg\mathfrak{C}_{3,<\omega}$ the set $\bigcup(\mathcal{A}\setminus\{\mathfrak{C}_{3,<\omega}\})$ and define similarly $\neg\mathfrak{V}_{<\omega}$, $\neg\mathfrak{T}_{<\omega}$ etc. Let τ_1 (resp.τ_2)be a tournament obtained by replacing every vertex of a 2-element tournament (resp. a vertex of a 3-cycle) by a 3-cycle. We prove successively that (i) $\tau_1 \in \mathfrak{C}_{3,<\omega} \setminus \neg\mathfrak{C}_{3,<\omega}$; (ii) $\tau_2 \in \mathfrak{K}_{<\omega} \setminus \neg\mathfrak{K}_{<\omega}$. (iii) $T_5 \in \mathfrak{T}_{<\omega} \setminus \neg\mathfrak{T}_{<\omega}$; $V_7 \in \mathfrak{V}_{<\omega} \setminus \neg\mathfrak{V}_{<\omega}$; (iv) $U_7 \in \mathfrak{U}_{<\omega} \setminus \neg\mathfrak{U}_{<\omega}$; (v) $H_{[\underline{3}]} \in \mathfrak{H}_{<\omega} \setminus \neg\mathfrak{H}_{<\omega}$;

The proofs of the first and second assertions are immediate. Concerning the next three one, we may note that according to Lemma 15 and Corollary 9, $\mathfrak{V}_{<\omega}$, $\mathfrak{T}_{<\omega}$, and $\mathfrak{U}_{<\omega}$ are pairwise incomparable w.r.t. inclusion. In fact, we derive these three assertions from the following observations:

- The 3-cycle is the only indecomposable subtournament of the tournaments $C_3.\omega$ and $K_{[\omega]}$.
- Up to isomorphism, T_{2p+1} (resp. V_{2p+1}, U_{2p+1}) where $p \geq 2$, are the only finite indecomposable subtournaments on at least 5 vertices of $T_{[\omega]}$(resp. $V_{[\omega]}$, $U_{[\omega]}$).
- The tournaments T_5, V_7 and U_7 are not embeddable into $H_{[\omega]}$.

For the last assertion, we observe that the tournament $H_{[3]}$ is indecomposable and use the previous observations. □

Lemma 18. *Members of $\mathfrak{B}$ are pairwise incomparable with respect to embeddability.*

Proof. According to Fact 2 and Lemma 17 it suffices to prove that:

Claim 3. *If $\alpha \in \{\omega, \omega^*\}$, $X_{[\alpha]}$ does not embed into $X_{[\alpha^*]}$.*

If $X_{[\alpha]}$ is $C_{3[\alpha]}$, $V_{[\alpha]}$, $T_{[\alpha]}$ or $H_{[\alpha]}$ this is obvious: α is embeddable in $X_{[\alpha]}$ but not in $X_{[\alpha^*]}$. If $X_{[\alpha]} = U_{[\alpha]}$, note that for an arbitrary acyclic tournament C, $U_{[C]}$ can be divided into two acyclic subsets A_0 and A_1 such that no 3-cycle contains more than one vertex of A_0 and every pair of distinct vertices of A_1 is included in to some 3-cycle (set $A_0 := A \times \{0\}$ and $A_1 := A \times \{1\}$). Since in $U_{[\omega]}$, A_0 has type ω, whereas in $U_{[\omega^*]}$, A_0 has type ω^*, $U_{[\omega]}$ is not embeddable in $U_{[\omega^*]}$. If $X_{[\alpha]} = K_{[\alpha]}$, note that each autonomous set of $K_{[\omega]}$ is finite whereas each autonomous set of $K_{[\omega^*]}$ is cofinite. Hence, $K_{[\omega]}$ is not embeddable in $K_{[\omega^*]}$. □

This is the first part of Lemma 1. We give the proof of the second part in the next section.

6. Profiles of members of $\mathfrak{B}$

According to Fact 2, our twelve acyclically indecomposable tournaments yield only six ages, those of $C_{3[\omega]}$, $V_{[\omega]}$, $T_{[\omega]}$, $U_{[\omega]}$, $H_{[\omega]}$ and $K_{[\omega]}$. For three of these tournaments, the exact values of the profile are know or easy to compute. For the others, we make no attempt of an exact computation.

6.1. **Profile of $C_{3[\omega]}$.** The first values are

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189.$$

The sequence is A000930 in [26]. It satisfies the following recurrence $\varphi_{C_{3[\omega]}}(n) = \varphi_{C_{3[\omega]}}(n-1)+\varphi_{C_{3[\omega]}}(n-3)$ for $n \geq 3$. The Hibert series is $H_{\varphi_{C_{3[\omega]}}}(x) := 1/(1-x-x^3)$. According to [6], $\varphi_{C_{3[\omega]}}(n) = \lfloor d * c^n + 1/2 \rfloor$ where c is the real root of $x^3 - x^2 - 1$ and d is the real root of $31 * x^3 - 31 * x^2 + 9 * x - 1$ ($c = 1.465571231876768...$ and $d = 0.611491991950812...$).

6.2. **Profile of $V_{[\omega]}$.** The first values are:

$$1, 1, 1, 2, 4, 9, 21, 48.$$

Fact 3. $\varphi_{V_{[\omega]}}(n) \geq 2^{n-5}$.

Proof. In $V_{[\omega]}$, the vertex a (or more exactly $\{a\}$) is the intersection of two 3-cycles. We prove that the number of n-element restrictions of $V_{[\omega]}$ for which a is the intersection of two 3-cycles is at least 2^{n-5}. For that, let $n \geq 5$ be an integer. For each subset A of $\{1, \dots, n-5\}$, set $\overline{A} := \{a\} \cup ((A \cup \{0, n-4\}) \times \{0\}) \cup ((\{0, \dots n-4\} \setminus A) \times \{1\})$. As it is easy to see, the restrictions of $V_{[\omega]}$ to the 2^{n-5} subsets A of $\{1, \dots n-5\}$ are pairwise non isomorphic. □

6.3. **Profile of $T_{[\omega]}$.** The tournament $T_{[\omega]}$ is diamond-free. Its age is the collection of finite diamond-free tournaments. There is a countable homogeneous tournament L whose age is this collection. Thus $T_{[\omega]}$ and L have the same profile. According to Cameron [3]:

$$\varphi_L(n) = \frac{1}{2n} \sum_{d|n, d \text{odd}} \phi(d) 2^{n/d} \tag{4}$$

where ϕ is the Euler's totient function.

As an immediate corollary we have $T_{[\omega]}(n) \geq (2-\epsilon)^n$.

6.4. **Profile of $U_{[\omega]}$.**

Lemma 19. *$\varphi_{U_{[\omega]}}(n) \geq (1-\epsilon)2^{n-2}$ for every $\epsilon > 0$ and n large enough.*

Proof. Let h and n be two integers with $5 \leq 2h+1 \leq n$. Denote by $\Sigma_h(n)$ be the collection of tournaments on n vertices which are a lexicographical sum of non-empty acyclic tournaments indexed by U_{2h+1}. Let $\Sigma(n) := \bigcup\{\Sigma_h(n) : 5 \leq 2h+1 \leq n\}$ and let $N_h(n)$, resp. $N(n)$, be the number of members of $\Sigma_h(n)$, resp. $\Sigma(n)$.

It is easy to check that each member of $\Sigma(n)$ is embeddable in $U_{[\omega]}$. We claim that two members of $\Sigma(n)$ are isomorphic if and only if they are equal. Indeed, first $\Sigma_h(n)$ and $\Sigma_{h'}(n)$ are disjoint whenever $h \neq h'$ (If they are not disjoint, then since U_{2h+1} and $U_{2h'+1}$ are indecomposable there are isomorphic, hence $h = h'$). Next, observe that U_{2h+1} is *rigid*, that is the identity map is the unique automorphism of U_{2h+1}. This observation follows readily from the fact that $\{h+1, \dots, 2h\}$ is the set of centers of diamonds of U_{2h+1} [1]. From the rigidity of U_{2h+1} follows that a lexicographic sum $\sum_{i \in U_{2h+1}} \underline{m}_i$ of non-empty acyclic tournaments $\underline{m}_i$ determines entirely the sequence $m_0, \dots, m_{2h}$.This proves our claim. From this claim, we deduce first that: $\varphi_{U_{[\omega]}}(n) \geq \sum\limits_{h=2}^{p} N_h(n)$ where $p = \lfloor \frac{n-1}{2} \rfloor$. We deduce next that $N_h(n)$ is the number of integer solutions of the equation: $n_1 + \cdots + n_{2h+1} = n - 2h - 1$, that is $N_h(n) = \binom{n-1}{2h}$. Combining these two facts, we have $\varphi_{U_{[\omega]}}(n) \geq \sum\limits_{h=2}^{p} \binom{n-1}{2h} = \sum\limits_{h=0}^{p} \binom{n-1}{2h} - 1 - \binom{n-1}{2} = 2^{n-2} - 1 - \binom{n-1}{2}$. This proves the lemma. □

6.5. **Profile of $H_{[\omega]}$.**

Lemma 20. *$\varphi_{H_{[\omega]}}(n) \geq (1-\epsilon)2^{n-4}$ for every $\epsilon > 0$ and n large enough.*

Proof. The proof is somewhat similar to the proof of Lemma 19.

Let $h \geq 3$ be an integer. Set $A_h := \{(3k, i) : k < h\}$ and $Z_h := \{(3k+1, 0), (3k+2, 1) : k < h\}$. Let $Z \subseteq Z_h$; set $H_h(Z) := H_{[\omega]\restriction V_h \cup Z}$. Let n be an integer with $n \geq 2h$. Denote by $\Sigma'_h(Z, n)$ the collection of tournaments T on n vertices which are a lexicographical sum $\sum_{i \in H_h(Z)} \underline{m}_i$ of non-empty acyclic tournaments $\underline{m}_i$, subject to the requirement that $m_i = 1$ for each $i \not\in Z$. Let $\Sigma'_h(n) := \bigcup\{\Sigma'_h(Z, n) : Z \subseteq Z_h\}$, $\Sigma'(n) := \bigcup\{\Sigma'_h(n) : 6 \leq 2h \leq n\}$ and let $N'_h(Z, n)$, resp. $N'_h(n)$, resp. $N'(n)$be the size of $\Sigma'_h(Z, n)$, resp. $\Sigma'_h(n)$, resp. $\Sigma'(n)$.

It is easy to check that each member of $\Sigma'(n)$ is embeddable in $H_{[\omega]}$. We claim that two members of $\Sigma'(n)$ are isomorphic if and only if they are equal. This claim follows from the fact that $H_h(Z)$ is indecomposable and rigid for every $Z \subseteq Z_h$. Indeed, note that from this fact $H_h(Z)$ and $H'_h(Z')$ are isomorphic if and only if there are equal, in particular $\Sigma'_h(n)$ and $\Sigma'_{h'}(n)$ are disjoint whenever $h \neq h'$. We leave the checking of the fact mentionned above to the reader (we only note that $H_h(\emptyset)$ is isomorphic to $H_{\underline{h}}$. From this claim, we deduce first that $\varphi_{H_{[\omega]}}(n) \geq \sum\limits_{h=2}^{p} N_h(n)$ where $p = \lfloor \frac{n}{2} \rfloor$. We deduce next that $N_h(n)$ is the number of integer solutions of the equation: $n_1 + \cdots + n_{2h-2} = n - 2h$, that is $N_h(n) = \binom{n-3}{2h-3}$. Combining these two facts, we have $\varphi_{H_{[\omega]}}(n) \geq \sum\limits_{h=3}^{p} \binom{n-3}{2h-3} = \sum\limits_{j=1}^{\lfloor \frac{n-4}{2} \rfloor} \binom{n-3}{2j+1} \geq 2^{n-4} - \binom{n-3}{1} - 1$. The conclusion of the lemma follows. □

6.6. **Profile of $K_{[\omega]}$.**

Lemma 21. *$\varphi_{K_{[\omega]}}(n) = 2^{n-2}$ for every $n \geq 2$.*

Proof. We have $\varphi_{K_{[\omega]}}(0) = \varphi_{K_{[\omega]}}(1) = \varphi_{K_{[\omega]}}(2) = 1$. We prove that:

$$\varphi_{K_{[\omega]}}(n) = 1 + \sum_{j=1}^{n-2} (n-j-1)\varphi_{K_{[\omega]}}(j) \tag{5}$$

for $n \geq 3$. The lemma follows by induction on n. Since $\varphi_{K_{[\omega]}}(3) = 2$, formula (5) holds. Hence we may suppose $n \geq 4$.

Denote by $f_{K_{[\omega]}}(n)$ (resp. $g_{K_{[\omega]}}(n)$) the number of strongly connected (resp. non strongly connected) subtournaments of $K_{[\omega]}$ having n vertices, these tournaments being counted up to isomorphism. We have $f_{K_{[\omega]}}(0) = f_{K_{[\omega]}}(1) = f_{K_{[\omega]}}(2) = 0$, $f_{K_{[\omega]}}(3) = 1$. More generally, we have $f_{K_{[\omega]}}(n) = \varphi_{K_{[\omega]}}(n-2)$ for $n \geq 4$. Indeed, every strongly connected subtournament of $K_{[\omega]}$ having n vertices, is obtained by dilating some vertex of a 3-cycle by a subtournament of $K_{[\omega]}$ having $(n-2)$ vertices of $K_{[\omega]}$. On an other hand, $g_{K_{[\omega]}}(n) = 1 + \sum\limits_{p=3}^{n-1} (n-p+1) f_{K_{[\omega]}}(p)$ for $n \geq 4$. Indeed, every non acyclic and non strongly connected subtournament of $K_{[\omega]}$ has exactly one strongly connected component which is not a singleton. Hence, the number of non strongly connected subtournaments of $K_{[\omega]}$ on n vertices having a strongly connected component on p vertices of a given isomorphy type is the number of integer solutions of the equation: $n_1 + n_2 = n - p$. This number being $\binom{n-p+1}{1} = n - p + 1$ the above formula follows. With the fact that $\varphi_{K_{[\omega]}}(n) = f_{K_{[\omega}}(n) + g_{K_{[\omega]}}(n)$, this yield formula (5). □

7. Proof of Theorem 2

Let T be an infinite acyclically indecomposable tournament. We prove that some member of $\mathfrak{B}$ is embeddable in T. The first step is:

Claim 4. *$V(T)$ contains an infinite subset A such that:*

(1) Either every pair of distinct elements of A is included into a 3-cycle of T.
(2) Or $V(T)$ contains no infinite subset whose pairs of distinct elements are included into a 3-cycle of T and either:
  (a) every pair of distinct elements of A is included into a diamond of T or
  (b) every pair of elements of A forms the end-vertices of some double diamond of T but A contains no infinite subset whose pairs of distinct elements are included into some diamond.

Proof of Claim 4. Suppose that neither (1) nor (2-a) holds. Let $f : \mathbb{N} \to V$ a one-to-one map. We define successively three subsets X_1, X_2, X_3 made of pairs $\{n, m\}$ of $[\mathbb{N}]^2$, such that $n < m$, depending wether $\{f(n), f(m)\}$ is contained into:

(a) some 3-cycle of T;
(b) some diamond;
(c) the endpoint of a self dual double diamond.

According to Lemma 9, $[\mathbb{N}]^2 = X_1 \cup X_2 \cup X_3$. Hence, from Ramsey's Theorem, there is an infinite subset Y of $\mathbb{N}$ and $i \in \{1, 2, 3\}$ such that $[Y]^2 \subseteq X_i$. Set $A := \{f(n) : n \in Y\}$. With our supposition, Case (a) and (b) are impossible. Thus A satisfies condition (2-b) as claimed. □.

Next, we prove that in case (1) some member of $\mathfrak{B} \setminus \{V_{[\omega]}, V_{[\omega^*]}\}$ is embeddable in T. In case (2-a), some member of $\{C_{3[\omega]}, C_{3[\omega^*]}, V_{[\omega]}, V_{[\omega^*]}\}$ is embeddable in T and, in case (2-b), $C_{3[\omega]}$ or $C_{3[\omega^*]}$ is embeddable in T. The following lemmas take care of each case.

Lemma 22. *If a tournament T contains an infinite subset A such that every pair of distinct elements of A is included into a 3-cycle then some member of $\mathfrak{B} \setminus \{V_{[\omega]}, V_{[\omega^*]}\}$ is embeddable in T.*

Proof. Let $f : \mathbb{N} \to A$ be a one-to-one map. The hypothesis on A allows to define a map $g : [\mathbb{N}]^2 \to V(T)$ such that $\{f(n), f(m), g(n, m)\}$ is a 3-cycle of T for every $n < m$. Let $\Phi := \{f, g\}$, let $\mathfrak{L} :=< \omega, T, \Phi >$ and for a subset X of $\mathbb{N}$, let $\Phi_{\restriction X} := \{f_{\restriction X}, g_{\restriction [X]^2}\}$ and let $\mathfrak{L}_{\restriction X} :=< \omega_{\restriction X}, T, \Phi_{\restriction X} >$. According to Theorem 6, there is an infinite subset X of $\mathbb{N}$ such that $\mathfrak{L}_{\restriction X}$ is invariant.

Via a relabelling of X with the integers, we may suppose that $X = \mathbb{N}$. Hence $\mathfrak{L}$ is invariant. Let A' be the image of f.

Claim 5. *1. $T(f(n), f(m))$ is constant on pairs (n, m) such that $n < m$.*
2. $T(f(n), g(m, k))$ is constant on triples (n, m, k) such that $n < m < k$.
3. $g(n, m) \notin \{f(k), g(n', m')\}$ for all $k, n < m < n' < m'$.

4. If $T(g(n,m), g(m,n')) \neq T(g(n,m), g(n',m'))$ for some $n < m < n' < m'$ then for $D := \{g(4k+i, 4k+i+1) : k \in \mathbb{N}, i \in \{0,1,2\}\}$, $T_{\restriction D}$ is isomorphic to $C_3.\omega$ or to $C_3.\omega^*$.

Proof of Claim 5. Item 1. Since $\mathfrak{L}$ is invariant, if $T(f(n), f(m)) = 1$ for some $n < m$ then $T(f(n'), f(m'))$=1 for all $n' < m'$.
Item 2. Same argument that in Item 1.
Item 3. According to Item 1, if $T(f(n), f(m)) = 1$, f is an isomorphism from ω onto $T_{\restriction A'}$. Similarly, if $T(f(n), f(m)) = 0$ for some $n < m$ then f is an isomorphism from ω onto $T^*_{\restriction A'}$. In both cases, $T_{\restriction A'}$ is acyclic, hence it cannot contain a 3-cycle. This proves that there are no k and no $n < m$ such that $g(n,m) = f(k)$. Now, suppose that $g(n,m) = g(n',m')$ for some $n < m < n' < m'$; pick m'' with $m' < m''$. Let h be the local isomorphism from ω to ω defined by $h(n) = n$, $h(m) = m$, $h(n') = m'$, $h(m') = m''$. Since $\mathfrak{L}$ is invariant, $g(n,m) = g(m',m'')$, hence $g(n',m') = g(m',m'')$. But, as we have seen above, $T(f(n'), f(m')) = T(f(m'), f(m''))$. Since $\{f(n'), f(m'), g(n',m')\}$ is a 3-cycle, $T(f(n'), f(m')) = T(f(m'), g(n',m'))$. A similar argument yields $T(g(m',m''), f(m')) = T(f(m'), f(m''))$, hence $T(g(m',m''), f(m')) = T(f(m'), g(n',m'))$. Since T is a tournament, $g(n,m) \neq g(n',m')$, a contradiction.
Item 4. Let $k \in \mathbb{N}$. Set $x_{i,k} := g(4k+i, 4k+i+1)$ for $i \in \{0,1,2\}$ and $k \in \mathbb{N}$ and set $D_k := \{x_{i,k} : i \in \{0,1,2\}\}$. Since $\mathfrak{L}$ is invariant, we have:

$$T(x_{i,k}, x_{i+1,k}) \neq T(x_{i,k}, x_{i+2,k})$$

Again by the invariance of $\mathfrak{L}$, we have:

$$T(x_{i,k}, x_{i+1,k}) = T(x_{i+1,k}, x_{i+2,k})$$

hence $T_{\restriction D_k}$ is a 3-cycle. By the invariance of $\mathfrak{L}$, $T(x_{i,k}, x_{i',k'})$ is constant on the pairs $(x_{i,k}, x_{i',k'})$ such that $k < k'$. If the value is 1, $T_{\restriction D}$ is isomorphic to $C_3.\omega$ and if the value is 0, $T_{\restriction D}$ is isomorphic to $C_3.\omega^*$. □

In order to get the conclusion of Lemma 22, we may suppose that neither $C_3.\omega$, nor $C_3.\omega^*$, is embeddable in T. According to Item 1 of Claim 5, $T(f(n), f(m))$ is constant on the pairs (n,m) such that $n < m$. We may suppose that $T(f(n), f(m)) = 1$ (otherwise, it suffices to replace T by T^*). We will consider two cases:
Case 1. There are some $n < m < k$ such that $T(f(k), g(n,m)) = 1$.

In this case, since $\mathfrak{L}$ is invariant, $T(f(k'), g(n',m')) = 1$ for all $n' < m' < k'$.
Case 2. $T(f(k), g(n,m)) = 0$ for all $n < m < k$.

Let $F : \mathbb{N} \times \{0,1\} \to V(T)$ defined by setting $F(n,1) := f(n), F(n,0) := g(n,n+1)$. According to Item 3 of Claim 5, F is one to one. Let T' be the tournament with vertex set $\mathbb{N} \times \{0,1\}$, such that $T'(x,y) = T(F(x), F(y))$ for every pair of vertices of $\mathbb{N} \times \{0,1\}$.

Claim 6. *1. $T'((n,1),(m,1)) = 1$ for $n < m$.*
2. $T'((n,0),(n+1,0)) = T'((n,0),(m,0))$ for $n < m$.
3. $T'((n,0), T'(n,1)) = 1$
4. $T((n,1),(n+1,0)) = T((n,1),(m,0))$ for $n < m$.

Proof of Claim 6. Item1. $T'((n,1), T'(m,1)) = T(f(n), f(m)) = 1$.
Item 2. Since neither $C_{3[\omega]}$, nor $C_{3[\omega^*]}$, is embeddable in T, we have $T'((n,0),(n+1,0)) = T'((n,0),(m,0))$ for $n < m$.

Item 3. Since $\{f(n), f(n+1), g(n, n+1)\}$ is a 3-cycle of T, $\{(n,1),(n+1,1),(n,0)\}$ is a 3-cycle of T'. It follows that $T'((n,0),(n,1)) = T'((n+1,1),(n,0)) = 1$.
Item 4. Item 2 of Claim 1. □

Suppose that Case 1 holds. We have $T'((m,1),(n,0)) = 1$ for all m, $n+1 < m$. Since $T'((n+1,1),(n,0)) = 1$, we have $T'((n,1),(m,0)) = 1$ for all $n < m$. This added to Claim 2 insures that:

Claim 7. *T' is a skew product of the 2-element tournament $\underline{2}$ by the chain ω.*

In order to conclude the proof of Lemma 22 there are four cases to consider. Subcase 1.1. $T'((n,1),(n+1,0)) = 1$. Subcase 1.1.1. $T'((n,0),(n+1,0)) = 1$. In this case $T' := H_{[\omega]}$. Subcase 1.1.2. $T'((n,0),(n+1,0)) = 0$. In this case T' is isomorphic to $K_{[\omega^*]}$. Subcase 1.2. $T'((n,1),(n+1,0)) = 0$. Subcase 1.2.1. $T((n,0),(n+1,0)) = 1$. In this case $T' = T_{[\omega]}$. Subcase 1.2.2. $T((n,0),(n+1,0)) = 0$. In this case T' is isomorphic to $U_{[\omega^*]}$.

Suppose that Case 2 holds. We have $T'((m,1),(n,0)) = 0$ for all m, $n+1 < m$.

Claim 8. $T'((n,0),(n+1,0)) = 0$.

Proof of Claim 8. Let $k \in \mathbb{N}$. Set $D_k := \{(2k,0),(2k,1),(2k+1,1))$. Set $D := \cup\{D_k : k \in \mathbb{N}\}$. The tournament $T'_{\restriction D_k}$ is a 3-cycle. If $T'((n,0),(n+1,0)) = 1$ then $T'((n,0),(m,0)) = 1$ for $n < m$. We have then $T'(x,y) = 1$ whenever $x \in D_k, y \in D'_k$, $k < k'$. Hence $T'_{\restriction D_k}$ is isomorphic to $C_3.\omega$. Contradicting our assumption. □

Claim 9. *Let $E' := \mathbb{N} \times 2 \setminus \{(0,1)\}$. Then $T'_{\restriction E'}$ is isomorphic to $K_{[\omega]}$.*

Proof of Claim 9. Let $G : \mathbb{N} \times 2 \to \mathbb{N} \times 2$ defined by seting $G(n,0) = F(n,1)$ and $G(n,1) = F(n,0)$. Let T'' on $\mathbb{N} \times 2$ defined by $T''(x,y) = T'(x,y)$. One can easily check that $T'' = K_{[\omega]}$. □.

With Claim 9 the proof of Lemma 22 is complete. □

Lemma 23. *Let T be a tournament containing no infinite subset whose pairs of distinct elements are included into a 3-cycle. If T contains an infinite subset A such that every pair of distinct element of A is included into a diamond then some member of $\{C_3.\omega, C_3.\omega^*, V_{[\omega]}, V_{[\omega^*]}\}$ is embeddable in T.*

Proof. Let $f : \mathbb{N} \to A$ be a one-to-one map. We may define $f_i : [\mathbb{N}]^2 \to V(T)$ for $i \in \{0,1\}$ so that for $n < m$, the set $\{f(n), f(m), f_i(n,m) : i \in \{0,1\}\}$ forms a diamond and either $f(n)$ or $f(m)$ does not belong to the 3-cycle of this diamond.

Let $\Phi := \{f, f_i : i \in \{0,1\}\}$ and let $\mathfrak{L} :=< \omega, T, \Phi >$.For a subset X of $\mathbb{N}$, let $\Phi_{\restriction X} := \{f_{\restriction X}, f_{i\restriction [X]^2} : i \in \{0,1,\}\}$ and let $\mathfrak{L}_{\restriction X} :=< \omega_{\restriction X}, T, \Phi_{\restriction X} >$. According to Theorem 6, there is an infinite subset X of $\mathbb{N}$ such that $\mathfrak{L}_{\restriction X}$ is invariant.

Via a relabelling of X with the integers, we may suppose that $X = \mathbb{N}$, that is $\mathfrak{L}$ is invariant.

Since $\mathfrak{L}$ is invariant, $T(f(n), f(m))$ is constant on pairs (n,m) such that $n < m$. With no loss of generality, we may suppose that

$$T(f(n), f(m)) = 1. \tag{6}$$

Case 1. Suppose that there is a pair (n_0, m_0) such that $n_0 < m_0$ and $\{f(n_0), f_i(n_0, m_0) : i \in \{0,1\}\}$ forms a 3-cycle. With no loss of generality we may suppose that:

$$T(f(n_0), f_1(n_0, m_0)) = T(f_1(n_0, m_0), f_0(n_0, m_0)) = T(f_0(n_0, m_0), f(n_0)) = 1.$$

Claim 10. *Let $n < m < n' \leq m'$*
1. $T(f_i(m, n'), f(m')) = 1$ *for* $i \in \{0, 1\}$.
2. $T(f(n), f_1(m, n')) = 1$.
3. $f_1(n, m) \neq f_1(n', m')$.
4. $f_i(n, m) \neq f(k)$ *for* $k \leq n$ *or* $m \leq k$.

Proof of Claim 10. Item 1. Since $\mathfrak{L}$ is invariant, $\{f(m), f_i(m, n') : i \in \{0, 1\}\}$ is the 3-cycle of $\{f(m')f(n'), f_i(m, n') : i \in \{0, 1\}\}$. Hence $T(f_i(m, n'), f(n')) = T(f(m), f(n')) = 1$. Now we may suppose $m' > n'$. Since $T(f_i(m, n'), f(n')) = T(f(n'), f(m')) = 1$ and A does not contains a pair of distinct elements forming a 3-cycle, $T(f_i(m, n'), f(m')) = 1$.
Item 2. We have $T(f(n), f(m)) = T(f(m), f_1(m, n')) = 1$. Since A does not contains a pair of distinct elements forming a 3-cycle, $T(f(n), f_1(m, n')) = 1$.
Item 3. Suppose $f_1(n, m) = f_1(n', m')$. We have $T(f(n'), f_1(n', m')) = 1$, whereas from Item 2, $T(f_i(n, m), f(n')) = 1$. Hence $f_1(n, m) \neq f_1(n', m')$.
Item 4. If $k \geq m$, we have $T(f_i(n, m), f(k))$ hence the result. If $k = n$, $f_i(n, m) = f(n)$ is impossible by definition of f_i. If $k < n$ and $f_i(n, m) = f(k)$ then select $k' < k'' < n'' < m''$. By the invariance of $\mathfrak{L}$, get $f_i(n'', m'') = f(k')$ and $f_i(n'', m'') = f(k'')$, hence $f(k') = f(k'')$. A contradiction with the hypothesis that f is one to one. □

Note that from Item 1 follows that one could choose $f_i(n, m)$ independent of m.

Let $F : \mathbb{N} \times \{0, 1, 2\} \to V(T)$ defined by $F(n, 0) = f(3n)$, $F(n, i+1) := f_i(3n, 3n+1)$ for $i \in \{0, 1\}$.

Claim 11. *1. For $i, j \in \{0, 1, 2\}$, $T(F(n, i), (m, j))$ is constant on pairs (n, m) such that $n < m$.*
2. $T(F(n, 2), F(m, 2)) = 1$ *for* $n < m$.
3. $T(F(n, 2), F(m, 1)) = 1$ *for* $n < m$

Proof of Claim 11. Item 1. $\mathfrak{L}$ is invariant.
Item 2. If $T(F(n, 2), F(m, 2)) = 0$ then $T(f_1(3m, 3m+1), f_1(3n, 3n+1)) = 1$. Since $T(f(3m), f_1(3m, 3m+1)) = 1$ and $T(f_1(3n, 3n+1), f(3m)) = 1$, $\{f_1(3m, 3m+1), f_1(3n, 3n+1), f(m)\}$ is a 3-cycle. By the invariance of $\mathfrak{L}$, every pair of distinct elements of $E' := \{f_1(3n', 3n'+1) : n' \in \mathbb{N}\}$ contains a 3-cycle, contradicting the hypothese of the lemma.
Item 3. If $T(F(m, 1), F(n, 2)) = 1$ then the set $\{F(m, 1), F(n, 2), F(m, 2)\}$ forms a 3-cycle and as in the item above we contradict the hypothesis in the lemma. □.
Subcase 1.1. $T(F(m, 2), F(n, 1)) = 1$ for some $n < m$. Then this equality holds for all pairs (n', m') such that $n' < m'$. Let $a \notin \mathbb{N}$ and $G : \mathbb{N} \cup \{a\} \to V(T)$ defined by $G(a) := F(0, 1)$, $G(n, 0) := F(n, 0)$, $G(n, 1) := F(n+1, 2)$.

Claim 12. *G is an embedding of $V_{[\omega]}$ into T.*

Proof of Claim 12. Let $n \leq n'$. We have $T(G(n, 0), G(n', 0)) = 1$ from equation 6, $T(G(n, 1), G(n', 1) = 1$ from Item 2 of Claim11, $T(G(n, 0), G(n', 1)) = 1$ from Item 2 and Item 3 of Claim10 and $T(G((n, 1), G(n', 0)) = 1$ from Item 1 of Claim10 if $n < n'$. We have $T(G(a), G(n, 0)) = T(F(0, 1), F(n, 0)) = T(F(n, 1), F(n, 0)) = 1$ from the hypothese of Case 1. And we have $T(G(n, 1), G(a)) = T(F(n+1, 2), F(0, 1)) = 1$. This proves our claim. □

Subcase 1.2. $T(F(n,1), F(m,2)) = 1$ for every $n < m$.

Claim 13. *F is one-to-one and $T(F(n,1), F(m,1) = 1$ for every $n < m$.*

Proof of Claim 13. The first part of Claim 13 is obvious.If the second part does not hold, $\{F(n,1), F(m,2), F(m,1)\}$ forms a 3-cycle and every pair of distinct elements of $A' := \{F(n',1) : n' \in \mathbb{N}\}$ is included into a 3-cycle, contradicting the hypothesis of the lemma. $\square$.

Let $G' : \mathbb{N} \times 2 \cup \{a\} \to V(T)$ defined by $G'(a) := F(0,0)$, $G'(n,0) := F(n,2)$, $G'(n,1) := F(n+1,1)$.

Claim 14. *If $T(F(m,1), F(n,0)) = 1$ for some $n < m$, G' is an embedding from $V_{[\omega]}$ into T.*

Straightforward verification.

Let T' be defined on $\mathbb{N} \times \{0,1,2\}$ by $T'(x,y) := T(F(x), F(y))$. Let $T'_n := T'_{\restriction \{n\} \times \mathbb{N}}$.

Claim 15. *If $T(F(n,0), F(m,1)) = 1$ for some $n < m$, T' is the ω-sum of the T_n's.*

Indeed, we have $T(F(n,i), F(m,j)) = 1$ for $n < m$.

Case 2. Case 1 does not hold. In this case $\{f(m_0), f_i(n_0, m_0) : i \in \{0,1\}\}$ forms a 3-cycle. The treatment of this case is similar and left to the reader. $\square$

Lemma 24. *Let T be a tournament containing no infinite subset whose pairs of distinct vertices are included into a 3-cycle. If T contains an infinite subset A such that every pair of distinct vertices of A forms the end-vertices of some self-dual double diamond and is not included into a diamond then either $C_{3[\omega]}$ or $C_{3[\omega^*]}$ is embeddable in T.*

Proof. With Ramsey theorem, we may suppose that no pair of elements of A is included into a 3-cycle. Let $f : \mathbb{N} \to A$ be a one-to-one map. We may define $f_i : [\mathbb{N}]^2 \to V(T)$ for $i \in \{0,1,2\}$ so that $f(n)$ and $f(m)$ are the end-vertices of the self-dual double diamond $\{f(n), f(m), f_i(n,m) : i \in \{0,1,2\}\}$.

According to our construction, we have:

Claim 16. *1. $T(f(n), f_i(n,m)) = T(f(n), f(m)) = T(f(_i(n,m), f(m))$.*
2. $T(f_0(n,m), f_1(n,m)) = T(f_1(n,m), f_2(n,m)) = T(f_2(n,m), f_0(n,m))$.

Let $\Phi := \{f, f_i : i \in \{0,1,2\}\}$ and let $\mathfrak{L} :=< \omega, T, \Phi >$.For a subset X of $\mathbb{N}$, let $\Phi_{\restriction X} := \{f_{\restriction X}, f_{i\restriction [X]^2} : i \in \{0,1,2\}\}$ and let $\mathfrak{L}_{\restriction X} :=< \omega_{\restriction X}, T, \Phi_{\restriction X} >$. According to Theorem 6, there is an infinite subset X of $\mathbb{N}$ such that $\mathfrak{L}_{\restriction X}$ is invariant. Via a relabelling of X with the integers, we may suppose that $X = \mathbb{N}$, that is $\mathfrak{L}$ is invariant.

Let $F : \mathbb{N} \times \{0,1,2,3\} \to V(T)$ defined by $F(n,0) = f(3n)$, $F(n,i+1) := f_i(3n, 3n+1)$ for $i \in \{0,1,2\}$.

Claim 17. *F is one-to-one.*

Proof of Claim 17. Suppose first that $F_{i+1}(n) = F_{j+1}(m)$ for some $i,j \in \{0,1,2\}$, $n < m$. This means $f_i(3n, 3n+1) = f_j(3m, 3m+1)$. Since $\mathfrak{L}$ is invariant $f_i(3n, 3n+1) = f_j(3m+1, 3m+2)$. Hence $f_j(3m, 3m+1) = f_j(3m+1, 3m+2)$. From our construction, $T(f(3m), f(3m+1)) = T(f_j(3m, 3m+1), f(3m+1))$ and $T(f(3m+1), f_j(3m+1, 3m+2)) = T(f(3m+1), f(3m+2))$. Since $\mathfrak{L}$ is invariant, $T(f(n), f(m))$

is constant on pair (n, m) such that $n < m$. In particular, $T(f(3m), f(3m+1)) = T(f(3m+1), f(3m+2))$. Hence $T(f_j(3m, 3m+1), f(3m+1)) = T(f(3m+1), f_j(3m+1, 3m+2))$. Thus $T(f_j(3m, 3m+1), f(3m+1)) = T(f(3m+1), f_j(3m, 3m+1))$, which is impossible since $f_j(3m, 3m+1)$ and $f(3m+1)$ are distinct. Next, suppose that $F_0(n) = F_{i+1}(m)$ for some n, m, $n \neq m$. If $n < m$, choose $n'' < n' < m'$. Since $\mathfrak{L}$ is invariant, we have $f(3n'') = f_i(3m', 3m'+1)$ and $f(3n') = f_i(3m', 3m'+1)$ hence $f(3n'') = f(3n')$, contradicting the fact that f is one-to-one. If $m < n$, choose $m' < n' < n''$ and use the same argument. □

According to the above claim, we may define a tournament T' with vertex set $\mathbb{N} \times \{0, 1, 2, 3\}$ such that $T'(x, y) = T(x, y)$. Indeed, let $T'_n := T'_{\restriction \{n\} \times \{0,1,2,3\}}$ for $n \in \mathbb{N}$:

Claim 18. *T' is the lexicographic sum of the T_n's, this sum being either an ω-sum or an ω^*-sum.*

Proof of Claim 18. Let $i, j \in \{0, 1, 2, 3\}$. Since $\mathfrak{L}$ is invariant, $T(F(n, i), F(m, j))$ is constant on pairs (n, m) such that $n < m$. Hence T' is a skew product. We proceed directly. With no loss of generality, we may suppose $T(F(n, 0), F(m, 0)) = 1$ for $n < m$ (otherwise, replace T by T^*), hence $T'((n, 0), (m, 0)) = 1$. According to Item 1 of Claim 16, we have $T(F(n, 0), T(F(n, i+1)) = 1$. With this and the fact that no pair of distinct elements of A belong to a 3-cycle, we also have $T(F(n, 0), F(m, i+1)) = T(F(n, i+1), F(m, 0))$ for all pairs n, m such that $n < m$. Let $\mathbb{N}_{i+1} := \{(n, i+1) : n \in \mathbb{N}\}$. If some pair of elements of $\mathbb{N}_{i+1}$ is included into a 3-cycle of T', then all pairs are included into a 3-cycle. Hence, every pair of the infinite set $A_{i+1} := \{F(n, i+1) : n \in \mathbb{N}\}$ would be included into some 3-cycle of elements of T, which is excluded. It follows that $T'((n, i+1), (m, i+1)) = 1$ if $n < m$. If $T'((n, i+1), (m, j+1)) = 0$ for some $n < m$, $i \neq j$, then the set $\{F(n, 0), F(m, 0), F(n, i+1), F(m, j+1)\}$ forms a diamond of T' hence every pair of the infinite set $A_0 := \{F(n, 0) : n \in \mathbb{N}\}$ would be included into some diamond of T, which is excluded. From this T' is the ω-sum of the T_n's, as claimed. □

Since each T_n contains a 3-cycle, with Claim 18 the proof of the lemma is complete. □

References

[1] Y. Boudabbous, J. Dammak, P. Ille. Indecomposability and duality of tournaments. *Discrete Math.,*223 (2000) 55-82.

[2] P.J. Cameron. *Oligomorphic permutation groups.* London Math. Soc. Lecture Note Series. 152pp., Cambridge University Press, Cambridge, 1990.

[3] P.J. Cameron. Sequences realized by oligomorphic permutation groups. *J. Integer Seq.*, 3(1):Article 00.1.5, 1 HTML document (electronic), 2000.

[4] P.J. Cameron. Some counting problems related to permutation groups. *Discrete Math.*, 225(1-3)(2000)77-92. Formal power series and algebraic combinatorics (Toronto, ON, 1998).

[5] C. Charretton, M. Pouzet. Les chaînes dans les modèles d'Ehrenfeucht-Mostowski. *C.R.Acad.Sci. Paris, Série A*, t.290 (1981), 715-717.

[6] B. Cloitre, Sequence A000930 in The On-Line Encyclopedia of Integer Sequences, Nov 30 2002.

[7] J-F. Culus, B. Jouve. Tournois sans intervalle acyclique. *Comptes Rendus Acad. Sci. Paris, Série A,* t.341 (2005), 465-468.

[8] J-F. Culus, B. Jouve. Décomposition de tournois en classes acycliques. SFS 2004, 11i 'emes rencontres de la Société Francophone de Classification, 8-10 septembre 2004, Bordeaux, France.

[9] A. Ehrenfeucht, T. Harju and G. Rozenberg. *The theory of 2-structures. A framework for decomposition and transformation of graphs*. World Scientific, 1999.
[10] P. Erdös, E. Fried, A. Hajnal and E.C. Milner. Some remarks on simple tournaments. *Algebra Universalis*, 2(1972) 238-245.
[11] R. Fraïssé. *Cours de logique mathématique. Tome 1: Relation et formule logique*. Gauthier-Villars Éditeur, Paris, 1971.
[12] R. Fraïssé, Theory of relations, second edition, North-Holland Publishing Co., Amsterdam, p.ii+451, 2000.
[13] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar.*, 18(1967), 25-66.
[14] B. Jouve. Transitive convex subsets in large tournaments. *Electronic Notes in Discrete Mathematics,* (22):433-437 (7th International Colloquium on Graph Theory ICGT 05, 12-16 sept. 2005, Hyière, France.
[15] F. Maffray, M. Preissmann. A translation of Tibor Gallai's paper: Transitiv orientierbare Graphen, in: J.L. Ramirez-Alfonsin and B.A. Reed (eds.). *Perfect Graphs.* Wiley, New York, 2001, pp. 25–66.
[16] V. Müller, J. Nesetril and J. Pelant. Either tournaments or algebras? *Discrete Math.*, 11(1975)35-66.
[17] M. Pouzet, Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations. *Math. Z.* **150(2)** (1976), pp. 117–134.
[18] M. Pouzet. *Sur la théorie des relations*. Thèse d'état, Université Claude-Bernard, Lyon 1, N 78-05, 1978.
[19] M. Pouzet. Application de la notion de relation presque-enchaînable au dénombrement des restrictions finies d'une relation. *Z. Math. Logik Grundlag. Math.*, 27(4)(1981) 289–332.
[20] M. Pouzet, N. Thiéry. Some relational structures with polynomial growth and their associated algebras. May 10th, 2005, 19pages, presented at FPSAC for the 75 birthday of A.Garsia. http://arxiv.org/math.CO/0601256.
[21] M. Pouzet, H. Sikaddour, N. Zaguia, Which posets have a scattered MacNeille completion? *Algebra Universalis*, 53(2005) 287-299.
[22] M. Pouzet. The profile of relations. *Global Journal of Pure and Applied Mathematics*, Proceedings of the 14^{th} Symposium of the Tunisian Mathematical Society held in Hammamet, March 20-23, 2006, 2(2006) 237-272.
[23] M. Pouzet. When is the orbit algebra of a group an integral domain? Proof of a conjecture of P.J. Cameron.*Theoretical Informatics and applications*, 83 (2008),83-103.
[24] F.P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30 (1930) 264-286.
[25] J.H. Schmerl, W.T. Trotter. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* 113 (1993) 191-205.
[26] N. Sloane. *The On-Line Encyclopedia of Integer Sequences,* URL: http://www.research.att.com.

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