The edge-flipping group of a graph*

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Abstract

Let $X = (V, E)$ be a finite simple connected graph with n vertices and m edges. A configuration is an assignment of one of two colors, black or white, to each edge of X. A move applied to a configuration is to select a black edge $\epsilon \in E$ and change the colors of all adjacent edges of ϵ. Given an initial configuration and a final configuration, try to find a sequence of moves that transforms the initial configuration into the final configuration. This is the edge-flipping puzzle on X, and it corresponds to a group action. This group is called the edge-flipping group $\mathbf{W}_E(X)$ of X. This paper shows that if X has at least three vertices, $\mathbf{W}_E(X)$ is isomorphic to a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^k$ and the symmetric group S_n of degree n, where $k = (n-1)(m-n+1)$ if n is odd, $k = (n-2)(m-n+1)$ if n is even, and $\mathbb{Z}$ is the additive group of integers.

Keywords: group actions; orbits; semidirect products.

1 Introduction

An ordered pair $X = (V, E)$ is a *finite simple graph* if V is a finite set and E is a set of some 2-element subsets of V. The elements of V are called *vertices* of X and the elements of E are called *edges* of X. Two vertices

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u, v of X are *neighbors* if $\{u, v\} \in E$. A finite simple graph $X = (V, E)$ is *connected* if for any two distinct vertices $u, v \in V$ there exists a subset $\{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{k-1}, u_k\}\}$ of E with $u_0 = u$ and $u_k = v$.

Throughout this paper, $X = (V, E)$ is a finite simple connected graph with $|V| = n$ and $|E| = m$. This paper focuses on two flipping puzzles defined on the graph X as follows. A *configuration* of the first puzzle (second puzzle, respectively) is an assignment of one of two colors, black or white, to each edge of X (vertex of X, respectively). A move applied to a configuration is to select a black edge $\epsilon \in E$ (black vertex $v \in V$, respectively) and change the colors of all edges ϵ' with $|\epsilon' \cap \epsilon| = 1$ (all neighbors of v, respectively). Given an initial configuration and a final configuration, try to find a sequence of moves that transforms the initial configuration into the final configuration. This is the *edge-flipping puzzle* (the *vertex-flipping puzzle*, respectively) on X. A set O of some configurations is an *orbit* of the edge-flipping puzzle (the vertex-flipping puzzle, respectively) on X if for any two configurations in O, one can reach the other by a sequence of moves. The edge-flipping puzzle corresponds to a group action, and this group is called the edge-flipping group $\mathbf{W}_E(X)$ of $X = (V, E)$. See Section 3 for details.

The orbits of the edge-flipping puzzle on X have been determined. If X is a tree with at least three vertices, the edge-flipping group $\mathbf{W}_E(X)$ of X is isomorphic to the symmetric group S_n of degree n. Wu [18] illustrated both of these results. The main goal of the current study is to produce the complement part of his second result, and show that if X has at least three vertices, then $\mathbf{W}_E(X)$ is isomorphic to

$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{(n-1)(m-n+1)} \rtimes S_n, & \text{if } n \text{ is odd;} \\ (\mathbb{Z}/2\mathbb{Z})^{(n-2)(m-n+1)} \rtimes S_n, & \text{if } n \text{ is even,} \end{cases}$$

where $\mathbb{Z}$ is the additive group of integers. Section 5 explains the structure of $\mathbf{W}_E(X)$ in greater detail.

The development and history of vertex-flipping puzzles can be found in the literature [4, 5, 6, 8, 10, 12, 15, 17]. For example, the vertex-flipping puzzles implicitly appear in Chuah and Hu's papers [5, 6] when they study the equivalence classes of Vogan diagrams and extended Vogan diagrams [1, 2, 14]. Note that the vertex-flipping puzzles are called lit-only σ-games in [12, 15, 17].

The vertex-flipping puzzle on X also corresponds to a group action [15], and this group is called the vertex-flipping group $\mathbf{W}_V(X)$ of X. Some properties of the vertex-flipping group $\mathbf{W}_V(X)$ of X have been known. For example,

$\mathbf{W}_V(X)$ has the trivial center, and $\mathbf{W}_V(X)$ is a homomorphic image of the Coxeter group W of X. If X is a simply-laced Dynkin diagram, $\mathbf{W}_V(X)$ is isomorphic to the quotient group of W by its center $Z(W)$; moreover, $|Z(W)| = 1$ or 2. See [9] for details. In other words, $\mathbf{W}_V(X)$ can be treated as a combinatorial version of the reflection groups on real vector spaces. Although J. Humphreys gave a faithful geometric representation of any Coxeter group W in a real vector space [11, Section 5.3], the group structures of W and $\mathbf{W}_V(X)$ are worthy of further study.

On the other hand, the edge-flipping puzzle on X is the vertex-flipping puzzle played on the line graph $L(X)$ of X. The edge-flipping group $\mathbf{W}_E(X)$ of X is also the vertex-flipping group of $L(X)$. The main result of this study can be used to find the group structures of the vertex-flipping groups of some graphs, which do not need to be line graphs. See Section 6 for details. Note that line graphs are classified in terms of nine forbidden induced subgraphs [3, 16].

2 Edge spaces and bond spaces

Let $X = (V, E)$ denote a finite simple connected graph with $|V| = n$ and $|E| = m$. In this section we give some basic definitions and properties about the edge space and the bond space of X. The reader may refer to [7, p.23–p.28] for details. Let $\mathcal{E}$ denote the power set of E. Let $\mathbf{F}_2 = \{0, 1\}$ denote the 2-element field. For $F, F' \in \mathcal{E}$, define $F + F' := \{\epsilon \in E \mid \epsilon \in F \cup F', \epsilon \notin F \cap F'\}$; i.e., the symmetric difference of F and F', and define $1 \cdot F := F$ and $0 \cdot F := \emptyset$, the empty set. Then $\mathcal{E}$ forms a vector space over $\mathbf{F}_2$ and is called the *edge space* of X. Note that the zero element of $\mathcal{E}$ is $\emptyset$ and $-F = F$ for $F \in \mathcal{E}$. Since $\{\{\epsilon\} \mid \epsilon \in E\}$ is a basis of $\mathcal{E}$, we have $\dim \mathcal{E} = m$. In the same way as above, the power set $\mathcal{V}$ of V also forms a vector space over $\mathbf{F}_2$ with symmetric difference as vector addition, and we call $\mathcal{V}$ the *vertex space* of X. Clearly, $\dim \mathcal{V} = n$.

For a subset U of V, let $E(U)$ denote the subset of E consisting of all edges of X that have exactly one element in U. In graph theory, $E(U)$ is often called an *edge cut* of X if U is a nonempty and proper subset of V. Let $E(v) := E(\{v\})$ for $v \in V$ and notice that $E(\epsilon) = E(\{x, y\})$ for $\epsilon = \{x, y\} \in E$.

Proposition 2.1. *Let $X = (V, E)$ be a finite simple connected graph. Then the following (i),(ii) hold.*

(i) *Each* $\epsilon = \{x, y\} \in E$ *lies in exactly two edge cuts* $E(x)$ *and* $E(y)$ *among* $E(v)$ *for all* $v \in V$.

(ii) *For* $U \subseteq V$, $E(U) = \sum_{v \in U} E(v)$.

Proof. (i) is immediate from the definition of $E(v)$ for $v \in V$. (ii) is immediate from (i) and the definition of $E(U)$. $\square$

The *bond space* $\mathcal{B}$ of X is the subspace of $\mathcal{E}$ spanned by $E(v)$ for all $v \in V$. In the following, we give some basic properties of $\mathcal{B}$.

Proposition 2.2. *Let* $X = (V, E)$ *be a finite simple connected graph with* n *vertices, and let* $\mathcal{B}$ *be the bond space of* X. *Then the following (i)-(iv) hold.*

(i) $\mathcal{B} = \{E(U) \mid U \subseteq V\}$.

(ii) $\dim \mathcal{B} = n - 1$.

(iii) *For* $u \in V$, $E(u) = \sum_{v \in V - \{u\}} E(v)$.

(iv) *For* $u \in V$, *the set* $\{E(v) \mid v \in V - \{u\}\}$ *is a basis of* $\mathcal{B}$.

Proof. (i) follows immediately from Proposition 2.1(ii). Note that the map from the vertex space $\mathcal{V}$ onto the bond space $\mathcal{B}$ of X, defined by

$$U \mapsto E(U) \text{ for } U \in \mathcal{V},$$

is a linear transformation with kernel $\{\emptyset, V\}$. Hence $\dim \mathcal{B} = n - 1$ and this prove (ii). Let $u \in V$. Since $E(V) = \emptyset$, we have

$$\begin{aligned} E(u) &= E(u) + E(V) \\ &= \sum_{v \in V - \{u\}} E(v). \end{aligned}$$

This proves (iii). Also, the set $\{E(v) \mid v \in V - \{u\}\}$ is a basis of $\mathcal{B}$ since it has at most $n - 1$ elements and spans $\mathcal{B}$ by (iii). This proves (iv). $\square$

It is not hard to see that there exists an $(n-1)$-element subset T of E such that (V, T) is connected since $X = (V, E)$ is connected. We call such T a *spanning tree* of E. The following proposition says that $\{F \mid F \subseteq E - T\}$ is a set of coset representatives of $\mathcal{B}$ in $\mathcal{E}$.

Proposition 2.3. *Let $X = (V, E)$ be a finite simple connected graph with n vertices and m edges. Let $\mathcal{E}$ and $\mathcal{B}$ be the edge space and bond space of X respectively. Then the subset $\{F \mid F \subseteq E - T\}$ of $\mathcal{E}$ is a set of coset representatives of $\mathcal{B}$ in $\mathcal{E}$, where T is a spanning tree of E.*

Proof. Note that there are 2^{m-n+1} cosets of $\mathcal{B}$ in $\mathcal{E}$ because of $\dim \mathcal{B} = n-1$ and $\dim \mathcal{E} = m$. It is clear that $|\{F \mid F \subseteq E - T\}| = 2^{m-n+1}$. For any two distinct $F, F' \subseteq E - T$, the graph $(V, E - (F - F'))$ is still connected since $T \subseteq E - (F - F')$, which implies that $F - F'$ is not an edge cut of X; i.e., $F - F' \notin \mathcal{B}$. Hence $\{F \mid F \subseteq E - T\}$ is a set of coset representatives of $\mathcal{B}$ in $\mathcal{E}$. ∎

3 Edge-flipping groups and invariant subsets

In this and next sections, we rephrase some results of [18] in order to facilitate our work. Let $X = (V, E)$ denote a finite simple connected graph with $|V| = n$ and $|E| = m$. Let $\mathcal{E}$ and $\mathcal{B}$ denote the edge space and bond space of X respectively. We regard every configuration of the edge-flipping puzzle on X as an element G of $\mathcal{E}$, where G consists of all black edges. We give a new interpretation of the moves in the edge-flipping puzzle on X: on each round, we select an edge $\epsilon \in E$. If ϵ is a black edge, then we change the colors of all edges ϵ' with $|\epsilon' \cap \epsilon| = 1$; otherwise, we do nothing. Clearly, the orbits of the edge-flipping puzzle of X under the new moves are unchanged. However, the new move by selecting an edge ϵ of X is corresponding to the map $\boldsymbol{\rho}_\epsilon : \mathcal{E} \to \mathcal{E}$ defined by

$$\boldsymbol{\rho}_\epsilon G = \begin{cases} G + E(\epsilon), & \text{if } \epsilon \in G; \\ G, & \text{otherwise} \end{cases} \tag{3.1}$$

for $G \in \mathcal{E}$. From the definition of $\boldsymbol{\rho}_\epsilon$, we see that $\boldsymbol{\rho}_\epsilon^2$ is the identity map on $\mathcal{E}$. In particular, $\boldsymbol{\rho}_\epsilon$ is invertible. It is straightforward to check that $\boldsymbol{\rho}_\epsilon$ is a linear transformation on $\mathcal{E}$. Hence $\boldsymbol{\rho}_\epsilon$ is an element in the general linear group $\mathrm{GL}(\mathcal{E})$ of $\mathcal{E}$.

Definition 3.1. The *edge-flipping group* $\mathbf{W}_E(X)$ of $X = (V, E)$ is the subgroup of the general linear group $\mathrm{GL}(\mathcal{E})$ of $\mathcal{E}$ generated by the set $\{\boldsymbol{\rho}_\epsilon \mid \epsilon \in E\}$.

Recall that I is an *invariant subset* of $\mathcal{E}$ under $\mathbf{W}_E(X)$ if $I \subseteq \mathcal{E}$ and $\mathbf{W}_E(X)I \subseteq I$. In the following, we give some significant invariant subsets of $\mathcal{E}$ under $\mathbf{W}_E(X)$.

Proposition 3.2. *([18]) Let $X = (V, E)$ be a finite simple connected graph. Let $\mathcal{E}$ and $\mathcal{B}$ be the edge space and bond space of X respectively. Then each coset of $\mathcal{B}$ in $\mathcal{E}$ is an invariant subset of $\mathcal{E}$ under $\mathbf{W}_E(X)$.*

Proof. It suffices to show that $\boldsymbol{\rho}_\epsilon G$ is in $G + \mathcal{B}$ for any $\epsilon \in E$ and $G \in \mathcal{E}$. By (3.1), $\boldsymbol{\rho}_\epsilon G$ is equal to either $G + E(\epsilon)$ or G. Hence, $\boldsymbol{\rho}_\epsilon G \in G + \mathcal{B}$ since $E(\epsilon) \in \mathcal{B}$. $\square$

From now to the end of this section, we shall study the group action of $\mathbf{W}_E(X)$ on the bond space $\mathcal{B}$ of X. Recall that the set $\{E(v) \mid v \in V\}$ spans $\mathcal{B}$. We first determine the cardinality of the set $\{E(v) \mid v \in V\}$ as follows.

Lemma 3.3. *Let $X = (V, E)$ be a finite simple connected graph with $|V| = n$. Then*

$$|\{E(v) \mid v \in V\}| = \begin{cases} n, & \text{if } n \geq 3; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. If X is a one-vertex graph, then $\{E(v) \mid v \in V\} = \{\emptyset\}$, and if X is a connected graph with two vertices, say x and y, then $E(x) = E(y)$ and hence $|\{E(v) \mid v \in V\}| = 1$. Now suppose $n \geq 3$. Pick two distinct vertices $u, v \in V$, we show $E(u) \neq E(v)$. Since the edge cut $E(\{u, v\})$ is nonempty and by Proposition 2.1(ii), $E(u) + E(v) = E(\{u, v\}) \neq \emptyset$; i.e., $E(u) \neq E(v)$. $\square$

Throughout the remainder of this section, we assume $n \geq 3$ and we denote the symmetric group on $\{E(v) \mid v \in V\}$ of degree n by S_n. Suppose $\epsilon = \{x, y\} \in E$. From Proposition 2.1(i) and the definition of $\boldsymbol{\rho}_\epsilon$ in (3.1), we know that $\boldsymbol{\rho}_\epsilon$ fixes all $E(v)$'s except $E(x)$ and $E(y)$. Also from Proposition 2.1(ii), we know that $E(\epsilon) = E(x) + E(y)$, and hence $\boldsymbol{\rho}_\epsilon E(x) = E(y)$ and $\boldsymbol{\rho}_\epsilon E(y) = E(x)$. In brief, the mapping of $\boldsymbol{\rho}_\epsilon$ on $\{E(v) \mid v \in V\}$ is the transposition $(E(x), E(y))$ in S_n. Since each element $\mathbf{g} \in \mathbf{W}_E(X)$ is generated by $\boldsymbol{\rho}_\epsilon$ for $\epsilon \in E$, the mapping of $\mathbf{g}$ on $\{E(v) \mid v \in V\}$ is like a permutation in S_n. Hence we have the following definition.

Definition 3.4. Let $\alpha : \mathbf{W}_E(X) \to S_n$ denote the group homomorphism defined by

$$\alpha(\mathbf{g})(E(v)) = \mathbf{g}E(v)$$

for $v \in V$ and $\mathbf{g} \in \mathbf{W}_E(X)$.

Let T be a spanning tree of E, and let $\mathbf{W}_E(X)_T$ denote the subgroup of $\mathbf{W}_E(X)$ generated by the set $\{\boldsymbol{\rho}_\epsilon \mid \epsilon \in T\}$. We say that X is a *tree* if $E = T$. The following lemma shows that $\mathbf{W}_E(X)$ is isomorphic to S_n if X is a tree.

Lemma 3.5. *([18, Theorem 8]) Let $X = (V, E)$ be a finite simple connected graph with $|V| = n \geq 3$. Let S_n be the symmetric group on $\{E(v) \mid v \in V\}$ of degree n. Then $\alpha(\mathbf{W}_E(X)) = \alpha(\mathbf{W}_E(X)_T) = S_n$, where T is a spanning tree of E. Moreover, $\mathbf{W}_E(X)$ is isomorphic to S_n if X is a tree.*

Proof. Note that $\alpha(\epsilon)$ is the transposition $(E(x), E(y))$ for every $\epsilon = \{x, y\} \in E$. Let $A = \{(E(x), E(y)) \in S_n \mid \{x, y\} \in T\}$. Pick any two distinct vertices $u, v \in V$. Then there exists a subset $\{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{k-1}, u_k\}\}$ of T with $u_0 = u$ and $u_k = v$. Note that

$$\begin{aligned}(E(u), E(v)) =& (E(u_{k-1}), E(u_k)) \cdots (E(u_1), E(u_2))(E(u_0), E(u_1)) \\ & (E(u_1), E(u_2)) \cdots (E(u_{k-1}), E(u_k)).\end{aligned}$$

Hence A generates all transpositions in S_n and then A generates S_n. Thus, the first assertion holds. For the second assertion, let X be a tree. Since $m = n-1$, the edge space $\mathcal{E}$ is the bond space $\mathcal{B}$ of X by Proposition 2.2(ii). Hence, for $\mathbf{g} \in \mathbf{W}_E(X)$, if $\mathbf{g}E(v) = E(v)$ for every v then $\mathbf{g}$ is the identity map on $\mathcal{E}$. This shows that the kernel of α is trivial. From this and the first assertion, the second assertion holds. ☐

Example 3.6. Let $X = (V, E)$ be the star graph of $n \geq 3$ vertices. By Lemma 3.5, the edge-flipping group $\mathbf{W}_E(X)$ of X is isomorphic to S_n.

4 Orbits

Let $X = (V, E)$ denote a finite simple connected graph with $|V| = n$ and $|E| = m$, and let T denote a spanning tree of E. Let $\mathbf{W}_E(X)$ denote the edge-flipping group of X. In this section, we give a description of the orbits of the edge-flipping puzzle on X in terms of our language. For this purpose, we fix a vertex u in V for this whole section and choose a nice basis of the bond space $\mathcal{B}$ of X. Recall that from Proposition 2.2(iii) $E(u) = \sum_{v \in V - \{u\}} E(v)$, and from Proposition 2.2(iv)

$$\Delta := \{E(v) \mid v \in V - \{u\}\}$$

is a basis of $\mathcal{B}$. We call Δ the *simple basis* of $\mathcal{B}$. For each element G in $\mathcal{B}$, let $\Delta(G)$ denote the subset of Δ such that the sum of all elements in $\Delta(G)$ is equal to G, and let the *simple weight* $sw(G)$ of G be the cardinality of $\Delta(G)$. For example, $\Delta(E(u)) = \{E(v) \mid v \in V - \{u\}\}$ and $sw(E(u)) = n-1$.

Recall that an *orbit* of $\mathcal{E}$ under $\mathbf{W}_E(X)$ is the set $\{\mathbf{g}G \mid \mathbf{g} \in \mathbf{W}_E(X)\}$ for some $G \in \mathcal{E}$. Note that the orbits of $\mathcal{E}$ under $\mathbf{W}_E(X)$ are corresponding to the orbits of the edge-flipping puzzle on X and are the minimal nonempty invariant subsets of $\mathcal{E}$ under $\mathbf{W}_E(X)$. Therefore, by Proposition 2.3 and Proposition 3.2, every orbit of $\mathcal{E}$ under $\mathbf{W}_E(X)$ is contained in $F + \mathcal{B}$ for some $F \subseteq E - T$. In the following lemma, we give a description of the orbits of $\mathcal{B}$ under $\mathbf{W}_E(X)$ in terms of simple weights.

Lemma 4.1. *([18, Theorem 10]) Let $X = (V, E)$ be a finite simple connected graph with $|V| = n \geq 3$ and let T be a spanning tree of E. Then the orbits of $\mathcal{B}$ under $\mathbf{W}_E(X)$ are the same as the orbits of $\mathcal{B}$ under $\mathbf{W}_E(X)_T$. More precisely, these orbits are*

$$\Omega_i := \{G \in \mathcal{B} \mid sw(G) = i, n-i\} \text{ for } i = 0, 1, \ldots, \lceil \frac{n-1}{2} \rceil,$$

where $\mathbf{W}_E(X)_T$ is the subgroup of $\mathbf{W}_E(X)$ generated by $\{\boldsymbol{\rho}_\epsilon \mid \epsilon \in T\}$.

Sketch of Proof. Recall that from Proposition 2.1(ii) and Proposition 2.2(i), the bond space $\mathcal{B}$ of X consists of $E(U) = \sum_{v \in U} E(v)$ for all $U \subseteq V$. By Lemma 3.5, both $\mathbf{W}_E(X)$ and $\mathbf{W}_E(X)_T$ act on $\{E(v) \mid v \in V\}$ as the symmetric group on $\{E(v) \mid v \in V\}$. Hence every orbit of $\mathcal{B}$ under $\mathbf{W}_E(X)$ (or $\mathbf{W}_E(X)_T$) is one of $\{E(U) \mid |U| = i\}$ for $0 \leq i \leq n$. Since $E(u) = \sum_{v \in V - \{u\}} E(v)$, both $\{E(U) \mid |U| = i\}$ and $\{E(U) \mid |U| = n-i\}$ are equal to Ω_i for $0 \leq i \leq \lceil \frac{n-1}{2} \rceil$. $\square$

For nonempty $F \subseteq E - T$, the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)$ in terms of simple weights is given in the following.

Lemma 4.2. *([18, Theorem 12]) Let $X = (V, E)$ be a finite simple connected graph with $|V| = n \geq 3$ and let T be a spanning tree of E. Let F be a nonempty subset of $E - T$ and $\epsilon \in F$. Then the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)$ are the same as the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)_{T \cup \{\epsilon\}}$. More precisely, these orbits are*

$$\begin{cases} F + \mathcal{B}, & \textit{if } n \textit{ is odd;} \\ F + \mathcal{B}_e \text{ and } F + \mathcal{B}_o, & \textit{if } n \textit{ is even,} \end{cases} \tag{4.1}$$

where $\mathbf{W}_E(X)_{T\cup\{\epsilon\}}$ *is the subgroup of* $\mathbf{W}_E(X)$ *generated by* $\{\boldsymbol{\rho}_{\epsilon'} \mid \epsilon' \in T \cup \{\epsilon\}\}$, $\mathcal{B}_e := \{G \in \mathcal{B} \mid sw(G) \text{ is even}\}$ *and* $\mathcal{B}_o := \{G \in \mathcal{B} \mid sw(G) \text{ is odd}\}$.

Sketch of Proof. We now determine the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)_{T\cup\{\epsilon\}}$. Since $F \cap T = \emptyset$ and by (3.1), $\boldsymbol{\rho}_{\epsilon'}F = F$ for any $\epsilon' \in T$. Hence $\mathbf{W}_E(X)_T F = F$, and the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)_T$ are

$$F + \Omega_i \text{ for } i = 0, 1, \ldots, \lceil \frac{n-1}{2} \rceil \tag{4.2}$$

by Lemma 4.1. It thus remains to consider the action of additional map $\boldsymbol{\rho}_\epsilon$ on $F+\mathcal{B}$. Let $F+G \in F+\mathcal{B}$ with $G \in \mathcal{B}$ and $sw(G) = i$ for some $0 \leq i \leq n-1$. Note that $\boldsymbol{\rho}_\epsilon(F + G) = F + (E(\epsilon) + \boldsymbol{\rho}_\epsilon G)$ and $E(\epsilon) + \boldsymbol{\rho}_\epsilon G \in \mathcal{B}$. We discuss the simple weight of $E(\epsilon) + \boldsymbol{\rho}_\epsilon G$ as follows. If $u \notin \epsilon$ then $sw(E(\epsilon)) = 2$ and

$$sw(E(\epsilon) + \boldsymbol{\rho}_\epsilon G) = \begin{cases} i+2, & \text{if } |\Delta(G) \cap \Delta(E(\epsilon))| = 0; \\ i, & \text{if } |\Delta(G) \cap \Delta(E(\epsilon))| = 1; \\ i-2, & \text{otherwise,} \end{cases} \tag{4.3}$$

and if $u \in \epsilon$ then $sw(E(\epsilon)) = n-2$ and

$$sw(E(\epsilon) + \boldsymbol{\rho}_\epsilon G) = \begin{cases} i, & \text{if } |\Delta(G) \cap \Delta(E(\epsilon))| = i-1; \\ n-i-2, & \text{otherwise.} \end{cases} \tag{4.4}$$

By (4.3) and (4.4), some sets in (4.2) are further put together to become an orbit of $F+\mathcal{B}$ under $\mathbf{W}_E(X)_{T\cup\{\epsilon\}}$ and we have the result as described in (4.1). Since ϵ is an arbitrary element of F, the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)_{T\cup F}$ and under $\mathbf{W}_E(X)_{T\cup\{\epsilon\}}$ are the same, where $\mathbf{W}_E(X)_{T\cup F}$ is the subgroup of $\mathbf{W}_E(X)$ generated by the set $\{\boldsymbol{\rho}_{\epsilon'} \mid \epsilon' \in T \cup F\}$. Note that $\boldsymbol{\rho}_{\epsilon'}F = F$ for all $\epsilon' \in E - (T \cup F)$. Hence the orbits of $F + \mathcal{B}$ under $\mathbf{W}_E(X)$ and under $\mathbf{W}_E(X)_{T\cup F}$ are also the same. ☐

5 More on the edge-flipping groups

Let $X = (V, E)$ denote a finite simple connected graph with $|V| = n \geq 3$ and $|E| = m$. In this section, we investigate the structure of the edge-flipping group $\mathbf{W}_E(X)$ of X. Let $\mathcal{B}_i$ be a copy of the bond space $\mathcal{B}$ of X for $1 \leq i \leq m-n+1$. Recall that their *direct sum*

$$\mathcal{B}^{m-n+1} := \bigoplus_{i=1}^{m-n+1} \mathcal{B}_i$$

is the set of all $(m-n+1)$-tuples $(G_i)_{i=1}^{m-n+1}$ where $G_i \in \mathcal{B}_i$ and where the addition is defined componentwise; i.e., $(G_i)_{i=1}^{m-n+1} + (H_i)_{i=1}^{m-n+1} = (G_i + H_i)_{i=1}^{m-n+1}$. Let $\mathrm{Aut}(\mathcal{B}^{m-n+1})$ denote the automorphism group of $\mathcal{B}^{m-n+1}$.

Definition 5.1. Let $\beta : \mathbf{W}_E(X) \to \mathrm{Aut}(\mathcal{B}^{m-n+1})$ denote the group homomorphism defined by

$$\beta(\mathbf{g})(G_i)_{i=1}^{m-n+1} = (\mathbf{g}G_i)_{i=1}^{m-n+1}$$

for $\mathbf{g} \in \mathbf{W}_E(X)$ and $(G_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$.

Recall that from Lemma 3.5 the group homomorphism α from $\mathbf{W}_E(X)$ into the symmetric group S_n on $\{E(v) \mid v \in V\}$ is surjective. The following lemma shows that there exists a unique group homomorphism $\theta : S_n \to \mathrm{Aut}(\mathcal{B}^{m-n+1})$ such that the diagram

$$\begin{tikzcd}
\mathbf{W}_E(X) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S_n \arrow[d, "\theta"] \\
 & \mathrm{Aut}(\mathcal{B}^{m-n+1})
\end{tikzcd}$$

Figure 1.

is commutative.

Lemma 5.2. *There exists a unique group homomorphism* $\theta : S_n \to \mathrm{Aut}$ $(\mathcal{B}^{m-n+1})$ *such that* $\beta = \theta \circ \alpha$. *Moreover,*

$$\theta(\sigma)(E(v_i))_{i=1}^{m-n+1} = (\sigma(E(v_i)))_{i=1}^{m-n+1} \tag{5.1}$$

for $v_1, v_2, \ldots, v_{m-n+1} \in V$ *and* $\sigma \in S_n$.

Proof. Since α is surjective, if θ exists then θ is unique. To show the existence of θ, it suffices to show the kernel $\mathrm{Ker}\ \alpha$ of α is contained in the kernel $\mathrm{Ker}\ \beta$ of β. If $\mathbf{g} \in \mathrm{Ker}\ \alpha$, then $\mathbf{g}E(v) = E(v)$ for all $v \in V$ and hence $\mathbf{g} \in \mathrm{Ker}\ \beta$ since $\{E(v) \mid v \in V\}$ spans $\mathcal{B}$. Pick $\sigma \in S_n$ and choose an element $\mathbf{h}$ in $\mathbf{W}_E(X)$ such that $\alpha(\mathbf{h}) = \sigma$. To prove (5.1), it suffices to show that

$$\beta(\mathbf{h})(E(v_i))_{i=1}^{m-n+1} = (\alpha(\mathbf{h})(E(v_i)))_{i=1}^{m-n+1}$$

for $v_1, v_2, \ldots, v_{m-n+1} \in V$, since $\beta(\mathbf{h}) = \theta(\sigma)$ and $\alpha(\mathbf{h}) = \sigma$. By Definition 3.4 and Definition 5.1, we obtain that both sides of the above equation are equal to $(\mathbf{h}E(v_1), \mathbf{h}E(v_2), \ldots, \mathbf{h}E(v_{m-n+1}))$ as desired. $\square$

In view of Lemma 5.2, there is a *semidirect product* of $\mathcal{B}^{m-n+1}$ and S_n with respect to θ [13, p.155], denoted by $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$; i.e., $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$ is the set $\mathcal{B}^{m-n+1} \times S_n$ with the group operation defined by

$$\begin{aligned} &((G_i)_{i=1}^{m-n+1}, \sigma)\ ((H_i)_{i=1}^{m-n+1}, \tau) \\ =&((G_i)_{i=1}^{m-n+1} + \theta(\sigma)(H_i)_{i=1}^{m-n+1}, \sigma\tau) \end{aligned} \tag{5.2}$$

for all $(G_i)_{i=1}^{m-n+1}, (H_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$ and $\sigma, \tau \in S_n$. Recall that T denotes a spanning tree of E and $|T| = n-1$. Let $E-T = \{\epsilon_1, \epsilon_2, \dots, \epsilon_{m-n+1}\}$. Since $\{\epsilon_i\} + \mathbf{W}_E(X)\{\epsilon_i\} \subseteq \mathcal{B}$ for $1 \leq i \leq m-n+1$, we can define a map from $\mathbf{W}_E(X)$ into $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$ as follows.

Definition 5.3. Let $\gamma : \mathbf{W}_E(X) \to \mathcal{B}^{m-n+1} \rtimes_\theta S_n$ denote the map defined by

$$\gamma(\mathbf{g}) = ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g}))$$

for $\mathbf{g} \in \mathbf{W}_E(X)$.

The following lemma shows that γ is a group monomorphism.

Lemma 5.4. *γ is a group monomorphism from $\mathbf{W}_E(X)$ into $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$.*

Proof. For $\mathbf{g}, \mathbf{h} \in \mathbf{W}_E(X)$,

$$\begin{aligned} \gamma(\mathbf{g})\gamma(\mathbf{h}) &= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g}))((\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{h})) \\ &= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + \theta(\alpha(\mathbf{g}))(\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{g})\alpha(\mathbf{h})) \\ &= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + \beta(\mathbf{g})(\{\epsilon_i\} + \mathbf{h}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\ &= ((\{\epsilon_i\} + \mathbf{g}\{\epsilon_i\})_{i=1}^{m-n+1} + (\mathbf{g}\{\epsilon_i\} + \mathbf{gh}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\ &= ((\{\epsilon_i\} + \mathbf{gh}\{\epsilon_i\})_{i=1}^{m-n+1}, \alpha(\mathbf{gh})) \\ &= \gamma(\mathbf{gh}). \end{aligned}$$

This shows that γ is a group homomorphism. Since each $\mathbf{g} \in \mathrm{Ker}\ \gamma$ fixes the spanning set $\{\{\epsilon_1\}, \{\epsilon_2\}, \dots, \{\epsilon_{m-n+1}\}\} \cup \{E(v) \mid v \in V\}$ of the edge space $\mathcal{E}$ of X, $\mathbf{g}$ is the identity map on $\mathcal{E}$. Hence Ker γ is trivial. $\square$

By Lemma 5.4, $\mathbf{W}_E(X)$ is isomorphic to the subgroup $\gamma(\mathbf{W}_E(X))$ of $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$. Fortunately, $\gamma(\mathbf{W}_E(X))$ is knowable. Recall that $\mathcal{B}_e$, defined in Lemma 4.2, is an $(n-2)$-dimensional subspace of $\mathcal{B}$. Let $\mathcal{B}_e^{m-n+1}$ denote the subgroup

$$\bigoplus_{i=1}^{m-n+1} \mathcal{B}_{e,i}$$

of $\mathcal{B}^{m-n+1}$, where $\mathcal{B}_{e,i} := \mathcal{B}_e$ for $1 \leq i \leq m-n+1$.

Theorem 5.5. *Let $X = (V, E)$ be a finite simple connected graph with $n \geq 3$ vertices and m edges. Then the edge-flipping group $\mathbf{W}_E(X)$ is isomorphic to*

$$\begin{cases} \mathcal{B}^{m-n+1} \rtimes_\theta S_n, & \text{if } n \text{ is odd;} \\ \mathcal{B}_e^{m-n+1} \rtimes_\theta S_n, & \text{if } n \text{ is even.} \end{cases}$$

Proof. It suffices to show that for any $\sigma \in S_n$, there exists $\mathbf{g} \in \mathbf{W}_E(X)$ such that

$$\gamma(\mathbf{g}) = ((\emptyset)_{i=1}^{m-n+1}, \sigma), \tag{5.3}$$

and for each $1 \leq i \leq m - n + 1$, for any

$$G \in \begin{cases} B_i, & \text{if } n \text{ is odd;} \\ B_{e,i} & \text{if } n \text{ is even,} \end{cases}$$

there exists $\mathbf{h} \in \mathbf{W}_E(X)$ such that

$$\gamma(\mathbf{h}) = (\emptyset, \ldots, \emptyset, G, \emptyset, \ldots, \emptyset, \alpha(\mathbf{h})), \tag{5.4}$$

where G is in the ith coordinate. From Lemma 3.5, (5.3) follows by choosing $\mathbf{g} \in \mathbf{W}_E(X)_T$ with $\alpha(\mathbf{g}) = \sigma$, since $\mathbf{g}\{\epsilon_j\} = \{\epsilon_j\}$ for each $1 \leq j \leq m - n + 1$. From Lemma 4.2, (5.4) follows by choosing $\mathbf{h} \in \mathbf{W}_E(X)_{T \cup \{\epsilon_i\}}$ with $\mathbf{h}\{\epsilon_i\} = \{\epsilon_i\} + G$, since $\mathbf{h}\{\epsilon_j\} = \{\epsilon_j\}$ for $j \neq i$. □

Since dim $\mathcal{B} = n - 1$ and dim $\mathcal{B}_e = n - 2$, the additive groups of $\mathcal{B}$ and $\mathcal{B}_e$ are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ and $(\mathbb{Z}/2\mathbb{Z})^{n-2}$ respectively, where $\mathbb{Z}$ is the additive group of integers.

Example 5.6. Let X be a cycle of n vertices. Then the edge-flipping group $\mathbf{W}_E(X)$ of X is isomorphic to

$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n, & \text{if } n \text{ is odd;} \\ (\mathbb{Z}/2\mathbb{Z})^{n-2} \rtimes S_n, & \text{if } n \text{ is even} \end{cases}$$

by Theorem 5.5.

The following corollary says that there is a unique (up to isomorphism) edge-flipping group $\mathbf{W}_E(X)$ of all finite simple connected graphs $X = (V, E)$ with $|V| = n \geq 3$ and $|E| = m$.

Corollary 5.7. *Let $X = (V, E)$ and $X' = (V', E')$ be two finite simple connected graphs with $|V| = |V'| = n \geq 3$ and $|E| = |E'|$. Then the edge-flipping group $\mathbf{W}_E(X)$ of X is isomorphic to the edge-flipping group $\mathbf{W}_{E'}(X')$ of X'.*

Proof. We may assume that $V' = V$. Recall that the simple basis Δ of $\mathcal{B}$ is the set $\{E(v) \mid v \in V - \{u\}\}$ for some fixed vertex $u \in V$. Define $E'(v)$, $\mathcal{B}'$, $\Delta' := \{E'(v) \mid v \in V - \{u\}\}$, $\mathcal{B}'_e$, S'_n and θ' correspondingly. From Theorem 5.5, it suffices to show that $\mathcal{B}^{m-n+1} \rtimes_\theta S_n$ and $\mathcal{B}_e^{m-n+1} \rtimes_\theta S_n$ are isomorphic to $\mathcal{B}'^{m-n+1} \rtimes_{\theta'} S'_n$ and $\mathcal{B}_e'^{m-n+1} \rtimes_{\theta'} S'_n$ respectively. Let $\mu : \mathcal{B} \to \mathcal{B}'$ denote the invertible linear transformation defined by

$$\mu(E(v)) = E'(v)$$

for $v \in V - \{u\}$. Note that there exists a unique group isomorphism $\mu_* : S_n \to S'_n$ such that

$$\mu_*(\sigma)(E'(v)) = \mu(\sigma(E(v)))$$

for all $\sigma \in S_n$ and $v \in V$. Hence, there exists a unique bijective map $\phi : \mathcal{B}^{m-n+1} \rtimes_\theta S_n \to \mathcal{B}'^{m-n+1} \rtimes_{\theta'} S'_n$ such that

$$\phi((G_i)_{i=1}^{m-n+1}, \sigma) = ((\mu(G_i))_{i=1}^{m-n+1}, \mu_*(\sigma))$$

for all $(G_i)_{i=1}^{m-n+1} \in \mathcal{B}^{m-n+1}$ and $\sigma \in S_n$. The map ϕ sends $\mathcal{B}_e^{m-n+1} \rtimes_\theta S_n$ to $\mathcal{B}_e'^{m-n+1} \rtimes_{\theta'} S'_n$ since $\mu(\mathcal{B}_e) = \mathcal{B}'_e$. By (5.1) and (5.2), it is straightforward to verify that ϕ is a group isomorphism, as desired. $\square$

6 Applications

Let $X = (V, E)$ denote a finite simple connected graph with $|E| = m$. In this section, we investigate the vertex-flipping group $\mathbf{W}_V(X)$ of X, which is a vertex version of the edge-flipping group $\mathbf{W}_E(X)$ of X. The vertex-flipping group $\mathbf{W}_V(X)$ of X is also a subgroup of the general linear group $\mathrm{GL}(\mathcal{V})$ of the vertex space $\mathcal{V}$ of X. The orbits of $\mathcal{V}$ under $\mathbf{W}_V(X)$ are corresponding to the orbits of the vertex-flipping puzzle on X. See [9, 15] for details. For $v \in V$, let $N(v)$ denote the set consisting of all neighbors of v. In the following, we give a formal definition of $\mathbf{W}_V(X)$.

Definition 6.1. The *vertex-flipping group* $\mathbf{W}_V(X)$ of $X = (V, E)$ is the subgroup of the general linear group $\mathrm{GL}(\mathcal{V})$ of $\mathcal{V}$ generated by $\{\boldsymbol{s}_v \mid v \in V\}$, where $\boldsymbol{s}_v : \mathcal{V} \to \mathcal{V}$ is defined by

$$\boldsymbol{s}_v U = \begin{cases} U + N(v), & \text{if } v \in U; \\ U, & \text{otherwise} \end{cases}$$

for $U \in \mathcal{V}$.

The *line graph* $L(X)$ of $X = (V, E)$ is a finite simple connected graph with vertex set E and edge set $\{\{\epsilon, \epsilon'\} \mid |\epsilon \cap \epsilon'| = 1 \text{ for } \epsilon, \epsilon' \in E\}$. From this definition, the edge-flipping group of X and the vertex-flipping group of $L(X)$ are the same.

For this whole section, let $Y = (Z, H)$ denote the finite simple connected graph with $Z = \{0, 1, 2, \ldots, m-1\}$ and $H = \{\{1, 2\}, \{2, 3\}, \ldots, \{m-2, m-1\}, \{0, i_1\}, \{0, i_2\}, \ldots, \{0, i_\ell\}\}$, where $m \geq 2$ and $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m-1$. See Figure 2. For example, the simply-laced Dynkin diagrams and extended Dynkin diagrams $\widetilde{A}_n, \widetilde{E}_7, \widetilde{E}_8$ are such graphs.

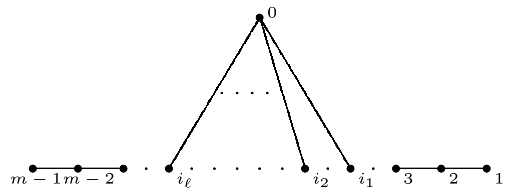

Figure 2. The graph Y

Let the value π_1 of Y be

$$\begin{cases} \sum_{t=1}^{\ell} (-1)^t i_t, & \text{if } \ell \text{ is even;} \\ \sum_{t=1}^{\ell} (-1)^t i_t + m, & \text{otherwise.} \end{cases} \tag{6.1}$$

Note that $1 \leq \pi_1 \leq m-1$.

Theorem 6.2. *([10, Theorem 3.9]) Let $1 \leq k \leq m-1$ be an integer. Then the vertex-flipping group of $Y = (Z, H)$ is unique (up to isomorphism) among those graphs Y with $\pi_1 = k$.* □

The aim of this section is to determine the structures of the vertex-flipping groups $\mathbf{W}_Z(Y)$ of some graphs Y. For this purpose, we define some terms. Two finite simple graphs $X = (V, E)$ and $X' = (V', E')$ are *isomorphic* if there exists a bijective map $\phi : V \to V'$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E'$ for all $x, y \in V$. We shall denote that two finite simple graphs X and X' are isomorphic by writing $X \cong X'$. A graph (U, F) is a *subgraph* of $X = (V, E)$ if $U \subseteq V$ and $F \subseteq E$, and a subgraph (U, F) of $X = (V, E)$ is *induced* if $F = \{\{x, y\} \in E \mid x, y \in U\}$. An (induced) subgraph (U, F) of $X = (V, E)$ is an *(induced) path* if $U = \{v_0, v_1, \ldots, v_k\}$ and $F = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$ for some nonnegative integer k, where v_i are all distinct. The following easy fact will be used later.

Lemma 6.3. *Let $X = (V, E)$ be a finite simple connected graph with $|E| = m$. Let $1 \le k \le m$ be an integer. Then X contains a path of k edges if and only if the line graph $L(X)$ of X contains an induced path of k vertices.* $\square$

We determine the structure of the vertex-flipping group $\mathbf{W}_Z(Y)$ of $Y = (Z, H)$ in some special cases.

Corollary 6.4. *Let $Y = (Z, H)$ be a finite simple connected graph with $Z = \{0, 1, 2, \ldots, m-1\}$ and $H = \{\{1, 2\}, \{2, 3\}, \ldots, \{m-2, m-1\}, \{0, i_1\}, \{0, i_2\}, \ldots, \{0, i_\ell\}\}$, where $1 \le i_1 < i_2 < \cdots < i_\ell \le m-1$. Let the value π_1 of Y be defined in (6.1). Then the following (i)-(iii) hold.*

(i) If Y is isomorphic to a line graph $L(X)$ for some finite simple connected graph X, then $\pi_1 \in \{1, 2, m-2, m-1\}$.

(ii) If $\pi_1 \in \{1, m-1\}$, then the vertex-flipping group $\mathbf{W}_Z(Y)$ of Y is isomorphic to the symmetric group S_{m+1} of degree $m+1$.

(iii) If $\pi_1 \in \{2, m-2\}$, then $\mathbf{W}_Z(Y)$ is isomorphic to

$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes S_m, & \textit{if } m \textit{ is odd;} \\ (\mathbb{Z}/2\mathbb{Z})^{m-2} \rtimes S_m, & \textit{if } m \textit{ is even,} \end{cases}$$

where $\mathbb{Z}$ is the additive group of integers.

Proof. Suppose that Y is isomorphic to $L(X)$ for some finite simple connected graph X. Since Y contains the induced path

$$(\{1, 2, \ldots, m-1\}, \{\{1, 2\}, \{2, 3\}, \ldots, \{m-2, m-1\}\}),$$

X contains a path of $m-1$ edges by Lemma 6.3. Note that X has m edges, one more edge besides the path. Hence the left column of Figure 3 completely lists all such graphs X, and the right column is the corresponding line graph $L(X)$ of X. By computing the value π_1 of $Y \cong L(X)$ and using (6.1), we find $\pi_1 = 1, 2, m-2$, or $m-1$. This proves (i). Note that the vertex-flipping group $\mathbf{W}_Z(Y)$ of Y is the edge-flipping group of X, and its group structure only depends on π_1 by Theorem 6.2. Hence we find $\mathbf{W}_Z(Y)$ as listed in (ii),(iii) by Theorem 5.5.

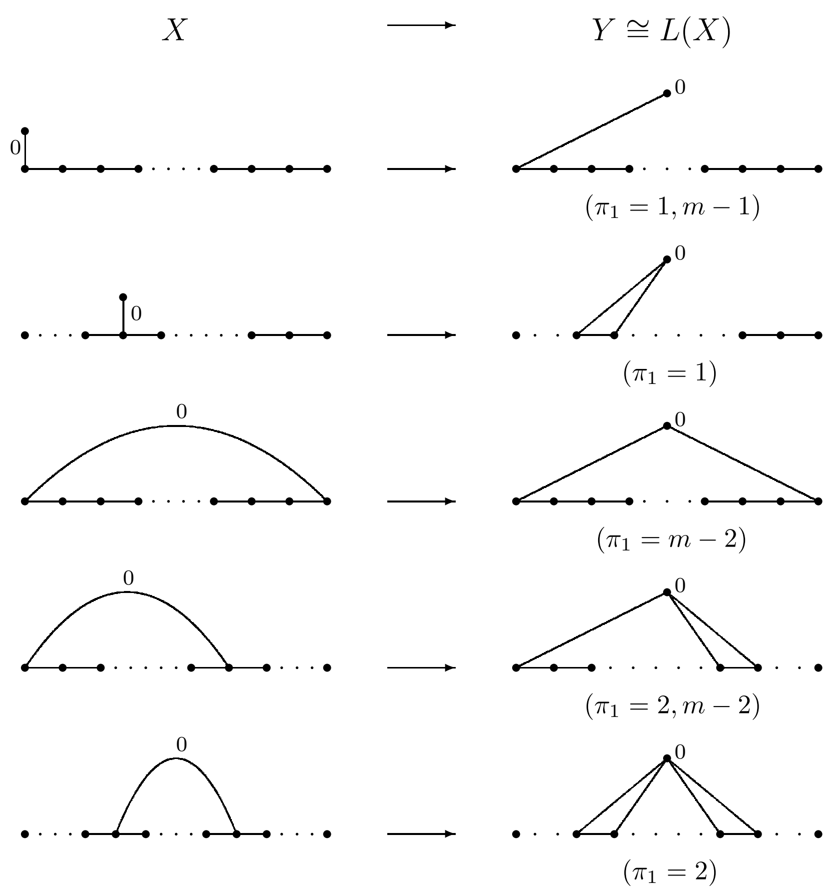

Figure 3. All graphs Y are isomorphic to line graphs

$\square$

Example 6.5. The graph $Y = (Z, H)$ in Figure 4

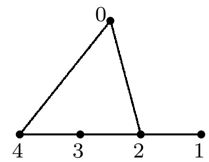

Figure 4.

is a five-vertex graph containing an induced path of four vertices. By (6.1), its value π_1 is 2. Hence, $\mathbf{W}_Z(Y)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ by Corollary 6.4(iii).

In the case of $\pi_1 \in \{1, 2, m-2, m-1\}$, we use (6.1) to find all such graphs, and there are only two graphs that are not isomorphic to line graphs. We show both of them in Figure 5. Note that their possible values π_1 are in $\{2, m-2\}$. By Corollary 6.4(iii), if $Y = (Z, H)$ is one of two graphs in Figure 5, the vertex-flipping group $\mathbf{W}_Z(Y)$ of Y is isomorphic to

$$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes S_m, & \text{if } m \text{ is odd;} \\ (\mathbb{Z}/2\mathbb{Z})^{m-2} \rtimes S_m, & \text{if } m \text{ is even.} \end{cases}$$

Figure 5. All graphs Y with $\pi_1 \in \{1, 2, m-2, m-1\}$ are not isomorphic to line graphs

Remark 6.6. Theorem 6.2 implies that in the class of 2^{m-1} graphs $Y = (Z, H)$, the number of the vertex-flipping groups $\mathbf{W}_Z(Y)$ of Y is at most $m-1$ up to isomorphism. Together with Corollary 5.7, it seems that for a given vertex number, the number of non-isomorphic vertex-flipping groups is not too large, and a classification of them seems to be visible.

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