Cumulants and convolutions via Abel polynomials

E. Di Nardo, P. Petrullo, D. Senato

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Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, via dell'Ateneo Lucano 10, 85100 Potenza, Italia.
elvira.dinardo@unibas.it, p.petrullo@gmail.com, domenico.senato@unibas.it

Abstract

We provide an unifying polynomial expression giving moments in terms of cumulants, and viceversa, holding in the classical, boolean and free setting. This is done by using a symbolic treatment of Abel polynomials. As a by-product, we show that in the free cumulant theory the volume polynomial of Pitman and Stanley plays the role of the complete Bell exponential polynomial in the classical theory. Moreover via generalized Abel polynomials we construct a new class of cumulants, including the classical, boolean and free ones, and the convolutions linearized by them. Finally, via an umbral Fourier transform, we state a explicit connection between boolean and free convolution.

keywords: umbral calculus, free cumulant, boolean cumulant, classical cumulant, volume polynomial, Abel polynomials.

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1 Introduction

Cumulants linearize the convolution of probability measures in the three universal probability theories: classical, boolean and free. The last is a noncommutative probability theory introduced by Voiculescu [33] with a view to tackle certain problems in operator algebra. More precisely, a new kind of independence is defined by replacing tensor products with free products and this can help understand the Von Neumann algebras of free groups. The combinatorics underlying this subject is based on the notion of noncrossing partition, whose first systematical study is due to Kreweras [10] and Poupard [20]. Within free probability, noncrossing partitions are extensively used by Speicher [16]. Speicher takes his lead from the definition of classical multilinear cumulants in terms of the Möbius function. However, he changes the lattice where the Möbius inversion formula is applied.

Instead of using the lattice of all partitions of a finite set, he uses the smaller lattice of noncrossing partitions. Such a new family of cumulants, known as free cumulants, turns out to be the semi-invariants of Voiculescu, originally introduced via the R-transform. Biane [2] has shown how free cumulants can be used to obtain asymptotical estimations of the characters of large symmetric groups.

As is well known, some results of noncrossing partition theory can be recovered via Lagrange inversion formula. Recently, a simple expression of Lagrange inversion formula has been given by Di Nardo and Senato [4] within classical umbral calculus. This paper arises from this new Lagrange symbolic formula.

The classical umbral calculus [24] is a renewed version of the celebrated umbral calculus of Roman and Rota [21]. It consists of a symbolic technique to deal with sequences of numbers, indexed by nonnegative integers, where the subscripts are treated as powers. Recently Di Nardo and Senato [4, 5] have developed this umbral language in view of probabilistic applications. Moreover the umbral syntax has been fruitfully used in computational k-statistics and their generalizations [6]. The first algebraic approach to this topic was given by McCullagh [14] and Speed [26]. Applications to bilinear generating functions for polynomial sequences are given by Gessel [9].

Rota and Shen [22] have already used umbral methods in exploring some algebraic properties of cumulants, only in the classical theory. They have proved that the umbral handling of cumulants encodes and simplifies their combinatorics properties. In this paper, we go further showing how the umbral syntax allows us to explore the more hidden connection between the theory of free cumulants and that of classical and boolean cumulants.

As pointed out in [17], the recent results of Belinschi and Nica [3] revealed a deeper connection between free and boolean convolution that deserves a further clarification. Indeed, this connection cannot be encoded in a straight way in the formal power series language. We provide this connection via an umbral Fourier transform. Moreover, quite surprisingly, the umbral methods bring to the light that the key to manage all these families of cumulants is the connection between binomial sequences and Abel polynomials [23]. This connection gives the chance to find a new and very simple parametrization of free cumulants in terms of moments. If α is the umbra representing the moments and $\mathfrak{K}_\alpha$ is the umbra representing the free cumulants, then $\bar{\mathfrak{K}}_\alpha^n \simeq \bar{\alpha}(\bar{\alpha} - n.\bar{\alpha})^{n-1}$. This parametrization closely parallels the one connecting cumulants and moments, either in the classical or in the boolean setting, which are respectively $\kappa_\alpha^n \simeq \alpha(\alpha - \mathbf{1}.\alpha)^{n-1}$ and $\bar{\eta}_\alpha^n \simeq \bar{\alpha}(\bar{\alpha} - \mathbf{2}.\bar{\alpha})^{n-1}$, where κ_α^n denotes the n-th classical cumulant and $\bar{\eta}_\alpha^n$ the n-th boolean cumulant.

The inverse expression giving moments in terms of free cumulants is obtained (up to a sign) simply by swapping the umbra representing moments with the umbra representing its free cumulants, $\bar{\alpha}^n \simeq \bar{\mathfrak{K}}_\alpha(\bar{\mathfrak{K}}_\alpha + n.\bar{\mathfrak{K}}_\alpha)^{n-1}$. It is remarkable that the polynomial, on the right side of the previous expression, looks like the volume polynomials of Pitman and Stanley [19] obtained when the indeterminates are replaced by scalars, $V_n(a, a, \ldots, a) = a(a + n\,a)^{n-1}$. So we prove that moments of an umbra can be recovered from volume polynomials of Pitman and Stanley [19] when the indeterminates are replaced with the uncorrelated and similar free cumulant umbrae. In other words, in the free cumulant theory the volume polynomials are the analogs of the complete Bell exponential polynomials in the classical cumulant theory.

The paper is structured as follows. In Section 2, we recall the combinatorics of classical, boolean and free cumulants with the aim to demonstrate how the umbral syntax provides an unifying framework to deal with these number sequences. Indeed, in Section 3, after recalling the umbral syntax, a theorem embedding the algebras of multiplicative functions on the posets of all partitions and of all interval partitions of a finite set in the classical umbral calculus is proved. In this section we also recall the umbral theory of classical cumulants and we show how the umbral theory of boolean cumulants is easily deduced from the classical one by introducing the boolean unity umbra. A symbolic theory of free cumulants closes the section. We also show that Catalan numbers are the moments of the unique umbra whose free cumulants are all equal to 1. In Section 4, we state the connection between volume polynomials and free cumulants. In the last section we introduce a new class of cumulants, including the classical, boolean and free ones, and the convolutions linearized by them.

2 Cumulants and convolutions

The combinatorics of classical, free and boolean cumulants were studied by Lehner [12, 13], Speicher [28], and Speicher and Wouroudi [29]. In the following we recall the main results of their approach.

Denote by $[n]$ the set of positive integers $\{1, 2, \ldots, n\}$ and by Π_n the set of all partitions of $[n]$. The algebra of the multiplicative functions on the poset $(\Pi_n, \leq)$ (see [8] or [31]), where $\leq$ is the *refinement* order, provides nice formulae when the coefficients of the exponential formal power series $f[g(t) - 1]$ are expressed in terms of the coefficients of $f(t)$ and $g(t)$, where $f(t) = 1 + \sum_{n \geq 1} f_n \frac{t^n}{n!}$, and $g(t) = 1 + \sum_{n \geq 1} g_n \frac{t^n}{n!}$. For example, the coefficients of $\log f(t)$, expanded in an exponential power series, are known as *formal cumulants* of $f(t)$. When $f(t)$ is the moment generating function of a random variable X, this sequence has a meaning which is not purely formal. For example, cumulants of order $2, 3$ and 4 concur in characterizing the variance, the skewness and the curtosis of a random variable.

Let us recall some well known facts on multiplicative functions. We denote the minimum and the maximum of the poset Π_n by $\mathbf{0}_n$ and $\mathbf{1}_n$ respectively. The number of blocks of a given $\pi \in \Pi_n$ will be denoted by $\ell(\pi)$.

If $\sigma, \pi \in \Pi_n$ and $[\sigma, \pi] = \{\tau \in \Pi_n \,|\, \sigma \leq \tau \leq \pi\}$, then there is an unique sequence of non-negative integers $(k_1, k_2, \ldots, k_n)$ with $k_1 + 2k_2 + \cdots + nk_n = \ell(\sigma)$ and $k_1 + k_2 + \cdots + k_n = \ell(\pi)$ such that

$$[\sigma, \pi] \cong \Pi_1^{k_1} \times \Pi_2^{k_2} \times \cdots \times \Pi_n^{k_n}. \tag{2.1}$$

In particular, if π has exactly m_i blocks of cardinality i, then

$$[\mathbf{0}_n, \pi] \cong \Pi_1^{m_1} \times \Pi_2^{m_2} \times \cdots \times \Pi_n^{m_n}. \tag{2.2}$$

Then $[\pi, \mathbf{1}_n] \cong \Pi_{\ell(\pi)}$. The sequence $(k_1, k_2, \ldots, k_n)$ is called the *type* of the interval $[\sigma, \pi]$, where

$$k_i = \text{number of blocks of } \pi \text{ that are the union of } i \text{ blocks of } \sigma. \tag{2.3}$$

The vector $sh(\pi) = (c_1, c_2, \ldots, c_l)$ whose entries are the cardinalities of the blocks of π, arranged in nondecreasing order, will be called the *shape* of π.

A function $\mathfrak{f} : \Pi_n \times \Pi_n \to \mathbb{C}$ is said to be *multiplicative* if $\mathfrak{f}(\sigma, \pi) = f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n}$, whenever (2.1) holds and $f_n := \mathfrak{f}(\mathbf{0}_n, \mathbf{1}_n)$. The Möbius function μ, the Zeta function ζ and the Delta function δ are multiplicative functions with $\mu_n = (-1)^{n-1}(n-1)!$, $\zeta_n = 1$, and $\delta_n = \delta_{1,n}$ (the Kronecker delta).

A convolution $\star$ is defined between two multiplicative functions $\mathfrak{f}$ and $\mathfrak{g}$. We have

$$(\mathfrak{f} \star \mathfrak{g})(\sigma, \pi) := \sum_{\sigma \leq \tau \leq \pi} \mathfrak{f}(\sigma, \tau)\, \mathfrak{g}(\tau, \pi). \tag{2.4}$$

The function $\mathfrak{f} \star \mathfrak{g}$ is also multiplicative. In particular, if $\mathfrak{h} = \mathfrak{f} \star \mathfrak{g}$, then $h_n = (\mathfrak{f} \star \mathfrak{g})(\mathbf{0}_n, \mathbf{1}_n)$ so that

$$h_n = \sum_{\tau \in \Pi_n} f_\tau\, g_{\ell(\tau)}, \tag{2.5}$$

where $f_\tau := f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$ and m_i is the number of blocks of τ of cardinality i. The function δ is the identity with respect to the convolution $\star$. Furthermore, μ and ζ are inverse each other with respect to $\star$, that is $\mu \star \zeta = \zeta \star \mu = \delta$.

Theorem 2.1. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be three multiplicative functions on the lattice $(\Pi_n, \leq)$ with $f_n = \mathfrak{f}(\mathbf{0}_n, \mathbf{1}_n)$, $g_n = \mathfrak{g}(\mathbf{0}_n, \mathbf{1}_n)$ and $h_n = \mathfrak{h}(\mathbf{0}_n, \mathbf{1}_n)$. If $f(t) = 1 + \sum_{n \geq 1} f_n \frac{t^n}{n!}$, $g(t) = 1 + \sum_{n \geq 1} g_n \frac{t^n}{n!}$ and $h(t) = 1 + \sum_{n \geq 1} h_n \frac{t^n}{n!}$, then*

$$h(t) = f[g(t) - 1] \Longleftrightarrow \mathfrak{h} = \mathfrak{g} \star \mathfrak{f}. \tag{2.6}$$

The formulae expressing cumulants c_n in terms of moments m_n, and viceversa, are easily recovered from (2.5). Indeed, let $F(t) = 1 + \sum_{n \geq 1} m_n \frac{t^n}{n!}$ and $C(t) = \log F(t) = 1 + \sum_{n \geq 1} c_n \frac{t^n}{n!}$ its cumulant generating function. If $\mathfrak{m}$ and $\mathfrak{k}$ denote two multiplicative functions on $(\Pi_n, \leq)$ such that $\mathfrak{m}(\mathbf{0}_n, \mathbf{1}_n) = m_n$ and $\mathfrak{k}(\mathbf{0}_n, \mathbf{1}_n) = c_n$, then from Theorem 2.1 we have

$$\begin{cases} \mathfrak{k} &= \mathfrak{m} \star \mu, \\ \mathfrak{m} &= \mathfrak{k} \star \zeta. \end{cases} \tag{2.7}$$

Free cumulants occur in noncommutative context of probability theory (see for instance [32]). A *noncommutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital noncommutative algebra and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a unital linear functional. An element X of $\mathcal{A}$ is called *noncommutative random variable.* The n-th *moment* of X is the complex number $m_n = \varphi(X^n)$, the *distribution* of X is the collection of its moments $(\varphi(X), \varphi(X^2), \varphi(X^3), \ldots)$. The moment generating function of X is the formal power series

$$M(t) = 1 + \sum_{n \geq 1} m_n t^n. \tag{2.8}$$

The *noncrossing (or free) cumulants* of X are the coefficients r_n of the ordinary power series $R(t) = 1 + r_1 t + r_2 t^2 + \ldots$ such that

$$M(t) = R[tM(t)]. \tag{2.9}$$

This relation between cumulants and moments of a noncommutative random variable has been found by Speicher [27] and characterize the free cumulants introduced by Voiculescu [32], so we assume (2.9) as a definition of free cumulants (see also [15] and [28]). Moreover, Speicher [27] has shown that an identity analogous to (2.7) holds between free cumulants $\{r_n\}_{n\geq 1}$ and moments $\{m_n\}_{n\geq 1}$ of a (noncommutative) random variable X, if we change the lattice of partitions of a set into the lattice of noncrossing partitions $(\mathcal{NC}_n, \leq)$.

A *noncrossing partition* $\pi = \{B_1, B_2, \ldots, B_s\}$ of the set $[n]$ is a partition such that if $1 \leq h < l < k < m \leq n$, with $h, k \in B_j$, and $l, m \in B_{j'}$, then $j = j'$ (see [10], [25] and [20] for a detailed handling). Let $\mathcal{NC}_n$ denote the set of all noncrossing partitions of $[n]$. Its cardinality is equal to the n-th Catalan number $\mathcal{C}_n$. The convolution $*$ defined on the multiplicative functions on the the lattice $(\mathcal{NC}_n, \leq)$ is given by

$$(\mathfrak{f} * \mathfrak{g})(\sigma, \pi) := \sum_{\substack{\sigma \leq \tau \leq \pi \\ \tau \in \mathcal{NC}_n}} \mathfrak{f}(\sigma, \tau)\, \mathfrak{g}(\tau, \pi). \tag{2.10}$$

Following Nica and Speicher [15], if $\tilde{\tau}$ is the Kreweras complement of a noncrossing partition τ, then $\mathfrak{f}(\mathbf{0}_n, \tau) = \mathfrak{f}_\tau$ and $\mathfrak{g}(\tau, \mathbf{1}_n) = \mathfrak{g}_{\tilde{\tau}}$. Hence, if $\mathfrak{h} = \mathfrak{f} * \mathfrak{g}$ then

$$h_n = \sum_{\tau \in \mathcal{NC}_n} f_\tau\, \mathfrak{g}_{\tilde{\tau}}. \tag{2.11}$$

If we denote by $\zeta_{\mathcal{NC}}$ and $\mu_{\mathcal{NC}}$ the Zeta function and the Möbius function on the noncrossing partition lattice respectively, then we have $\mathfrak{h} = \mathfrak{f} * \zeta_{\mathcal{NC}}$ if and only if $\mathfrak{f} = \mathfrak{h} * \mu_{\mathcal{NC}}$.

Theorem 2.2 (Speicher [27])**.** *Let X be a noncommutative random variable with moment generating function $M(t)$ and free cumulant generating function $R(t)$ as in (2.8) and (2.9). If $\mathfrak{m}$ and $\mathfrak{r}$ are two multiplicative functions on the lattice $(\mathcal{NC}_n, \leq)$ such that $\mathfrak{m}(\mathbf{0}_n, \mathbf{1}_n) = m_n$ and $\mathfrak{r}(\mathbf{0}_n, \mathbf{1}_n) = r_n$, then*

$$\begin{cases} \mathfrak{r} &= \mathfrak{m} * \mu_{\mathcal{NC}}, \\ \mathfrak{m} &= \mathfrak{r} * \zeta_{\mathcal{NC}}. \end{cases}$$

The notion of *boolean cumulants* arises from considering the *boolean convolution* of probability measures [29]. Within stochastic differential equations, this family of cumulants is also known as "partial cumulants". The boolean cumulants of X are the coefficients h_n of the ordinary delta series $H(t) = h_1 t + h_2 t^2 + \ldots$ such that

$$M(t) = \frac{1}{1 - H(t)}, \tag{2.12}$$

where $M(t)$ is the same as (2.8). From a combinatorial point of view, the formulae involving moments and boolean cumulants are recovered by defining a convolution $\diamond$ on the multiplicative functions on the lattice of interval partitions (see [34]). This lattice turns out to be isomorphic to the boolean lattice of a n-set, from which the name of boolean convolution has been derived. A partition π of Π_n is said to be an *interval partition* if each block B_i of π is an interval $[a_i, b_i]$ of $[n]$, that is $B_i = [a_i, b_i] = \{x \in [n] \,|\, a_i \leq x \leq b_i\}$, where $a_i, b_i \in [n]$. We denote by $\mathcal{I}_n$ the subset of Π_n of all the interval partitions. The pair $(\mathcal{I}_n, \leq)$, where $\leq$ is the refinement order, is a lattice. The type of each interval $[\sigma, \tau]$ in $\mathcal{I}_n$ is the same as (2.3). Let $\mathfrak{f}$ and $\mathfrak{g}$ be two multiplicative functions on the interval partition lattice. We define the convolution $\mathfrak{h} = \mathfrak{f} \diamond \mathfrak{g}$, which is also multiplicative, by

$$(\mathfrak{f} \diamond \mathfrak{g})(\sigma, \pi) := \sum_{\substack{\sigma \leq \tau \leq \pi \\ \tau \in \mathcal{I}_n}} \mathfrak{f}(\sigma, \tau) \mathfrak{g}(\tau, \pi). \tag{2.13}$$

So if $\mathfrak{h} = \mathfrak{f} \diamond \mathfrak{g}$, then

$$h_n = \sum_{\tau \in \mathcal{I}_n} f_\tau \, g_{\ell(\tau)}. \tag{2.14}$$

Given the power series $H(t)$ and $M(t)$ in (2.12), if $\mathfrak{h}$ and $\mathfrak{m}$ are two multiplicative functions on the lattice of interval partitions, with $\mathfrak{m}(\mathbf{0}_n, \mathbf{1}_n) = m_n$ and $\mathfrak{h}(\mathbf{0}_n, \mathbf{1}_n) = h_n$, then we have

$$\begin{cases} \mathfrak{h} &= \mathfrak{m} \diamond \mu_{\mathcal{I}}, \\ \mathfrak{m} &= \mathfrak{h} \diamond \zeta_{\mathcal{I}}, \end{cases} \tag{2.15}$$

where $\mu_{\mathcal{I}}$ and $\zeta_{\mathcal{I}}$ are the Möbius function and the zeta function on $(\mathcal{I}_n, \leq)$.

Theorem 2.3. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be three multiplicative functions on the lattice $(\mathcal{I}_n, \leq)$ with $f_n = \mathfrak{f}(\mathbf{0}_n, \mathbf{1}_n)$, $g_n = \mathfrak{g}(\mathbf{0}_n, \mathbf{1}_n)$ and $h_n = \mathfrak{h}(\mathbf{0}_n, \mathbf{1}_n)$. If $f(t) = 1 + \sum_{n \geq 1} f_n t^n$, $g(t) = 1 + \sum_{n \geq 1} g_n t^n$ and $h(t) = 1 + \sum_{n \geq 1} h_n t^n$, then*

$$h(t) = f[g(t) - 1] \Longleftrightarrow \mathfrak{h} = \mathfrak{g} \diamond \mathfrak{f}. \tag{2.16}$$

Theorems 2.1 and 2.3 state that the convolutions $\star$ and $\diamond$ express the composition of exponential power series and ordinary power series respectively. So these convolutions are noncommutative. This is not true for the convolution (2.10). In fact, the map $\tau \to \tilde{\tau}$ is an order-reversing bijection such that $sh(\tilde{\tilde{\tau}}) = sh(\tau)$, and by virtue of (2.11) we obtain $\mathfrak{f} * \mathfrak{g} = \mathfrak{g} * \mathfrak{f}$ (see [15] for more details).

3 Symbolic methods for classical, boolean and free cumulants

We start this section recalling the necessary tools of the umbral syntax; main references are [4, 5, 7] and [24].

Let us denote by X the set of variables $X = \{x_1, x_2, \ldots, x_n\}$. A classical umbral calculus consists of the following data: a set $A = \{\alpha, \beta, \ldots\}$, called the *alphabet*, whose elements are named *umbrae*; a linear functional E, called the *evaluation*, defined on the polynomial ring $R[X][A]$ and taking value in $R[X]$ (a ring whose quotient field is of characteristic zero), such that $E[1] = 1$ and $E[x_1^s x_2^m \cdots x_n^t \alpha^i \beta^j \cdots \gamma^k] = x_1^s x_2^m \cdots x_n^t E[\alpha^i]E[\beta^j] \cdots E[\gamma^k]$ (*uncorrelation* property) for all nonnegative integers s, m, t, i, j, k; two special umbrae ε (*augmentation*) and u (*unity*) such that $E[\varepsilon^i] = \delta_{0,i}$, and $E[u^i] = 1$, for $i = 0, 1, 2, \ldots$.

A sequence $(1, a_1, a_2, \ldots)$ of elements of R is *represented* by a scalar umbra α if $E[\alpha^i] = a_i$, for $i = 0, 1, 2, \ldots$. In this case we say that a_i is the i-th moment of α. In the following the powers of an umbra α will be also called moments, if this does not give rise to misunderstandings. A sequence $(1, p_1, p_2, \ldots)$ of elements of $R[X]$, such that p_n is of degree n for all n, is *represented* by a polynomial umbra ψ if $E[\psi^i] = p_i$ for $i = 0, 1, 2, \ldots$.

The factorial moments of a scalar umbra α are the elements $a_{(n)} \in R$ such that $a_{(0)} = 1$ and $a_{(n)} = E[(\alpha)_n] = E[\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)]$ for all $n \geq 1$. The polynomial $(\alpha)_n$ is an umbral polynomial. More general, an umbral polynomial is a polynomial $p \in R[X][A]$. The support of p is the set of all umbrae occurring in p. If p and q are two umbral polynomials, then p and q are uncorrelated if and only if their supports are disjoint. Moreover the polynomials p and q are *umbrally equivalent* if and only if $E[p] = E[q]$, in symbols $p \simeq q$.

Two umbrae are *similar*, in symbols $\alpha \equiv \gamma$, if and only if $E[\alpha^n] = E[\gamma^n]$ for all n. So, each sequence is represented by infinite many uncorrelated (i.e. distinct) umbrae. In the following, we shall denote by $\alpha, \alpha', \alpha'', \ldots$ a family of similar and uncorrelated umbrae. We extend the alphabet A with the so-called *auxiliary umbrae* obtained via operations among similar umbrae. This leads to the construction of a *saturated umbral calculus* in which auxiliary umbrae are treated as elements of a suitable alphabet. For example, the symbol $n.\alpha$ denotes an auxiliary umbra similar to the sum of n distinct umbrae, each one similar to the umbra α, that is $n.\alpha \equiv \alpha' + \alpha'' + \cdots + \alpha'''$. We remark that $n.(\alpha+\gamma) \equiv n.\alpha + n.\gamma$, and $\alpha.n \equiv n\alpha$, for every umbrae α and γ and for all nonnegative n.

The *generating function* $f(\alpha, t)$ of α is $f(\alpha, t) = 1 + \sum_{n \geq 1} a_n \frac{t^n}{n!}$. A formal construction is given in [4]. In particular, we have $f(\varepsilon, t) = 1, f(u, t) = e^t$ and $f(n.\alpha, t) = f(\alpha, t)^n$.

Special umbrae are the *Bell umbra* β and the *singleton umbra* χ. The Bell umbra β has moments given by the Bell numbers so that $f(\beta, t) = e^{e^t - 1}$. The singleton umbra χ has moments $E[\chi^n] = 1$, if $n = 0, 1$, and $E[\chi^n] = 0$ otherwise, so that $f(\chi, t) = 1 + t$. The derivative umbra α_D of an umbra α is the umbra whose moments are $(\alpha_D)^n \simeq \partial_\alpha \alpha^n \simeq n\alpha^{n-1}$ for $n = 1, 2, \ldots$ [7]. We have $f(\alpha_D, t) = 1 + t\, f(\alpha, t)$, and in particular $E[\alpha_D] = 1$.

Given an umbra α, the umbra denoted by $-1.\alpha$ is uniquely determined (up to similarity) by the condition $\alpha + (-1.\alpha) \equiv \varepsilon$. The umbra $-1.\alpha$ is said to be the *inverse* of α. Its generating function is $f(\alpha, t)^{-1}$. Then, the umbra $-n.\alpha$ is the inverse of $n.\alpha$ and $f(-n.\alpha) = f(\alpha, t)^{-n}$.

A generalization of the auxiliary umbra $n.\alpha$ (*dot operation*) is introduced when n is replaced by an umbra γ. We denote by $\gamma.\alpha$ an auxiliary umbra with moments

$$(\gamma.\alpha)^n \simeq \sum_{\lambda \vdash n} \mathrm{d}_\lambda \, (\gamma)_{\ell(\lambda)} \, (\alpha')^{\lambda_1} (\alpha'')^{\lambda_2} \cdots (\alpha''')^{\lambda_{\ell(\lambda)}}, \tag{3.1}$$

where the sum ranges over all the partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n, where $l = \ell(\lambda)$ is the number of positive parts of λ, and $\mathrm{d}_\lambda = \binom{n}{\lambda}/[m(\lambda)_1! m(\lambda)_2! \cdots m(\lambda)_n!]$, where $m(\lambda)_i$ denotes the number of parts of λ equal to i. From now on, we denote $(\alpha')^{\lambda_1}(\alpha'')^{\lambda_2} \cdots (\alpha''')^{\lambda_{\ell(\lambda)}}$ and $m(\lambda)_1!\ m(\lambda)_2! \cdots\ m(\lambda)_n!$ by α_λ and $m(\lambda)!$ respectively. The generating function of $\gamma.\alpha$ is $f(\gamma.\alpha, t) = f[\gamma, \log f(\alpha, t)]$. In particular, we have $\alpha.u \equiv u.\alpha \equiv \alpha$ for all α in A, and $\chi.\beta \equiv \beta.\chi \equiv u$.

The composition of $f(\gamma, t)$ and $f(\alpha, t)$ is the generating function of $\gamma.\beta.\alpha$, $f(\gamma.\beta.\alpha, t) = f[\gamma, f(\alpha, t) - 1]$. The umbra $\gamma.\beta.\alpha$ is said to be the *composition umbra* of γ and α. The moments of $\gamma.\beta.\alpha$ are

$$(\gamma.\beta.\alpha)^n \simeq \sum_{\lambda \vdash n} \mathrm{d}_\lambda \, \gamma^{\ell(\lambda)} \, \alpha_\lambda. \tag{3.2}$$

In particular $(\gamma.\beta).\alpha \equiv \gamma.(\beta.\alpha)$ and

$$\gamma_D.\beta.\alpha_D \equiv (\alpha + \gamma.\beta.\alpha_D)_D. \tag{3.3}$$

Finally the symbol $\alpha^{<-1>}$ denotes an umbra whose generating function is the compositional inverse $f^{<-1>}(\alpha, t)$ of $f(\alpha, t)$. Such an umbra is uniquely determined (up to similarity) by the relations $\alpha.\beta.\alpha^{<-1>} \equiv \alpha^{<-1>}.\beta.\alpha \equiv \chi$.

Theorem 3.1. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be three multiplicative functions on the lattice $(\Pi_n, \leq)$. If α, γ and ω are three umbrae with moments $\alpha^n \simeq f_n$, $\gamma^n \simeq g_n$, and $\omega^n \simeq h_n$, then we have $\mathfrak{h} = \mathfrak{f} \star \mathfrak{g} \Longleftrightarrow \omega \equiv \gamma.\beta.\alpha$.*

Proof. Note that in the equivalence (3.2), d_λ counts the number of partitions of Π_n of shape λ so that $(\gamma.\beta.\alpha)^n \simeq \sum_{\pi \in \Pi_n} \gamma^{\ell(\pi)} \alpha_\pi$, where $\pi = \{B_1, B_2, \ldots, B_l\} \in \Pi_n$ and we set $\alpha_\pi = (\alpha')^{|B_1|}(\alpha')^{|B_2|} \cdots (\alpha''')^{|B_l|}$. The result follows by comparing this last equivalence with (2.5). ∎

Remark 3.1. Theorem 3.1 states that multiplicative functions on $(\Pi_n, \leq)$ can be thought as umbrae, and the convolution $\star$ of two multiplicative functions corresponds to a composition umbra. The umbra $\chi.\chi$ is the umbral counterpart of the Möbius function μ. In fact, $f(\chi.\chi, t) = 1 + \log(1 + t)$ so that $(\chi.\chi)^n \simeq (-1)^{n-1}(n-1)! = \mu_n$. In addition, the umbral counterparts of the Zeta function ζ and the Delta function δ are respectively the unity umbra u and the singleton umbra χ. Hence the relations among multiplicative functions can be interpreted in the umbral syntax. For example, we have $\delta = \mu \star \zeta = \zeta \star \mu$ similarly to $\chi \equiv (\chi.\chi).\beta.u \equiv u.\beta.(\chi.\chi)$. Furthermore, an umbra α has a compositional inverse $\alpha^{<-1>}$ if and only if $E[\alpha] = a_1 \neq 0$. In analogy, a multiplicative function $\mathfrak{f}$ has an inverse respect to the convolution $\star$ if and only if $f_1 \neq 0$.

3.1 Classical cumulants

Classical cumulants have been studied via the classical umbral calculus in [4]. Here we state a new theorem concerning a parametrization of classical cumulants and moments. This parametrization represents the trait d'union with the umbral theory of boolean and free cumulants, that we introduce later on.

For each umbra α, the α-*cumulant umbra* is an umbra, denoted by κ_α, similar to $\chi . \alpha$. In particular we have $\kappa_\alpha \equiv (\chi . \chi) . \beta . \alpha$, and from Theorem 3.1, this similarity is the umbral version of (2.7), if we assume $F(t) = f(\alpha, t)$ and $C(t) = f(\kappa_\alpha, t)$. The formulae expressing the cumulants $\kappa_\alpha^n \simeq c_n$ in terms of their moments $\alpha^n \simeq m_n$ are easily recovered from (3.2):

$$c_n = \sum_{\lambda \vdash n} \mathrm{d}_\lambda (-1)^{\ell(\lambda)-1} (\ell(\lambda) - 1)!\, m_\lambda, \tag{3.4}$$

where $m_\lambda = E[\alpha_\lambda]$, so that $m_\lambda = m_{\lambda_1} m_{\lambda_2} \cdots m_{\lambda_{\ell(\lambda)}}$. The relation $\kappa_\alpha \equiv \chi . \alpha$ is inverted by $\alpha \equiv \beta . \kappa_\alpha$ by which we have $m_n = \sum_{\lambda \vdash n} \mathrm{d}_\lambda c_\lambda$. In particular $m_n = Y_n(c_1, c_2, \ldots, c_n)$, where Y_n is the complete Bell exponential polynomial. The Bell umbra β is the unique umbra, up to similarity, having the sequence of cumulants $\{1\}_{n \geq 1}$, being $\kappa_\beta \equiv \chi . \beta \equiv u$. Moreover, we have $\beta^n \simeq \mathcal{B}_n = |\Pi_n|$.

Compared with moments, cumulants are special sequences because of their properties of additivity and homogeneity. The following theorem states these properties in umbral terms. Recall that the *disjoint sum* of the umbrae α and γ is an auxiliary umbra such that $(\alpha \dot{+} \gamma)^n \simeq \alpha^n + \gamma^n$.

Theorem 3.2. *For all umbrae $\alpha, \gamma \in A$ and for all $c \in R$, the following properties hold:*

$$\begin{array}{lcll} \kappa_{\alpha+\gamma} & \equiv & \kappa_\alpha \dot{+} \kappa_\gamma & \textit{(additivity property)}; \\ \kappa_{\alpha+cu} & \equiv & \kappa_\alpha \dot{+} c\chi, & \textit{(semi-invariance for translation property)}; \\ \kappa_{c\alpha} & \equiv & c\kappa_\alpha. & \textit{(homogeneity property)}. \end{array}$$

Theorem 3.3 (Parametrization)**.** *If κ_α is the α-cumulant umbra, then*

$$\alpha^n \simeq \kappa_\alpha (\kappa_\alpha + \alpha)^{n-1} \quad \textit{and} \quad \kappa_\alpha^n \simeq \alpha(\alpha - 1.\alpha)^{n-1}. \tag{3.5}$$

Proof. Sine for any umbra $\alpha \in A$ we have $(\beta . \alpha)^n \simeq \alpha(\alpha + \beta . \alpha)^{n-1}$, see [5], we obtain the former in equivalence (3.5) replacing α by $\kappa_\alpha \equiv \chi . \alpha$. The latter can be proved as follows. We have

$$\alpha(\alpha - 1.\alpha)^{n-1} \simeq \sum_{\substack{1 \leq k \leq n \\ \lambda \vdash n-k}} \binom{n-1}{k-1} \mathrm{d}_\lambda (-1)_{\ell(\lambda)}\, \alpha^k\, \alpha_\lambda,$$

and, setting $\lambda \leftarrow \lambda \cup k$ (i.e. a part equal to k is joined with λ), we recover equation (3.4). □

3.2 Boolean cumulants

The notion of boolean cumulant requires the connection between umbrae and ordinary generating functions. We obtain this connection simply by multiplying an umbra by the boolean unity umbra $\bar{u}$, whose moments are $\bar{u}^n \simeq n!$. In fact, if α has moments $\alpha^n \simeq a_n$, the umbra $\bar{\alpha} \equiv \bar{u}\alpha$ has generating function $f(\bar{\alpha}, t) = 1 + a_1 t + a_2 t^2 + \cdots$. Note that, $\alpha \equiv \gamma$ if and only if $\bar{\alpha} \equiv \bar{\gamma}$. The following theorem is the analogous of Theorem 3.1 for the lattice $(\mathcal{I}_n, \leq)$.

Theorem 3.4. *Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ be three multiplicative functions on the lattice $(\mathcal{I}_n, \leq)$. If α, γ and ω are three umbrae with moments $\alpha^n \simeq f_n$, $\gamma^n \simeq g_n$, and $\omega^n \simeq h_n$, then we have $\mathfrak{h} = \mathfrak{f} \diamond \mathfrak{g} \Longleftrightarrow \bar{\omega} \equiv \bar{\gamma}.\beta.\bar{\alpha}$.*

Proof. Since $\bar{\alpha}_\lambda \simeq \lambda! \alpha_\lambda$, from (3.2) the moments h_n of $\bar{\gamma}.\beta.\bar{\alpha}$ are

$$h_n = \sum_{\lambda \vdash n} \frac{\ell(\lambda)!}{m(\lambda)!} \, g_{\ell(\lambda)} \, f_\lambda. \tag{3.6}$$

But $\ell(\lambda)!/m(\lambda)!$ is the number of interval partitions of shape λ, so that $\mathfrak{h} = \mathfrak{f} \diamond \mathfrak{g}$. $\square$

Since the α-cumulant umbra is such that $\kappa_\alpha \equiv \chi.\alpha \equiv u^{<-1>}.\beta.\alpha$, we define the *$\alpha$-boolean cumulant umbra* by taking the "bar version"of the previous similarity.

Definition 3.1. The *α-boolean cumulant umbra* is the umbra η_α such that $\bar{\eta}_\alpha \equiv \bar{u}^{<-1>}.\beta.\bar{\alpha}$.

If h_n denotes the n-th moment of α-boolean cumulant umbra, from (3.6) we have

$$h_n = \sum_{\lambda \vdash n} \frac{\ell(\lambda)!}{m(\lambda)!} \, (-1)^{\ell(\lambda)-1} \, m_\lambda. \tag{3.7}$$

Definition 3.1 is based on the following proposition that states that h_n in (3.7) are the same as the coefficients of $H(t)$ in (2.12).

Proposition 3.5. *If η_α is the α-boolean cumulant umbra, then $f(\bar{\eta}_\alpha, t) = 2 - f(\bar{\alpha}, t)^{-1}$ and $f(\bar{\alpha}, t) = (1 - [f(\bar{\eta}_\alpha, t) - 1])^{-1}$.*

Proof. We have $f(\bar{u}, t) = (1-t)^{-1}$ so $\bar{u} \equiv -1.-\chi$ and $-1.\bar{u} \equiv -\chi$. Moreover, we have $\bar{u}^{<-1>}.\beta \equiv -\chi.-\beta$, since $-\chi.-\beta.\bar{u} \equiv -\chi.-\beta.-1.-\chi \equiv -\chi.\beta.-1.-1.-\chi \equiv -\chi.\beta.-\chi \equiv \chi$, this because $-\chi$ is the compositional inverse of itself and $\bar{u}^{<-1>}.\beta.\bar{u} \equiv \chi$. Therefore we have $f(\bar{u}^{<-1>}.\beta, t) = 2 - e^{-t}$, by which the results follow. $\square$

Theorem 3.6 (Boolean Inversion Theorem)**.** *If η_α is the α-boolean cumulant, then $\bar{\alpha} \equiv \bar{u}.\beta.\bar{\eta}_\alpha$.*

Proof. The result follows from (3.1) by left dot product of both sides with $\bar{u}.\beta$. $\square$

The unique umbra (up to similarity) having sequence of boolean cumulants $\{1\}_{n\geq 1}$ is an umbra α such that $\bar{\alpha} \equiv \bar{u}.\beta.\bar{u} \equiv (2\bar{u})_D$. Since $\bar{\alpha}^n \simeq (2\bar{u})_D^n \simeq n(2\bar{u})^{n-1} \simeq n! 2^{n-1}$, then such an umbra has moments 2^{n-1}, that is the number of interval partitions $\mathcal{I}_n$. The following theorem gives a parametrization of boolean cumulants and moments. The proof is omitted.

Theorem 3.7 (Boolean parametrization)**.** *If η_α is the α-boolean cumulant umbra, then*

$$\bar{\alpha}^n \simeq \bar{\eta}_\alpha (\bar{\eta}_\alpha + 2.\bar{\alpha})^{n-1} \quad \textit{and} \quad \bar{\eta}_\alpha^n \simeq \bar{\alpha}(\bar{\alpha} - 2.\bar{\alpha})^{n-1}. \tag{3.8}$$

Similarly to the α-cumulant umbra, we can state additivity and homogeneity properties also for the α-boolean cumulant umbra.

Theorem 3.8 (Homogeneity property)**.** *If η_α is the α-boolean cumulant umbra, then $\eta_{c\alpha} \equiv c\eta_\alpha$.*

Proof. Since $\bar{u}^{<-1>}\boldsymbol{.}\beta\boldsymbol{.}c\bar{\alpha} \equiv c(\bar{u}^{<-1>}\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha})$ and $\overline{c\alpha} \equiv c\bar{\alpha}$, then from (3.1) we have $\bar{\eta}_{c\alpha} \equiv c\bar{\eta}_\alpha$ and finally $\eta_{c\alpha} \equiv c\eta_\alpha$. □

Theorem 3.9 (Additivity property)**.** *If η_α, η_γ and η_ξ are the boolean cumulant umbrae of α, γ and ξ respectively, then*

$$\eta_\xi \equiv \eta_\alpha \dot{+} \eta_\gamma \Leftrightarrow -1\boldsymbol{.}\bar{\xi} \equiv -1\boldsymbol{.}\bar{\alpha} \dot{+} -1\boldsymbol{.}\bar{\gamma}. \tag{3.9}$$

Proof. Let $-1\boldsymbol{.}\bar{\xi} \equiv -1\boldsymbol{.}\bar{\alpha} \dot{+} -1\boldsymbol{.}\bar{\gamma}$. Due to $-1\boldsymbol{.}\bar{\alpha} \equiv (-\chi\boldsymbol{.}\beta)\boldsymbol{.}\bar{\eta}_\alpha$, we have $-\chi\boldsymbol{.}\beta\boldsymbol{.}\bar{\eta}_\xi \equiv -\chi\boldsymbol{.}\beta\boldsymbol{.}\bar{\eta}_\alpha \dot{+} -\chi\boldsymbol{.}\beta\boldsymbol{.}\bar{\eta}_\gamma \equiv -\chi\boldsymbol{.}(\beta\boldsymbol{.}\bar{\eta}_\alpha + \beta\boldsymbol{.}\bar{\eta}_\gamma)$ so that $\beta\boldsymbol{.}\bar{\eta}_\xi \equiv \beta\boldsymbol{.}\bar{\eta}_\alpha + \beta\boldsymbol{.}\bar{\eta}_\gamma$. Taking the left product of both sides for χ, the result follows. □

We define the *boolean convolution* of α and γ to be the umbra $\alpha \uplus \gamma$ such that $\overline{\alpha \uplus \gamma} \equiv -1\boldsymbol{.}(-1\boldsymbol{.}\bar{\alpha} \dot{+} -1\boldsymbol{.}\bar{\gamma})$. Theorem 3.9 assures this is the unique convolution linearized by boolean cumulants. In this way, from (3.9) we express the additivity property of the boolean cumulant umbra with respect to the boolean convolution as follows $\eta_{\alpha \uplus \gamma} \equiv \eta_\alpha \dot{+} \eta_\gamma$. Since $\bar{\eta}_{cu} \equiv (\bar{u}^{<-1>}\boldsymbol{.}\beta\boldsymbol{.}\bar{u})\boldsymbol{.}c \equiv \chi\boldsymbol{.}c \equiv c\chi$, from (3.9) we have $\eta_{\alpha \uplus cu} \equiv \eta_\alpha \dot{+} c\,\chi$ that gives the *semi-invariance property.*

Once more, note the analogy with the convolution linearized by classical cumulants, that is $\alpha + \gamma \equiv -1\boldsymbol{.}(-1\boldsymbol{.}\alpha + -1\boldsymbol{.}\gamma)$.

3.3 Free cumulants

Definition 3.2 (Free cumulant umbra)**.** For a given umbra α, the unique umbra $\mathfrak{K}_\alpha$ (up to similarity) such that $(-1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)_D \equiv \bar{\alpha}_D^{<-1>}$ is called the *free cumulant umbra* of α.

The moments of $\mathfrak{K}_\alpha$ will be called *free cumulants* of the umbra α. Definition 3.2 is based on the following proposition that states that the free cumulants of an umbra α, whose moments are m_n, are the coefficients of $R(t)$ in (2.9).

Proposition 3.10. *If $\mathfrak{K}_\alpha$ is the free cumulant umbra of α, then $\bar{\alpha} \equiv \bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D$.*

Proof. By using Definition 3.2, we have $\bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>} \equiv \bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}(-1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)_D$ and via (3.3) we obtain $\bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}(-1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)_D \equiv (\bar{\alpha} - 1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D)_D$. As $\bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>} \equiv \chi$, then $\bar{\alpha} - 1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D \equiv \varepsilon \Leftrightarrow \bar{\alpha} \equiv \bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D$. □

Proposition 3.10 gives (2.9), if we set $f(\bar{\alpha}, t) = M(t)$, $f(\bar{\mathfrak{K}}_\alpha, t) = R(t)$ and observe that $f(\bar{\alpha}_D, t) = 1 + t f(\bar{\alpha}, t)$.

Theorem 3.11. *If $\mathfrak{K}_\alpha$ is the free cumulant umbra of α, then $\bar{\mathfrak{K}}_\alpha \equiv \bar{\alpha}\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>}$ and $\bar{\alpha} \equiv \bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}(-1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)_D^{<-1>}$.*

Proof. The former similarity follows from Theorem 3.10 as we have $\bar{\alpha}\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>} \equiv \bar{\mathfrak{K}}_\alpha\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>}$ and $\bar{\alpha}_D\boldsymbol{.}\beta\boldsymbol{.}\bar{\alpha}_D^{<-1>} \equiv \chi$. The latter similarity follows from Definition 3.2, by observing that $\beta\boldsymbol{.}\bar{\alpha}_D \equiv \beta\boldsymbol{.}(-1\boldsymbol{.}\bar{\mathfrak{K}}_\alpha)_D^{<-1>}$. □

A parametrization of free cumulants and moments can be constructed by using the so-called *umbral Abel polynomials* [7]

$$A_n(x,\alpha) \simeq \begin{cases} u & \text{if } n=0, \\ x(x-n.\alpha)^{n-1} & \text{if } n \geq 1. \end{cases} \tag{3.10}$$

Note that if the umbra α is replaced by the umbra $a.u$, with u the unity umbra and $a \in R$, then $E[A_n(x,a.u)] = A_n(x,a)$ for all $n \geq 1$, where $\{A_n(x,a)\}$ denotes the Abel polynomial sequence, $A_n(x,a) = x(x-na)^{n-1}$.

Theorem 3.12 (Free parametrization). *If $\mathfrak{K}_\alpha$ is the free cumulant umbra of α, then*

$$\bar{\alpha}^n \simeq \bar{\mathfrak{K}}_\alpha(\bar{\mathfrak{K}}_\alpha + n.\bar{\mathfrak{K}}_\alpha)^{n-1} \quad \textit{and} \quad \bar{\mathfrak{K}}_\alpha^n \simeq \bar{\alpha}(\bar{\alpha} - n.\bar{\alpha})^{n-1}. \tag{3.11}$$

Proof. In [7], the following equivalence $A_n(x,\alpha) \simeq (x.\beta.\alpha_D^{<-1>})^n$, is proved for all $n \geq 1$, so that $A_n(\bar{\mathfrak{K}}_\alpha, -1.\bar{\mathfrak{K}}_\alpha) \simeq [\bar{\mathfrak{K}}_\alpha.\beta.(-1.\bar{\mathfrak{K}}_\alpha)_D^{<-1>}]^n$. From the latter similarity in Theorem 3.11, we have $\bar{\alpha}^n \simeq A_n(\bar{\mathfrak{K}}_\alpha, -1.\bar{\mathfrak{K}}_\alpha) \simeq \bar{\mathfrak{K}}_\alpha(\bar{\mathfrak{K}}_\alpha - n.(-1.\bar{\mathfrak{K}}_\alpha))^{n-1}$ by which the former equivalence (3.11) follows. From the latter similarity of Theorem 3.11, we have $\bar{\mathfrak{K}}_\alpha^n \simeq (\bar{\alpha}.\beta.\bar{\alpha}_D^{<-1>})^n \simeq A_n(\bar{\alpha}, \bar{\alpha})$. The latter equivalence (3.11) follows by replacing x with $\bar{\alpha}$ in (3.10). $\square$

Corollary 3.13. *With $\{r_n\}_{n\geq 1}$ and $\{m_n\}_{n \geq 1}$ given in (2.9), we have $m_n = \sum_{\lambda \vdash n} (n)_{\ell(\lambda)-1} r_\lambda / m(\lambda)!$ and $r_n = \sum_{\lambda \vdash n} (-n)_{\ell(\lambda)-1} m_\lambda / m(\lambda)!$.*

The Abel parametrization allows us to prove the homogeneity property of the free cumulant umbra, since for any $c \in R$ and for any $\alpha \in A$ we have $-n.(c\alpha) \equiv c(-n.\alpha)$, see [4].

Theorem 3.14 (Homogeneity property). *If $\mathfrak{K}_\alpha$ is the free cumulant umbra of α, then we have $\mathfrak{K}_{c\alpha} \equiv c\mathfrak{K}_\alpha$, for all $c \in R$.*

In order to prove the additivity property of the free cumulant umbra we introduce an umbra δ_P such that $(\delta_P)^n \simeq \delta^{n+1}/(n+1)$ for $n = 1, 2, \ldots$. Thanks to this device, Definition 3.2 gives $\bar{\mathfrak{K}}_\alpha \equiv -1.(\bar{\alpha}_D^{<-1>})_P$. Denote by $\mathfrak{L}_{\bar{\alpha}}$ the umbra $(\bar{\alpha}_D^{<-1>})_P$.

Consider the multiplicative function $\mathfrak{f}$ on the noncrossing partition lattice defined by $\alpha^{n-1} \simeq f_n$. Note that $\mathfrak{f}$ is unital, that is $f_1 = 1$. The generating function $f(\mathfrak{L}_{\bar{\alpha}}, t)$ is exactly the Fourier transform $(\mathcal{F}\mathfrak{f})(t)$ considered by Nica and Speicher [15]. In particular, being $[\mathcal{F}(\mathfrak{f} * \mathfrak{g})](t) = (\mathcal{F}\mathfrak{f})(t)(\mathcal{F}\mathfrak{g})(t)$ for all $\mathfrak{f}$ and $\mathfrak{g}$ unital, if $\gamma^{n-1} \simeq g_n$ and $\omega^{n-1} \simeq h_n$, then we obtain $\mathfrak{h} = \mathfrak{f} * \mathfrak{g} \Leftrightarrow \mathfrak{L}_{\bar{\omega}} \equiv \mathfrak{L}_{\bar{\alpha}} + \mathfrak{L}_{\bar{\gamma}}$. This way, an analog of Theorem 3.1 and Theorem 3.4 for unital multiplicative functions on the noncrossing partitions lattice is given.

Theorem 3.15 (Additivity property). *If $\mathfrak{K}_\alpha, \mathfrak{K}_\gamma$ and $\mathfrak{K}_\xi$ are the free cumulant umbrae of α, γ and ξ respectively, then*

$$\mathfrak{K}_\xi \equiv \mathfrak{K}_\alpha \dot{+} \mathfrak{K}_\gamma \Leftrightarrow -1.\mathfrak{L}_{\bar{\xi}} \equiv -1.\mathfrak{L}_{\bar{\alpha}} \dot{+} -1.\mathfrak{L}_{\bar{\gamma}}. \tag{3.12}$$

Remark 3.2 (Connection between boolean and free convolution)**.** Write $\bar{\alpha}_D{}^{<-1>}{}_P$ for $(\bar{\alpha}_D^{<-1>})_P$. By virtue of Theorem 3.15, the *free convolution* $\alpha \boxplus \gamma$ of α and γ has to be defined by $\overline{\alpha \boxplus \gamma}_D{}^{<-1>}{}_P \equiv -1.[-1.\bar{\alpha}_D{}^{<-1>}{}_P \dot{+} -1.\bar{\gamma}_D{}^{<-1>}{}_P]$, so that $\mathfrak{K}_{\alpha \boxplus \gamma} \equiv \mathfrak{K}_\alpha \dot{+} \mathfrak{K}_\gamma$. Moreover, thanks to the umbra $\mathfrak{L}_{\bar{\alpha}}$ we have

$$\mathfrak{L}_{\overline{\alpha \boxplus \gamma}} \equiv \mathfrak{L}_{\bar{\alpha}} \uplus \mathfrak{L}_{\bar{\gamma}},$$

which gives the connection between boolean and free convolution.

Semi-invariance property can be proved by observing that $\mathfrak{K}_{\alpha \boxplus cu} \equiv \mathfrak{K}_\alpha \dot{+} c \mathfrak{K}_u$ so that $\mathfrak{K}_{\alpha \boxplus cu} \equiv \mathfrak{K}_\alpha \dot{+} c\chi$, being $\bar{\mathfrak{K}}_u \equiv \bar{u}.\beta.\bar{u}_D^{<-1>} \equiv \chi$.

Definition 3.3 (Catalan umbra)**.** The *Catalan umbra* is the unique umbra ς such that $\mathfrak{K}_\varsigma \equiv u$, that is $\bar{\varsigma} \equiv \bar{u}.\beta.(-1.\bar{u})_D^{<-1>}$.

As it is well known, Catalan numbers count the noncrossing partitions of a set. So in the free setting, the Catalan umbra plays the same role played by the Bell umbra β in the classical framework.

Proposition 3.16 (Catalan numbers)**.** *If $\mathcal{C}_n$ is the n-th Catalan number, then $\varsigma^n \simeq \mathcal{C}_n$.*

Proof. We have $n!\varsigma^n \simeq \bar{\varsigma}^n \simeq n! \sum_{\mu \vdash n} (n)_{\ell(\mu)-1}/m(\mu)!$. As well known (see for instance [10]), $(n)_{\ell(\mu)-1}/m(\mu)!$ is the number of noncrossing partitions of shape μ and $|\mathcal{NC}_n| = \mathcal{C}_n$, so that $\varsigma^n \simeq |\mathcal{NC}_n| = \mathcal{C}_n$. ☐

4 Volume polynomial

In this section we provide an explicit connection between free cumulants and parking functions via volume polynomials. Moreover we prove that in the free setting the volume polynomials play the same role played by the complete Bell exponential polynomials in the classical settings.

Recall that a *parking function* of length n is a sequence $(p_1, p_2, \ldots, p_n)$ of n positive integers, whose nondecreasing arrangement $(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$ is such that $p_{i_j} \leq j$. We denote by park(n) the set of all parking functions of length n; its cardinality is $(n+1)^{n-1}$. The symmetric group $\mathfrak{S}_n$ acts on the set park(n) by permuting the entries of parking functions. As well known, the number of orbits in park$(n)^{\mathfrak{S}_n}$ is equal to the n-th Catalan number $\mathcal{C}_n$. It is also known that a map τ can be defined from park(n) to $\mathcal{NC}_n$ whose restriction to park$(n)^{\mathfrak{S}_n}$ is bijective. The n-volume polynomial $V_n(x_1, x_2, \ldots, x_n)$, introduced by Pitman and Stanley [19], is the following homogeneous polynomial of degree n:

$$V_n(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{p \in \text{park}(n)} x_p, \tag{4.1}$$

where $x_p = x_{p_1} x_{p_2} \cdots x_{p_n}$ whenever $p = (p_1, p_2, \ldots, p_n)$. For each $p \in$ park(n) let $m(p) = (m_1, m_2, \ldots, m_n)$ be the vector of the multiplicities of p, that is $m_j =$

$|\{i \,|\, p_i = j\}|$. If λ is a partition of n, then we say that the parking function p is of type λ if the nonzero entries of $m(p)$ consists of a rearrangement of the parts of λ. The orbit $\mathcal{O}_p = \{\omega(p) \,|\, \omega \in \mathfrak{S}_n\}$ of a parking function of type λ has cardinality $n!/\lambda!$. The map τ has the following property: p is of type λ if and only if $\tau(p) = \{B_1, B_2, \ldots, B_l\}$ is of shape λ. Hence, the polynomial $V_n(x_1, x_2, \ldots, x_n)$ can be written as

$$V_n(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash n} \frac{1}{\lambda!} \frac{(n)_{\ell(\lambda)-1}}{m(\lambda)!} x^{\lambda}, \tag{4.2}$$

being $x^{\lambda} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$. In particular when x_i are replaced by similar and uncorrelated umbrae we have $n! V_n(\alpha', \alpha'', \ldots, \alpha''') \simeq \alpha(\alpha + n\boldsymbol{.}\alpha)^{n-1}$, for all $\alpha \in A$ (see [18]). By using this last result and Theorem 3.12, the following theorem provides an explicit connection between free cumulants and parking functions.

Theorem 4.1. *Let α be an umbra and let $\mathfrak{K}_{\bar{\alpha}}$ be its free cumulant umbra. If $\mathfrak{K}'$, $\mathfrak{K}''$,..., $\mathfrak{K}'''$ are n uncorrelated umbrae similar to $\mathfrak{K}_{\alpha}$ and $V_n(x_1, x_2, \ldots, x_n)$ is the n-volume polynomial (4.1), then $\bar{\alpha}^n \simeq V_n(\bar{\mathfrak{K}}', \bar{\mathfrak{K}}', \ldots, \bar{\mathfrak{K}}''')$.*

Corollary 4.2. *If ς is the Catalan umbra and $u', u'', \ldots, u'''$ are uncorrelated umbrae similar to the unity u, then $\bar{\varsigma}^n \simeq V_n(\bar{u}', \bar{u}'', \ldots, \bar{u}''')$, or equivalently $n!\mathcal{C}_n = E[\bar{u}(\bar{u} + n\boldsymbol{.}\bar{u})^{n-1}]$.*

Observe that, from (4.1) we have $n! V_n(x_1, x_2, \ldots, x_n) = \sum_{p \in \mathrm{park}(n)} x_p$. If we restrict the sum to the quotient $\mathrm{park}(n)^{\mathfrak{S}_n}$ (i.e. if we take only a parking function per orbit) we obtain polynomials $R_n(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash n} (n)_{\ell(\lambda)-1} x_p / m(\lambda)!$ such that $R_n(\mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}''') \simeq m_n$. Thanks to the parametrization given in Theorem 3.3 and Theorem 3.7 we can also construct polynomials $C_n(x_1, x_2, \ldots, x_n)$ and $H_n(x_1, x_2, \ldots, x_n)$ such that $m_n = C_n(\kappa', \kappa'', \ldots, \kappa''') = H_n(\eta', \eta'', \ldots, \eta''')$, $\kappa', \kappa'', \ldots, \kappa'''$ and $\eta', \eta'', \ldots, \eta'''$ being uncorrelated umbrae similar to κ_{α} and η_{α} respectively. This will be done in the next section for a more general class of cumulants.

Finally, since we have $E[C_n(\kappa', \kappa'', \ldots, \kappa''')] = Y_n(c_1, c_2, \ldots, c_n)$, then the analog of the complete Bell polynomials in the boolean and free case are the polynomials $E[H_n(\eta', \eta'', \ldots, \eta''')]$ and $E[R_n(\mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}''')]$ respectively.

5 Linear cumulants and Abel polynomials

Let $\{g_n\}_{n \geq 1}$ be a sequence of nonnegative integers represented by an umbra γ. Let us define the *generalized Abel polynomials* as the umbral polynomials $A_n^{(\gamma)}(\delta, \alpha)$ such that $A_n^{(\gamma)}(\delta, \alpha) \simeq \delta(\delta - g_n\boldsymbol{.}\alpha)^{n-1}$ for $n \geq 1$. In particular, when $\alpha \equiv \delta$ we will write $A_n^{(\gamma)}(\alpha)$ instead of $A_n^{(\gamma)}(\alpha, \alpha)$. It can be shown that (see [18], Theorem 3.1)

$$A_n^{(\gamma)}(\alpha) \simeq \sum_{\lambda \vdash n} \mathrm{d}_{\lambda} (-g_n)_{\ell(\lambda)-1} (\alpha')^{\lambda_1} (\alpha'')^{\lambda_2} \cdots (\alpha''')^{\lambda_{\ell(\lambda)}}. \tag{5.1}$$

Generalized Abel polynomials allow us to express classical, boolean and free cumulants in terms of moments. Indeed for the classical cumulants from Theorem 3.3 we have $\kappa_\alpha^n \simeq A_n^{(u)}(\alpha)$, since the sequence $\{1\}_{n\geq 1}$ is represented by the unity umbra u. Since the sequence $\{2\}_{n\geq 1}$ is represented by the umbra $2{\bf .}u$, from Theorem 3.7 we have $\bar{\eta}_\alpha^n \simeq A_n^{(2.u)}(\bar{\alpha})$ for the boolean cumulants. Since the sequence $\{n\}_{n\geq 1}$ is represented by the umbra u_D, from Theorem 3.12 we have $\bar{\mathfrak{K}}_\alpha^n \simeq A_n^{(u_D)}(\bar{\alpha})$ for the free cumulants. In this section, by using generalized Abel polynomials, we show how to construct a more general family of cumulants possessing the additivity, homogeneity and semi-invariance properties. To the best of our knowledge, a previous attempt to give a unifying approach to cumulants families was given in [1], but the boolean case seems not fit in.

Definition 5.1. [Cumulant umbrae] The umbra $\mathfrak{K}_{\gamma,\alpha}$ such that $\mathfrak{K}_{\gamma,\alpha}^n \simeq A_n^{(\gamma)}(\alpha)$ for all $n \geq 1$ is called the *cumulant umbra* of α induced by the umbra γ.

Let $\mathbf{a} = (a_n)_{n\geq 1}$ and $\mathbf{g} = (g_n)_{n\geq 1}$ be the sequences of moments of α and γ respectively. Then the n-th *cumulant* of α induced by γ is $c_n(\mathbf{a};\mathbf{g}) = E[\mathfrak{K}_{\gamma,\alpha}^n]$. If we choose as umbra γ the umbra $k{\bf .}u$ and we set $c_{n,k} = E[A_n^{(k.u)}(\alpha)]$, then we may consider the infinite matrix

$$C(\mathbf{a}) = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots \\ c_{2,1} & c_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

The cumulants induced by the umbra $k{\bf .}u$ are the ones occurring in the k-th column. But we can construct different sequences of cumulants of α by extracting one entry from each row of $C(\mathbf{a})$. For example, suppose to define the umbra γ such that $g_n = (n + k - 1)$, for all $n \geq 1$ and for a fixed positive integer k. The cumulants induced by this umbra γ are the ones occurring in the k-th diagonal of $C(\mathbf{a})$.

By means of equivalence (5.1) and Definition 5.1, we have

$$\mathfrak{K}_{\gamma,\alpha}^n \simeq Q_n(\gamma; \alpha', \alpha'', \ldots, \alpha''') \tag{5.2}$$

where $Q_n(\gamma; x_1, x_2, \ldots, x_n) = \sum_{\lambda\vdash n} d_\lambda (-g_n)_{\ell(\lambda)-1} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ ($\lambda_i = 0$ if $i > \ell(\lambda)$) are homogeneous polynomial in $R[X]$ of degree n whose coefficients do not depend on α. This property of $Q_n(\gamma; x_1, \ldots, x_n)$ gives rise to the following theorem.

Theorem 5.1 (Homogeneity property)**.** *If $\mathfrak{K}_{\gamma,\alpha}$ is the cumulant umbra of α induced by the umbra γ, then $\mathfrak{K}_{\gamma,j\alpha} \equiv j\mathfrak{K}_{\gamma,\alpha}$ for all $j \in R$.*

If we set $\mathbf{j}\,\mathbf{a} = (j^n a_n)_{n\geq 1}$, then the homogeneity property states that $c_n(\mathbf{j}\,\mathbf{a};\mathbf{g}) = j^n c_n(\mathbf{a};\mathbf{g})$ for all n. In terms of the matrix $C(\mathbf{a})$, the homogeneity property can be restated as $C(\mathbf{j}\,\mathbf{a})^T = \mathrm{diag}(j, j^2, \cdots) C(\mathbf{a})^T$. It is also possible to express the moments of α in terms of its cumulants induced by any γ with positive integer moments.

Theorem 5.2 (Invertibility property)**.** *For all scalar umbrae γ whose moments $\{g_n\}_{n\geq 1}$ are positive integers, there exists a sequence $\{P_n(\gamma; x_1, \ldots, x_n)\}_{n\geq 1}$ of* *homogeneous umbral polynomials of degree n, such that for all n and for all $\alpha \in A$ we have $\alpha^n \simeq P_n(\gamma; \mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}''')$, for all n-sets $\{\mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}'''\}$ of umbrae similar to $\mathfrak{K}_{\gamma,\alpha}$.*

Proof. Suppose to denote by c_n the n-th moment of $\mathfrak{K}_{\gamma,\alpha}$. From (5.2), $c_n = a^n + q(a_1, a_2, \ldots, a_{n-1})$ where q is a suitable polynomial in $a_1, a_2, \ldots, a_{n-1}$. So a_n can be expressed in terms of $c_1, \ldots, c_n$ by recursions. By replacing occurrences of product of powers of the c_i's by suitable products of powers of the x_i's, the polynomials P_n such that $a_n = E[P_n(\gamma; \mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}''')]$ can be constructed from these expressions. Finally, from the homogeneity property 5.1, we have $P_n(\gamma; j\mathfrak{K}', j\mathfrak{K}'', \ldots, j\mathfrak{K}''') \simeq j^n \alpha^n$, which assures the homogeneity of the P_n's. □

Each sequence of cumulants linearizes a certain convolution of umbrae (i.e. of moments) and this is why we call the elements of the matrix $C(\mathbf{a})$ linear cumulants. More precisely, we define the *convolution* of two umbrae α and η induced by the umbra γ to be the auxiliary umbra $\alpha +_{(\gamma)} \eta$ such that

$$\mathfrak{K}_{\gamma,\alpha+_{(\gamma)}\omega} \equiv \mathfrak{K}_{\gamma,\alpha} \dot{+} \mathfrak{K}_{\gamma,\omega}, \qquad \text{(Additivity property).} \tag{5.3}$$

In particular, convolutions are commutative. The invertibility property 5.2 assures the existence of the convolution of any pair of umbrae induced by any umbra whose moments are positive integers.

Theorem 5.3. *For all scalar umbrae γ whose moments $\{g_n\}_{n\geq 1}$ are positive integers, there exists a sequence of polynomials $T_n(\gamma; x_1, \ldots, x_n, y_1, \ldots, y_n)$ homogeneous of degree n for all n, such that for all n and for all scalar umbrae $\alpha, \omega \in A$ we have $(\alpha +_{(\gamma)} \omega)^n \simeq T_n(\gamma; \alpha', \alpha'', \ldots, \alpha''', \omega', \omega'', \ldots, \omega''')$.*

Proof. Due to the invertibility property 5.2, there exists $P_n(\gamma; x_1, x_2, \ldots, x_n)$ such that $(\alpha +_{(\gamma)} \omega)^n \simeq P_n(\gamma; \mathfrak{K}', \mathfrak{K}'', \ldots, \mathfrak{K}''')$. Then, suppose to replace each occurrence of $x_i^{\lambda_i}$ in P_n with $x_i^{\lambda_i} + y_i^{\lambda_i}$ and denote by T_n the polynomial resulting of this replacement. By virtue of the additivity property (5.3), it is straightforward to prove that T_n satisfies all the properties of the theorem. □

In general, the cumulant umbrae $\mathfrak{K}_{\gamma,\alpha}$'s do not have the semi-invariance property. This is due to the fact that $\mathfrak{K}_{\gamma,u}$ is not similar to χ, so that $\mathfrak{K}_{\gamma,\alpha+_{(\gamma)}cu}$ is not similar to $\mathfrak{K}_{\gamma,\alpha} \dot{+} c\chi$. However, after a suitable normalization of cumulants, moments and convolutions it is possible to recover the semi-invariance property. More explicitly, for the first column (classical cumulants) no normalization is needed. For the second column the right normalization (which returns boolean cumulants) is obtained via the moments $n!$ of the boolean unity $\bar{u}$. Indeed, $\{\mathfrak{K}^n_{2.u,\alpha}/n!\}_{n\geq 1}$ is a sequence of cumulants for the moments $\{\alpha^n/n!\}_{n\geq 1}$ which is semi-invariant with respect to the convolution $\{(\alpha +_{(2.u)} \omega)^n/n!\}_{n\geq 1}$. For the main diagonal (free cumulants) it is again $\bar{u}$ giving a good normalization. More generally, for columns and diagonals the normalization is always possible and it is obtained via umbrae representing positive integer moments.

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