ENUMERATION RESULTS FOR ALTERNATING TREE FAMILIES

MARKUS KUBA AND ALOIS PANHOLZER

Abstract. We study two enumeration problems for *up-down alternating trees*, i.e., rooted labelled trees T, where the labels $v_1, v_2, v_3, \dots$ on every path starting at the root of T satisfy $v_1 < v_2 > v_3 < v_4 > \cdots$. First we consider various tree families of interest in combinatorics (such as unordered, ordered, d-ary and Motzkin trees) and study the number T_n of different up-down alternating labelled trees of size n. We obtain for all tree families considered an implicit characterization of the exponential generating function $T(z)$ leading to asymptotic results of the coefficients T_n for various tree families. Second we consider the particular family of up-down alternating labelled ordered trees and study the influence of such an alternating labelling to the average shape of the trees by analyzing the parameters *label of the root node*, *degree of the root node* and *depth of a random node* in a random tree of size n. This leads to exact enumeration results and limiting distribution results.

1. Introduction

The family $\mathcal{T}$ of unrooted unordered alternating trees (also called intransitive trees) consists of all unrooted unordered labelled trees T, where the nodes of T with $|T| = n$ (the number $|T|$ of nodes of T will be called the size of T) are labelled by distinct integers of $\{1, 2, \dots, n\}$ in such a way that for every path $v_1, v_2, v_3, \dots$ in T it holds $v_1 < v_2 > v_3 < v_4 > \cdots$ or $v_1 > v_2 < v_3 > v_4 > \cdots$ (we always identify a node $v \in T$ with its label).

This tree family appears in various contexts in combinatorics as in the enumeration of admissible bases of certain hypergeometric systems [5], in the enumeration of so called local binary search trees [11] and when enumerating the number of regions of certain hyperplane arrangements [12].

The enumeration problem for the number T_n of unrooted unordered alternating trees of size n has been solved by A. Postnikov in [11] by obtaining the following formula for T_n:

$$T_n = \frac{1}{n 2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}, \quad \text{for } n \ge 2.$$

The corresponding problem for rooted ordered alternating trees has been addressed and solved by C. Chauve, S. Dulucq and A. Rechnitzer in [1]. They considered trees T, where one node of T is distinguished as the root and where the subtrees of each node of T are linearly ordered, which are labelled by distinct integers of $\{1, 2, \dots, |T|\}$ in an “alternating way”, i.e., in such a way that for every path $v_1, v_2, v_3, \dots$ in T it holds $v_1 < v_2 > v_3 < v_4 > \cdots$ or

---

Date: May 28, 2008.

Key words and phrases. Alternating trees, Generating functions, Functional equations, Asymptotic enumeration results.

This work was supported by the Austrian Science Foundation FWF, grant S9608-N13.

An extended abstract of this paper will appear in the conference proceedings of FPSAC 2008.

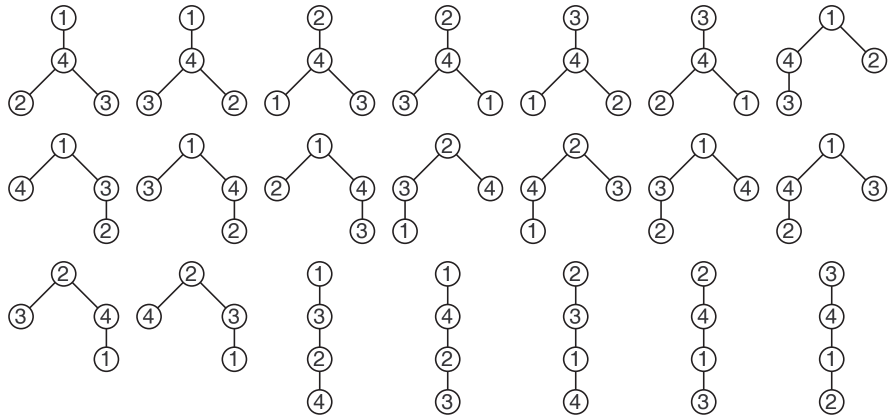

FIGURE 1. All 21 different up-down alternating labelled Motzkin trees of size 4.

$v_1 > v_2 < v_3 > v_4 > \cdots$. The authors of [1] found that the number T_n of rooted ordered alternating trees of size $n \geq 2$ is given by the surprisingly simple formula $T_n = 2(n-1)^{n-1}$.

The aim of the present work is to address and to give (up to some extent) solutions to the following two problems for alternating trees. First we consider the enumeration problem for other alternating labelled tree families, as, e.g., for binary trees, d-ary trees and Motzkin trees, where the corresponding families of unlabelled or arbitrary labelled trees appear frequently in combinatorics or computer science. We remark that all trees considered in this paper are rooted trees and we remark further that it is sufficient for the enumeration problem to count "up-down alternating labelled trees", i.e., it holds for every path $v_1, v_2, v_3, \ldots$ starting at the root of a tree: $v_1 < v_2 > v_3 < v_4 > \cdots$. Of course, the number of all alternating labelled trees of size $n \geq 2$ of a rooted tree family is twice the number of up-down alternating labelled trees of size n. As an example in Figure 1 all 21 up-down alternating labelled Motzkin trees (ordered trees, where each node has either 0, 1, or 2 children) of size 4 are given.

Our study relies on a description of the combinatorial decomposition of an up-down alternating tree T of a tree family considered with respect to the largest element $n = |T|$ in T. This decomposition leads to a recursive description of the enumeration problem and to quasi-linear first order partial differential equations for suitably defined multivariate generating functions. For all tree families considered the differential equation appearing can be solved implicitly, which also leads to an implicit characterization of the exponential generating function $T(z)$ of the number T_n of trees of size n by means of certain functional equations. With few exceptions, amongst them the already known results for rooted ordered and rooted (or unrooted) unordered alternating trees, it does not seem that there are explicit formulæ for T_n available. However, the appearing functional equations for the generating functions of the number T_n of up-down alternating labelled trees are particularly useful to obtain asymptotic results of T_n for various tree families.

Second we are interested in the influence of an alternating labelling to the average structure or shape of the trees in a tree family compared to an arbitrary labelling. We do this by considering one particular tree family, namely the family of up-down alternating labelled ordered trees, and studying several tree parameters for random trees of size n (i.e., each of the T_n different trees of size n is chosen with equal probability). In particular we obtain limiting distribution results for the label of the root node, the degree of the root node, and the depth (i.e., the distance to the root) of a randomly chosen node in a random tree of size n. Interestingly one even obtains exact formulæ for the number of up-down alternating labelled ordered trees of size n, where the root is labelled by j, as well as for the number of up-down alternating labelled ordered trees of size n, where the root has degree m. To show these results we again use the basic decomposition of an up-down alternating tree T with respect to the largest element $n = |T|$, which again leads to certain partial differential equations for suitably introduced bivariate generating functions. A study of these differential equations leads then to exact or asymptotic results for the parameters studied.

2. Results

2.1. **Enumeration results for up-down alternating labelled trees.** We give here our results for the number T_n of up-down alternating labelled trees of size n for various tree families $\mathcal{T}$ as described in Subsection 3.1.

Theorem 1. *The exponential generating function $T(z) := \sum_{n\geq 1} T_n \frac{z^n}{n!}$ of the numbers T_n is for the tree families considered implicitly given as solution of the following functional equations:*

$$\text{Ordered trees:} \quad z = (1-T(z)) \log \frac{1}{1-T(z)}, \quad \text{or explicitly} \quad T(z) = 1 - e^{-W(z)},$$

$$\text{Unordered trees:} \quad z = \frac{2T(z)}{1+e^{T(z)}}, \quad \text{or explicitly} \quad T(z) = \frac{z}{2} + W\big(\frac{ze^{\frac{z}{2}}}{2}\big),$$

$$d\text{-ary trees:} \quad z = \frac{2}{\big(1+(1+T(z))^{d+1}\big)^{\frac{d-1}{d+1}}} \int_0^{T(z)} \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}},$$

$$d\text{-bundled ordered trees:} \quad z = \frac{2}{\big(1+\big(\frac{1}{1-T(z)}\big)^{d-1}\big)^{\frac{d+1}{d-1}}} \int_0^{T(z)} \Big(1+\big(\frac{1}{1-x}\big)^{d-1}\Big)^{\frac{2}{d-1}} dx,$$

$$\text{Motzkin trees:} \quad z = \int_0^{T(z)} \frac{dx}{\frac{\left(4r(x)+4\sqrt{108+r^2(x)}\right)^{\frac{2}{3}}}{16} + \frac{9}{\left(4r(x)+4\sqrt{108+r^2(x)}\right)^{\frac{2}{3}}} - \frac{3}{4}},$$

$$\text{with} \quad r(x) = 8(T^3(z)-x^3) + 12(T^2(z)-x^2) + 24(T(z)-x) + 10,$$

$$\text{Strict binary trees:} \quad z = \int_0^{T(z)} \frac{dx}{\frac{\left(4r(x)+4\sqrt{4+r^2(x)}\right)^{\frac{2}{3}}}{4} + \frac{4}{\left(4r(x)+4\sqrt{4+r^2(x)}\right)^{\frac{2}{3}}} - 1},$$

$$\text{with} \quad r(x) = (T^3(z)-x^3) + 3(T(z)-x),$$

where the function $W(z) := \sum_{n\geq 1} n^{n-1} \frac{z^n}{n!}$ appearing is the so called tree function.

Theorem 2. *The numbers T_n are for each of the families of up-down alternating labelled ordered, unordered, d-ary and d-bundled ordered trees asymptotically given by*

$$T_n \sim C\rho^{-n} n^{-\frac{3}{2}} n!,$$

where ρ is the radius of convergence of the corresponding exponential generating function $T(z)$ and C is some computable constant, which may differ for every tree family considered. For these tree families the radius of convergence ρ and the constant C are given as follows:

$$\text{Ordered trees:} \quad \rho = \frac{1}{e} \approx 0.367879\ldots, \quad C = \frac{1}{\sqrt{2\pi}\, e} \approx 0.146762\ldots$$

$$\text{Unordered trees:} \quad \rho = -2W(-e^{-1}) \approx 0.556929\ldots, \quad C = \frac{\sqrt{2+\rho}}{2\sqrt{\pi}} \approx 0.451080\ldots$$

$$d\text{-ary trees:} \quad \rho = \frac{2}{(d-1)(1+\tau)^d}, \quad C = \sqrt{\frac{1+(1+\tau)^{d+1}}{2d(d-1)(1+\tau)^{d-1}\pi}},$$

with τ the positive real solution of the equation

$$\frac{\big(1+(1+\tau)^{d+1}\big)^{\frac{d-1}{d+1}}}{(d-1)(1+\tau)^d} = \int_0^\tau \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}},$$

$$d\text{-bundled ordered trees:} \quad \rho = \frac{2(1-\tau)^d}{d+1}, \quad C = \sqrt{\frac{(1-\tau)^{d+1}\big(1+(\frac{1}{1-\tau})^{d-1}\big)}{2d(d+1)\pi}},$$

with τ the positive real solution of the equation

$$\frac{\big(1+\big(\frac{1}{1-\tau}\big)^{d-1}\big)^{\frac{d+1}{d-1}}}{(d+1)\big(\frac{1}{1-\tau}\big)^d} = \int_0^\tau \Big(1+(\frac{1}{1-x})^{d-1}\Big)^{\frac{2}{d-1}} dx.$$

Furthermore the numbers T_n, $n \geq 1$, are for the families of ordered, unordered and 3-bundled ordered trees given by the following exact formulæ:

$$\text{Ordered trees:} \quad T_n = (n-1)^{n-1},$$

$$\text{Unordered trees:} \quad T_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} k^{n-1},$$

$$3\text{-bundled ordered trees:} \quad T_n = \frac{(n-1)!}{2^{n+1}} \sum_{k=0}^{2n} \binom{2n}{k} \binom{\frac{5n-3}{2}-k}{n-1}.$$

We remark that the exact formulæ of T_n for the families of ordered trees and unordered trees already appear (or are easily deduced from results) in [1] and [11], respectively.

We finish this subsection by collecting numerical values of the numbers T_n for small values of n for various tree families considered. These results are given in Table 1.

2.2. **Results for tree parameters in up-down alternating labelled ordered trees.** We give here our exact and asymptotic results of parameters described in Section 4 for the family of up-down alternating labelled ordered trees.

Theorem 3. *Let $T_{n,j}$ denote the number of up-down alternating labelled ordered trees of size n, where the root node has label j, with $1 \leq j \leq n$, and let L_n be the random variable, which gives the label of the root node of a randomly chosen up-down alternating labelled ordered tree of size-n. Then $T_{n,j}$ is given by the following exact formula:*

$$T_{n,j} = (n-j)(n-1)^{j-2} n^{n-j-1},$$

Tree family	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
Ordered trees	1	1	4	27	256	3125	46656	823543	16777216	387420489
Unordered trees	1	1	3	14	90	738	7364	86608	1173240	17990600
Binary trees	1	2	10	72	700	8560	126360	2187200	43452640	974721600
Ternary trees	1	3	24	285	4608	94311	2338560	68157369	2283724800	86502077739
2-bundled trees	1	2	14	160	2548	52064	1298840	38268736	1300468000	50071359296
3-bundled trees	1	3	30	483	10800	309375	10810800	445940775	21208884480	1142594883675
Motzkin trees	1	1	4	21	154	1409	15666	204049	3054946	51654981
Strict binary trees	1	0	2	0	36	0	1672	0	148576	0

TABLE 1. Numerical values of the numbers T_n of up-down alternating labelled trees of size $n \leq 10$ for various tree families considered.

and the normalized random variable $\frac{L_n}{n}$ converges for $n \to \infty$ in distribution to a random variable L, i.e., $\frac{L_n}{n} \xrightarrow{(d)} L$, with density function $f(x) = (1-x)e^{1-x}$, for $0 \leq x \leq 1$.

Theorem 4. *Let $T_{n,m}$ denote the number of up-down alternating labelled ordered trees of size n, where the root node has degree m, with $0 \leq m < n$, and let R_n be the random variable, which gives the degree of the root node of a randomly chosen up-down alternating labelled ordered tree of size-n. Then $T_{n,m}$ is given by the following exact formula:*

$$T_{n,m} = H_m (n-1)^{n-1} + \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^{\ell} \frac{\ell+1}{\ell} (n-1-\ell)^{n-1},$$

and the random variable R_n converges for $n \to \infty$ in distribution to a discrete random variable R, i.e., $R_n \xrightarrow{(d)} R$, whose distribution is given by

$$\mathbb{P}\{R = m\} = \big(\frac{e-1}{e}\big)^m - 1 + \sum_{\ell=1}^{m} \frac{\big(\frac{e-1}{e}\big)^{\ell}}{\ell}, \quad \textit{for } m \in \mathbb{N}.$$

Here $H_m := \sum_{k=1}^{m} \frac{1}{k}$ denotes the m-th harmonic number.

Theorem 5. *Let D_n be the random variable, which counts the depth (i.e., the distance to the root) of a randomly chosen node in a random up-down alternating labelled ordered tree of size n. Then the normalized random variable $\frac{D_n}{\sqrt{n}}$ converges for $n \to \infty$ in distribution to a Rayley-distributed random variable R_α, i.e., $\frac{D_n}{\sqrt{n}} \xrightarrow{(d)} R_\alpha$, with parameter $\alpha = \frac{2}{3}$, where R_α has density function $f(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}}$, for $x \geq 0$.*

3. Enumeration of up-down alternating trees

3.1. **Tree families considered.** In the following we describe the combinatorial families $\mathcal{T}$ of trees that we consider here. Basically all trees contained in the tree families are up-down alternating labelled rooted trees. This means that we only consider labelled trees: the nodes in a tree T of size $|T| = n$ are labelled by distinct integers of $\{1, 2, \dots, n\}$, where one of the n nodes of T is distinguished as the root node. Furthermore the labelling of any tree T is an "up-down alternating labelling", i.e., it must hold for any sequence of nodes $v_1, v_2, v_3, \dots$ lying on the path from the root to an arbitrary node in T that $v_1 < v_2 > v_3 < v_4 > \cdots$, where we identify a node with its label.

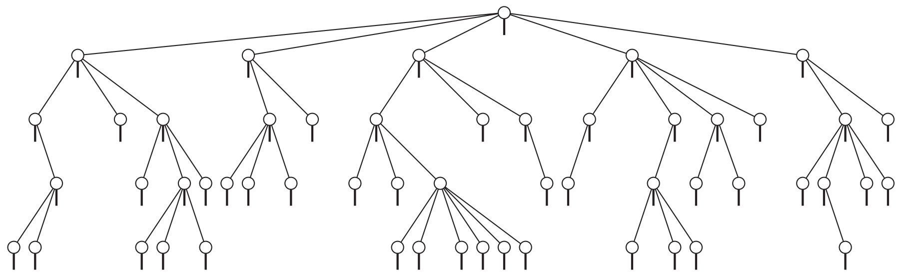

FIGURE 2. An example of a 2-bundled tree. A bar separates the subtrees into left and right ones.

To specify the up-down alternating labelled tree families $\mathcal{T}$ we are dealing with, we describe the corresponding arbitrary labelled tree families $\tilde{\mathcal{T}}$ and define that $\mathcal{T} \subseteq \tilde{\mathcal{T}}$ contains exactly those trees $T \in \tilde{\mathcal{T}}$ with an up-down alternating labelling.

Labelled ordered trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, where a (possibly empty) sequence of labelled ordered trees is attached and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way.

Labelled unordered trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, where a (possibly empty) set of labelled unordered trees is attached and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way.

Labelled d-ary trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, which has d positions, where either a labelled d-ary tree is attached or not and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way.

Labelled d-bundled trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, which has d positions, where a (possibly empty) sequence of labelled d-bundled trees is attached and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way. Alternatively one might think of a d-bundled tree as an ordered tree, where the sequence of subtrees attached to any node in the tree is separated by $d - 1$ bars into d bundles.

Labelled Motzkin trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, where a sequence of 0, 1 or 2 labelled Motzkin trees is attached and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way.

Labelled strict binary trees: Every tree $T \in \tilde{\mathcal{T}}$ of size n consists of a root node, where a sequence of 0 or 2 labelled strict binary trees is attached and where the whole tree is relabelled with the labels $\{1, 2, \dots, n\}$ in an order preserving way.

Whereas the families of ordered, unordered, d-ary, strict binary and Motzkin-trees are well-known tree families with a lot of applications (see, e.g., [3, 13]), we remark that d-bundled trees appear, e.g., in the context of certain "preferential attachment" growth models for trees (see [7]) and furthermore, that they are satisfying some randomness preservation properties when studying cutting-down procedures for random trees (see [10]). An example of a 2-bundled tree is given in Figure 2.

3.2. **Combinatorial decompositions.** Fundamental to our approach is the description of the decomposition of an up-down alternating tree T of size $|T| = n$ in a tree family $\mathcal{T}$ with respect to node n. If we cut-off all edges incident with node n and relabel the resulting trees with labels from 1 up to their sizes in an order-preserving way we obtain for $n \geq 2$ an alternating tree $\widehat{T}$ of size $k \geq 1$ that contains the original root of the tree T and alternating trees $T_1, T_2, \dots, T_r$ of sizes $k_1, \dots, k_r \geq 1$, which correspond to the subtrees originally attached to node n. Of course, it holds that $k + k_1 + \cdots + k_r = n - 1$.

If $\mathcal{T}$ is one of the families of up-down alternating labelled unordered, ordered, d-ary, d-bundled ordered or Motzkin trees it follows that all the resulting trees $\widehat{T}, T_1, \dots, T_r$ are again alternating trees of the family $\mathcal{T}$. This is not true if $\mathcal{T}$ is the family of up-down alternating labelled strict binary trees: although $T_1, \dots, T_r$ are alternating trees of $\mathcal{T}$ this does not hold for the tree $\widehat{T}$, since there exists now a node in $\widehat{T}$, where only one subtree is attached. We first consider the decomposition for those tree families $\mathcal{T}$, where all the resulting trees are again contained in $\mathcal{T}$ and discuss the decomposition for the family of strict binary trees later.

In order to use the decomposition of an up-down alternating labelled tree of a family $\mathcal{T}$ to get a recursive description of the numbers T_n of different size-n trees of $\mathcal{T}$ we are now interested in an answer to the following question. What is the number of different trees $T \in \mathcal{T}$ of size $|T| = n$ that we can obtain by starting with alternating trees $\widehat{T}, T_1, \dots, T_r \in \mathcal{T}$ of corresponding sizes $k, k_1, \dots, k_r$, with $k + k_1 + \cdots + k_r = n - 1$, distributing the labels $\{1, 2, \dots, n-1\}$ amongst the nodes of $\widehat{T}, T_1, \dots, T_r$ and relabelling all these trees in an order-preserving way and afterwards attaching $T_1, \dots, T_r$ to a new vertex labelled by n and attaching node n to a node in $\widehat{T}$?

Obviously for all tree families we have a contribution of $\binom{n-1}{k, k_1, \dots, k_r}$ stemming from the distribution of the labels to the trees. Furthermore, depending on the tree family considered, we obtain a certain factor reflecting the number of possibilities of attaching the subtrees $T_1, \dots, T_r$ to node n. E.g., for d-ary trees this factor is $\binom{d}{r}$, for unordered trees this factor is $\frac{1}{r!}$, whereas for ordered trees this factor is 1.

However, the most interesting contribution is the factor coming from the number of possible positions, where node n can be attached to a node in $\widehat{T}$ of size $|\widehat{T}| = k$, such that the up-down alternating labelling is preserved for the resulting tree. This contribution, which will be denoted here by w, depends on the specific tree family considered and will be studied now. We require there the notion of the depth $h(v)$ of a node v in a tree $\widehat{T}$, which is given by the distance of v to the root of $\widehat{T}$, i.e., the number of edges lying on the unique path from the root of $\widehat{T}$ to node v. It holds then that node n can only be attached to nodes $v \in \widehat{T}$, which are at an even level, i.e., where it holds $h(v) \equiv 0 \mod 2$. The set of nodes of $\widehat{T}$ at an even level will be denoted by $V := \{v \in \widehat{T} : h(v) \equiv 0 \mod 2\}$ and its cardinality by $\ell := |V|$. Furthermore we denote by $d^+(v)$ the out-degree (the number of children) of a node v in a tree $\widehat{T}$.

Ordered trees: The number of positions w of attaching node n to one of the nodes of V is given by

$$w = \sum_{v \in V} (d^+(v) + 1) = \sum_{v \in V} 1 + \sum_{v \in V} d^+(v) = |V| + |\widehat{T} \setminus V| = |\widehat{T}| = k,$$

since $\sum_{v \in V} d^+(v)$ gives exactly the number of nodes in $\widehat{T}$ at an odd level. Thus we obtain that, independent of the specific tree $\widehat{T}$, there are always $|\widehat{T}| = k$ positions of attaching node n, such that the resulting tree is again up-down alternating labelled.

Unordered trees: The number of positions w of attaching node n to one of the nodes of V is now simply given by $w = |V| = \ell$.

d-ary trees: We obtain now for w:

$$w = \sum_{v \in V} (d - d^+(v)) = d \sum_{v \in V} 1 - \sum_{v \in V} d^+(v) = d|V| - |\widehat{T} \setminus V| = d\ell - (k - \ell) = (d+1)\ell - k.$$

d-bundled ordered trees: Now w is given as follows:

$$w = \sum_{v \in V} (d^+(v) + d) = d \sum_{v \in V} 1 + \sum_{v \in V} d^+(v) = d|V| + |\widehat{T} \setminus V| = d\ell + (k - \ell) = (d-1)\ell + k.$$

Motzkin trees: To count the number of positions w of attaching node n to one of the nodes of V we define the set of nodes in V with out-degree 0 and 1, respectively, by $V^{[0]} := \{v \in V : d^+(v) = 0\}$, $V^{[1]} := \{v \in V : d^+(v) = 1\}$, and use the notation $\ell^{[0]} := |V^{[0]}|$, $\ell^{[1]} := |V^{[1]}|$ for their cardinalities. We obtain then:

$$w = \sum_{v \in V^{[0]}} 1 + \sum_{v \in V^{[1]}} 2 = |V^{[0]}| + 2|V^{[1]}| = \ell^{[0]} + 2\ell^{[1]}.$$

Thus this combinatorial decomposition only leads for the family of ordered trees directly to a recurrence for the number T_n of alternating labelled trees of size n, whereas we have to store additional information for the other tree families considered: for unordered trees, d-ary trees and d-bundled ordered trees we will introduce the number $T_{n,m}$ of alternating labelled trees of size n, where exactly m nodes are at an even level, and for Motzkin trees we will introduce the number $T_{n,m^{[0]},m^{[1]}}$ of alternating labelled trees, where exactly $m^{[0]}$ nodes at an even level have out-degree 0 and $m^{[1]}$ nodes at an even level have out-degree 1. Of course, it holds $T_n = \sum_{m \ge 0} T_{n,m}$ and $T_n = \sum_{m^{[0]},m^{[1]} \ge 0} T_{n,m^{[0]},m^{[1]}}$, respectively.

Strict binary trees: As mentioned above when applying this decomposition to an up-down alternating tree T of the family $\mathcal{T}$ of strict binary trees the resulting tree $\widehat{T}$ containing the original root of T is no more a strict binary tree. To treat the family of strict binary trees with the same approach as before we consider a larger tree family $\mathcal{S} \supseteq \mathcal{T}$ containing $\mathcal{T}$. The family $\mathcal{S}$ consists now of all up-down alternating labelled rooted trees T, where every node $v \in T$ at an odd level ($h(v) \equiv 1 \mod 2$) has a sequence of 0 or 2 children and where every node $v \in T$ at an even level ($h(v) \equiv 0 \mod 2$) has a sequence of 0, 1 or 2 children. Of course, the family $\mathcal{T}$ of alternating labelled strict binary trees contains exactly those trees of $\mathcal{S}$, which do not contain any nodes of out-degree 1. As immediately seen the basic decomposition of an up-down alternating labelled tree $T \in \mathcal{S}$ of size $|T| = n$ with respect to n leads to up-down alternating labelled trees $\widehat{T}, T_1, \dots, T_r$, which are all elements of $\mathcal{S}$. Thus we can repeat the considerations made above for the family $\mathcal{S}$. It remains to study the number w of possible positions, where node n can be attached to a node in $\widehat{T}$ of size $|\widehat{T}| = k$, such that the up-down alternating labelling is preserved for the resulting tree. We denote as above the set of nodes in $\widehat{T}$ at an even level with V and define the set of nodes in V with out-degree 0 and 1, respectively, by $V^{[0]} := \{v \in V : d^+(v) = 0\}$, $V^{[1]} := \{v \in V : d^+(v) = 1\}$, and use the notation $\ell^{[0]} := |V^{[0]}|$, $\ell^{[1]} := |V^{[1]}|$ for their cardinalities. We obtain then:

$$w = \sum_{v \in V^{[0]}} 2 + \sum_{v \in V^{[1]}} 1 = 2|V^{[0]}| + |V^{[1]}| = 2\ell^{[0]} + \ell^{[1]}.$$

Thus in order to treat the enumeration problem for strict binary trees we will introduce the number $T_{n,m^{[0]},m^{[1]}}$ of alternating labelled trees of the family $\mathcal{S}$ defined above, where exactly $m^{[0]}$ nodes at an even level have out-degree 0 and $m^{[1]}$ nodes at an even level have out-degree 1. The basic combinatorial decomposition leads then to a recursive description of these numbers $T_{n,m^{[0]},m^{[1]}}$. Of course, we are interested in particular in the number T_n of up-down alternating labelled strict binary trees, which are given by $T_n = \sum_{m^{[0]}\ge 0} T_{n,m^{[0]},0}$.

General tree families: We remark that one could use this basic decomposition to obtain a recursive description of the number of up-down alternating labelled trees for any family $\mathcal{T}$ of so called simply generated trees (for a definition see, e.g., [4]; all tree families considered here are special instances of simply generated tree families), where the out-degree of a node $v \in T$ is bounded a priori for all trees $T \in \mathcal{T}$ by some fixed bound d.

As we have seen for the families of up-down alternating labelled unordered, ordered and d-bundled ordered trees the approach also works for some instances of simply generated tree families, where the degree of a node $v \in T$ is not bounded by some universal constant for all $T \in \mathcal{T}$; however, in general one would be forced to store then the whole sequence $(m^{[0]}, m^{[1]}, m^{[2]}, \dots)$ of the numbers $m^{[i]}$ of nodes v in a tree T at an even level with out-degree $d^+(v) = i$.

3.3. **Recurrences.** The basic decomposition of an up-down alternating labelled tree T of size $|T| = n$ with respect to node n described in Subsection 3.2 immediately leads to recurrences for the numbers T_n (ordered trees), $T_{n,m}$ (unordered trees, d-ary trees and d-bundled ordered trees) or $T_{n,m^{[0]},m^{[1]}}$ (Motzkin trees and strict binary trees) introduced there. In the following we collect the recurrences for these tree families, where we use the standard notation $\delta_{i,j}$ for the Kronecker δ-function. Furthermore we remark that the appearing numbers are all zero for values of n, m, $m^{[0]}$, $m^{[1]}$, which are not listed below.

Ordered trees:

$$T_n = \sum_{r\ge 0} \sum_{\substack{k+k_1+\dots+k_r=n-1,\\ k,k_1,\dots,k_r\ge 1}} k \binom{n-1}{k,k_1,\dots,k_r} T_k \cdot T_{k_1}\cdots T_{k_r}, \quad \text{for } n\ge 2, \qquad T_1 = 1.$$

Unordered trees:

$$T_{n,m} = \sum_{r\ge 0} \frac{1}{r!} \sum_{\substack{k+k_1+\dots+k_r=n-1,\\ k,k_1,\cdots,k_r\ge 1}} \sum_{\substack{\ell+\ell_1+\dots+\ell_r=m,\\ \ell,\ell_1,\cdots,\ell_r\ge 0}} \ell \binom{n-1}{k,k_1,\dots,k_r} T_{k,\ell}\cdot T_{k_1,\ell_1}\cdots T_{k_r,\ell_r},$$

for $n \ge 2$ and $1 \le m \le n$, $\qquad T_{1,m} = \delta_{1,m}$.

d-ary trees:

$$T_{n,m} = \sum_{r=0}^{d} \binom{d}{r} \sum_{\substack{k+k_1+\dots+k_r=n-1,\\ k,k_1,\cdots,k_r\ge 1}} \sum_{\substack{\ell+\ell_1+\dots+\ell_r=m,\\ \ell,\ell_1,\cdots,\ell_r\ge 0}} ((d+1)\ell - k) \binom{n-1}{k,k_1,\dots,k_r} \times$$

$$\times T_{k,\ell}\cdot T_{k_1,\ell_1}\cdots T_{k_r,\ell_r}, \quad \text{for } n\ge 2 \text{ and } 1\le m\le n, \qquad T_{1,m} = \delta_{1,m}.$$

d-bundled trees:

$$T_{n,m} = \sum_{r\ge 0}\binom{r+d-1}{r} \sum_{\substack{k+k_1+\cdots+k_r=n-1,\\ k,k_1,\cdots,k_r\ge 1}} \sum_{\substack{\ell+\ell_1+\cdots+\ell_r=m,\\ \ell,\ell_1,\cdots,\ell_r\ge 0}} \big((d-1)\ell+k\big)\binom{n-1}{k,k_1,\dots,k_r}\times$$
$$\times T_{k,\ell}\cdot T_{k_1,\ell_1}\cdots T_{k_r,\ell_r}, \quad \text{for } n\ge 2 \text{ and } 1\le m\le n, \qquad T_{1,m}=\delta_{1,m}.$$

Motzkin trees:

$$T_{n,m^{[0]},m^{[1]}} = \sum_{r=0}^{2} \sum_{\substack{k+k_1+\cdots+k_r=n-1,\\ k,k_1,\cdots,k_r\ge 1}} \sum_{\substack{\ell^{[0]}+\ell_1^{[0]}+\cdots+\ell_r^{[0]}=m^{[0]},\\ \ell^{[0]},\ell_1^{[0]},\cdots,\ell_r^{[0]}\ge 0}} \sum_{\substack{\ell^{[1]}+\ell_1^{[1]}+\cdots+\ell_r^{[1]}=m^{[1]},\\ \ell^{[1]},\ell_1^{[1]},\cdots,\ell_r^{[1]}\ge 0}}$$
$$\big(\ell^{[0]}+2\ell^{[1]}\big)\binom{n-1}{k,k_1,\dots,k_r} T_{k,\ell^{[0]},\ell^{[1]}}\cdot T_{k_1,\ell_1^{[0]},\ell_1^{[1]}}\cdots T_{k_r,\ell_r^{[0]},\ell_r^{[1]}},$$
$$\text{for } n\ge 2 \text{ and } 0\le m^{[0]},m^{[1]}\le n, \qquad T_{1,m^{[0]},m^{[1]}}=\delta_{1,m^{[0]}}\cdot\delta_{0,m^{[1]}}.$$

Strict binary trees:

$$T_{n,m^{[0]},m^{[1]}} = \big(2m^{[0]}+m^{[1]}\big)T_{n-1,m^{[0]},m^{[1]}}$$
$$+ \sum_{\substack{k+k_1+k_2=n-1,\\ k,k_1,k_2\ge 1}} \sum_{\substack{\ell^{[0]}+\ell_1^{[0]}+\ell_2^{[0]}=m^{[0]},\\ \ell^{[0]},\ell_1^{[0]},\ell_2^{[0]}\ge 0}} \sum_{\substack{\ell^{[1]}+\ell_1^{[1]}+\ell_2^{[1]}=m^{[1]},\\ \ell^{[1]},\ell_1^{[1]},\ell_2^{[1]}\ge 0}}$$
$$\big(2\ell^{[0]}+\ell^{[1]}\big)\binom{n-1}{k,k_1,k_2} T_{k,\ell^{[0]},\ell^{[1]}}\cdot T_{k_1,\ell_1^{[0]},\ell_1^{[1]}}\cdot T_{k_2,\ell_2^{[0]},\ell_2^{[1]}},$$
$$\text{for } n\ge 2 \text{ and } 0\le m^{[0]},m^{[1]}\le n, \qquad T_{1,m^{[0]},m^{[1]}}=\delta_{1,m^{[0]}}\cdot\delta_{0,m^{[1]}}.$$

With this recurrences one can compute the numbers T_n of up-down alternating tree families considered for small values of n by summation for values m, or $m^{[0]}$, $m^{[1]}$, respectively; see the remarks given in Subsection 3.2. However, for this purpose it seems to be easier to extract coefficients from the generating functions solutions stated in Theorem 1. E.g., this has been carried out to generate the values of T_n presented in Table 1.

3.4. **Generating functions.** We treat the recurrences for the numbers T_n, $T_{n,m}$, and $T_{n,m^{[0]},m^{[1]}}$, respectively, appearing in Subsection 3.3 by introducing suitable generating functions:

$$T(z) := \sum_{n\ge 1} T_n\frac{z^n}{n!}, \qquad F(z,u) := \sum_{n\ge 1}\sum_{m\ge 0} T_{n,m}\frac{z^n}{n!}u^m,$$
$$\text{and} \qquad F(z,u_0,u_1) := \sum_{n\ge 1}\sum_{m^{[0]}\ge 0}\sum_{m^{[1]}\ge 0} T_{n,m^{[0]},m^{[1]}}\frac{z^n}{n!}u_0^{m^{[0]}}u_1^{m^{[1]}}.$$

This leads, apart from the instance of ordered trees, where a nonlinear ordinary differential equation occurs, to first order quasilinear partial differential equations for the generating functions introduced. We will omit here these straightforward computations and just state the resulting equations below, where we use the abbreviation $F := F(z,u_0,u_1)$; additionally the initial conditions $T(0) = F(0,u) = F(0,u_0,u_1) = 0$ hold:

$$\text{Ordered trees:} \quad T'(z) - 1 = \frac{zT'(z)}{1-T(z)}, \tag{1a}$$

$$\text{Unordered trees:} \quad F_z(z,u) - u = ue^{F(z,u)} F_u(z,u), \tag{1b}$$

$$d\text{-ary trees:} \quad F_z(z,u) - u = (1+F(z,u))^d \big((d+1)uF_u(z,u) - zF_z(z,u)\big), \tag{1c}$$

$$d\text{-bundled ordered trees:} \quad F_z(z,u) - u = \frac{1}{(1-F(z,u))^d}\big((d-1)uF_u(z,u) + zF_z(z,u)\big), \tag{1d}$$

$$\text{Motzkin trees:} \quad F_z - u_0 = \big(1+F+F^2\big)\big(u_1F_{u_0} + 2F_{u_1}\big), \tag{1e}$$

$$\text{Strict binary trees:} \quad F_z - u_0 = \big(1+F^2\big)\big(2u_1F_{u_0} + F_{u_1}\big). \tag{1f}$$

For all the differential equations appearing in (1) we obtain implicit solutions of the generating functions considered by using the method of characteristics for first order quasilinear partial differential equations, see, e.g., [14] for a description of this method. These solutions are given as follows, where we again use the abbreviation $F := F(z,u_0,u_1)$:

$$\text{Ordered trees:} \quad z = \big(1-T(z)\big)\log\frac{1}{1-T(z)}, \tag{2a}$$

$$\text{Unordered trees:} \quad z = \frac{1}{u+e^{F(z,u)}}\log\Big(\frac{e^{F(z,u)}\big(u-1+e^{F(z,u)}\big)}{u}\Big), \tag{2b}$$

$$d\text{-ary trees:} \quad z = \frac{1}{u^{\frac{1}{d+1}}}\int_0^{F(z,u)} \frac{dx}{\big(u+(1+F(z,u))^{d+1}-(1+x)^{d+1}\big)^{\frac{d}{d+1}}}, \tag{2c}$$

$$d\text{-bundled ordered trees:} \quad z = u^{\frac{1}{d-1}}\int_0^{F(z,u)} \frac{dx}{\big(u+\big(\frac{1}{1-F(z,u)}\big)^{d-1}-\big(\frac{1}{1-x}\big)^{d-1}\big)^{\frac{d}{d-1}}}, \tag{2d}$$

$$\text{Motzkin trees:} \quad z = \int_0^{F(z,u_0,u_1)} \frac{dx}{u_0 - \frac{u_1^2}{4} + \frac{s^2(x)}{4}}, \tag{2e}$$

where $s(x) = s(x,F,u_0,u_1)$ and $r(x) = r(x,F,u_0,u_1)$ are given as follows:

$$s(x) = \frac{\big(4r(x)+4\sqrt{256(u_0-\frac{u_1^2}{4})^3+r^2(x)}\big)^{\frac{1}{3}}}{2} - \frac{8(u_0-\frac{u_1^2}{4})}{\big(4r(x)+4\sqrt{256(u_0-\frac{u_1^2}{4})^3+r^2(x)}\big)^{\frac{1}{3}}},$$

$$r(x) = 8(F^3-x^3) + 12(F^2-x^2) + 24(F-x) + 12u_0u_1 - 2u_1^3.$$

$$\text{Strict binary trees:} \quad z = \int_0^{F(z,u_0,u_1)} \frac{dx}{u_0 - u_1^2 + s^2(x)}, \tag{2f}$$

where $s(x) = s(x,F,u_0,u_1)$ and $r(x) = r(x,F,u_0,u_1)$ are given as follows:

$$s(x) = \frac{\big(4r(x)+4\sqrt{4(u_0-u_1^2)^3+r^2(x)}\big)^{\frac{1}{3}}}{2} - \frac{2(u_0-u_1^2)}{\big(4r(x)+4\sqrt{4(u_0-u_1^2)^3+r^2(x)}\big)^{\frac{1}{3}}},$$

$$r(x) = (F^3-x^3) + 3(F-x) + 3u_0u_1 - 2u_1^3.$$

Remark 1. The functions $s(x)$ appearing in equation (2e) for Motzkin trees and in equation (2f) for strict binary trees are solutions of the following third order polynomial equations, with $r(x)$ as given in equation (2e) and (2f), respectively:

$$\text{Motzkin trees:} \quad s^3 + 12\big(u_0 - \frac{u_1^2}{4}\big)s - r(x) = 0,$$
$$\text{Strict binary trees:} \quad s^3 + 3\big(u_0 - u_1^2\big)s - r(x) = 0.$$

It is not difficult to check, e.g., by using a computer algebra system, that the functions given in equation (2) are satisfying the differential equations and the initial conditions given in equation (1) and are thus indeed the required solutions. However, as an example we demonstrate for one particular case how one may solve the differential equations appearing by using the method of characteristics. This is carried out in Example 1, where the corresponding differential equation (1c) for d-ary trees is treated and thus the derivation of (2c) is shown.

Example 1. In order to solve the differential equation (1c) for d-ary trees we use the substitution $Q(z,u) := F(z,u) + 1$ and obtain after simple manipulations the following PDE (we use the abbreviation $Q = Q(z,u)$):

$$(1 + zQ^d)Q_z - (d+1)uQ^dQ_u - u = 0. \tag{3}$$

We assume now that we have an implicit description of a solution $Q = Q(z,u)$ of (3) via the equation

$$f(z,u,Q) = c = \text{const.},$$

with a certain differentiable function f. Taking derivatives of this equation w.r.t. z and u we obtain $f_z + f_QQ_z = 0$ and $f_u + f_QQ_u = 0$. Plugging these equations into (3) we obtain the following linear PDE in reduced form for the function $f(z,u,Q)$:

$$(1 + zQ^d)f_z - (d+1)uQ^d f_u + uf_Q = 0. \tag{4}$$

To solve equation (4) we apply the method of characteristics and consider thus the following system of ordinary differential equations, the so called system of characteristic differential equations:

$$\dot{z} = (1 + zQ^d), \quad \dot{u} = -(d+1)uQ^d, \quad \dot{Q} = u, \tag{5}$$

where we regard here z, u, and Q as dependent variables of t, namely, $z = z(t)$, $u = u(t)$, $Q = Q(t)$, and $\dot{z} = \frac{dz(t)}{dt}$, etc. We are searching now for first integrals of the system of characteristic differential equations (5), i.e., for functions $\xi(z,u,Q)$, which are constant along any solution curve (a so called characteristic curve) of (5).

One might proceed as follows. From the second and third equation of (5) we obtain the differential equation

$$\frac{du}{dQ} = -(d+1)Q^d,$$

leading to the general solution

$$u = -Q^{d+1} + c_1.$$

This immediately gives the following first integral of (5):

$$\xi_1(z,u,Q) = c_1 = u + Q^{d+1}. \tag{6}$$

From the first and third equation of (5) we obtain, after the substitution $u = c_1 - Q^{d+1}$, the differential equation

$$\frac{dz}{dQ} = \frac{1}{c_1 - Q^{d+1}} + z \frac{Q^d}{c_1 - Q^{d+1}}.$$

The general solution of this first order linear differential equation is given as follows:

$$z = \frac{1}{(c_1 - Q^{d+1})^{\frac{1}{d+1}}} \int_1^Q \frac{1}{(c_1 - t^{d+1})^{\frac{d}{d+1}}} dt + \frac{c_2}{(c_1 - Q^{d+1})^{\frac{1}{d+1}}},$$

which leads, after backsubstituting $c_1 = u + Q^{d+1}$, to the following first integral of (5), which is independent of (6):

$$\xi_2(z, u, Q) = c_2 = u^{\frac{1}{d+1}} z - \int_1^Q \frac{dt}{\big(u + Q^{d+1} - t^{d+1}\big)^{\frac{d}{d+1}}}. \tag{7}$$

Thus the general solution of (4) is given as follows:

$$f(z, u, Q) = G\big(\xi_1(z, u, Q), \xi_2(z, u, Q)\big) = \text{const.}, \tag{8}$$

with arbitrary differentiable functions G in two variables and $\xi_1(z, u, Q)$, $\xi_2(z, u, Q)$ given in (6) and (7). One can also solve (8) w.r.t. the variable z and obtains then

$$z = \frac{1}{u^{\frac{1}{d+1}}} \int_1^Q \frac{dt}{\big(u + Q^{d+1} - t^{d+1}\big)^{\frac{d}{d+1}}} + \frac{1}{u^{\frac{1}{d+1}}} g(u + Q^{d+1}), \tag{9}$$

with arbitrary differentiable functions $g(x)$ in one variable. To characterize the function $g(x)$ in (9) we use the initial condition $Q(0, u) = 1$. We obtain then, after plugging $z = 0$ and $Q = 1$ into (9), that $g(x) = 0$. Therefore we get the following implicit solution of $Q(z, u)$ and hence the solution of $F(z, u) = Q(z, u) - 1$ as stated in equation (2c):

$$\begin{aligned} z &= \frac{1}{u^{\frac{1}{d+1}}} \int_1^{Q(z,u)} \frac{dt}{\big(u + (Q(z,u))^{d+1} - t^{d+1}\big)^{\frac{d}{d+1}}} \\ &= \frac{1}{u^{\frac{1}{d+1}}} \int_0^{F(z,u)} \frac{dx}{\big(u + (1 + F(z,u))^{d+1} - (1+x)^{d+1}\big)^{\frac{d}{d+1}}}. \end{aligned}$$

The exponential generating functions $T(z) = \sum_{n \ge 1} T_n \frac{z^n}{n!}$ of the number T_n of up-down alternating trees of size n that we are mainly interested in, can be obtained from the solutions given in (2) via $T(z) = F(z, 1)$ (for unordered trees, d-ary trees and d-bundled ordered trees), $T(z) = F(z, 1, 1)$ (for Motzkin trees) and $T(z) = F(z, 1, 0)$ for strict binary trees. We obtain then that the functions $T(z)$ are given implicitly as solutions of certain functional equations, which are collected in Theorem 1.

Remark 2. After plugging $u = 1$ into equation (2c) we obtain the formula

$$z = \int_0^{T(z)} \frac{dx}{\big(1 + (1 + T(z))^{d+1} - (1+x)^{d+1}\big)^{\frac{d}{d+1}}}, \tag{10}$$

which does not match with the expression

$$z = \frac{2}{\big(1+(1+T(z))^{d+1}\big)^{\frac{d-1}{d+1}}} \int_0^{T(z)} \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}} \tag{11}$$

stated in Theorem 1 as solution of $T(z)$ for d-ary trees. However, it can be checked easily that both equations (10) and (11) characterize the same function $T(z)$, since they are solutions of the differential equation

$$T'(z) = \frac{\big(1+T(z)\big)^{d+1}+1}{2-(d-1)z\big(1+T(z)\big)^{d}} \tag{12}$$

and satisfying also the initial condition $T(0)=0$.

Also one can check easily that the formula

$$z = \int_0^{T(z)} \frac{dx}{\Big(1+\big(\frac{1}{1-T(z)}\big)^{d-1} - \big(\frac{1}{1-x}\big)^{d-1}\Big)^{\frac{d}{d-1}}}$$

obtained after plugging $u=1$ into equation (2d) and the formula

$$z = \frac{2}{\Big(1+\big(\frac{1}{1-T(z)}\big)^{d-1}\Big)^{\frac{d+1}{d-1}}} \int_0^{T(z)} \Big(1+\big(\frac{1}{1-x}\big)^{d-1}\Big)^{\frac{2}{d-1}} dx$$

stated in Theorem 1 as solution of $T(z)$ for d-bundled trees characterize the same function $T(z)$, since they are solutions of the differential equation

$$T'(z) = \frac{\big(\frac{1}{1-T(z)}\big)^{d-1}+1}{2-(d+1)z\big(\frac{1}{1-T(z)}\big)^{d}}$$

satisfying the initial condition $T(0)=0$.

The implicit characterization given in Theorem 1 turns out to be advantageous when studying the asymptotic behaviour of the numbers $T_n = n![z^n]T(z)$ of up-down alternating labelled d-ary trees and d-bundled trees, respectively.

Remark 3. The solution of $T(z)$ for Motzkin trees given in Theorem 1 can also be described as

$$z = \int_0^{T(z)} \frac{4dx}{3+s^2(x)},$$

with $s(x) = s(x,T(z))$ the suitable root of the equation

$$s^3 + 9s - r(x) = 0,$$

and $r(x) = r(x,T(z))$ as stated in the corresponding part of Theorem 1.

Also the solution of $T(z)$ for strict binary trees given in Theorem 1 can be described as

$$z = \int_0^{T(z)} \frac{dx}{1+s^2(x)},$$

with $s(x) = s(x,T(z))$ the suitable root of the equation

$$s^3 + 3s - r(x) = 0,$$

and $r(x) = r(x,T(z))$ as stated in the corresponding part of Theorem 1.

For the instance of ordered and unordered trees the functions $T(z)$ can be expressed via the so called tree function $W(z) = \sum_{n\geq 1} \frac{n^{n-1}}{n!} z^n$, which satisfies the functional equation

$$W(z) = z e^{W(z)},$$

and exact formulæ for the numbers T_n can be obtained by extracting coefficients. This leads to results obtained by [1] and [11]. For most of the other tree families considered it does not seem that there are simple explicit formulæ for the numbers T_n available. We only remark the somewhat curious fact that for the instance of 3-bundled ordered trees the equation for $T(z)$ as given in Theorem 1 can be simplified to

$$z = \frac{2T(z)(1-T(z))^3(2-T(z))}{(1+(1-T(z))^2)^2} = \frac{T(z)}{\frac{\left(1+(1-T(z))^2\right)^2}{2(1-T(z))^3(2-T(z))}},$$

which also leads to an exact formula for T_n by applying the Lagrange inversion formula (see, e.g., [13]) and a certain binomial identity that can be obtained by combining Pfaff's reflection law with Kummer's formula for hypergeometric series (see [6, p. 217]). The derivation of T_n for 3-bundled ordered trees is carried out in the following. First we obtain by extracting coefficients:

$$\begin{aligned}
[z^n]T(z) &= \frac{1}{n}[T^{n-1}]\Big(\frac{(1+(1-T)^2)^2}{2(1-T)^3(2-T)}\Big)^n = \frac{1}{n2^n}[T^{n-1}]\frac{(1+(1-T)^2)^{2n}}{(1-T)^{3n}(2-T)^n} \\
&= \frac{1}{n4^n}[T^{n-1}]\frac{(1+(1-T)^2)^{2n}}{(1-T)^{3n}}\sum_{k\geq 0}\binom{n-1+k}{k}\Big(\frac{T}{2}\Big)^k \\
&= \frac{1}{n4^n}\sum_{k\geq 0}\binom{n-1+k}{k}\frac{1}{2^k}[T^{n-1-k}]\frac{(1+(1-T)^2)^{2n}}{(1-T)^{3n}} \\
&= \frac{1}{n4^n}\sum_{k\geq 0}\binom{n-1+k}{k}\frac{1}{2^k}[T^{n-1-k}]\sum_{\ell\geq 0}\binom{2n}{\ell}(1-T)^{2\ell-3n} \\
&= \frac{1}{n4^n}\sum_{k=0}^{n-1}\binom{n-1+k}{k}\frac{1}{2^k}\sum_{\ell=0}^{2n}\binom{2n}{\ell}\binom{2\ell-3n}{n-1-k}(-1)^{n-1-k}.
\end{aligned}$$

This gives the following expression for $T_n = n![z^n]T(z)$:

$$T_n = \frac{(n-1)!}{4^n}\sum_{\ell=0}^{2n}\binom{2n}{\ell}\sum_{k=0}^{n-1}\binom{n-1+k}{k}\binom{2\ell-3n}{n-1-k}(-1)^{n-1-k}\frac{1}{2^k}. \tag{13}$$

We consider now the sum

$$s_n = \sum_{k=0}^{n-1}\binom{n-1+k}{k}\binom{m}{n-1-k}(-1)^{n-1-k}\frac{1}{2^k} = (-1)^{n-1}\binom{m}{n-1}\mathrm{F}\left(\left.\begin{matrix} n, -n+1 \\ m-n+2\end{matrix}\,\right|\,\frac{1}{2}\right).$$

In order to simplify s_n one can apply the following identity for hypergeometric series (see [6]):

$$2^{-a}\mathrm{F}\left(\left.\begin{matrix} a, 1-a \\ 1+b-a\end{matrix}\,\right|\,\frac{1}{2}\right) = \mathrm{F}\left(\left.\begin{matrix} a, b \\ 1+b-a\end{matrix}\,\right|\,-1\right) = \frac{(b/2)!}{b!}\frac{(b-a)!}{(b/2-a)!},$$

which gives, after specializing for $a = -n+1$ and $b = m-2n+2$, the following expression:

$$s_n = 2^{n-1}\binom{n-\frac{3}{2}-\frac{m}{2}}{n-1}. \tag{14}$$

With (14) equation (13) can be simplified and we obtain the following expression:

$$T_n = \frac{(n-1)!}{2^{n+1}}\sum_{\ell=0}^{2n}\binom{2n}{\ell}\binom{\frac{5n-3}{2}-\ell}{n-1}.$$

The exact results of T_n for up-down alternating labelled odered, unordered and 3-bundled trees are collected in Theorem 2.

3.5. **Asymptotic enumeration results.** An advantage of the implicit characterization of the generating functions $T(z) = \sum_{n\ge 1} T_n \frac{z^n}{n!}$ as the solution of certain functional equations (as obtained in Subsection 3.4) is that this often allows to apply analytic techniques leading to asymptotic results for the coefficients T_n, for $n \to \infty$.

We will not go into all the details here, but also refer to the very general treatment [4] and to the survey [2]. Basically one has to determine the radius of convergence ρ of the analytic function $T(z)$, which already leads to some growth estimates of T_n. However, for a detailed description of the growth of the coefficients T_n one has to locate all singularities on the radius of convergence (the so called dominant singularities) and describe the behaviour of $T(z)$ locally in a complex neighbourhood of their dominant singularities. By applying transfer lemmata, i.e., singularity analysis [4], this leads then in many instances to precise asymptotic results for T_n.

In the following we briefly show the asymptotic enumeration results of T_n for up-down alternating labelled ordered, unordered, d-ary and d-bundled ordered trees that are collected in Theorem 2.

Ordered trees: Of course, there is nothing better than the explicit formula $T_n = (n-1)^{n-1}$, which immediately leads, by applying Stirling's formula (see, e.g., [6]), to the following asymptotic equivalent:

$$T_n = (n-1)^{n-1} \sim \frac{n^{n-1}}{e} \sim \frac{1}{e\sqrt{2\pi}} e^n n^{-\frac{3}{2}} n!.$$

We just remark that this result could be obtained also by expanding $T(z) = 1 - e^{-W(z)}$ around the unique dominant singularity $\rho = \frac{1}{e}$ of the tree function $W(z)$ (and thus also of $T(z)$) and applying singularity analysis.

Unordered trees: Here we use the explicit formula

$$T(z) = \frac{z}{2} + W\big(\frac{ze^{\frac{z}{2}}}{2}\big)$$

and the well-known facts (see, e.g., [4]) that the tree function $W(z)$ has a unique dominant singularity at $z = \frac{1}{e}$, it is analytically continuable to a so called Δ region, and it admits the following local expansion in a complex neighbourhood of $z = \frac{1}{e}$:

$$W(z) = 1 - \sqrt{2}\sqrt{1-ez} + \frac{2}{3}(1-ez) + \mathcal{O}\big((1-ez)^{\frac{3}{2}}\big). \tag{15}$$

The unique dominant singularity ρ of $T(z)$ is then given as the real solution of the equation $\frac{\rho}{2}e^{\frac{\rho}{2}} = e^{-1}$, which can also be described explicitly as $\rho = -2W(-e^{-1})$.

Furthermore $T(z)$ has the following local expansion in a complex neighbourhood of $z=\rho$:

$$T(z) = 1 + \frac{\rho}{2} - \sqrt{2+\rho}\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\big(\rho - z\big).$$

Singularity analysis immediately gives then the following asymptotic equivalent of T_n:

$$T_n = n![z^n]T(z) \sim \frac{\sqrt{2+\rho}}{2\sqrt{\pi}}\rho^{-n}n^{-\frac{3}{2}}n!.$$

d-ary trees: We consider the implicit equation

$$z = \frac{2}{\big(1+(1+T)^{d+1}\big)^{\frac{d-1}{d+1}}}\int_0^T \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}} =: F(T). \tag{16}$$

According to the implicit function theorem a function $T(z)$ satisfying (16) is analytic around z, whenever $F'(T) \neq 0$. We consider now the equation $F'(T) = 0$, which leads to

$$\frac{\big(1+(1+T)^{d+1}\big)^{\frac{d-1}{d+1}}}{(d-1)(1+T)^d} = \int_0^T \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}}. \tag{17}$$

It can be shown easily that equation (17) has a unique positive real solution $\tau > 0$. One dominant singularity of $T(z)$ is then given by the positive real value $\rho = F(\tau)$, which, of course, satisfies $\tau = T(\rho)$. Since τ satisfies (17) we can simplify the relation between ρ and τ and obtain:

$$\rho = F(\tau) = \frac{1}{\big(1+(1+\tau)^{d+1}\big)^{\frac{d-1}{d+1}}}\int_0^\tau \frac{dx}{\big(1+(1+x)^{d+1}\big)^{\frac{2}{d+1}}} = \frac{2}{(d-1)(1+\tau)^d}, \tag{18}$$

or equivalently $\tau = -1 + \big(\frac{2}{(d-1)\rho}\big)^{\frac{1}{d}}$.

To show that ρ is the only dominant singularity of $T(z)$ we consider an arbitrary value $\tilde{\rho} = \rho e^{i\varphi}$, with $0 < \varphi < 2\pi$, on the circle of convergence ρ. Let us denote by $\tilde{\tau}$ the value $\tilde{\tau} := T(\tilde{\rho})$. Since $T(z) = \sum_{n\ge1} T_n \frac{z^n}{n!}$ and $T_n > 0$, for $n \ge 1$, it trivially holds:

$$|\tilde{\tau}| = |T(\tilde{\rho})| \le T(|\tilde{\rho}|) = T(\rho) = \tau.$$

We assume now that there exists a value $\tilde{\rho} = \rho e^{i\varphi}$, with $0 < \varphi < 2\pi$, on the circle of convergence, which is also a singularity of $T(z)$. Then, due to the implicit function theorem, $\tilde{\tau}$ has to satisfy $F'(\tilde{\tau}) = 0$, which would also lead to the relation

$$\tilde{\tau} = -1 + \big(\frac{2}{(d-1)\tilde{\rho}}\big)^{\frac{1}{d}}.$$

But from this relation we easily obtain for $0 < \varphi < 2\pi$:

$$\begin{aligned} |\tilde{\tau}|^2 &= \Big(-1 + \big(\frac{2}{(d-1)\rho}\big)^{\frac{1}{d}}\Big)^2 + 2\big(\frac{2}{(d-1)\rho}\big)^{\frac{1}{d}}\big(1-\cos\frac{\varphi}{d}\big) \\ &> \Big(-1 + \big(\frac{2}{(d-1)\rho}\big)^{\frac{1}{d}}\Big)^2 = \tau^2, \end{aligned}$$

and thus, that $|\tilde{\tau}| > \tau$, which is a contradiction.

Therefore we obtain that ρ is the unique dominant singularity of $T(z)$. General considerations using the Weierstrass preparation theorem, see [2], show then that

$T(z)$ has a local representation around $z=\rho$ of the kind $T(z) = g(z) - h(z)\sqrt{1-\frac{z}{\rho}}$, with functions $g(z)$, $h(z)$ that are analytic around $z=\rho$. A Taylor series expansion of (16) easily gives then the following local expansion of $T(z)$ around $z=\rho$:

$$T(z) = \tau - \sqrt{\frac{2\big(1+(1+\tau)^{d+1}\big)}{d(d-1)(1+\tau)^{d-1}}}\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\big(\rho - z\big),$$

which, after applying singularity analysis, leads to the following asymptotic equivalent of the numbers T_n:

$$T_n \sim \sqrt{\frac{1+(1+\tau)^{d+1}}{2d(d-1)(1+\tau)^{d-1}\pi}}\rho^{-n} n^{-\frac{3}{2}} n!.$$

d-bundled ordered trees: Since the considerations for d-bundled ordered trees are very similar to the corresponding ones for d-ary trees, we will be more brief. We consider now the implicit equation

$$z = \frac{2}{\Big(1+\big(\frac{1}{1-T(z)}\big)^{d-1}\Big)^{\frac{d+1}{d-1}}} \int_0^{T(z)} \Big(1+\big(\frac{1}{1-x}\big)^{d-1}\Big)^{\frac{2}{d-1}} dx =: F(T) \tag{19}$$

and study the equation $F'(T) = 0$, which leads to

$$\frac{\Big(1+\big(\frac{1}{1-T}\big)^{d-1}\Big)^{\frac{d+1}{d-1}}}{(d+1)\big(\frac{1}{1-T}\big)^{d}} = \int_0^{T} \Big(1+\big(\frac{1}{1-x}\big)^{d-1}\Big)^{\frac{2}{d-1}} dx. \tag{20}$$

Equation (20) has a unique positive real solution $\tau > 0$ and one dominant singularity of $T(z)$ is given by the positive real value $\rho = F(\tau)$, which, of course, satisfies $\tau = T(\rho)$. Due to (20) we obtain further that

$$\rho = \frac{2(1-\tau)^d}{d+1} \quad \text{or equivalently} \quad \tau = 1 - \big(\frac{(d+1)\rho}{2}\big)^{\frac{1}{d}}.$$

A further singularity $\tilde{\rho} = \rho e^{i\varphi}$, with $0 < \varphi < 2\pi$, on the circle of convergence ρ would also lead to the relation

$$\tilde{\tau} = 1 - \big(\frac{(d+1)\tilde{\rho}}{2}\big)^{\frac{1}{d}},$$

with $\tilde{\tau} := T(\tilde{\rho})$ and thus to

$$\begin{aligned} |\tilde{\tau}|^2 &= \Big(1-\big(\frac{(d+1)\rho}{2}\big)^{\frac{1}{d}}\Big)^2 + 2\big(\frac{(d+1)\rho}{2}\big)^{\frac{1}{d}}\big(1-\cos\frac{\varphi}{d}\big) \\ &> \Big(1-\big(\frac{(d+1)\rho}{2}\big)^{\frac{1}{d}}\Big)^2 = \tau^2. \end{aligned}$$

This would imply $|\tilde{\tau}| > \tau$, which is a contradiction.

Furthermore one gets the following local expansion of $T(z)$ around the unique dominant singularity $z = \rho$:

$$T(z) = \tau - \sqrt{\frac{2(1-\tau)^{d+1}\big(1+(\frac{1}{1-\tau})^{d-1}\big)}{d(d+1)}}\sqrt{1-\frac{z}{\rho}} + \mathcal{O}\big(\rho - z\big),$$

which, after applying singularity analysis, leads to the following asymptotic equivalent of the numbers T_n:

$$T_n \sim \sqrt{\frac{(1-\tau)^{d+1}\big(1+(\frac{1}{1-\tau})^{d-1}\big)}{2d(d+1)\pi}}\,\rho^{-n} n^{-\frac{3}{2}} n!.$$

Remark 4. For the families $\mathcal{T}$ of up-down alternating labelled Motzkin trees and strict binary trees we also determined the radius ρ of convergence, but it seems to be involved to show that $z = \rho$ is the unique dominant singularity (in the instance of Motzkin trees) or that there are exactly two dominant singularities $z = \rho$ and $z = -\rho$ (in the instance of strict binary trees), respectively.

4. Parameters in up-down alternating labelled rooted trees

We study now certain parameters for the family $\mathcal{T}$ of up-down alternating labelled ordered trees. We recall that the number of size-n trees in $\mathcal{T}$ is given by $T_n = (n-1)^{n-1}$ and that its exponential generating function $T(z) = \sum_{n\ge 1} T_n \frac{z^n}{n!}$ is given by $T(z) = 1 - e^{-W(z)}$, where $W(z) = \sum_{n\ge 1} n^{n-1}\frac{z^n}{n!}$ is the tree function.

4.1. **Label of the root node.** First we want to count the number $T_{n,j}$ of up-down alternating labelled ordered trees of size n, where the root node has label j, with $1 \le j \le n$. In a probabilistic setting we introduce the random variable L_n, where $\mathbb{P}\{L_n = j\} = \frac{T_{n,j}}{T_n}$ gives the probability that the root node of a random size-n alternating tree has label j.

By using the basic decomposition of an alternating tree of size n with respect to the largest element n and counting the number of ways, where the root in the tree labelled by j will get, after an order preserving relabelling with elements $\{1, 2, \dots, k\}$, the label ℓ in the subtree of size k containing the original root, one obtains the following recurrence for the numbers $T_{n,j}$ (where the appearing numbers are all zero for values of n and j, which are not listed below):

$$\begin{aligned} T_{n,j} &= \sum_{r\ge 0} \sum_{\substack{k+k_1+\dots+k_r = n-1,\\ k,k_1,\dots,k_r \ge 1}} \sum_{\ell=1}^{j} k \binom{j-1}{\ell-1}\binom{n-1-j}{k-\ell}\binom{n-1-k}{k_1,\dots,k_r} \times \\ &\qquad \times T_{k,\ell}\cdot T_{k_1}\cdots T_{k_r}, \quad \text{for } n\ge 2 \text{ and } 1 \le j \le n-1, \qquad (21)\\ T_{n,n} &= \delta_{1,n}, \quad \text{for } n \ge 1. \end{aligned}$$

Recurrence (21) will be treated by introducing the bivariate generating function

$$F(z,v) := \sum_{n\ge 1}\sum_{1\le j\le n} T_{n,j}\frac{z^{j-1}}{(j-1)!}\frac{v^{n-j}}{(n-j)!},$$

which leads to the following first order linear partial differential equation with initial condition $F(z, 0) = 1$ (we omit here these straightforward, but lengthy computations):

$$\Big(1 - \frac{v}{1-T(z+v)}\Big)F_v(z,v) - \frac{z}{1-T(z+v)}F_z(z,v) - \frac{1}{1-T(z+v)}F(z,v) = 0. \qquad (22)$$

Using the explicit solution $T(z) = 1 - e^{-W(z)}$, with $W(z)$ the tree function, which has been stated in Theorem 1 we can write equation (22) as follows:

$$\big(1 - ve^{W(z+v)}\big)F_v(z,v) - ze^{W(z+v)}F_z(z,v) - e^{W(z+v)}F(z,v) = 0. \qquad (23)$$

To solve the PDE (23) we apply the method of characteristics. In the following we briefly give the occurring computations. First we consider the corresponding reduced PDE

$$\big(1 - ve^{W(z+v)}\big)F_v(z,v) - ze^{W(z+v)}F_z(z,v) = 0 \tag{24}$$

and thus the following system of characteristic equations:

$$\dot{v} = 1 - ve^{W(z+v)}, \quad \dot{z} = -ze^{W(z+v)}. \tag{25}$$

Introducing $u := v + z$ we further get, after adding the two characteristic equations, the system

$$\dot{u} = 1 - ue^{W(u)}, \quad \dot{z} = -ze^{W(u)}.$$

From these equations we obtain the separable differential equation

$$\frac{dz}{du} = -\frac{ze^{W(u)}}{1 - ue^{W(u)}},$$

whose general solution is given as

$$\log z = -W(u) + c_1.$$

This gives, after backsubstituting $u = v + z$, the following first integral of the system (25):

$$\xi(z,v) = c_1 = \log z + W(z+v).$$

In order to treat the PDE (23) we use now the following transformation from (z,v)-coordinates to (ξ,η)-coordinates:

$$\xi = \log z + W(z+v), \;\; \eta = z + v, \quad \text{or equivalently} \quad z = e^{\xi - W(\eta)}, \;\; v = \eta - e^{\xi - W(\eta)},$$

which leads to the differential equation:

$$\big(1 - \eta e^{W(\eta)}\big)F_\eta(\xi,\eta) - e^{W(\eta)}F(\xi,\eta) = 0. \tag{26}$$

The general solution of the separable equation (26) is given as

$$F(\xi,\eta) = C(\xi)e^{W(\eta)},$$

with arbitrary differentiable functions $C(x)$. Applying the inverse transformation to (z,v)-coordinates we obtain thus the general solution of (23):

$$\begin{aligned} F(z,v) &= C\big(\log z + W(z+v)\big)e^{W(z+v)} = C\big(\log z + W(z+v)\big)e^{\log z + W(z+v)}e^{-\log z} \\ &= \frac{\tilde{C}\big(\log z + W(z+v)\big)}{z} = \frac{G\big(ze^{W(z+v)}\big)}{z} = \frac{G\big(\frac{zW(z+v)}{z+v}\big)}{z}, \end{aligned} \tag{27}$$

with arbitrary differentiable functions $G(x)$ (and $\tilde{C}(x)$). Since we are interested in the particular solution of (23) satisfying the initial condition $F(z,0) = 1$ we will characterize the function $G(x)$ appearing in (27) by evaluating at $v = 0$, which gives

$$z = G\big(W(z)\big).$$

Thus $G(x)$ is given as the functional inverse of the tree function $W(x)$: $G(x) = W^{-1}(x)$. Of course, due to $z = W(z)e^{-W(z)}$, one obtains

$$G(x) = W^{-1}(x) = xe^{-x}.$$

Thus eventually we get the solution of (23) satisfying the initial condition:

$$F(z,v) = \frac{W(z+v)}{z+v}e^{-\frac{zW(z+v)}{z+v}} = e^{ve^{W(z+v)}}. \tag{28}$$

In order to obtain explicit formulæ for the numbers $T_{n,j}$ of size-n up-down alternating labelled ordered trees, where the root node has label j, or, equivalently, for the probabilities $\mathbb{P}\{L_n = j\} = \frac{T_{n,j}}{(n-1)^{n-1}}$ that the root node of a randomly chosen up-down alternating labelled ordered tree of size n is labelled by j, we extract coefficients from $F(z,v)$ given by (28). First we obtain:

$$\begin{aligned}
T_{n,j} &= (j-1)!(n-j)![z^{j-1}v^{n-j}]F(z,v) = (j-1)!(n-j)![z^{j-1}v^{n-j}]\sum_{k\geq 0} v^k \frac{e^{kW(z+v)}}{k!} \\
&= (j-1)!(n-j)!\sum_{k=0}^{n-j}\frac{1}{k!}[z^{j-1}v^{n-j-k}]e^{kW(z+v)} \\
&= (j-1)!(n-j)!\sum_{k=0}^{n-j}\frac{1}{(n-j-k)!}[z^{j-1}v^{k}]e^{(n-j-k)W(z+v)}.
\end{aligned}$$

In order to proceed we use that, for a power series $A(z) = \sum_{n\geq 1} a_n z^n$, the coefficients of $A(z+v)$ are given as follows: $[z^n v^k]A(z+v) = \binom{n+k}{k} a_{n+k}$. Furthermore we use

$$[z^m]e^{(n-j-k)W(z)} = (n-j-k)\frac{(n-m-j-k)^{m-1}}{m!},$$

which immediately follows by applying the Lagrange inversion formula. This gives then, for $n \geq 2$ and $1 \leq j \leq n$, the following explicit formula for the numbers $T_{n,j}$ stated in Theorem 3:

$$\begin{aligned}
T_{n,j} &= (j-1)!(n-j)!\sum_{k=0}^{n-j}\frac{1}{(n-j-k)!}\binom{j-1+k}{k}(n-j-k)\frac{(n-1)^{j-2+k}}{(j-1+k)!} \\
&= (n-j)(n-1)^{j-2}\sum_{k=0}^{n-j-1}\binom{n-j-1}{k}(n-1)^k = (n-j)\,(n-1)^{j-2}n^{n-j-1}. \qquad (29)
\end{aligned}$$

Of course, this immediately also shows the following explicit formula for the probabilities $\mathbb{P}\{L_n = j\}$:

$$\mathbb{P}\{L_n = j\} = \frac{(n-j)\,n^{n-j-1}}{(n-1)^{n-j+1}}, \quad \text{for } n \geq 2 \text{ and } 1 \leq j \leq n. \qquad (30)$$

The limiting distribution result for the probabilities $\mathbb{P}\{L_n = j\}$ as given also in Theorem 3 easily follows from the exact result (30) when considering $j \sim xn$, with $0 < x < 1$, and the straightforward computations are thus omitted here.

Using the explicit formula (30) one also easily obtains an exact expression for the expected value $\mathbb{E}(L_n)$ of L_n:

$$\mathbb{E}(L_n) = \sum_{j=1}^{n} j\,\mathbb{P}\{L_n = j\} = \sum_{j=1}^{n}\frac{j(n-j)\,n^{n-j-1}}{(n-1)^{n-j+1}} = 3n - 1 - \frac{n^n}{(n-1)^{n-1}},$$

which gives $\mathbb{E}(L_n) \sim (3-e)n = (0.281718\ldots)n$. Thus the result matches with the intuition that smaller labels are preferred to become the label of the root node in up-down alternating labelled ordered trees, but the exact amount of this preference is covered in the findings above.

4.2. **Root degree.** Next we are interested in the behaviour of the root degree in up-down alternating labelled ordered trees. To do this we introduce the random variable R_n, which counts the root degree of a randomly chosen up-down alternating labelled ordered tree of size n. If we denote now by $T_{n,m}$ the number of up-down alternating labelled ordered trees of size n, where the root node has degree m, with $0 \le m \le n$, the probabilities $\mathbb{P}\{R_n = m\}$ are just given by $\mathbb{P}\{R_n = m\} = \frac{T_{n,m}}{T_n}$.

By using the basic decomposition of an up-down alternating labelled ordered tree of size n with respect to the largest element n and using the fact that by cutting off node n the subtree of size k containing the root has either also degree m (this is obtained for $k-m-1$ positions of n) or has degree $m-1$ (this is obtained for m positions of n) we obtain the following recurrence for the numbers $T_{n,m}$ (where the appearing numbers are all zero for values of n and m, which are not listed below):

$$T_{n,m} = \sum_{r\ge 0} \sum_{\substack{k+k_1+\cdots+k_r = n-1,\\ k,k_1,\dots,k_r \ge 1}} \big((k-m-1)T_{k,m} + mT_{k,m-1}\big) T_{k_1} \cdot T_{k_2} \cdots T_{k_r} \binom{n-1}{k,k_1,\dots,k_r},$$
$$\text{for } n \ge 2, \qquad T_{1,m} = \delta_{0,m}. \tag{31}$$

Recurrence (31) can be treated by introducing the bivariate generating function

$$F(z,v) := \sum_{n\ge 1}\sum_{m\ge 0} T_{n,m} \frac{z^n}{n!} v^m.$$

This leads then to the following first order linear partial differential equation with initial condition $F(0,v) = 0$:

$$\big(1 - \frac{z}{1-T(z)}\big) F_z(z,v) + \frac{v(1-v)}{1-T(z)} F_v(z,v) = 1 - \frac{1-v}{1-T(z)} F(z,v). \tag{32}$$

The solution of this differential equation, which can be obtained again by applying the method of characteristics, is given as follows, where $W(z)$ denotes the tree function:

$$F(z,v) = \frac{W(z)e^{-W(z)}}{1-v} - \frac{e^{-W(z)}}{1-v} \log\Big(\frac{1}{1-v\big(1-e^{-W(z)}\big)}\Big). \tag{33}$$

We omit now the computations, but it can be checked easily that (33) is indeed the desired solution of (32).

Extracting coefficients from (33) by using the Lagrange inversion formula leads to an exact formula for $T_{n,m}$, which is given in Theorem 4. We first obtain

$$\begin{aligned}[v^m]F(z,v) &= W(z)e^{-W(z)} - e^{-W(z)} \sum_{k=1}^{m} [v^k] \log\Big(\frac{1}{1-v\big(1-e^{-W(z)}\big)}\Big) \\ &= W(z)e^{-W(z)} - \sum_{k=1}^{m} \frac{1}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^\ell e^{-(\ell+1)W(z)}\end{aligned} \tag{34}$$

and further

$$\begin{aligned}T_{n,m} &= n![z^n v^m]F(z,v) \\ &= (n-1)![W^{n-1}]e^{nW}\Big(e^{-W} - We^{-W} + \sum_{k=1}^{m}\frac{1}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell(\ell+1)e^{-(\ell+1)W}\Big)\end{aligned}$$

$$
\begin{aligned}
&= (n-1)!\Big(\frac{(n-1)^{n-1}}{(n-1)!} - \frac{(n-1)^{n-2}}{(n-2)!} + \sum_{k=1}^{m}\frac{1}{k}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}(\ell+1)\frac{(n-\ell-1)^{n-1}}{(n-1)!}\Big) \\
&= (n-1)!\sum_{k=1}^{m}\frac{1}{k}\frac{(n-1)^{n-1}}{(n-1)!} + (n-1)!\sum_{\ell=1}^{m}(-1)^{\ell}\frac{\ell+1}{\ell}\frac{(n-\ell-1)^{n-1}}{(n-1)!}\sum_{k=\ell}^{m}\binom{k-1}{\ell-1} \\
&= (n-1)^{n-1}H_m + \sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{\ell}\frac{\ell+1}{\ell}(n-\ell-1)^{n-1}, \qquad (35)
\end{aligned}
$$

with $H_m := \sum_{k=1}^{m}\frac{1}{k}$ the m-th harmonic number.

Of course, the exact distribution of R_n is then determined by

$$\mathbb{P}\{R_n = m\} = \frac{T_{n,m}}{(n-1)^{n-1}} = H_m + \sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{\ell}\frac{\ell+1}{\ell}\big(1-\frac{\ell}{n-1}\big)^{n-1}. \qquad (36)$$

The discrete limiting distribution result for R_n as given in Theorem 5 can be obtained either from this exact result or easier (due to the alternating sum involved in this exact expression) by applying singularity analysis to a local expansion of the functions $F_m(z) := [v^m]F(z,v)$ as given in (34) around the dominant singularity $\rho = e^{-1}$. Using the local expansion (15) of the tree function $W(z)$ around the dominant singularity $\rho = e^{-1}$ we get, for m fixed, the following local expansion of $F_m(z)$:

$$F_m(z) = \frac{1}{e} - \frac{1}{e}\sum_{k=1}^{m}\frac{1}{k}\big(\frac{e-1}{e}\big)^{k} - \frac{\sqrt{2}}{e}\Big[-1+\big(\frac{e-1}{e}\big)^{m} + \sum_{k=1}^{m}\frac{1}{k}\big(\frac{e-1}{e}\big)^{k}\Big]\sqrt{1-ez} + \mathcal{O}\big(1-ez\big),$$

which leads, by applying singularity analysis, to the following limiting distribution result of R_n:

$$\mathbb{P}\{R_n = m\} = \frac{n!}{(n-1)^{n-1}}[z^n]F_m(z) \to \big(\frac{e-1}{e}\big)^{m} - 1 + \sum_{k=1}^{m}\frac{1}{k}\big(\frac{e-1}{e}\big)^{k}, \quad \text{for } n \to \infty.$$

We further remark that the expected value of R_n can be computed easily and is given by the following exact formula:

$$\mathbb{E}(R_n) = \frac{1}{2}\Big[\big(\frac{n+1}{n-1}\big)^{n-1} - 1\Big],$$

which gives that $\mathbb{E}(R_n) \sim \frac{e^2-1}{2} \approx 3.194528\dots$. If we compare this result with the corresponding result (see, e.g., [4]) for unlabelled (or equivalently randomly labelled) ordered trees, where it holds $\mathbb{E}(R_n) \sim 3$, we obtain that on average the root of an alternating tree has a slightly higher degree than the root of a randomly labelled tree.

4.3. **Depth of nodes.** An important parameter when analysing the structure of random trees in rooted tree families is the depth of a randomly chosen node. Thus we are studying the random variable D_n, which counts the depth of a randomly chosen node in a random up-down alternating labelled ordered tree of size n. Due to the nature of the basic decomposition of alternating trees w.r.t. the node with label n in a size-n tree, we require for a study of D_n an auxiliary parameter, which we call "the depth of a random insertion point": if we choose an alternating ordered tree T of size n there are exactly n possibilities to attach a node with label $n+1$ to one of the nodes in T in such a way that the resulting tree is again an alternating ordered tree, now of size $n+1$. The depth of node $n+1$ when attached to a node in T in an appropriate way is then of interest here. We introduce thus the auxiliary random variable X_n, which counts the depth of node $n+1$ in a tree obtained by choosing a random alternating tree of size n and attaching a node with label $n+1$ at random at one of the n possibilities such that the resulting tree is an alternating tree of size $n+1$.

Using the basic decomposition of an up-down alternating labelled ordered tree T with respect to the node with the largest label n as described in Subsection 3.2 leading to trees $\widehat{T}$, T_1, ..., T_r, we obtain a system of recurrences for the probabilities $\mathbb{P}\{X_n = m\}$ and $\mathbb{P}\{D_n = m\}$. When computing the recurrence for the probabilities $\mathbb{P}\{X_n = m\}$ we only have to take into account the insertion points in $\widehat{T}$, T_1, ..., T_r, and the additional insertion point of T obtained after attaching node n to $\widehat{T}$. Analogous, when computing the recurrence for the probabilities $\mathbb{P}\{D_n = m\}$, we have to take into account the depth of nodes in $\widehat{T}$, T_1, ..., T_r and the depth of node n after attaching to $\widehat{T}$. We obtain then the following system of recurrences, where $T_n = (n-1)^{n-1}$ denote the number of up-down alternating labelled trees of size n (the appearing probabilities are all zero for values of n and m that are not listed below):

$$\begin{aligned}
\mathbb{P}\{X_n = m\} = \sum_{r\ge 0} \sum_{\substack{k+k_1+\cdots+k_r=n-1,\\ k,k_1,\dots,k_r\ge 1}} k \binom{n-1}{k, k_1, \dots, k_r} \frac{T_k \cdot T_{k_1} \cdots T_{k_r}}{T_n} \times \\
\times \Big(\frac{k}{n}\mathbb{P}\{X_k = m\} + \frac{1}{n}\mathbb{P}\{X_k = m\} \qquad\qquad (37a) \\
+ \sum_{i=1}^{r} \frac{k_i}{n} \sum_{\substack{m_1+m_2=m-1,\\ m_1,m_2\ge 0}} \mathbb{P}\{X_k = m_1\}\mathbb{P}\{X_{k_i} = m_2\}\Big), \quad \text{for } n \ge 2, \\
\mathbb{P}\{X_1 = m\} = \delta_{1,m}, \\
\mathbb{P}\{D_n = m\} = \sum_{r\ge 0} \sum_{\substack{k+k_1+\cdots+k_r=n-1,\\ k,k_1,\dots,k_r\ge 1}} k \binom{n-1}{k, k_1, \dots, k_r} \frac{T_k \cdot T_{k_1} \cdots T_{k_r}}{T_n} \times \\
\times \Big(\frac{k}{n}\mathbb{P}\{D_k = m\} + \frac{1}{n}\mathbb{P}\{X_k = m\} \qquad\qquad (37b) \\
+ \sum_{i=1}^{r} \frac{k_i}{n} \sum_{\substack{m_1+m_2=m-1,\\ m_1,m_2\ge 0}} \mathbb{P}\{X_k = m_1\}\mathbb{P}\{D_{k_i} = m_2\}\Big), \quad \text{for } n \ge 2, \\
\mathbb{P}\{D_1 = m\} = \delta_{0,m}.
\end{aligned}$$

We treat the system of recurrences (37) by introducing the bivariate generating functions

$$F(z,v) := \sum_{n\ge 1} \sum_{m\ge 0} n T_n \mathbb{P}\{X_n = m\} \frac{z^n}{n!} v^m,$$

$$G(z,v) := \sum_{n\ge 1} \sum_{m\ge 0} n T_n \mathbb{P}\{D_n = m\} \frac{z^n}{n!} v^m,$$

which leads to the following system of differential equations with initial conditions $F(0,v) = G(0,v) = 0$:

$$\big(1 - \frac{z}{1-T(z)}\big)F_z(z,v) - \frac{1}{1-T(z)}F(z,v) - \frac{v}{(1-T(z))^2}F(z,v)^2 - v = 0, \tag{38a}$$

$$\big(1 - \frac{z}{1-T(z)}\big)G_z(z,v) - \frac{1}{1-T(z)}F(z,v) - \frac{vF(z,v)}{(1-T(z))^2}G(z,v) - 1 = 0, \tag{38b}$$

with $T(z) = 1 - e^{-W(z)}$ and $W(z)$ the tree function.

In order to study the asymptotic behaviour of the depth D_n of a random node in a random alternating ordered tree of size n we use the so called method of moments. By studying the evaluation at $v = 1$ of the r-th derivatives of $F(z,v)$ and $G(z,v)$ w.r.t. v we are able to show that, for r fixed and $n \to \infty$, the r-th moment of the normalized random variable $\frac{D_n}{\sqrt{n}}$ converges to the r-th moment of a Rayley-distributed random variable R_α, with parameter $\alpha = \frac{2}{3}$. Since the Rayleigh-distribution is fully characterized by its moments an application of the Theorem of Fréchet and Shohat (see, e.g., [8]) shows the convergence in distribution of $\frac{D_n}{\sqrt{n}}$ to R_α, which is stated as Theorem 5.

We will here only sketch the appearing computations; see, e.g., [10] for another application of this method, which is figured out there in more detail. First we consider the differential equation (38a) for the generating function $F(z,v)$ of the probabilities $\mathbb{P}\{X_n = m\}$ of the auxiliary random variable X_n and introduce, for integers $r \geq 0$, the functions

$$F_r(z) := \frac{\partial^r}{\partial v^r}F(z,v)\bigg|_{v=1}.$$

For $F_0(z)$ we obtain from the definition:

$$F_0(z) = F(z,1) = \sum_{n\geq 1} nT_n \frac{z^n}{n!} = zT'(z) = \frac{W(z)e^{-W(z)}}{1-W(z)}, \tag{39}$$

whereas differentiating (38a) leads to the following differential equation of $F_r(z)$, $r \geq 1$:

$$F_r'(z) - \frac{(1+W(z))e^{W(z)}}{(1-W(z))^2}F_r(z) = S_r(z), \tag{40}$$

where the inhomogeneous part $S_r(z)$ is given as follows:

$$S_1(z) = \frac{W(z)^2}{(1-W(z))^3} + \frac{1}{1-W(z)},$$

and, for $r \geq 2$:

$$S_r(z) = \frac{1}{1-W(z)}\Big(re^{2W(z)}\sum_{k=0}^{r-1}\binom{r-1}{k}F_k(z)F_{r-1-k}(z) + e^{2W(z)}\sum_{k=1}^{r-1}\binom{r}{k}F_k(z)F_{r-k}(z)\Big).$$

One easily gets that the solution of $F_r(z)$ satisfying (40) and the initial condition $F_r(0) = 0$ is, for $r \geq 1$, given as follows:

$$F_r(z) = \frac{e^{-W(z)}}{(1-W(z))^2}\int_0^{W(z)} S_r\big(We^{-W}\big)(1-W)^3 dW. \tag{41}$$

It is now a crucial observation that the functions $F_r(z)$, $r \geq 0$, behave locally in a complex neighbourhood of the unique dominant singularity $\rho = e^{-1}$ (or equivalently around $W(z) = 1$) as follows, with certain constants c_r:

$$F_r(z) \sim \frac{c_r e^{-W(z)}}{(1 - W(z))^{r+1}}. \tag{42}$$

For $r = 0$ it follows directly from (39) and the local behaviour (15) of the tree function $W(z)$ that the asymptotic equivalent (42) is true with constant $c_0 = 1$:

$$F_0(z) = \frac{W(z)e^{-W(z)}}{1 - W(z)} \sim \frac{e^{-W(z)}}{1 - W(z)}.$$

For $r = 1$ the differential equation (40) leads to the following solution

$$F_1(z) = \Big(\frac{2}{3(1 - W(z))^2} + \frac{1}{1 - W(z)} + 1 - \frac{2}{3}\big(1 - W(z)\big)\Big)e^{-W(z)} \sim \frac{\frac{2}{3}e^{-W(z)}}{(1 - W(z))^2},$$

which shows (42) with $c_1 = \frac{2}{3}$. For general r, the asymptotic relation (42) can be shown by induction using (41) and theorems for singular integration, see [4]; as a byproduct one also obtains a recurrence for the constants c_r:

$$\begin{aligned}
F_r(z) &\sim \frac{e^{-W(z)}}{(1 - W(z))^2} \int_0^{W(z)} \Big[r \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{c_k}{(1 - W)^{k+1}} \frac{c_{r-1-k}}{(1 - W)^{r-k}} \\
&\qquad + \sum_{k=1}^{r-1} \binom{r}{k} \frac{c_k}{(1 - W)^{k+1}} \frac{c_{r-k}}{(1 - W)^{r-k+1}}\Big](1 - W)^2 dW \\
&\sim \sum_{k=1}^{r-1} \binom{r}{k} c_k c_{r-k} \frac{e^{-W(z)}}{(1 - W(z))^2} \int_0^{W(z)} \frac{1}{(1 - W)^r} dW \\
&\sim \frac{1}{r-1} \sum_{k=1}^{r-1} \binom{r}{k} c_k c_{r-k} \frac{e^{-W(z)}}{(1 - W(z))^{r+1}},
\end{aligned}$$

and thus

$$c_r = \frac{1}{r-1} \sum_{k=1}^{r-1} \binom{r}{k} c_k c_{r-k}, \quad \text{for } r \geq 2, \quad c_1 = \frac{2}{3}. \tag{43}$$

Recurrence (43) can be treated by standard methods leading to the solution

$$c_r = r!\big(\frac{2}{3}\big)^r, \quad \text{for } r \geq 1. \tag{44}$$

This gives the following local behaviour of the functions $F_r(z)$ around the dominant singularity $z = e^{-1}$:

$$F_r(z) \sim \big(\frac{2}{3}\big)^r r! \frac{e^{-W(z)}}{(1 - W(z))^{r+1}} \sim \big(\frac{2}{3}\big)^r r! \frac{e^{-1}}{(\sqrt{2}\sqrt{1 - ez})^{r+1}}. \tag{45}$$

We use now this description (45) of the local behaviour of $F_r(z)$ around $z = e^{-1}$ for a study of the differential equation (38b) for the generating function $G(z, v)$ of the probabilities

$\mathbb{P}\{D_n = m\}$ of the random variable D_n of interest. To do this we introduce, for integers $r \geq 0$, the functions

$$G_r(z) := \frac{\partial^r}{\partial z^r} G(z,v)\Big|_{v=1}.$$

Again, by using the definition, we immediately obtain for $G_0(z)$:

$$G_0(z) = G(z,1) = \sum_{n\geq 1} nT_n \frac{z^n}{n!} = zT'(z) = \frac{W(z)e^{-W(z)}}{1-W(z)}, \tag{46}$$

whereas differentiating (38b) leads, for $r \geq 1$, to the following differential equation for $G_r(z)$:

$$G_r'(z) - \frac{W(z)e^{W(z)}}{(1-W(z))^2} G_r(z) = \tilde{S}_r(z), \tag{47}$$

with

$$\tilde{S}_r(z) = \frac{1}{1-W(z)} \Big(e^{W(z)} F_r(z) + re^{2W(z)} \sum_{k=0}^{r-1} \binom{r-1}{k} F_k(z) G_{r-1-k}(z) + e^{2W(z)} \sum_{k=1}^{r} \binom{r}{k} F_k(z) G_{r-k}(z) \Big).$$

One easily gets that the solution of $G_r(z)$ satisfying (47) and the initial condition $G_r(0) = 0$ is, for $r \geq 1$, given as follows:

$$G_r(z) = \frac{e^{-W(z)}}{1-W(z)} \int_0^{W(z)} \tilde{S}_r\big(We^{-W}\big)(1-W)^2 dW. \tag{48}$$

Again it is a crucial observation that the functions $G_r(z)$, $r \geq 0$, behave locally in a complex neighbourhood of the unique dominant singularity $\rho = e^{-1}$ (or equivalently around $W(z) = 1$) as follows, with certain constants d_r:

$$G_r(z) \sim \frac{d_r e^{-W(z)}}{(1-W(z))^{r+1}}. \tag{49}$$

The asymptotic relation (49) can be proven by induction, where we additionally require the corresponding relation (45) for the auxiliary functions $F_r(z)$, leading also to a recurrence for the constants d_r:

$$\begin{aligned} G_r(z) &\sim \frac{e^{-W(z)}}{1-W(z)} \int_0^{W(z)} \Big[\frac{c_r}{(1-W)^{r+1}} + r \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{c_k}{(1-W)^{k+1}} \frac{d_{r-1-k}}{(1-W)^{r-k}} \\ &\qquad + \sum_{k=1}^{r} \binom{r}{k} \frac{c_k}{(1-W)^{k+1}} \frac{d_{r-k}}{(1-W)^{r-k+1}} \Big] (1-W) dW \\ &\sim \sum_{k=1}^{r} \binom{r}{k} c_k d_{r-k} \frac{e^{-W(z)}}{1-W(z)} \int_0^{W(z)} \frac{1}{(1-W)^{r+1}} dW \\ &\sim \frac{1}{r} \sum_{k=1}^{r} \binom{r}{k} c_k d_{r-k} \frac{e^{-W(z)}}{(1-W(z))^{r+1}}, \end{aligned}$$

and thus

$$d_r = \frac{1}{r} \sum_{k=1}^{r} \binom{r}{k} c_k d_{r-k}, \quad \text{for } r \geq 1. \tag{50}$$

Using $d_0 = 1$, which follows immediately from (46), and the solution (44) for c_r we can solve recurrence (50) by standard methods and obtain

$$d_r = r! \big(\frac{2}{3}\big)^r, \quad \text{for } r \geq 1.$$

This gives the following local behaviour of the functions $G_r(z)$ around the dominant singularity $z = e^{-1}$:

$$G_r(z) \sim \big(\frac{2}{3}\big)^r r! \frac{e^{-W(z)}}{(1 - W(z))^{r+1}} \sim \big(\frac{2}{3}\big)^r r! \frac{e^{-1}}{(\sqrt{2}\sqrt{1-ez})^{r+1}}. \tag{51}$$

Via singularity analysis we obtain then the following asymptotic equivalent of the r-th factorial moments $\mathbb{E}(D_n^{\underline{r}}) := \mathbb{E}\big(D_n(D_n - 1) \cdots (D_n - r + 1)\big)$:

$$\mathbb{E}(D_n^{\underline{r}}) = \frac{n!}{nT_n} [z^n] G_r(z) \sim \frac{\sqrt{\pi} r! \big(\frac{2}{3}\big)^r}{2^{\frac{r}{2}} \Gamma\big(\frac{r+1}{2}\big)} n^{\frac{r}{2}}. \tag{52}$$

Due to the relation $\mathbb{E}(D_n^r) = \sum_{k=0}^r {r \brace k} \mathbb{E}(D_n^{\underline{k}})$ between the r-th ordinary and the r-th factorial moments of D_n, where ${r \brace k}$ denote the Stirling numbers of the second kind, it holds $\mathbb{E}(D_n^r) \sim \mathbb{E}(D_n^{\underline{r}})$. By using the duplication formula of the Gamma-function:

$$\Gamma(\frac{r+1}{2}) \Gamma(\frac{r}{2} + 1) = \frac{r! \sqrt{\pi}}{2^r},$$

we obtain from (52), for every r fixed and $n \to \infty$, the following asymptotic equivalent of the r-th moment of the scaled random variable $\frac{D_n}{\sqrt{n}}$:

$$\mathbb{E}\Big(\big(\frac{D_n}{\sqrt{n}}\big)^r\Big) \sim \big(\frac{2}{3}\big)^r 2^{\frac{r}{2}} \Gamma\Big(1 + \frac{r}{2}\Big).$$

Since every r-th moment of $\frac{D_n}{\sqrt{n}}$ converges to the r-th moment of a Rayleigh distributed random variable R_α with parameter $\alpha = \frac{2}{3}$ we have indeed shown Theorem 5.

If one compares this result with the depth D_n of a random node in a randomly labelled ordered tree of size n (see, e.g., [9]), where $\frac{D_n}{\sqrt{n}}$ is also asymptotically Rayleigh distributed, but with a larger parameter $\alpha = 1$, one gets that on average the depth of a randomly chosen node in a randomly chosen alternating ordered tree is about 1/3 smaller than the depth of a randomly chosen node in a random labelled tree of the same size.

References

[1] C. Chauve, S. Dulucq and A. Rechnitzer, Enumerating alternating trees, *Journal of Combinatorial Theory, Series A* 94, 142–151, 2001.

[2] M. Drmota, Combinatorics and asymptotics on trees, *Cubo* 6, 105–136, 2004.

[3] P. Flajolet and R. Sedgewick, *An introduction to the analyis of algorithms*, Addison-Wesley, Reading, 1996.

[4] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, to appear.
Draft available at `http://algo.inria.fr/flajolet/Publications/books.html`

[5] I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, in: *"Arnold-Gelfand mathematical seminars: Geometry and Singularity Theory"*, 205–221, Birkhuser, Basel, 1997.
[6] R. Graham, D. Knuth and O. Patashnik, *Concrete Mathematics*, Second Edition, Addison-Wesley, Boston, 1994.
[7] M. Kuba and A. Panholzer, On the degree distribution of the nodes in increasing trees, *Journal of Combinatorial Theory, Series A* 114, 597–618, 2007.
[8] M. Loève, *Probability Theory I*, 4th Edition, Springer, New York, 1977.
[9] A. Meir and J. W. Moon, On the altitude of nodes in random trees, *Canadian Journal of Mathematics* 30, 997–1015, 1978.
[10] A. Panholzer, Cutting down very simple trees, *Quaestiones Mathematicae* 29, 211–227, 2006.
[11] A. Postnikov, Intransitive trees, *Journal of Combinatorial Theory, Series A* 79, 360–366, 1997.
[12] A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, *Journal of Combinatorial Theory, Series A* 91, 544–597, 2000.
[13] R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.
[14] M. E. Taylor, *Partial differential equations. Basic theory. Texts in Applied Mathematics, 23*, Springer, New York, 1996.

Markus Kuba, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8-10/104, A-1040 Wien, Austria

E-mail address: markus.kuba@tuwien.ac.at

Alois Panholzer, Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstr. 8-10/104, A-1040 Wien, Austria

E-mail address: Alois.Panholzer@tuwien.ac.at