Locally subquadrangular hyperplanes in symplectic and Hermitian dual polar spaces

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Abstract

In [11] all locally subquadrangular hyperplanes of finite symplectic and Hermitian dual polar spaces were determined with the aid of counting arguments and divisibility properties of integers. In the present note we extend this classification to the infinite case. We prove that symplectic dual polar spaces and certain Hermitian dual polar spaces cannot have locally subquadrangular hyperplanes if their rank is at least three and if their lines contain more than three points.

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1 Introduction

Let Π be a nondegenerate polar space (Tits [14]; Veldkamp [15]) of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is called a *dual polar space of rank n* (Cameron [1]). The dual polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles.

If x_1 and x_2 are two points of the dual polar space Δ, then $\mathrm{d}(x_1, x_2)$ denotes the distance between x_1 and x_2 in the collinearity graph of Δ. If x is a point of Δ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. For every point x of Δ, we define $x^\perp := \{x\} \cup \Gamma_1(x)$. The dual polar space Δ is a *near polygon* which means that for every point x and every line

L, there exists a unique point on L nearest to x. A set X of points of Δ is called a *subspace* if it contains all the points of a line as soon as it contains at least two points of it. X is called *convex* if it contains all the points on a shortest path between any two of its points.

There exists a bijective correspondence between the nonempty convex subspaces of Δ and the possibly empty singular subspaces of Π: if α is an $(n-1-k)$-dimensional singular subspace of Π, then the set of all maximal singular subspaces of Π containing α is a convex subspace of diameter k of Δ. The convex subspaces of diameter 2, 3, respectively $n-1$ are called the *quads*, *hexes*, respectively *maxes* of Δ. If F is a convex subspace of diameter $k \geq 2$, then the points and lines of Δ contained in F define a dual polar space $\widetilde{F}$ of rank k. In particular, if Q is a quad of Δ, then $\widetilde{Q}$ is a generalized quadrangle. Every two points x_1 and x_2 of Δ at distance k are contained in a unique convex subspace $\langle x_1, x_2 \rangle$ of diameter k. Every two distinct intersecting lines L_1 and L_2 of Δ are contained in a unique quad which we will denote by $\langle L_1, L_2 \rangle$. If M is a max of Δ, then every point x of Δ not contained in M is collinear with a unique point $\pi_M(x)$ of M. If M_1 and M_2 are two disjoint maxes, then the map $M_1 \to M_2; x \mapsto \pi_{M_2}(x)$ defines an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$.

Let Δ be a thick dual polar space of rank $n \geq 2$. A *hyperplane* of Δ is a proper subspace of Δ which meets every line. By Shult [12, Lemma 6.1], every hyperplane of Δ is a maximal subspace. If x is a point of Δ, then the set of points at distance at most $n-1$ from x is a hyperplane of Δ, the so-called *singular hyperplane with deepest point* x. A set of points of Δ is called an *ovoid* if it intersects every line of Δ in a unique point. Clearly ovoids are hyperplanes. If H is a hyperplane of Δ and if Q is a quad of Δ, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of $\widetilde{Q}$. Hence, either (i) $Q \subseteq H$, (ii) $Q \cap H$ is a singular hyperplane of $\widetilde{Q}$, (iii) $Q \cap H$ is a subquadrangle of $\widetilde{Q}$, or (iv) $Q \cap H$ is an ovoid of $\widetilde{Q}$. If case (i), (ii), (iii), respectively (iv) occurs, then we say that Q is *deep*, *singular*, *subquadrangular*, respectively *ovoidal* with respect to H.

In this paper we will meet three classes of (dual) polar spaces.

(I) For every $n \in \mathbb{N} \setminus \{0,1\}$ and every field $\mathbb{K}$, $Q(2n, \mathbb{K})$ denotes the orthogonal polar space associated to a nonsingular parabolic quadric of Witt index n of $\mathrm{PG}(2n, \mathbb{K})$. The corresponding dual polar space is denoted by $DQ(2n, \mathbb{K})$. If $\mathbb{K}$ is isomorphic to the finite field $\mathbb{F}_q$ with q elements, then we denote $DQ(2n, \mathbb{K})$ also by $DQ(2n, q)$.

(II) For every $n \in \mathbb{N} \setminus \{0,1\}$ and every field $\mathbb{K}$, let $W(2n-1, \mathbb{K})$ denote the symplectic polar space whose singular subspaces are the subspaces of $\mathrm{PG}(2n-1, \mathbb{K})$ which are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(2n-1,\mathbb{K})$. The corresponding dual polar space is denoted by $DW(2n-1,\mathbb{K})$. The dual polar space $DW(2n-1,\mathbb{K})$ is isomorphic to $DQ(2n,\mathbb{K})$ if and only if $\mathbb{K}$ is a perfect field of characteristic 2. If $\mathbb{K} \cong \mathbb{F}_q$, then we denote $DW(2n-1,\mathbb{K})$ also by $DW(2n-1,q)$.

(III) Let $n \in \mathbb{N} \setminus \{0,1\}$, let $\mathbb{K},\mathbb{K}'$ be two fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$ and let θ denote the unique nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$. Let $H(2n-1,\mathbb{K}',\theta)$ denote the Hermitian polar space associated to a nonsingular θ-Hermitian variety of Witt index n of $\mathrm{PG}(2n-1,\mathbb{K}')$. The corresponding dual polar space is denoted by $DH(2n-1,\mathbb{K}',\theta)$. If $\mathbb{K} \cong \mathbb{F}_q$ and $\mathbb{K}' \cong \mathbb{F}_{q^2}$, then we denote $DH(2n-1,\mathbb{K}',\theta)$ also by $DH(2n-1,q^2)$.

Now, let Π be one of the polar spaces $W(2n-1,\mathbb{K})$ or $H(2n-1,\mathbb{K}',\theta)$, where $n \geq 2$, $\mathbb{K}$, $\mathbb{K}'$ and θ are as above. We denote by Δ the dual polar space associated to Π. Let $\mathcal{P}$ denote the ambient projective space of Π. So, $\mathcal{P} \cong \mathrm{PG}(2n-1,\mathbb{K})$ if $\Pi = W(2n-1,\mathbb{K})$ and $\mathcal{P} \cong \mathrm{PG}(2n-1,\mathbb{K}')$ if $\Pi = H(2n-1,\mathbb{K}',\theta)$. If x_1 and x_2 are two noncollinear points of Π, then the unique line of $\mathcal{P}$ containing x_1 and x_2 intersects Π in a set of $|\mathbb{K}|+1$ points. Such a set of points is called a *hyperbolic line* of Π. The set of maxes of Δ corresponding to the points of a hyperbolic line of Π is called a *nice set of maxes of* Δ. Every two disjoint maxes M_1 and M_2 of Δ are contained in a unique nice set of maxes of Δ which we will denote by $\Omega(M_1,M_2)$. If L is a line meeting M_1 and M_2, then L meets every max of $\Omega(M_1,M_2)$. Moreover, every point of L is contained in precisely one of the maxes of $\Omega(M_1,M_2)$.

If x is a point of Δ, then the set of convex subspaces of Δ containing x defines a projective space $Res(x)$ isomorphic to either $\mathrm{PG}(n-1,\mathbb{K})$ (symplectic case) or $\mathrm{PG}(n-1,\mathbb{K}')$ (Hermitian case). If H is a hyperplane of Δ and $x \in H$, then $\Lambda_H(x)$ denotes the set of lines through x which are contained in H. We will regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If $\Lambda_H(x)$ coincides with the whole point-set of $Res(x)$, then x is called *deep* with respect to H.

A hyperplane H of a thick dual polar space Δ of rank $n \geq 3$ is called *locally singular*, *locally ovoidal*, respectively *locally subquadrangular*, if every non-deep quad of Δ is singular, ovoidal, respectively subquadrangular with respect to H.

The locally ovoidal hyperplanes of Δ are precisely the ovoids of Δ. The Hermitian dual polar space $DH(2n-1,q^2)$, $n \geq 3$, has no ovoids because its quads do not have ovoids. The symplectic dual polar space $DW(2n-1,q)$, $n \geq 3$, has no ovoids by Cooperstein and Pasini [5]. With the aid of transfinite recursion, it is easy to construct ovoids in infinite dual polar spaces (in particular in infinite symplectic and Hermitian dual polar spaces), see Cameron [2] or Cardinali and De Bruyn [3, Section 4].

If Δ is not isomorphic to $DQ(2n, \mathbb{K})$ for some field $\mathbb{K}$, then every locally singular hyperplane of Δ is singular by Cardinali, De Bruyn and Pasini [4, Theorem 3.5]. By De Bruyn [6, Theorem 1.3] (see also Shult and Thas [13]), the locally singular hyperplanes of $DQ(2n, \mathbb{K})$ are precisely those hyperplanes of $DQ(2n, \mathbb{K})$ which arise from the so-called spin-embedding of $DQ(2n, \mathbb{K})$.

Pasini and Shpectorov [11] classified all locally subquadrangular hyperplanes of finite thick dual polar spaces. They proved that the symplectic dual polar space $DW(2n-1, q)$, $n \geq 3$, has locally subquadrangular hyperplanes if and only if $q = 2$, in which case there is up to isomorphism a unique locally subquadrangular hyperplane. They also proved that the Hermitian dual polar space $DH(2n-1, q^2)$, $n \geq 3$, has locally subquadrangular hyperplanes if and only if $n = 3$ and $q = 2$, in which case there is up to isomorphism a unique locally subquadrangular hyperplane. The reasoning given in [11] makes use of counting arguments and divisibility properties of integers and can therefore not automatically be extended to the infinite case. The aim of this note is to prove that locally subquadrangular hyperplanes cannot exist in the infinite case.

The following is the main result of this note. We will prove it in Sections 2 and 3.

Main Theorem. *Let Δ be one of the dual polar spaces $DW(2n-1, \mathbb{K})$ or $DH(2n-1, \mathbb{K}', \theta)$, where n, $\mathbb{K}$, $\mathbb{K}'$ and θ are as above. If $n \geq 3$ and $|\mathbb{K}| \geq 3$, then Δ has no locally subquadrangular hyperplanes.*

2 The symplectic case

The following proposition is precisely the main theorem in the case $\Delta = DW(2n-1, \mathbb{K})$.

Proposition 2.1 *The dual polar space $DW(2n-1, \mathbb{K})$, $n \geq 3$ and $|\mathbb{K}| \geq 3$, has no locally subquadrangular hyperplanes.*

Proof. If H is a locally subquadrangular hyperplane of $DW(2n-1, \mathbb{K})$, $n \geq 3$, then there exists a hex F such that $H \cap F$ is a locally subquadrangular hyperplane of $\widetilde{F} \cong DW(5, \mathbb{K})$. So, it suffices to prove the proposition in the case $n = 3$. Suppose therefore that $n = 3$ and that H is a locally subquadrangular hyperplane of $DW(5, \mathbb{K})$, where $|\mathbb{K}| \geq 3$.

Claim I. *For every point x of H, $\Lambda_H(x)$ is one of the following sets of points of $Res(x) \cong \mathrm{PG}(2, \mathbb{K})$:*

(1) *a set of points of $Res(x)$ meeting each line of $Res(x)$ in a set of two points;*

(2) *the whole set of points of $Res(x)$.*

PROOF. Every quad through x is either deep or subquadrangular with respect to H. So,

> $(*)$ every line of $Res(x)$ meets $\Lambda_H(x)$ in either 2 points or the whole line.

If every line of $Res(x)$ meets $\Lambda_H(x)$ in precisely two points, then case (1) occurs. Suppose therefore that $\Lambda_H(x)$ contains a line M of $Res(x)$. Let $m \in M$, let M_1 and M_2 be lines of $Res(x)$ through m such that M, M_1 and M_2 are mutually distinct. By $(*)$, for every $i \in \{1,2\}$ there exists a point $m_i \in M_i \cap \Lambda_H(x)$ distinct from m. Also by $(*)$, the line $N := m_1m_2$ is contained in $\Lambda_H(x)$ since it contains at least three points of $\Lambda_H(x)$. By applying $(*)$ to some line through $\{u\} := M \cap N$ distinct from M and N, we see that there exists a point $v \in \Lambda_H(x) \setminus (M \cup N)$. Any line of $Res(x)$ through v distinct from uv contains at least three points of $\Lambda_H(x)$ and hence is completely contained in $\Lambda_H(x)$. It follows that every point of $Res(x)$ not contained on uv belongs to $Res(x)$. Suppose there exists a point w in $Res(x)$ not contained in $\Lambda_H(x)$. Then necessarily $w \in uv$. On every line of $Res(x)$ through w distinct from uv, there exists a unique point (namely w) not contained in $\Lambda_H(x)$, contradicting $(*)$ and the fact that $|\mathbb{K}| \geq 3$. (qed)

Claim II. *There exists a quad which is deep with respect to H.*

PROOF. Suppose to the contrary that every quad is subquadrangular with respect to H. Let Q_1 be an arbitrary quad of $DW(5,\mathbb{K})$, let x be an arbitrary point of $Q_1 \cap H$ and let L_1 and L_2 denote the two lines of Q_1 through x which are contained in H. By Claim I, there exists a line $L \subseteq H$ through x not contained in Q_1. Put $R_i := \langle L_i, L\rangle$, $i \in \{1,2\}$, and let y be an arbitrary point of $L \setminus \{x\}$. Since R_i is subquadrangular with respect to H, there exists a unique line $M_i \subseteq R_i \cap H$ through y distinct from L. The unique quad Q_2 through M_1 and M_2 is subquadrangular with respect to H. Let x_1 be an arbitrary point of $(x^\perp \cap Q_1) \setminus (L_1 \cup L_2)$, let U denote the unique line through $x_1 \notin H$ meeting Q_2 and let Q_3 be the unique element of $\Omega(Q_1,Q_2)$ containing $U \cap H$. Since $Q_3 \cap H$ is a subquadrangle (i.e. a full subgrid) of $\widetilde{Q_3}$, $\pi_{Q_1}(Q_3 \cap H)$ is a full subgrid of $\widetilde{Q_1}$. Clearly, $x_1 \in \pi_{Q_1}(Q_3 \cap H)$. If x_1' is a point of $L_1 \cup L_2$, and if x_i', $i \in \{2,3\}$, denotes the unique point of Q_i collinear with x_1', then $x_2' \in M_1 \cup M_2$ and hence $x_2' \in H$. Since $x_3' \in x_1'x_2'$, we have $x_3' \in Q_3 \cap H$ and hence $x_1' = \pi_{Q_1}(x_3') \in \pi_{Q_1}(Q_3 \cap H)$. Since x_1' was an arbitrary point of $L_1 \cup L_2$, $L_1 \cup L_2 \subseteq \pi_{Q_1}(Q_3 \cap H)$. Now, $\pi_{Q_1}(Q_3 \cap H)$ is a full subgrid of $\widetilde{Q_1}$ containing the three lines L_1, L_2 and xx_1 through x, clearly a contradiction. Hence, there exists a quad which is deep with respect to H. (qed)

We are now ready to derive a contradiction. Suppose x is a deep point of H, and let $y \in \Gamma_1(x)$. If Q is a quad through x and y, then $x^\perp \cap Q \subseteq H$ implies that Q is deep with respect to H. By Claim I, it then follows that y is also deep with respect to H. By the connectedness of $DW(5, \mathbb{K})$, every point of $DW(5, \mathbb{K})$ is deep with respect to H, which is clearly impossible.

So, there are no points of H which are deep with respect to H. By Claim I, every quad is then subquadrangular. But this is impossible by Claim II. $\square$

Remark. As already mentioned in Section 1, the conclusion of Proposition 2.1 is not valid if $n \geq 3$ and $\mathbb{K} = \mathbb{F}_2$. We needed the condition $|\mathbb{K}| \geq 3$ at one place in the proof of Proposition 2.1, namely in the proof of Claim I. If $\mathbb{K} = \mathbb{F}_2$, then a quick inspection of the proof of Claim I learns that there is one other possibility for $\Lambda_H(x)$ which is not in contradiction with condition $(*)$, namely $\Lambda_H(x)$ might also be a punctured projective plane.

Locally subquadrangular hyperplanes of $DQ(2n, 2)$, $n \geq 2$, are easily constructed. Let $Q(2n, 2)$ be the nonsingular parabolic quadric of $\mathrm{PG}(2n, 2)$ associated with $DQ(2n, 2)$, and let α be a hyperplane of $\mathrm{PG}(2n, 2)$ which intersects $Q(2n, 2)$ in a nonsingular hyperbolic quadric $Q^+(2n-1, 2)$. The maximal singular subspaces of $Q(2n, 2)$ not contained in α then define a locally subquadrangular hyperplane of $DQ(2n, 2)$. By Pasini and Shpectorov [11], every locally subquadrangular hyperplane of $DQ(2n, 2)$ is obtained in this way.

3 The Hermitian case

In this section, $\mathbb{K}'$ and $\mathbb{K}$ are fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$ and θ denotes the unique nontrivial element in the Galois group $Gal(\mathbb{K}'/\mathbb{K})$. Let $n \geq 3$ and let $DH(2n-1, \mathbb{K}', \theta)$ denote the Hermitian dual polar space as defined in Section 1. We will prove that if $|\mathbb{K}| \geq 3$, then $DH(2n-1, \mathbb{K}', \theta)$ has no locally subquadrangular hyperplanes. If H is a locally subquadrangular hyperplane of $DH(2n-1, \mathbb{K}', \theta)$, $n \geq 3$, then there exists a hex F such that $H \cap F$ is a locally subquadrangular hyperplane of $\widetilde{F} \cong DH(5, \mathbb{K}', \theta)$. So, it suffices to prove that $DH(5, \mathbb{K}', \theta)$ has no locally subquadrangular hyperplanes if $|\mathbb{K}| \geq 3$.

3.1 A useful lemma

Lemma 3.1 *Let V_i, $i \in \{1, 2\}$, be a 3-dimensional vector space over $\mathbb{K}'$, let p_i be a point of $\pi_i := \mathrm{PG}(V_i)$, let $\mathcal{L}_i$ be the set of lines of π_i through p_i, let X_i be a set of points of π_i containing p_i such that $L_i \cap X_i$ is a Baer-$\mathbb{K}$-subline* *of L_i for every $L_i \in \mathcal{L}_i$, and let $\mathcal{A}_i$, $i \in \{1,2\}$, be the set of all subsets A of $\mathcal{L}_i$ such that $\bigcup_{L_i \in A}(L_i \cap M_i)$ is a Baer-$\mathbb{K}$-subline of M_i for at least one and hence every line $M_i \notin \mathcal{L}_i$ of π_i. Suppose ϕ be a bijection between $\mathcal{L}_1$ and $\mathcal{L}_2$ inducing a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$. Then for every two distinct lines U and V of $\mathcal{L}_1$, there exists a line $W \in \mathcal{L}_1 \setminus \{U,V\}$ such that* (1) *$M_1 \cap U$, $M_1 \cap V$, $M_1 \cap W$ are contained in X_1 for at least one line $M_1 \notin \mathcal{L}_1$ of π_1,* (2) *$M_2 \cap \phi(U)$, $M_2 \cap \phi(V)$, $M_2 \cap \phi(W)$ are contained in X_2 for at least one line $M_2 \notin \mathcal{L}_2$ of π_2.*

Proof. Let $\bar{e}_0$, $\bar{e}_1$ and $\bar{e}_2$ be three vectors of V_1 such that $p_1 = \langle \bar{e}_0 \rangle$, $U \cap X_1 = \{\langle \bar{e}_0 \rangle\} \cup \{\langle \bar{e}_1 + \lambda \bar{e}_0 \rangle \mid \lambda \in \mathbb{K}\}$ and $V \cap X_1 = \{\langle \bar{e}_0 \rangle\} \cup \{\langle \bar{e}_2 + \lambda \bar{e}_0 \rangle \mid \lambda \in \mathbb{K}\}$. Let A_1^* denote the unique element of $\mathcal{A}_1$ consisting of all lines of the form $\langle \bar{e}_0, k\bar{e}_1 + l\bar{e}_2 \rangle$ where $(k,l) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Then $U, V \in A_1^*$. If $W \in \mathcal{L}_1 \setminus A_1^*$, then $W \cap X_1 = \{\langle \bar{e}_0 \rangle\} \cup \{\langle k\bar{e}_1 + l\bar{e}_2 + m\bar{e}_0 + \lambda \bar{e}_0 \rangle \mid \lambda \in \mathbb{K}\}$ for some $k,l,m \in \mathbb{K}'$ with $(k,l) \notin \mathbb{K}^2$. Since $(k,l) \notin \mathbb{K}^2$ and $\mathbb{K}'$ can be regarded as a 2-dimensional vector space over $\mathbb{K}$, there exists a $(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{K}^3$ such that

$$\begin{vmatrix} \lambda_1 & 1 & 0 \\ \lambda_2 & 0 & 1 \\ m + \lambda_3 & k & l \end{vmatrix} = -k \cdot \lambda_1 - l \cdot \lambda_2 + \lambda_3 + m = 0,$$

i.e. there exists a line $M_1 \notin \mathcal{L}_1$ of π_1 such that $M_1 \cap U$, $M_1 \cap V$ and $M_1 \cap W$ are contained in X_1. Similarly, there exists an $A_2^* \in \mathcal{A}_2$ containing $\phi(U)$ and $\phi(V)$ such that for every $W' \in \mathcal{L}_2 \setminus A_2^*$, there exists a line $M_2 \notin \mathcal{L}_2$ of π_2 such that $M_2 \cap \phi(U)$, $M_2 \cap \phi(V)$ and $M_2 \cap W'$ are contained in X_2.

Since ϕ induces a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$, $\phi^{-1}(A_2^*) \in \mathcal{A}_1$. Clearly, U and V are contained in A_1^* and $\phi^{-1}(A_2^*)$. The lines U and V of $\mathcal{L}_1$ are contained in precisely $|\mathbb{K}|+1$ elements of $\mathcal{A}_1$. These elements of $\mathcal{A}_1$ determine a partition of $\mathcal{L}_1 \setminus \{U,V\}$. Since $|\mathbb{K}|+1 \geq 3$, there exists a line $W \in \mathcal{L}_1$ not contained in $A_1^* \cup \phi^{-1}(A_2^*)$. The line W satisfies the required conditions. □

3.2 The nonexistence of locally subquadrangular hyperplanes of $DH(5, \mathbb{K}', \theta)$ when $|\mathbb{K}| \geq 3$

Every quad of the dual polar space $DH(5, \mathbb{K}', \theta)$ is isomorphic to the generalized quadrangle $Q^-(5, \mathbb{K}) \cong DH(3, \mathbb{K}', \theta)$ associated to a nonsingular quadric of Witt index 2 of $\mathrm{PG}(5, \mathbb{K})$ which becomes a nonsingular quadric of Witt index 3 when regarded over the quadratic extension $\mathbb{K}'$ of $\mathbb{K}$.

Suppose Q is a quad of $DH(5, \mathbb{K}', \theta)$ and H is a hyperplane of $DH(5, \mathbb{K}', \theta)$ such that $Q \cap H$ is a subquadrangle of $\widetilde{Q} \cong Q^-(5, \mathbb{K})$. Since $Q \cap H$ is a hyperplane of $\widetilde{Q}$, the point-line geometry $\widetilde{Q \cap H}$ induced on $Q \cap H$ is isomorphic to the generalized quadrangle $Q(4,\mathbb{K})$. Let x be an arbitrary point of $Q \cap H$. Then Q defines a line L_Q of $Res(x) \cong \mathrm{PG}(2,\mathbb{K}')$ and the set of lines through x contained in $Q \cap H$ defines a Baer-$\mathbb{K}$-subline L'_Q of L_Q (see e.g. [3, Corollary 1.5(4)]). We make the latter statement more detailed. Suppose V is a 6-dimensional vector space over $\mathbb{K}'$ such that $\mathrm{PG}(V)$ contains the θ-Hermitian variety $H(5,\mathbb{K}',\theta)$ which defines $DH(5,\mathbb{K}',\theta)$. Let W denote the 3-dimensional subspace of V such that x corresponds to $\mathrm{PG}(W)$ and let p denote the point of $\mathrm{PG}(W)$ corresponding to Q. Then the lines of $DH(5,\mathbb{K}',\theta)$ through x correspond to the lines of $\mathrm{PG}(W)$, L_Q corresponds to the set of lines of $\mathrm{PG}(W)$ through p and L'_Q corresponds to a set A of lines of $\mathrm{PG}(W)$ through p such that $\bigcup_{L\in A} L \cap M$ is a Baer-$\mathbb{K}$-subline of $\mathrm{PG}(W)$ for every line M of $\mathrm{PG}(W)$ not containing p.

Lemma 3.2 *If H is a locally subquadrangular hyperplane of $DH(5,\mathbb{K}',\theta)$, then there is a quad which is deep with respect to H.*

Proof. Suppose to the contrary that every quad is subquadrangular with respect to H. Let L be an arbitrary line of $DH(5,\mathbb{K}',\theta)$ contained in H and let x_1, x_2 be two distinct points of L.

For every $i \in \{1,2\}$, we define $\pi_i := Res(x_i)$ and $p_i := L$, X_i denotes the set of lines through x_i contained in H and $\mathcal{L}_1 = \mathcal{L}_2$ denotes the set of quads of $DH(5,\mathbb{K}',\theta)$ containing L. The projective plane π_i admits a system of Baer-$\mathbb{K}$-sublines as explained before this lemma. Let $\mathcal{A}_i$ be the set of subsets of $\mathcal{L}_i$ as defined in Lemma 3.1. If we regard L as a line of $\mathrm{PG}(V)$, then the elements of $\mathcal{L}_1 = \mathcal{L}_2$ correspond to the points of L and the elements of $\mathcal{A}_1$ and $\mathcal{A}_2$ correspond to the Baer-$\mathbb{K}$-sublines of $L \subseteq \mathrm{PG}(V)$. So, $\mathcal{A}_1 = \mathcal{A}_2$ and the trivial permutation ϕ of $\mathcal{L}_1 = \mathcal{L}_2$ induces a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$. Lemma 3.1 now implies that there exist 3 distinct quads R_1, R_2 and R_3 through L, a quad Q_1 through x_1 not containing L and a quad Q_2 through x_2 not containing L such that each of the lines $R_1 \cap Q_1$, $R_2 \cap Q_1$, $R_3 \cap Q_1$, $R_1 \cap Q_2$, $R_2 \cap Q_2$, $R_3 \cap Q_2$ are contained in H.

Now, $Q_1 \cap H$ and $\pi_{Q_1}(Q_2 \cap H)$ are two $Q(4,\mathbb{K})$-subquadrangles of $\widetilde{Q_1}$ containing the lines $R_1 \cap Q_1$, $R_2 \cap Q_1$ and $R_3 \cap Q_1$. These two subquadrangles of $\widetilde{Q_1}$ define two Baer-$\mathbb{K}$-sublines of $Res(x_1)$. Since there is a unique Baer-$\mathbb{K}$-subline of $Res(x_1)$ containing $R_1 \cap Q_1$, $R_2 \cap Q_1$ and $R_3 \cap Q_1$, we have $x_1^\perp \cap Q_1 \cap H = x_1^\perp \cap \pi_{Q_1}(Q_2 \cap H) = \pi_{Q_1}(x_2^\perp \cap Q_2 \cap H)$. Now, let x_3 be an arbitrary point of $x_1^\perp \cap (Q_1 \setminus H)$, let U denote the unique line through x_3 meeting Q_2 and let Q_3 denote the unique element of $\Omega(Q_1,Q_2)$ containing $U \cap H$. Since Q_3 is subquadrangular with respect to H, $\pi_{Q_1}(Q_3 \cap H)$ is a $Q(4,\mathbb{K})$-subquadrangle of $\widetilde{Q_1}$ containing x_3. If x'_1 is a point of $x_1^\perp \cap Q_1 \cap H = \pi_{Q_1}(x_2^\perp \cap Q_2 \cap H)$ and if x'_i, $i \in \{2,3\}$, denotes the unique point of Q_i collinear with x_1', then $x_2' \in x_2^\perp \cap Q_2 \cap H$ and hence $x_2' \in H$. Since $x_3' \in x_1'x_2'$, we have $x_3' \in Q_3 \cap H$ and hence $x_1' = \pi_{Q_1}(x_3') \in \pi_{Q_1}(Q_3 \cap H)$. Since x_1' was an arbitrary point of $x_1^\perp \cap Q_1 \cap H$, we have $x_1^\perp \cap Q_1 \cap H \subseteq \pi_{Q_1}(Q_3 \cap H)$. The lines through x_1 contained in $\pi_{Q_1}(Q_3 \cap H)$ define a Baer-$\mathbb{K}$-subline of $Res(x_1)$ containing the Baer-$\mathbb{K}$-subline of $Res(x_1)$ corresponding to $Q_1 \cap H$ and the extra point x_1x_3, clearly a contradiction.

We conclude that there must be at least one deep quad. □

Lemma 3.3 *If H is a locally subquadrangular hyperplane of $DH(5, \mathbb{K}', \theta)$, then no point of H is deep with respect to H.*

Proof. We prove that if $x \in H$ is deep with respect to H, then also every point $y \in \Gamma_1(x)$ is deep with respect to H. But this is easy. Take an arbitrary line L through y distinct from xy and consider the quad $\langle L, xy\rangle$. Since $x^\perp \cap \langle L, xy\rangle \subseteq H$, $\langle L, xy\rangle$ must be deep with respect to H. So, $L \subseteq H$.

By the connectedness of $DH(5, \mathbb{K}', \theta)$, it now follows that every point of $DH(5, \mathbb{K}', \theta)$ is contained in H, which is clearly absurd. □

Proposition 3.4 *If $|\mathbb{K}| \geq 3$, then the dual polar space $DH(5, \mathbb{K}', \theta)$ has no locally subquadrangular hyperplanes.*

Proof. Suppose to the contrary that H is a locally subquadrangular hyperplane of $DH(5, \mathbb{K}', \theta)$. By Lemma 3.2, there exists a quad which is deep with respect to H. Let x be an arbitrary point of such a deep quad. Then the set $\Lambda_H(x)$, regarded as a set of points of $Res(x)$, satisfies the following properties:

(A) Every line L of $Res(x)$ intersects $\Lambda_H(x)$ in either a Baer-$\mathbb{K}$-subline of L or the whole of L.

(B) There exists a line M^* of $Res(x)$ which is contained in $\Lambda_H(x)$.

By (A) if a line L of $Res(x)$ contains three distinct points l_1, l_2 and l_3 of $\Lambda_H(x)$, then the unique Baer-$\mathbb{K}$-subline of L containing l_1, l_2 and l_3 is also contained in $\Lambda_H(x)$.

Step 1. *There exists a Baer-$\mathbb{K}$-subplane $\mathcal{B}$ of $Res(x) \cong \mathrm{PG}(2, \mathbb{K}')$ which intersects M^* in $|\mathbb{K}| + 1$ points and is completely contained in $\Lambda_H(x)$.*
PROOF. Let l^* be a point of $\Lambda_H(x)$ not contained in M^* and let M_1^*, M_2^* be two distinct lines of $Res(x)$ through l^*. By (A), there exists a Baer-$\mathbb{K}$-subline B_i, $i \in \{1, 2\}$, of M_i^* containing l^* and $M_i^* \cap M^*$ which is itself contained in $\Lambda_H(x)$. Let $\mathcal{B}$ denote the unique Baer-$\mathbb{K}$-subplane of $Res(x)$ containing $B_1 \cup B_2$.

We prove that every point u of $\mathcal{B}$ is contained in $\Lambda_H(x)$. Obviously, this holds if $u \in M_1^* \cup M_2^* \cup M^*$. So, suppose $u \notin M_1^* \cup M_2^* \cup M^*$. Since $|\mathbb{K}| \geq 3$, there exists a line L_u of $\mathcal{B}$ through u which intersects M_1^*, M_2^* and M^* in three distinct points of $\Lambda_H(x)$. The unique Baer-$\mathbb{K}$-subline of L_u containing these points is contained in $\Lambda_H(x)$. In particular, $u \in \Lambda_H(x)$. So, every point of $\mathcal{B}$ is contained in $\Lambda_H(x)$. (qed)

Definition. For every point $w \in \mathcal{B}$, let $\mathcal{L}_w$ denote the set of lines of $Res(x)$ through w which intersect $\mathcal{B}$ in $|\mathbb{K}| + 1$ points.

Step 2. *There exists a point $w^* \in \mathcal{B} \cap M^*$ such that every line of $\mathcal{L}_{w^*}$ is contained in $\Lambda_H(x)$.*
PROOF. Let u be an arbitrary point of $\mathcal{B} \setminus M^*$, let L_u be an arbitrary line through u intersecting $\mathcal{B}$ in the singleton $\{u\}$, let v be an arbitrary point of $(L_u \cap \Lambda_H(x)) \setminus (\{u\} \cup M^*)$ and let L_v denote the unique line through v intersecting $\mathcal{B}$ in $|\mathbb{K}|+1$ points. Since $L_v \cap \Lambda_H(x)$ contains the Baer-$\mathbb{K}$-subline $L_v \cap \mathcal{B}$ of L_v and the extra point v, L_v is completely contained in $\Lambda_H(x)$. Put $\{w^*\} := L_v \cap M^*$. We prove that every line of $\mathcal{L}_{w^*}$ is contained in $\Lambda_H(x)$. Let r be an arbitrary point distinct from w^* which is contained in a line of $\mathcal{L}_{w^*}$. Since $|\mathbb{K}| + 1 \geq 4$, there exists a line $W \in \mathcal{L}_{w^*}$ distinct from w^*r, M^* and L_v. Let w' denote an arbitrary point of $W \cap \mathcal{B}$ distinct from w^*. For every line M of $Res(x)$ not containing w^*, $\bigcup_{L \in \mathcal{L}_{w^*}} L \cap M$ is a Baer-$\mathbb{K}$-subline of M. In particular, $\bigcup_{L \in \mathcal{L}_{w^*}} L \cap w'r$ is a Baer-$\mathbb{K}$-subline of $w'r$. The line $w'r$ contains three points of $\Lambda_H(x)$, namely w', $w'r \cap M^*$ and $w'r \cap L_v$. Hence, the unique Baer-$\mathbb{K}$-subline $\bigcup_{L \in \mathcal{L}_{w^*}} L \cap w'r$ of $w'r$ through these three points is contained in $\Lambda_H(x)$. Since $w^*r \in \mathcal{L}_{w^*}$ and $\{r\} = w'r \cap w^*r$, we have $r \in \Lambda_H(x)$. Hence, every line of $\mathcal{L}_{w^*}$ is contained in $\Lambda_H(x)$. (qed)

Step 3. *Every point u of $Res(x)$ is contained in $\Lambda_H(x)$.*
PROOF. By Step 2, we may suppose that u is not contained in a line of $\mathcal{L}_{w^*}$. Let W denote a line through w^* not belonging to $\mathcal{L}_{w^*} \cup \{w^*u\}$ and let u' be a point of $W \cap \Lambda_H(x)$ distinct from w^*. The intersection $uu' \cap \Lambda_H(x)$ contains the Baer-$\mathbb{K}$-subline $uu' \cap \left(\bigcup_{L \in \mathcal{L}_{w^*}} L \right)$ of uu' and the extra point u'. Hence, uu' is completely contained in $\Lambda_H(x)$. In particular $u \in \Lambda_H(x)$. (qed)

Step 3 says that x is deep with respect to H. But this is impossible by Lemma 3.3. So, our assumption that $DH(5, \mathbb{K}', \theta)$ has locally subquadrangular hyperplanes was wrong. $\square$

As explained in the beginning of Section 3, Proposition 3.4 allows us to conclude the following:

Corollary 3.5 *If $|\mathbb{K}| \geq 3$ and $n \geq 3$, then the dual polar space $DH(2n-1, \mathbb{K}', \theta)$ has no locally subquadrangular hyperplanes.*

3.3 Appendix: Locally subquadrangular hyperplanes of $DH(5,4)$

The conclusion of Proposition 3.4 is not valid if $\mathbb{K} = \mathbb{F}_2$ and $\mathbb{K}' = \mathbb{F}_4$. Pasini and Shpectorov [11] proved that the dual polar space $DH(2n-1,4)$, $n \geq 3$, has locally subquadrangular hyperplanes if and only if $n = 3$ in which case there exists up to isomorphism a unique locally subquadrangular hyperplane.

Let $H(5,4)$ be a nonsingular Hermitian variety of PG$(5,4)$. A *hyperoval* of $H(5,4)$ is a nonempty set of points of $H(5,4)$ intersecting each line of $H(5,4)$ in either 0 or 2 points. Pasechnik [10] used a computer backtrack search to prove that $H(5,4)$ has a unique isomorphism class of hyperovals of size 126 (see also De Bruyn [9] for a computer free proof). If X is a hyperoval of size 126 of $H(5,4)$, then Pasini and Shpectorov [11] proved that the set H_X of all maximal singular subspaces of $H(5,4)$ which meet X is a locally subquadrangular hyperplane of $DH(5,4)$. Moreover, every locally subquadrangular hyperplane of $DH(5,4)$ can be obtained in this way.

We give another construction for the locally subquadrangular hyperplanes of $DH(5,4)$. Consider in PG$(6,2)$ a nonsingular parabolic quadric $Q(6,2)$, let k denote the kernel of this quadric and let π be a hyperplane of PG$(6,2)$ intersecting $Q(6,2)$ in an elliptic quadric $Q^-(5,2)$. The projection from the kernel k on the hyperplane π defines an isomorphism between the polar space $Q(6,2)$ and the symplectic polar space $W(5,2)$ associated to a suitable symplectic polarity of π. Any set of points of $W(5,2)$ which is isomorphic to the subset $\pi \setminus Q^-(5,2)$ of π is called an *elliptic set of points* of $W(5,2)$. A set of quads of $DW(5,2)$ corresponding to an elliptic set of points of $W(5,2)$ is called an *elliptic set of quads* of $DW(5,2)$.

It is well-known that the dual polar space $DW(5,2)$ can be isometrically embedded into the dual polar space $DH(5,4)$. In fact up to isomorphism there exists a unique such isometric embedding, see De Bruyn [7, Theorem 1.5]. Now, let $DW(5,2)$ be isometrically embedded into $DH(5,4)$ and for every quad Q of $DW(5,2)$, let $\overline{Q}$ denote the unique quad of $DH(5,4)$ containing all points of Q. If A is an elliptic set of quads of $DW(5,2)$, then by De Bruyn [8], $H := DW(5,2) \cup \left(\bigcup_{Q \in A} \overline{Q} \right)$ is a locally subquadrangular hyperplane of $DH(5,4)$.

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