# The largest and the smallest fixed points of permutations 

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#### Abstract

We give a new interpretation of the derangement numbers $d_{n}$ as the sum of the values of the largest fixed points of all non-derangements of length $n-1$. We also show that the analogous sum for the smallest fixed points equals the number of permutations of length $n$ with at least two fixed points. We provide analytic and bijective proofs of both results, as well as a new recurrence for the derangement numbers.


## 1 Largest fixed point

Let $[n]=\{1,2, \ldots, n\}$, and let $\mathcal{S}_{n}$ denote the set of permutations of $[n]$. Throughout the paper, we will represent permutations using cycle notation unless specifically stated otherwise. Recall that $i$ is a fixed point of $\pi \in \mathcal{S}_{n}$ if $\pi(i)=i$. Denote by $\mathcal{D}_{n}$ the set of derangements of $[n]$, i.e., permutations with no fixed points, and let $d_{n}=\left|\mathcal{D}_{n}\right|$. Given $\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}$, let $\ell(\pi)$ denote the largest fixed point of $\pi$. Let

$$
a_{n, k}=\left|\left\{\pi \in \mathcal{S}_{n}: \ell(\pi)=k\right\}\right| .
$$

Clearly,

$$
\begin{equation*}
a_{n, 1}=d_{n-1} \quad \text { and } \quad a_{n, n}=(n-1)!. \tag{1}
\end{equation*}
$$

It also follows from the definition that

$$
\begin{equation*}
a_{n, k}=d_{n-1}+\sum_{j=1}^{k-1} a_{n-1, j}, \tag{2}
\end{equation*}
$$

since by removing the largest fixed point $k$ of a permutation in $\mathcal{S}_{n} \backslash \mathcal{D}_{n}$, we get a permutation of $\{1, \ldots, k-1, k+1, \ldots, n\}$ whose largest fixed point (if any) is less than $k$. If in (2) we replace $k$ by $k-1$, then by subtraction we obtain

$$
\begin{equation*}
a_{n, k}=a_{n, k-1}+a_{n-1, k-1} \tag{3}
\end{equation*}
$$

for $k \geq 2$, or equivalently, $a_{n, k}=a_{n, k+1}-a_{n-1, k}$ for $k \geq 1$. Together with the second equation in (1), it follows that the numbers $a_{n, k}$ form Euler's difference table of the factorials (see [2, 3, 4]). Table 1 shows the values of $a_{n, k}$ for small $n$. The combinatorial interpretation given in [2, 3] is that $a(n, k)$ is the number of permutations of $[n-1$ ] where none of $k, k+1, \ldots, n-1$ is a fixed point. This interpretation is clearly equivalent to ours using the same reasoning behind equation (2).

[^0]| $n \backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 1 |  |  |  |  |  |
| 2 | 0 | 1 |  |  |  |  |
| 3 | 1 | 1 | 2 |  |  |  |
| 4 | 2 | 3 | 4 | 6 |  |  |
| 5 | 9 | 11 | 14 | 18 | 24 |  |
| 6 | 44 | 53 | 64 | 78 | 96 | 120 |

Table 1: The values of $a_{n, k}$ for $n$ up to 6 .

We point out that it is possible to give a direct combinatorial proof of the recurrence (3) from our definition of the $a_{n, k}$. Indeed, let $\pi \in \mathcal{S}_{n}$ with $\ell(\pi)=k$. If $\pi(1)=m \neq 1$, then the permutation of $[n]$ obtained from the one-line notation of $\pi$ by moving $m$ to the end, replacing 1 with $n+1$, and subtracting one from all the entries has largest fixed point $k-1$. If $\pi(1)=1$, then removing 1 and subtracting one from the remaining entries of $\pi$ we get a permutation of $[n-1]$ whose largest fixed point is $k-1$.

Define

$$
\begin{equation*}
\alpha_{n}=\sum_{k=1}^{n} k a_{n, k}=\sum_{\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}} \ell(\pi) . \tag{4}
\end{equation*}
$$

We now state our main result, which we prove analytically and bijectively in the next two subsections.

Theorem 1.1 For $n \geq 1$, we have

$$
\alpha_{n}=d_{n+1} .
$$

### 1.1 Analytic proof

Replacing $n$ by $n+1$, from (4) we have

$$
\begin{equation*}
\alpha_{n+1}=a_{n+1,1}+2 a_{n+1,2}+\cdots+n a_{n+1, n}+(n+1) a_{n+1, n+1} . \tag{5}
\end{equation*}
$$

Adding (4) and (5) and taking into account (3), we obtain

$$
\begin{equation*}
\alpha_{n}+\alpha_{n+1}=a_{n+1,2}+2 a_{n+1,3}+\cdots+n a_{n+1, n+1}+(n+1)!. \tag{6}
\end{equation*}
$$

Adding (6) with the obvious equality

$$
(n+1)!-d_{n+1}=a_{n+1,1}+a_{n+1,2}+\cdots+a_{n+1, n}+a_{n+1, n+1},
$$

we obtain

$$
\alpha_{n}+\alpha_{n+1}+(n+1)!-d_{n+1}=\alpha_{n+1}+(n+1)!,
$$

whence $\alpha_{n}=d_{n+1}$.

### 1.2 Bijective proof

To find a bijective proof of Theorem 1.1, we first construct a set whose cardinality is $\alpha_{n}$. Let $\mathcal{M}_{n} \subset\left(\mathcal{S}_{n} \backslash \mathcal{D}_{n}\right) \times[n]$ be the set of pairs $(\pi, i)$ where $\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}$ and $i \leq \ell(\pi)$. We underline the number $i$ in $\pi$ to indicate that it is marked. For example, we write $(2)(3)(7)(8)(1, \underline{4}, 9)(5,6)$ instead of the pair $((2)(3)(7)(8)(1,4,9)(5,6), 4)$. It is clear that

$$
\left|\mathcal{M}_{n}\right|=\sum_{k=1}^{n} k a_{n, k}=\alpha_{n} .
$$

To prove Theorem 1.1, we give a bijection between $\mathcal{D}_{n+1}$ and $\mathcal{M}_{n}$.
Given $\pi \in \mathcal{D}_{n+1}$, we assign to it an element $\widehat{\pi} \in \mathcal{M}_{n}$ as follows. Write $\pi$ as a product of cycles, starting with the one containing $n+1$, say

$$
\pi=\left(n+1, i_{1}, i_{2}, \ldots, i_{r}\right) \sigma
$$

Let $q$ be the largest index, $1 \leq q \leq r$, such that $i_{1}<i_{2}<\cdots<i_{q}$. We define

$$
\widehat{\pi}= \begin{cases}\left(i_{1}\right)\left(i_{2}\right) \ldots\left(\underline{i_{r}}\right) \sigma & \text { if } q=r \\ \left(i_{1}\right)\left(i_{2}\right) \ldots\left(i_{q}\right)\left(\underline{i_{q+1}}, i_{q+2}, \ldots, i_{r}\right) \sigma & \text { if } q<r\end{cases}
$$

Now we describe the inverse map. Given $\widehat{\pi} \in \mathcal{M}_{n}$, let its unmarked fixed points be $i_{1}<$ $i_{2}<\cdots<i_{q}$, and let $j_{1}$ be the marked element. We can write $\widehat{\pi}=\left(i_{1}\right) \ldots\left(i_{q}\right)\left(\underline{j_{1}}, j_{2}, \ldots, j_{t}\right) \sigma$. Notice that $t=1$ if the marked element is a fixed point. Define

$$
\pi=\left(n+1, i_{1}, i_{2}, \ldots, i_{q}, j_{1}, j_{2}, \ldots, j_{t}\right) \sigma
$$

Here are some examples of the bijection between $\mathcal{D}_{n+1}$ and $\mathcal{M}_{n}$ :

$$
\begin{aligned}
\pi=(12,2,4,9,7,5,6)(1,3)(8,11,10) & \leftrightarrow \widehat{\pi}=(2)(4)(9)(\underline{7}, 5,6)(1,3)(8,11,10) \\
\pi=(10,2,7,8,3)(1,4,9)(5,6) & \leftrightarrow \widehat{\pi}=(2)(7)(8)(\underline{3})(1,4,9)(5,6) \\
\pi=(10,2,3,7,8,4,9,1)(5,6) & \leftrightarrow \widehat{\pi}=(2)(3)(7)(8)(\underline{4}, 9,1)(5,6)
\end{aligned}
$$

## 2 Smallest fixed point

In a symmetric fashion to the statistic $\ell(\pi)$, we can define $s(\pi)$ to be the smallest fixed point of $\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}$. Let

$$
b_{n, k}=\left|\left\{\pi \in \mathcal{S}_{n}: s(\pi)=k\right\}\right|
$$

The numbers $b_{n, k}$ appear in [1, pp. 174-176,185] as $R_{n, k}$ (called rank). Define

$$
\begin{equation*}
\beta_{n}=\sum_{k=1}^{n} k b_{n, k}=\sum_{\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}} s(\pi) . \tag{7}
\end{equation*}
$$

It is not hard to see by symmetry that

$$
\begin{equation*}
b_{n, k}=a_{n, n+1-k} \tag{8}
\end{equation*}
$$

Indeed, one can use the involution $\pi \mapsto \pi^{\prime}$ on $S_{n}$ where $\pi^{\prime}(i)=n+1-\pi(n+1-i)$. Alternatively, another involution that proves (8) consists of replacing each entry $i$ in the cycle representation of $\pi \in \mathcal{S}_{n}$ by $n+1-i$; for example, (183)(2)(4975)(6) is mapped to (927)(8)(6135)(4).

To find a combinatorial interpretation of $\beta_{n}$, let $\mathcal{E}_{n+1}$ be the set of permutations of $[n+1]$ that have at least two fixed points. We have that

$$
\begin{equation*}
\left|\mathcal{E}_{n+1}\right|=(n+1)!-d_{n+1}-(n+1) d_{n}, \tag{9}
\end{equation*}
$$

since out of the $(n+1)$ ! permutations of $[n+1]$, there are $d_{n+1}$ derangements and $(n+1) d_{n}$ permutations having exactly one fixed point.

The following result is the analogue of Theorem 1.1 for the statistic $s(\pi)$. We give an analytic proof based on that theorem, and a directive bijective proof as well.

Theorem 2.1 For $n \geq 1$, we have

$$
\beta_{n}=\left|\mathcal{E}_{n+1}\right| .
$$

### 2.1 Analytic proof

From the definitions of $\alpha_{n}$ and $\beta_{n}$, and equation (8), it follows that

$$
\alpha_{n}+\beta_{n}=(n+1) \sum_{k=1}^{n} a_{n, k}=(n+1)\left(n!-d_{n}\right) .
$$

Using Theorem 1.1, we have

$$
\beta_{n}=(n+1)!-(n+1) d_{n}-d_{n+1},
$$

which by (9) is just the cardinality of $\mathcal{E}_{n+1}$ as claimed.
Note also the following identities involving $\beta_{n}$ which follow from the known recurrence $d_{n}=n d_{n-1}+(-1)^{n}$ :

$$
\begin{aligned}
& \beta_{n}=(n+1)!+(-1)^{n}-2(n+1) d_{n} \\
& \beta_{n}=(n+1) \beta_{n-1}+n(-1)^{n+1}
\end{aligned}
$$

The sequence $\beta_{n}$ starts $0,1,1,7,31,191, \ldots$ Using the well known fact that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{d_{n}}{n!}=\frac{1}{e} \tag{10}
\end{equation*}
$$

we see that

$$
\lim _{n \rightarrow \infty} \frac{\beta_{n}}{(n+1)!}=1-\frac{2}{e}
$$

### 2.2 Bijective proof

Let $\mathcal{M}_{n}^{\prime} \subset\left(\mathcal{S}_{n} \backslash \mathcal{D}_{n}\right) \times[n]$ be the set of pairs $(\pi, i)$ where $\pi \in \mathcal{S}_{n} \backslash \mathcal{D}_{n}$ and $i \leq s(\pi)$. As before, we underline the number $i$ in $\pi$ to indicate that it is marked. It is clear that

$$
\left|\mathcal{M}_{n}^{\prime}\right|=\sum_{k=1}^{n} k b_{n, k} .
$$

We now give a bijection between $\mathcal{E}_{n+1}$ and $\mathcal{M}_{n}^{\prime}$. Given $\pi \in \mathcal{E}_{n+1}$, let $i$ be its smallest fixed point. We can write

$$
\pi=(i)\left(n+1, j_{2}, \ldots, j_{t}\right) \sigma,
$$

where no $j$ s appear if $n+1$ is a fixed point. Define

$$
\widetilde{\pi}=\left(\underline{i}, j_{2}, \ldots, j_{t}\right) \sigma
$$

Note that $\widetilde{\pi} \in \mathcal{M}_{n}^{\prime}$, because if $\sigma$ has fixed points then they are all larger than $i$, and if it does not, then $t=1$ and $i$ is the smallest fixed point of $\widetilde{\pi}$. Essentially, $\pi$ and $\widetilde{\pi}$ are related by conjugation by the transposition $(i, n+1)$.

Conversely, given $\widetilde{\pi} \in \mathcal{M}_{n}^{\prime}$, let $i$ be the marked entry. We can write

$$
\tilde{\pi}=\left(\underline{i}, j_{2}, \ldots, j_{t}\right) \sigma
$$

where no $j$ s appear if $i$ is a fixed point. Then

$$
\pi=(i)\left(n+1, j_{2}, \ldots, j_{t}\right) \sigma
$$

Roughly speaking, we replace $\underline{i}$ with $n+1$ and add $i$ as a fixed point. Note that if $t \geq 2$ then $\sigma$ must have fixed points.

Here are some examples of the bijection between $\mathcal{E}_{n+1}$ and $\mathcal{M}_{n}$ :

$$
\begin{array}{ll}
\pi=(3)(10,1,7,2,8)(5)(6)(4,9) & \leftrightarrow \\
\pi & =(3,1,7,2,8)(5)(6)(4,9), \\
\pi=(5)(10)(6)(3,1,7,2,8)(4,9) & \leftrightarrow \\
\pi & =(\underline{5})(6)(3,1,7,2,8)(4,9) .
\end{array}
$$

## 3 Other remarks

### 3.1 A recurrence for the derangement numbers

An argument similar to the bijective proof of Theorem 1.1 can be used to prove the recurrence

$$
\begin{equation*}
d_{n}=\sum_{j=2}^{n}(j-1)\binom{n}{j} d_{n-j} \tag{11}
\end{equation*}
$$

combinatorially as follows.
A derangement $\pi \in \mathcal{D}_{n}$ can be written as a product of cycles, starting with the one containing $n$, say

$$
\pi=\left(n, i_{1}, i_{2}, \ldots, i_{r}\right) \sigma
$$

Consider two cases:

- If $i_{1}<i_{2}<\cdots<i_{r-1}$ (this is vacuously true for $r=1,2$ ), then the number of choices for the numbers $i_{1}, \ldots, i_{r}$ satisfying this condition is $r\binom{n-1}{r}$, since we can first choose an $r$-subset of $[n-1]$ and then decide which one is $i_{r}$. Now, the number of choices for $\sigma$ is $d_{n-r-1}$.
- Otherwise, there is an index $1 \leq q \leq r-1$ such that $i_{1}<i_{2}<\cdots<i_{q}>i_{q+1}$. In this case, there are $q\binom{n-1}{q+1}$ choices for the numbers $i_{1}, \ldots, i_{q+1}$, since we can first choose a $(q+1)$-subset of $[n-1]$ and then decide which element other than the maximum is $i_{q+1}$. Now, there are $d_{n-q-1}$ choices for $\left(i_{q+1}, \ldots, i_{r}\right) \sigma$.
The total number of choices is

$$
\begin{aligned}
\sum_{r=1}^{n-1} r\binom{n-1}{r} d_{n-r-1}+\sum_{q=1}^{n-1} q\binom{n-1}{q+1} d_{n-q-1}=\sum_{r=1}^{n-1} r\left(\binom{n-1}{r}\right. & \left.+\binom{n-1}{r+1}\right) d_{n-r-1} \\
& =\sum_{r=1}^{n-1} r\binom{n}{r+1} d_{n-r-1}
\end{aligned}
$$

which equals the right hand side of (11).
Alternatively, the recurrence (11) is relatively straightforward to prove using generating functions. Indeed, let

$$
D(x)=\sum_{n \geq 0} d_{n} \frac{x^{n}}{n!}=\frac{e^{-x}}{1-x}
$$

be the generating function for the number of derangements. The generating function for the right hand side of (11), starting from $n=1$, is

$$
\begin{aligned}
& \sum_{n \geq 1} \sum_{j=2}^{n}(j-1)\binom{n}{j} d_{n-j} \frac{x^{n}}{n!}=\left(\sum_{i \geq 0} d_{i} \frac{x^{i}}{i!}\right)\left(\sum_{j \geq 1}(j-1) \frac{x^{j}}{j!}\right) \\
&=\frac{e^{-x}}{1-x}\left(x e^{x}-e^{x}+1\right)=-1+\frac{e^{-x}}{1-x}=D(x)-1 .
\end{aligned}
$$

### 3.2 Probabilistic interpretation

Let $X_{n}$ be the random variable that gives the value of the largest fixed point of a random element of $\mathcal{S}_{n} \backslash \mathcal{D}_{n}$. Its expected value is then

$$
E\left[X_{n}\right]=\frac{\sum_{k=1}^{n} k a_{n, k}}{\left|\mathcal{S}_{n} \backslash \mathcal{D}_{n}\right|}
$$

Theorem 1.1 is equivalent to the fact that

$$
\begin{equation*}
E\left[X_{n}\right]=\frac{d_{n+1}}{n!-d_{n}} . \tag{12}
\end{equation*}
$$

Using (10), we get from equation (12) that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{E\left[X_{n}\right]}{n}=\frac{1}{e-1} . \tag{13}
\end{equation*}
$$

Occurrences of fixed points in a random permutation of $[n]$, normalized by dividing by $n$, approach a Poisson process in the interval $[0,1]$ with mean 1 as $n$ goes to infinity. An interpretation of equation (13) is that, in such a Poisson process, if we condition on the fact that there is at least one occurrence, then the largest event occurs at $1 /(e-1)$ on average.

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## References

[1] Ch. A. Charalambides, Enumerative Combinatorics, Chapman \& Hall/CRC, Boca Raton, FL, 2002.
[2] D. Dumont and A. Randrianarivony, Dérangements et nombres de Genocchi, Discrete Math. 132 (1994), 37-49.
[3] I. Gessel, Symmetric inclusion-exclusion, Sém. Lothar. Combin. 54 (2005/07), Art. B54b.
[4] F. Rakotondrajao, $k$-fixed-points-permutations, Integers 7 (2007), A36.


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