On the Nonexistence of Skew-symmetric Amorphous Association Schemes

Dedicated to the Memory of Robert A. Liebler

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Abstract

An association scheme is amorphous if it has as many fusion schemes as possible. Symmetric amorphous schemes were classified by A. V. Ivanov [A. V. Ivanov, Amorphous cellular rings II, in Investigations in algebraic theory of combinatorial objects, pages 39–49. VNIISI, Moscow, Institute for System Studies, 1985] and commutative amorphous schemes were classified by T. Ito, A. Munemasa and M. Yamada [T. Ito, A. Munemasa and M. Yamada, Amorphous association schemes over the Galois rings of characteristic 4, European J. Combin., 12(1991), 513–526]. A scheme is called skew-symmetric if the diagonal relation is the only symmetric relation. We prove the nonexistence of skew-symmetric amorphous schemes with at least 4 classes. We also prove that non-symmetric amorphous schemes are commutative.

1 Introduction

Let $\mathfrak{X} = (X, R = \{R_0, R_1, \ldots, R_d\})$ be a commutative association scheme with d classes. $\mathfrak{X}$ is called amorphous if it has as many fusion schemes as possible. If $\mathfrak{X}$ is symmetric, then it is amorphous if and only if every partition of R containing $\{R_0\}$ gives rise to a fusion scheme. However, if $\mathfrak{X}$ is non-symmetric, then in order for a partition of R containing $\{R_0\}$ to give rise to a fusion scheme, this partition has to be closed under taking inverse, i.e., it is admissible [6]. So, if $\mathfrak{X}$ is non-symmetric, then it is amorphous if and only if every admissible partition of R gives rise to a fusion scheme.

A. V. Ivanov ([8], see also [4]) classified symmetric amorphous association schemes with at least three classes: all basic graphs in such a scheme are strongly regular graphs of Latin square types, or they are all negative Latin square type. Association schemes with two classes are amorphous by definition and there are many examples in which the basic graphs are not either Latin square types. Hence the assumption "at least three classes" is essential. T. Ito, A. Munemasa and M. Yamada [6] classified commutative amorphous association schemes under the assumption $\theta + \phi \geq 3$, where θ is the number of pairs of non-symmetric relations, and ϕ is the number of non-diagonal symmetric relations (see Section 2). The assumption $\theta + \phi \geq 3$ garantees that their symmetrizations have at least three classes. What about association schemes with $\theta + \phi \leq 2$?

This paper addresses the case $(\theta, \phi) = (2, 0)$. Included in this case are four-class association schemes which have no non-diagonal symmetric relations. Association schemes with this property will be referred to as *skew-symmetric*. The symmetrizations of skew-symmetric schemes with four classes, as we will see, indeed have basic graphs of Latin square type or negative Latin square type. However, such schemes can not exist, due to the following theorem.

Theorem 1. *There is no skew-symmetric amorphous association scheme with 4 classes.*

Surprisingly, this simple result eleminates the existence of many amorphous association schemes.

Theorem 2 (Main Theorem)**.** *There is no skew-symmetric amorphous association scheme with at least 4 classes.*

Theorem 1 answers a question put forward by E. Bannai and S.Y. Song ([2, p.395]) regarding the existence of certain amorphous association schemes with 4 classes. The proofs of both theorems rely on Theorem 6, in which we determine the eigenmatrices of skew-symmetric schemes with 4 classes.

We note that all association schemes in other cases of $\theta + \phi < 3$ are trivially amorphous. Let $\mathfrak{X}$ be an association scheme with $\theta + \phi < 3$. If $(\theta, \phi) = (0, 1)$, $\mathfrak{X}$ is a complete graph. If $(\theta, \phi) = (1, 0)$, $\mathfrak{X}$ is a doubly regular tournament. If $(\theta, \phi) = (0, 2)$, $\mathfrak{X}$ is equivalent to a pair of complementary strong regular graphs. Many chapters of books have been devoted to strong regular graphs (e.g. [3]). If $(\theta, \phi) = (1, 1)$, $\mathfrak{X}$ is a non-symmetric association scheme with 3 classes, and examples of primitive ones are not abundant except the Liebler-Mena family [9] and some examples in [10] (see [7] and the references there).

It is natural to ask if there exist non-commutative amorphous schemes. We rule out this possibility with an algebraic argument.

The general references are [1, 3] for association schemes and strongly regular graphs, and [4, 6, 15] for amorphous association schemes. In the rest of this paper, all association schemes are assumed to be commutative unless otherwise stated.

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2 Preliminaries

Let X be a finite set with cardinality $n \geq 2$ and $R = \{R_0, R_1, \ldots, R_d\}$ be a set of binary relations on X. $\mathfrak{X} = (X, R)$ is called an *association scheme with d classes* (*a d-class association scheme*, or simply, *a scheme*) if the following axioms are satisfied:

(i) R is a partition of $X \times X$ and $R_0 = \{(x, x) \mid x \in X\}$ is the diagonal relation.

(ii) For $i = 0, 1, \ldots, d$, the inverse $R_i^{\mathsf{T}} = \{(y, x) | (x, y) \in R_i\}$ of R_i is also among the relations: $R_i^{\mathsf{T}} = R_{i'}$ for some i' $(0 \leq i' \leq d)$.

(iii) For any triple of $i, j, k = 0, 1, \ldots, d$, there exists an integer p_{ij}^k such that for all $(x, y) \in R_k$,

$$|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^k.$$

The integers p_{ij}^k are called the *intersection numbers*. The integer $k_i = p_{ii'}^0$ is called the *valency* of R_i. In fact, for any $x \in X$, $k_i = |\{y \in X | \ (x, y) \in R_i\}|$.

Furthermore, $\mathfrak{X}$ is called *commutative* if $p_{ij}^k = p_{ji}^k$ for all i, j, k.

R_i and $R_{i'}$ are called *paired relations*. If $i = i'$, then R_i is called *symmetric* or *self-paired*. Let

$$\theta = |\{\{i, i'\} | i \neq i', 1 \leq i \leq d\}|, \quad \phi = |\{i | i = i', 1 \leq i \leq d\}|.$$

$\mathfrak{X}$ is called *symmetric* if all relations R_i are symmetric: $\theta = 0$. Otherwise, $\mathfrak{X}$ is said to be *non-symmetric*. We call $\mathfrak{X}$ *skew-symmetric* if R_0 is the only self-paired relation: $\phi = 0$.

A partition $\Lambda_0, \Lambda_1, \ldots, \Lambda_e$ of index set $\{0, 1, \ldots, d\}$ of R is called *admissible* [6] if $\Lambda_0 = \{0\}$, $\Lambda_i \neq \emptyset$ and $\Lambda_i^{\mathrm{T}} = \Lambda_j$ for some j $(1 \leq i, j \leq e)$, where $\Lambda^{\mathrm{T}} = \{\alpha' | \alpha \in \Lambda\}$ is called the inverse of Λ. We may also talk about these properties in terms of the relations when it is convenient, which we did at the beginning of the Introduction.

Let $R_{\Lambda_i} = \cup_{\alpha \in \Lambda_i} R_\alpha$. If $(X, \{R_{\Lambda_i}\}_{i=0}^e)$ is an association scheme, it is called a *fusion scheme* of $\mathfrak{X}$. In particular, the fusion scheme $\mathfrak{X}^{\mathrm{sym}} = (X, \{R_0, R_i \cup R_i^{\mathrm{T}}\}_{i=0}^d)$ is called the *symmetrization* of $\mathfrak{X}$. $\mathfrak{X}$ is *amorphous* if every admissible partition gives a fusion scheme. Note that if $\mathfrak{X}$ is symmetric, then every partition containing $\{0\}$ is admissible by definition. Amorphous schemes are extremal in the sense they have as many fusion schemes as possible.

Let us recall the (first) eigenmatrix $P = (P_{ij})$ of a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$. Let A_i and E_i $(0 \leq i \leq d)$ be the *adjacency matrices* and *primitive idempotents* of $\mathfrak{X}$. Then the eigenmatrix P is a square matrix of order $d+1$ defined by

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \text{ for } j = 0, 1, \ldots, d.$$

P is characterized by $A_j E_i = P_{ij} E_i$ for all $i, j = 0, 1, \ldots d$. We may index the rows and columns of P by E_i and A_i $(i = 0, 1, 2)$, respectively. Moreover, $P_{0i} = k_i$ and $P_{i0} = 1$ $(0 \leq i \leq d)$. Let $m_i = \text{rank } E_i$. Then A_j has eigenvalues $P_{0j} = k_j, P_{1j}, \ldots, P_{dj}$ with multiplicities $m_0 = 1, m_1, \ldots m_d$, respectively. The rows and columns of P satisfy the *orthogonality relations*:

$$\sum_{i=0}^{d} \frac{1}{k_i} P_{ji} \overline{P}_{ki} = \frac{n}{m_j} \delta_{jk}, \qquad \sum_{i=0}^{d} m_i P_{ij} \overline{P}_{ik} = n k_j \delta_{jk}, \tag{2.1}$$

where $\overline{x}$ is the complex conjugate of x and δ is the Kronecker symbol.

The numbers m_j and p_{ij}^{ℓ} can be calculated from P:

$$m_j = \frac{n}{\sum\limits_{i=0}^{d} \frac{1}{k_i} P_{ji} \overline{P}_{ji}}, \tag{2.2}$$

$$p_{ij}^{\ell} = \frac{1}{n k_\ell} \sum_{h=0}^{d} m_h P_{hi} P_{hj} \overline{P}_{h\ell}. \tag{2.3}$$

In the rest of this paper, the following theorem, referred as the Bannai-Muzychuk criterion for fusion schemes, will be used repeatedly (see [2], [6]).

Theorem 3. *Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ be a commutative association scheme. Let $\{\Lambda_i\}_{i=0}^e$ be an admissible partition of the index set $\{0, 1, \ldots, d\}$. Then $\{\Lambda_i\}_{i=0}^e$ gives rise to a fusion scheme $(X, \{R_{\Lambda_i}\}_{i=0}^e)$ if and only if there exists a dual partition $\{\Lambda_i^*\}_{i=0}^e$ of $\{0, 1, \ldots, d\}$ with $\Lambda_0^* = \{0\}$ such that each (Λ_i^*, Λ_j) block of the eigenmatrix P has constant row sum. Moreover, the constant row sum of the (Λ_i^*, Λ_j)-block is the (i, j) entry of the eigenmatrix of the fusion scheme.*

Now we consider two-class association schemes. Let $\mathfrak{X} = (X, \{R_0, R_1, R_2\})$ be an association scheme. If $\mathfrak{X}$ is non-symmetric, then $R_2 = R_1^{\mathrm{T}}$. Its eigenmatrix is

$$P = \begin{bmatrix} 1 & k & k \\ 1 & \dfrac{-1+\sqrt{-n}}{2} & \dfrac{-1-\sqrt{-n}}{2} \\ 1 & \dfrac{-1-\sqrt{-n}}{2} & \dfrac{-1+\sqrt{-n}}{2} \end{bmatrix}, \tag{2.4}$$

where $n = |X| = 2k + 1$.

Two-class symmetric schemes are closely related to strongly regular graphs. A regular graph (X, F) with vertex set X, edge set F and valency k, is called *strongly regular* if

$$\lambda = |\{z|(x, z) \in F, (z, y) \in F\}|$$

is constant for all $(x, y) \in F$ and

$$\mu = |\{z|(x, z) \in F, (z, y) \in F\}|$$

is constant for all $(x, y) \notin F$ $(x \neq y)$. The numbers $n = |X|, k, \lambda, \mu$ are the parameters of this graph.

A strongly regular graph with parameters (n, k, λ, μ) is of (positive) *Latin square type* or *negative Latin square type* if $n = v^2$ (a square) and either (i)

$$k = g(v - 1), \quad \lambda = (g - 1)(g - 2) + v - 2, \quad \mu = g(g - 1),$$

or (ii)

$$k = g(v + 1), \quad \lambda = (g + 1)(g + 2) - v - 2, \quad \mu = g(g + 1).$$

They are denoted by $L_g(v)$ and $NL_g(v)$, respectively. Graphs with $L_g(v)$ parameters can be constructed with $g - 2$ mutually orthogonal Latin squares of order v. Graphs with $NL_g(v)$ parameters do exist: for example, the Clebsch graph is $NL_1(4)$.

For a symmetric amorphous scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$, each graph $(X, R_i)(i \neq 0)$ is strongly regular. It was shown in [8, 4] that if $d \geq 3$, all graphs $(X, R_i)(i \neq 0)$ are strongly regular graphs of Latin square type, or they are all negative Latin square type. The converse is also true ([14, Theorem 3]): if (X, R_i), $i = 1, \ldots, d$ are strongly regular graphs of all Latin square type or all negative Latin square type such that $\cup_{i=1}^d R_i = X \times X - R_0$ and $R_i \cap R_j = \emptyset$ $(i \neq j)$, then $(X, \{R_i\}_{i=0}^d)$ is an amorphous association scheme. In [6], T. Ito et al. classified commutative amorphous association schemes with $\theta + \phi \geq 3$ and determined their eigenmatrices and intersection numbers. They also constructed some amorphous schemes on Galois rings of characteristic 4.

If (X, R_1) is a strongly regular graph with parameters (n, k_1, λ, μ), the complement (X, R_2) of (X, R_1) is also strongly regular, where $R_2 = X \times X - R_0 - R_1$. Furthermore, $(X, \{R_0, R_1, R_2\})$ is a symmetric scheme, which has the following eigenmatrix:

$$P = \begin{bmatrix} 1 & k_1 & k_2 \\ 1 & r & t \\ 1 & s & u \end{bmatrix} \begin{array}{l} 1 \\ m_1 \\ m_2, \end{array} \tag{2.5}$$

where $t = -r - 1$, $u = -s - 1$. The numbers k_1, r, s are the eigenvalues of the adjacency matrix A_1 of R_1 and r, s may be expressed in terms of n, k_1, λ and μ. Here, we write the multiplicities to the right of P. Conversely, a two-class symmetric scheme gives rise to a pair of complementary strongly regular graphs.

Now we conclude this section with two lemmas that we will need later. In the rest of this paper, we always choose $r \geq 0 > s$.

Lemma 4. *Let Γ be a strong regular graph with eigenvalues k, r, s and multiplicities $1, m_1, m_2$.*

(i) *If $m_1 = m_2$ (hence $k = m_1$), Γ is a strong regular graph with parameters $n = 4\mu + 1$, $k = 2\mu$, $\lambda = \mu - 1$. Such a graph is called a conference graph, denoted by $C(n)$.*

(ii) *If $k = m_1$, Γ is $L_g(v)$ with $v = r - s$, $g = -s$.*

(iii) *If $k = m_2$, Γ is $NL_g(v)$ with $v = r - s$, $g = r$.*

One can prove this lemma directly using [3, exercise 5, p.244]) or see Theorem 2.1 of [11]. If v is odd, we note that $L_{\frac{1}{2}(v+1)}(v)$ and $NL_{\frac{1}{2}(v-1)}(v)$ have identical parameters and both agree with $C(n)$ with the argument $\mu = (v^2-1)/4$.

We state the next lemma without a proof since it is straightforward.

Lemma 5. *Let $\mathfrak{X}$ be a d-class association scheme with adjacency matrices A_i and primitive idempotents E_i.*

(i) *If $A_i^{\mathsf{T}} = A_j$, then $P_{\alpha i} = \overline{P}_{\alpha j}$ $(0 \le \alpha \le d)$. So, if $A_i^{\mathsf{T}} \ne A_i$, A_i has at least one pair of nonreal eigenvalues that are complex conjugates.*

(ii) *If $E_i^{\mathsf{T}} = E_j$, then $P_{i\alpha} = \overline{P}_{j\alpha}$ $(0 \le \alpha \le d)$. So, if $E_i^{\mathsf{T}} = E_i$, then P_{ik} are real for all k $(0 \le k \le d)$, and if $E_i^{\mathsf{T}} \ne E_i$, there are distinct α, β such that $P_{\alpha i}$ and $P_{\beta i}$ are nonreal complex conjugates.*

3 Eigenmatrices

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^4)$ be a skew-symmetric association scheme: $(\theta, \phi) = (2, 0)$. It is commutative since any association scheme with at most 4 classes is commutative [5]. Up to a permutation, we may assume $R_4 = R_1^{\mathsf{T}}$ and $R_3 = R_2^{\mathsf{T}}$. Let $\mathfrak{X}^{\mathrm{sym}} = (X, \{R_0, R_1 \cup R_4, R_2 \cup R_3\})$, the symmetrization of $\mathfrak{X}$. We will determine the eigenmatrix of $\mathfrak{X}$ from that of $\mathfrak{X}^{\mathrm{sym}}$.

Let $\widetilde{A}_i$ and $\widetilde{E}_i$ $(0 \le i \le 2)$ be the adjacency matrices and the primitive idempotents of $\mathfrak{X}^{\mathrm{sym}}$, respectively. Suppose that the eigenmatrix $\widetilde{P}$ of $\mathfrak{X}^{\mathrm{sym}}$ has form (2.5):

$$\widetilde{P} = \begin{bmatrix} 1 & k_1 & k_2 \\ 1 & r & t \\ 1 & s & u \end{bmatrix} \begin{matrix} 1 \\ m_1 \\ m_2 \end{matrix} .$$

In the rest of this paper, we assume that $r \ge 0 > s$.

S. Y. Song [13] mentioned that up to permutation of rows and columns, a feasible eigenmatrix of $\mathfrak{X}$ can be described as follows:

$$P = \begin{bmatrix} 1 & k/2 & (n-k-1)/2 & (n-1-k)/2 & k/2 \\ 1 & \rho & \tau & \overline{\tau} & \overline{\rho} \\ 1 & \sigma & \omega & \overline{\omega} & \overline{\sigma} \\ 1 & \overline{\sigma} & \overline{\omega} & \omega & \sigma \\ 1 & \overline{\rho} & \overline{\tau} & \tau & \rho \end{bmatrix} \begin{matrix} 1 \\ m \\ (n-m-1)/2 \\ (n-m-1)/2 \\ m \end{matrix} , \tag{3.1}$$

where the pair ρ and ω or the pair τ and σ are nonreal. He also gave a one-sentence explanation. Here, we will prove Song's observation and determined the entries of P. Let P be the eigenmatrix of $\mathfrak{X}$:

$$P = [P_{ij}]_{0 \le i,j \le 4}.$$

Since $A_4 = A_1^{\mathsf{T}}$, by Lemma 5, A_1 has at least one pair of nonreal eigenvalues. There are two cases to consider:

(1) A_1 has precisely one pair of nonreal eigenvalues $\rho, \overline{\rho}$. By Theorem 3, $\rho + \overline{\rho} = r$ or s.

Suppose $\rho + \overline{\rho} = r$. We may arrange the primitive idempotents E_i of $\mathfrak{X}$ such that $A_1 E_1 = \rho E_1$, $A_4 E_4 = \overline{\rho} E_4$. Hence $E_4 = E_1^{\mathsf{T}}$. By Lemma 5, P_{21}, P_{24}, P_{31} and P_{34} are all real, and $P_{21} = P_{24}, P_{31} = P_{34}$. Consider the remaining two primitive idempotents $E_2, E_3 \ne E_0$. Then we have either $E_3 = E_2^{\mathsf{T}}$ or $E_i = E_i^{\mathsf{T}}$ $(i = 2, 3)$, and the latter can not occur as we will see.

Suppose $E_i = E_i^{\mathsf{T}}$ $(i = 2, 3)$. By Lemma 5, the second and third rows of P have all real entries. Since $\widetilde{A}_1 = A_1 + A_4$, $P_{21} + P_{24}, P_{31} + P_{34} \in \{r, s\}$ again by Theorem 3. We must have $P_{21} + P_{24} \ne P_{31} + P_{34}$. Otherwise, the second and third rows of P are identical, which contradicts that P is nonsingular. We may assume without loss of generality that $P_{21} = P_{24} = r/2$,

$P_{31} = P_{34} = s/2$. Since A_2 and A_3 have a pair of nonreal eigenvalues τ and $\overline{\tau}$, P has the following form:

$$P = \begin{array}{c} \left[\begin{array}{ccccc} 1 & n_1 & n_2 & n_2 & n_1 \\ 1 & \rho & \tau & \overline{\tau} & \overline{\rho} \\ 1 & r/2 & t/2 & t/2 & r/2 \\ 1 & s/2 & u/2 & u/2 & s/2 \\ 1 & \overline{\rho} & \overline{\tau} & \tau & \rho \end{array} \right] \end{array} \begin{array}{c} 1 \\ f_1 \\ f_2 \\ f_3 \\ f_1 \end{array},$$

where n_i are the valencies of $\mathfrak{X}$ and $2n_i = k_i$. Now we calculate the multiplicity f_2 using (2.2):

$$f_2 = \frac{n}{\sum\limits_{i=0}^{4} \frac{1}{n_i} P_{2i} \overline{P}_{2i}} = \frac{n}{1 + \frac{2}{n_1}\left(\frac{r}{2}\right)^2 + \frac{2}{n_2}\left(\frac{t}{2}\right)^2} = \frac{n}{1 + \frac{r^2}{k_1} + \frac{t^2}{k_2}},$$

which is m_1 by (2.2). Similarly, we can obtain $f_3 = m_2$. So, $f_1 = 0$, impossible.

Now we have $E_3 = E_2^{\mathsf{T}}$. Since $E_2^{\mathsf{T}} \neq E_2$ and P_{21}, P_{41} are real, by Lemma 5, P_{22}, P_{23} are nonreal and $P_{22} = \overline{P}_{23}$. Since $A_2^{\mathsf{T}} = A_3$, $P_{32} = \overline{P}_{22} = \overline{P}_{33}$ again by Lemma 5. Let $\omega = P_{22}$. So P has the following form:

$$P = \left[\begin{array}{ccccc} 1 & n_1 & n_2 & n_2 & n_1 \\ 1 & \rho & \tau & \overline{\tau} & \overline{\rho} \\ 1 & s/2 & \omega & \overline{\omega} & s/2 \\ 1 & s/2 & \overline{\omega} & \omega & s/2 \\ 1 & \overline{\rho} & \overline{\tau} & \tau & \rho \end{array} \right], \tag{3.2}$$

where ρ and ω are both nonreal, $\rho + \overline{\rho} = r$, $\omega + \overline{\omega} = u$ and $\tau + \overline{\tau} = t$.

Suppose $\rho + \overline{\rho} = s$. Replacing s by r in (3.2), we obtain matrix P for this case, in which ρ and ω are both nonreal, $\rho + \overline{\rho} = s$, $\tau + \overline{\tau} = u$ and $\omega + \overline{\omega} = t$.

(2) A_1 has two pairs of nonreal eigenvalues: $\rho, \overline{\rho}$, and $\sigma, \overline{\sigma}$.

Without loss of generality, we may assume that $\rho + \overline{\rho} = r$, $\sigma + \overline{\sigma} = s$, and the first column of P is $(n_2, \rho, \sigma, \overline{\sigma}, \overline{\rho})^{\mathsf{T}}$. So P has form (3.1), where τ and ω can not be both real by Lemma 5.

We know from the above analysis that P has the form asserted in (3.1). Let

$$P = \begin{array}{c} \left[\begin{array}{ccccc} 1 & n_1 & n_2 & n_2 & n_1 \\ 1 & \rho & \tau & \overline{\tau} & \overline{\rho} \\ 1 & \sigma & \omega & \overline{\omega} & \overline{\sigma} \\ 1 & \overline{\sigma} & \overline{\omega} & \omega & \sigma \\ 1 & \overline{\rho} & \overline{\tau} & \tau & \rho \end{array} \right] \end{array} \begin{array}{c} 1 \\ f_1 \\ f_2 \\ f_2 \\ f_1 \end{array}.$$

Now we are ready to determine ρ, ω, τ and σ. Set

$$\rho = \frac{1}{2}(r + \sqrt{-y}), \quad \tau = \frac{1}{2}(t + \sqrt{-z}), \quad \sigma = \frac{1}{2}(s + \sqrt{-b}),$$

where $y, z, b \geq 0$. So, the first row and column of P are fixed, and hence the fourth row and column by Lemma 5. Therefore, $\omega = \frac{1}{2}(u \pm \sqrt{-c})$ for some $c \geq 0$. There are two cases to consider:

Case (i) $\omega = \frac{1}{2}(u + \sqrt{-c})$. Applying the first orthogonality relation to the first row of P, we obtain

$$1 + \frac{2\rho\overline{\rho}}{n_1} + \frac{2\tau\overline{\tau}}{n_2} = \frac{n}{f_1}.$$

Substituting ρ and τ into the above equation, we obtain

$$1 + \frac{1}{2n_1}(r^2 + y) + \frac{1}{2n_2}(t^2 + z) = \frac{n}{f_1}. \tag{3.3}$$

Applying the first orthogonality relation to the first row of $\widetilde{P}$, we obtain

$$1+\frac{r^2}{k_1}+\frac{t^2}{k_2}=\frac{n}{m_1}. \tag{3.4}$$

Note that $2n_i = k_i$ and $2f_i = m_i$. By (3.3) and (3.4), we have

$$\frac{y}{k_1}+\frac{z}{k_2}=\frac{n}{m_1}. \tag{3.5}$$

Similarly, we can obtain

$$\frac{b}{k_1}+\frac{c}{k_2}=\frac{n}{m_2}. \tag{3.6}$$

Applying the first orthogonality relation to the second and third rows of P and $\widetilde{P}$, we obtain

$$1+\frac{1}{n_1}(\rho\overline{\sigma}+\overline{\rho}\sigma)+\frac{1}{n_2}(\tau\overline{\omega}+\overline{\tau}\omega)=0, \qquad 1+\frac{rs}{k_1}+\frac{tu}{k_2}=0.$$

Substituting ρ, ω, τ, ω and the second equation into the first equation, we obtain

$$\frac{\sqrt{by}}{k_1}+\frac{\sqrt{cz}}{k_2}=0.$$

It follows that $by = 0$ and $cz = 0$. Since P is nonsingular, we must have either $(y, c) = (0, 0)$ or $(z, b) = (0, 0)$, but not both.

Suppose $(z, b) = (0, 0)$. By equations (3.5) and (3.6),

$$y=\frac{nk_1}{m_1}, \quad c=\frac{nk_2}{m_2}. \tag{3.7}$$

Suppose $(y, c) = (0, 0)$. By equations (3.5) and (3.6),

$$z=\frac{nk_2}{m_1}, \quad b=\frac{nk_1}{m_2}. \tag{3.8}$$

Note that in each case above, $\widetilde{P}$ determines P uniquely.

Case (ii) $\omega = \frac{1}{2}(u - \sqrt{-c})$. If $c = 0$, then $y = 0$. This case has been treated in Case (i). Now, assume $c > 0$. Since ω is nonreal, ρ is also nonreal: $y > 0$. Note that equations (3.5) and (3.6) still hold.

Now, applying the first orthogonality relation to the second and third rows of P and simplifying it in a similar way, we can obtain

$$\frac{\sqrt{by}}{k_1}-\frac{\sqrt{cz}}{k_2}=0. \tag{3.9}$$

Note if $z = 0$, then $b = 0$. This has been handled previously. Now we assume $z > 0$. Hence, $b > 0$. Therefore, ρ, ω, τ, and σ are all nonreal, and b, c, y, and z satisfy equations (3.5), (3.6), and (3.9).

We summarize the above discussion in the following theorem.

Theorem 6. *Let $\mathfrak{X} = (X, \{R_0, R_1, R_2, R_3, R_4\})$ be a skew-symmetric association scheme with $R_1 = R_4^{\mathrm{T}}$ and $R_2 = R_3^{\mathrm{T}}$. Let $\widetilde{P}$ be the eigenmatrix of the symmetrization $\mathfrak{X}^{\mathrm{sym}}$:*

$$\widetilde{P}=\begin{array}{c c} \begin{bmatrix} 1 & k_1 & k_2 \\ 1 & r & t \\ 1 & s & u \end{bmatrix} & \begin{array}{c} 1 \\ m_1 \\ m_2 \end{array} \end{array},$$

where $r \geq 0 > s$. Then the eigenmatrix of $\mathfrak{X}$ has the following form:

$$P = \begin{bmatrix} 1 & k_1/2 & k_2/2 & k_2/2 & k_1/2 \\ 1 & \rho & \tau & \overline{\tau} & \overline{\rho} \\ 1 & \sigma & \omega & \overline{\omega} & \overline{\sigma} \\ 1 & \overline{\sigma} & \overline{\omega} & \omega & \sigma \\ 1 & \overline{\rho} & \overline{\tau} & \tau & \rho \end{bmatrix}.$$

ρ, ω, τ and σ take values in one of three cases:

(i) $\sigma = \frac{s}{2}, \quad \tau = \frac{t}{2}, \quad \rho = \frac{1}{2}\left(r + \sqrt{-\frac{nk_1}{m_1}}\right), \quad \omega = \frac{1}{2}\left(u + \sqrt{-\frac{nk_2}{m_2}}\right).$

(ii) $\rho = \frac{r}{2}, \quad \omega = \frac{u}{2}, \quad \sigma = \frac{1}{2}\left(s + \sqrt{-\frac{nk_1}{m_2}}\right), \quad \tau = \frac{1}{2}\left(t + \sqrt{-\frac{nk_2}{m_1}}\right).$

(iii) $\rho = \frac{1}{2}\left(r + \sqrt{-y}\right), \quad \tau = \frac{1}{2}\left(t + \sqrt{-z}\right), \quad \sigma = \frac{1}{2}\left(s + \sqrt{-b}\right), \quad \omega = \frac{1}{2}\left(u - \sqrt{-c}\right),$

where b, c, y, and z are all positive and satisfy the following equations:

$$\frac{y}{k_1} + \frac{z}{k_2} = \frac{n}{m_1}, \quad \frac{b}{k_1} + \frac{c}{k_2} = \frac{n}{m_2}, \quad \frac{\sqrt{by}}{k_1} - \frac{\sqrt{cz}}{k_2} = 0.$$

4 Proof of the Main Theorem

In this section, we prove Theorems 1 and 2. Let $\mathfrak{X} = (X, \{R_0, R_1, R_2, R_3, R_4\})$ be a skew-symmetric amorphous scheme, whose eigenmatrix P is given in Theorem 6.

Suppose that P is given by Theorem 6(i). Since $\mathfrak{X}$ is amorphous, $R_0, R_1 \cup R_2, R_3 \cup R_4$ gives rise to a skew-symmetric association scheme, which has eigenmatrix (2.4). By Theorem 3,

$$\rho + \tau = \frac{1}{2}\left(r + t + \sqrt{-\frac{nk_1}{m_1}}\right) = \frac{-1 + \sqrt{-n}}{2}.$$

So $m_1 = k_1$. By Lemma 4(ii), $(X, R_1 \cup R_4)$ is $L_g(v)$ with $v = r - s$, $g = -s$.

The i-th intersection matrix B_i is a square matrix of order $d + 1$ whose (j, ℓ) entry is p_{ij}^{ℓ}. Using (2.3), we can obtain

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda + r}{4} & \frac{k_1(k_1 - \lambda - 1 - t)}{4k_2} & \frac{k_1(k_1 - \lambda - 1 - t)}{4k_2} & \frac{\lambda - 3r}{4} \\ 0 & \frac{k_1 - \lambda - 1 + t}{4} & \frac{k_1 - \mu + s}{4} & \frac{k_1 - \mu - s}{4} & \frac{k_1 - \lambda - 1 - t}{4} \\ 0 & \frac{k_1 - \lambda - 1 + t}{4} & \frac{k_1 - \mu - s}{4} & \frac{k_1 - \mu + s}{4} & \frac{k_1 - \lambda - 1 - t}{4} \\ \frac{k_1}{2} & \frac{\lambda + r}{4} & \frac{k_1(k_1 - \lambda - 1 + t)}{4k_2} & \frac{k_1(k_1 - \lambda - 1 + t)}{4k_2} & \frac{\lambda + r}{4} \end{bmatrix}.$$

From $p_{12}^1 - p_{12}^4 = 2t/4$ and $p_{12}^2 - p_{12}^3 = 2s/4$ we readily deduce that t and s are even integers. Since $r + t + 1 = 0$, r is odd. On the other hand, $p_{11}^1 = \frac{\lambda + r}{4} = \frac{s(s+2)+2r}{4}$. So r is even because $s(s + 2)$ is divisible by 4, a contradiction.

Suppose that P is given by Theorem 6(ii). In a similarly way, we can deduce $k_1 = m_2$ and hence $(X, R_1 \cup R_4)$ is $NL_g(v)$ with $v = r - s$, $g = r$. The first intersection matrix for this case can be obtained from the above B_1 by interchanging r and s, and t and u. We can readily deduce that r and u are even and s is odd. Since $p_{11}^1 = \frac{\lambda + s}{4} = \frac{r(r+2)+2s}{4}$, s is even, a contradiction.

Suppose that P is given by Theorem 6(iii). In this case, $\rho, \omega, \tau, \sigma$ are all nonreal. since $\mathfrak{X}$ is amorphous, $R_0, R_1 \cup R_2, R_3 \cup R_4$ determines a non-symmetric association scheme, which has eigenmatrix (2.4). So,

$$\rho + \tau = \frac{1}{2}(r + t + \sqrt{-y} + \sqrt{-z}) = \frac{-1 + \sqrt{-n}}{2},$$

and hence $\sqrt{-y} + \sqrt{-z} = \sqrt{-n}$. Similarly, $(X, \{R_0, R_1 \cup R_3, R_2 \cup R_4\})$ is non-symmetric association scheme and thus $\sqrt{-y} - \sqrt{-z} = \pm\sqrt{-n}$. These equations imply either $y = 0$ or $z = 0$, a contradiction. This completes the proof of Theorem 1. $\square$

Now we prove the main theorem. Suppose that $\mathfrak{X}$ is a skew-symmetric amorphous scheme with more than 4 classes. $\mathfrak{X}$ has a skew-symmetric fusion scheme with 4 classes, which can not exist by Theorem 1 because this fusion scheme is again amorphous. This completes the proof.

5 Concluding Remarks

As we mentioned in the Introduction that there does not exist any non-commutative amorphous scheme. We now give a short proof of this result. By the main theorem, we may assume $\phi \geq 1$. Since association schemes with at most 4 classes are commutative, we may assume $\theta + \phi \geq 3$. We first treat the minimal cases $(\theta, \phi) = (1, 3)$ or $(2, 1)$ and the general case will then follow. Suppose that $\mathfrak{X}$ is a non-symmetric amorphous scheme with $(\theta, \phi) = (1, 3)$ or $(2, 1)$. If $\mathfrak{X}$ is non-commutative, then the adjacency algebra generated by the adjacency matrices of $\mathfrak{X}$ over the complex numbers $\mathbb{C}$ is non-commutative of dimension 6. It is semisimple and thus is isomorphic to direct sum of full matrix algebras of degree 1,1 and 2:

$$\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}).$$

$\mathfrak{X}$ has a 4-class fusion scheme $\mathfrak{F}$, which is commutative. So the adjacency algebra of $\mathfrak{F}$ is commutative of dimension 5. On the other hand, $M_2(\mathbb{C})$ can not have commutative subalgebras of dimension 3 and thus the adjacency algebra of $\mathfrak{X}$ has no commutative subalgebras of dimension 5, which is a contradiction. Therefore, $\mathfrak{X}$ is commutative. (In fact, we have proved that a non-commutative scheme with 5 classes can not have a 4-class fusion scheme.)

Let $\mathfrak{X}$ be an amorphous non-symmetric scheme with $\theta + \phi > 4, \theta \geq 1$. Any two adjacency matrices of $\mathfrak{X}$ are among the adjacency matrices of some fusion scheme with $(\theta, \phi) = (1, 3)$ or $(2, 1)$. So the adjacency matrices of $\mathfrak{X}$ commute pairwisely and thus $\mathfrak{X}$ is commutative.

We note that the minimal cases can also be handled by a careful analysis of their intersection numbers, which shows that the intersection numbers in each case coincide with those of certain commutative amorphous scheme. In fact, using the notation in [6], amorphous schemes with $(\theta, \phi) = (1, 3)$ belong to $L_{g_1;g_2,g_3,g_4}(v)$ or $NL_{g_1;g_2,g_3,g_4}(v)$, and amorphous schemes with $(\theta, \phi) = (2, 1)$ belong to $L_{g_1,g_2;g_3}(v)$ or $NL_{g_1,g_2;g_3}(v)$.

We conclude with some remarks:

- E. R. van Dam and M. Muzychuk [15] gave an excellent survey of symmetric amorphous association schemes. Among many results, they gave all known constructions and enumeration of such schemes with vertices up to 49 vertices. However, there has not been much work done with the non-symmetric counterpart except [6].
- In light of Theorem 6, it is interesting to study skew-symmetric schemes with 4 classes. We are working on it in another paper.
- In the literature (e.g. [4],[8],[15]), an association scheme is call *amorphic* if every partition of R containing $\{R_0\}$ gives rise to a fusion scheme. The notion of admissible partition was introduced and the term amorphous was used by T. Ito, et al. in [6]. For symmetric schemes, the two notions are equivalent. Any association scheme with two classes is

trivially amorphic by definition. It is easy to see that any amorphic scheme with at least three classes is symmetric.

- I.N. Ponomarenko an A.R. Barghi [12] recently introduced amorphic C-algebras by axiomatizing the property that each partition of a standard basis leads to a fusion algebra. Just like association schemes, each amorphic C-algebra of dimension ≥ 4 is symmetric. They showed that each amorphic C-algebra is determined up to isomorphism by the multiset of its degrees (valencies in the case of association scheme) and an additional integer $\epsilon = \pm 1$ (reflecting the positive or negative Latin square type). Since our focus here is non-symmetric association schemes, our work has little overlap with that of [12] .

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