Maltsev digraphs have a majority polymorphism

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Abstract

We prove that when a digraph G has a Maltsev polymorphism, then G also has a majority polymorphism. We consider the consequences of this result for the structure of Maltsev digraphs and the complexity of the Constraint Satisfaction Problem.

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1. Introduction

In recent years, a marriage of universal algebra with graph theory brought about great advances in the study of the Constraint Satisfaction Problem (CSP) and related areas (see [2] or [1]).

Given G, the problem $\mathrm{CSP}(G)$ with input H consists of deciding whether there exists a homomorphism from H to G. (Note that in the general theory, G and H can be any relational structures, however we will consider only digraphs in this paper.) An important open question is how to determine the complexity of $\mathrm{CSP}(G)$ from the properties of G. In particular, the famous *dichotomy conjecture* by Feder and Vardi claims that if $\mathrm{CSP}(G)$ is not polynomial time solvable, then it is NP-complete (see [7]).

At the core of universal algebra's success in describing the complexity of CSP is the focus on algebras of polymorphisms. It turns out that the more polymorphisms G admits, the easier it is to solve $\mathrm{CSP}(G)$. More precisely, if we have two digraphs G and G' on the same vertex set and the algebra of polymorphisms of G' is contained in the algebra of polymorphisms of G, then $\mathrm{CSP}(G)$ can be reduced to $\mathrm{CSP}(G')$ in logarithmic space (see Theorem 2.16 in [2] for the idea, [8] for the logspace reduction proper).

Two often-encountered kinds of polymorphisms are the Maltsev and majority polymorphism. Existence of either kind of polymorphism guarantees a polynomial time algorithm for $\mathrm{CSP}(G)$ (for majority, see eg. [5], for the Maltsev polymorphism, see [4]), while if G has both these polymorphisms then $\mathrm{CSP}(G)$ is even solvable in deterministic logarithmic space (see [6]). The purpose of this paper is to prove that whenever a digraph G has a Maltsev polymorphism, then G has a majority polymorphism as well. The core idea of our proof is

☆Abbreviations used: CSP (Constraint Satisfaction Problem)

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that, given a Maltsev digraph G, we factorize G, obtain majority on the factorgraph G^+ by induction and then extend the majority to the original G.

We give an overview of the implications of our result for CSP in the Conclusions section.

2. Preliminaries

While our solution is rather elementary, the reader might still benefit from understanding the context in which we wrote this paper. A good summary of the combinatorics of digraph homomorphism can be found in [9], while [2] presents an overview of the algebraic techniques in CSP and [3] provides a good general introduction to universal algebra.

Through the paper, *digraph* will mean a finite directed graph with loops allowed. We will allow the null digraph, however the main result does not change if we demand that $V(G) \neq \emptyset$.

Definition 1. Let G be a digraph. The mapping $f : V(G)^n \to V(G)$ is a *polymorphism* if it is true that whenever we have (u_1, v_1), (u_2, v_2), $\ldots$, $(u_n, v_n) \in E(G)$ then

$$(f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)) \in E(G).$$

In this situation, we will also say that the mapping f is *compatible* with the edge relation $E(G)$.

Definition 2. Let G, H be digraphs. A mapping $f : V(G) \to V(H)$ is called a *homomorphism* if whenever $(u, v) \in E(G)$, we have $(f(u), f(v)) \in E(H)$.

Given a set of relations $R_1, \ldots R_n$, a primitive positive definition of a relation S is any formula using only variables, existential quantification, the equality relation, relations $R_1, \ldots, R_n$ and conjunctions. The following proposition is a part of the folklore of CSP (we invite the readers to prove the proposition as an exercise).

Proposition 3. *Let G be a digraph and R a relation defined by a primitive positive formula using the relation $E(G)$. Then all polymorphisms of G are compatible with R.*

Definition 4. A *Maltsev polymorphism* of a digraph G is any ternary polymorphism m such that the following equalities hold for all $x, y \in V(G)$:

$$m(x, y, y) = x$$
$$m(x, x, y) = y$$

Definition 5. A *majority* is any polymorphism M such that the following equalities hold for all $x, y \in V(G)$:

$$M(x, y, y) = y$$
$$M(y, x, y) = y$$
$$M(y, y, x) = y.$$

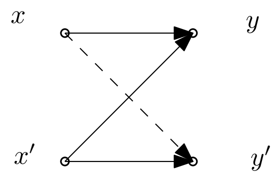

Figure 1: The vertices and edges in Observation 7.

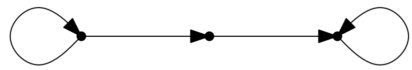

Figure 2: A rectangular digraph that is not Maltsev.

Definition 6. A digraph G is *Maltsev* if it has a Maltsev polymorphism. We say that G *has a majority* if there exists a majority polymorphism of G.

During the proof we will need the following notation: Let v be a vertex in a digraph G. We then denote by v^+ the vertex set $\{u \in V(G) : (v,u) \in E(G)\}$ and, similarly, by v^- the vertex set $\{u \in V(G) : (u,v) \in E(G)\}$. We will occasionally extend the mappings v^+ and v^- to whole sets of vertices.

We will call a vertex v a *source* if $v^- = \emptyset$ and a *sink* if $v^+ = \emptyset$. If v is neither a source nor a sink, we will call v *smooth*. For G a digraph, let $S^-(G)$ be the set of all sources of G and $S^+(G)$ be the set of all sinks in G.

3. Maltsev digraphs have majority

We begin with an easy but fundamental observation:

Observation 7. *Let G be a Maltsev digraph. If x, x', y, y' are (not necessarily all different) vertices of G and $(x,y), (x',y), (x',y') \in E(G)$ then also $(x,y') \in E(G)$ (see Figure 1).*

Proof. Let m be the Maltsev polymorphism of G. From the definition of a polymorphism we get $(m(x,x',x'), m(y,y,y')) \in E(G)$, but $m(x,x',x') = x$ and $m(y,y,y') = y'$, so $(x,y') \in E(G)$. □

Motivated by this observation we give the following definition:

Definition 8. Call a digraph G *rectangular* if whenever $(x,y), (x',y), (x',y') \in E(G)$ then also $(x,y') \in E(G)$.

All Maltsev digraphs are rectangular, however there are rectangular digraphs that are not Maltsev (see Figure 2; we will later see that this digraph violates Observation 12).

Definition 9. Let G be a digraph. We define two relations R^-, R^+ on $V(G)$ as follows:

$$xR^-y \Leftrightarrow \exists z, (z,x), (z,y) \in E(G)$$
$$xR^+y \Leftrightarrow \exists z, (x,z), (y,z) \in E(G).$$

Observe that the relations R^+ and R^- are symmetric. Also, the definition of R^+ and R^- is a primitive positive one, so any polymorphism of G is compatible with R^+ and R^-.

The following lemma is actually a collection of easy observations:

Lemma 10. *Let G be a rectangular digraph. Then the following holds:*

1. *If v is a sink then there is no x such that xR^+v.*
2. *If v is a source then there is no x such that xR^-v.*
3. *R^+ is an equivalence relation on $G \setminus S^+(G)$.*
4. *R^- is an equivalence relation on $G \setminus S^-(G)$.*
5. *Whenever xR^+y, we have $x^+ = y^+$ and x^+ is an equivalence class of R^-.*
6. *Whenever xR^-y, we have $x^- = y^-$ and x^- is an equivalence class of R^+.*
7. *The mapping $\phi : X \mapsto X^+$ is a bijection from the set of equivalence classes of R^+ to the set of equivalence classes of R^-.*

Proof. Parts (1) and (2) are easy: If $v^+ = \emptyset$, there is no z such that $(v, z) \in E(G)$ and therefore v can not be R^+-related to anything. Similarly for the dual case.

We know that the relation R^+ is symmetric on $V(G) \setminus S^+$. To prove reflexivity, consider $x \in V(G) \setminus S^+$. As $x \notin S^+$, there exists $z \in V(G)$ such that $(x, z) \in E(G)$ and so xR^+x.

From Observation 7, it follows that whenever xR^+y, we have $x^+ = y^+ \neq \emptyset$. From this we can easily get transitivity of R^+: If xR^+yR^+z, we have $x^+ = z^+ \neq \emptyset$ and so there exists t such that $(x, t), (z, t) \in E(G)$. Again, the proof of (4) is similar.

We already have half of (5), it remains to show that whenever $x^+ \neq \emptyset$, the set x^+ is an equivalence class of R^-. Obviously, x is a witness that all the vertices of x^+ are R^--related. If now $u \in x^+$ and vR^-u, then Observation 7 gives us that $(x, v) \in E(G)$ and so $v \in x^+$, concluding the proof. Once more, the statement (6) is a dual version of (5).

To prove (7), observe that if X is an equivalence class of R^+ then $X = (X^+)^-$ and similarly whenever Y is an equivalence class of R^-, we have $Y = (Y^-)^+$. The mapping ϕ is invertible and therefore is a bijection. ☐

Let G be a rectangular digraph. Denote by G^+ the digraph whose vertices are the equivalence classes of R^+ with $(X, Y) \in E(G^+)$ iff there exist vertices $x \in X$ and $y \in Y$ in $V(G)$ such that $(x, y) \in E(G)$. Similarly, let G^- be the digraph whose vertices are the R^- equivalence classes with $(X, Y) \in E(G^-)$ iff there exist vertices $x \in X$ and $y \in Y$ such that $(x, y) \in E(G)$.

Lemma 11. *Let G be a rectangular digraph and ϕ the mapping from part (7) of Lemma 10. Then the mapping ϕ is an isomorphism of G^+ to G^-.*

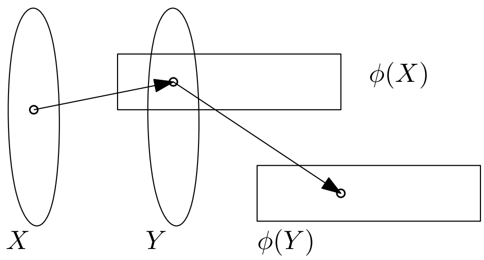

Figure 3: Picture proof of Lemma 11.

Proof. See Figure 3 for a picture of the proof.

First observe that we have $(X, Y) \in E(G^+)$ iff $X^+ \cap Y \neq \emptyset$ in G. But $X^+ = \phi(X)$, therefore $XY \in E(G^+)$ iff $\phi(X) \cap Y \neq \emptyset$. Similarly, $XY \in E(G^-)$ iff $X \cap \phi^{-1}(Y) \neq \emptyset$.

We know that ϕ is a bijection from $V(G^+)$ onto $V(G^-)$. We need to show that $(X, Y) \in E(G^+)$ iff $(\phi(X), \phi(Y)) \in E(G^-)$. However, we already have a chain of equivalent statements:

$$\begin{aligned}(X, Y) &\in E(G^+)\\ \phi(X) \cap Y &\neq \emptyset\\ \phi(X) \cap \phi^{-1}(\phi(Y)) &\neq \emptyset\\ (\phi(X), \phi(Y)) &\in E(G^-),\end{aligned}$$

which is precisely what we wanted. □

So far we have used only rectangularity. However, the following observation is not true for rectangular digraphs (try it for the digraph in Figure 2).

Observation 12. *Let G be a Maltsev digraph. Then G^+ is also Maltsev.*

Proof. Consider the Maltsev polymorphism m of G. Define the map t on G^+ by letting

$$t(x/_{R^+}, y/_{R^+}, z/_{R^+}) = m(x, y, z)/_{R^+}$$

for x, y, z vertices in $V(G) \setminus S^+$.

As the operation m is compatible with the relation R^+, t is well-defined. Moreover, t satisfies the Maltsev equations and a little thought gives us that t is a polymorphism of G^+. Therefore, G^+ is Maltsev. □

We are now ready to prove Theorem 13.

Theorem 13. *Any Maltsev digraph has a majority polymorphism.*

Proof. Let H be a vertex-minimal digraph such that H is Maltsev but has no majority. We will show that this leads to a contradiction.

Let us first consider the case $|V(H^+)| = |V(H^-)| = |V(H)|$. This is only possible when H has no sources or sinks and every R^+ or R^--class of H is a singleton. From Lemma 10 we obtain that H is then the digraph of the permutation ϕ and therefore is a disjoint union of directed cycles (we consider the null digraph to be an empty union of directed cycles). It is easy to verify that the mapping M defined as

$$M(x,y,z) = \begin{cases} y & \text{if } y = z, \\ x & \text{else,} \end{cases}$$

is a majority polymorphism of H.

We can thus assume that $|V(H^+)| < |V(H)|$. As H is the smallest counterexample and H^+ is Maltsev by Observation 12, there exists a majority polymorphism M^+ of H^+. Denote by M^- the polymorphism of H^- conjugated to M^+ via ϕ, i.e.

$$M^-(x,y,z) = \phi(M^+(\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(z))).$$

We now want to find a map $M(x,y,z)$ on H so that the following holds:

1. $M(x,x,y) = M(x,y,x) = M(y,x,x) = x$.
2. If $x,y,z \notin S^+$ then $M(x,y,z)/_{R^+} = M^+(x/_{R^+}, y/_{R^+}, z/_{R^+})$.
3. If $x,y,z \notin S^-$ then $M(x,y,z)/_{R^-} = M^-(x/_{R^-}, y/_{R^-}, z/_{R^-})$.

We will later prove that any such M is a polymorphism, concluding the proof. However, we will not explicitly demand M to be a polymorphism for now.

We can construct M by choosing, for each triple $(x,y,z) \in V(G)^3$ an image that satisfies (1)–(3). However, we need to show that the candidate set is nonempty for each choice of (x,y,z).

As M^+ and M^- are majority polymorphisms, the equalities (2) and (3) follow from (1) whenever two of the variables x,y,z are the same. Therefore, the only way that the value $M(x,y,z)$ can fail to exist is if for some x_1,x_2,x_3 vertices in $V(G) \setminus (S^+(G) \cup S^-(G))$ we would have

$$M^+(x_1/_{R^+}, x_2/_{R^+}, x_3/_{R^+}) \cap M^-(x_1/_{R^-}, x_2/_{R^-}, x_3/_{R^-}) = \emptyset.$$

Can such a thing happen? We know that for $i = 1,2,3$ the set $x_i/_{R^+} \cap x_i/_{R^-}$ is nonempty. Therefore, we have

$$\phi(\phi^{-1}(x_i/_{R^-})) \cap x_i/_{R^+} \neq \emptyset$$

and, as in the proof of Lemma 11, we obtain that

$$\left(\phi^{-1}(x_i/_{R^-}), x_i/_{R^+}\right) \in E(H^+)$$

for each $i = 1,2,3$. Applying the polymorphism M^+, we obtain

$$\left(M^+(\phi^{-1}(x_1/_{R^-}), \phi^{-1}(x_2/_{R^-}), \phi^{-1}(x_3/_{R^-})), M^+(x_1/_{R^+}, x_2/_{R^+}, x_3/_{R^+})\right) \in E(H^+),$$

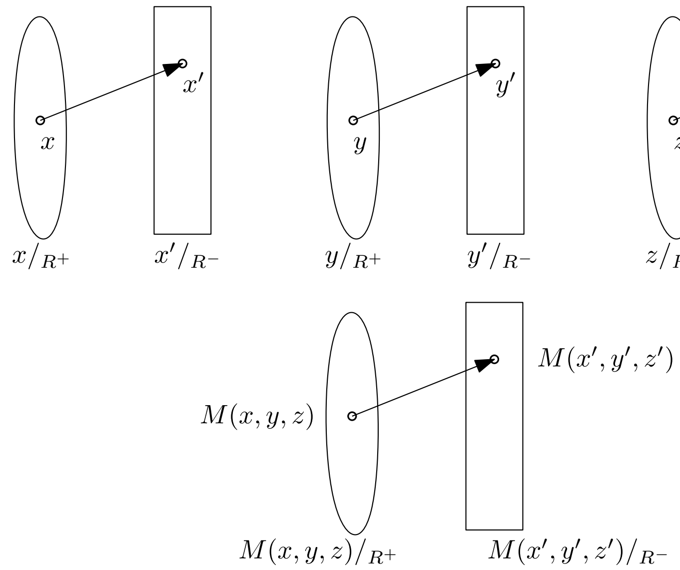

Figure 4: Showing that M is a polymorphism.

but this is precisely the same as

$$\phi(M^+(\phi^{-1}(x_1/_{R^-}), \phi^{-1}(x_2/_{R^-}), \phi^{-1}(x_3/_{R^-}))) \cap M^+(x_1/_{R^+}, x_2/_{R^+}, x_3/_{R^+}) \neq \emptyset.$$

Now recall the definition of M^- to see that we have just shown that

$$M^-(x_1/_{R^-}, x_2/_{R^-}, x_3/_{R^-}) \cap M^+(x_1/_{R^+}, x_2/_{R^+}, x_3/_{R^+}) \neq \emptyset.$$

Therefore, there exists a map M satisfying conditions (1)–(3). By (1) this M satisfies the majority equations. It remains to show that this M is in fact a polymorphism of H.

Let $(x, x'), (y, y'), (z, z') \in E(H)$. We want to show that $(M(x, y, z), M(x', y', z')) \in E(H)$. Obviously x, y, z are not sinks and x', y', z' are not sources. Moreover, a little thought gives us that (see Figure 4):

$$\phi^{-1}(x'/_{R^-}) = x/_{R^+}, \quad \phi^{-1}(y'/_{R^-}) = y/_{R^+}, \quad \phi^{-1}(z'/_{R^-}) = z/_{R^+}.$$

But then

$$\begin{aligned} M(x', y', z')/_{R^-} &= M^-(x'/_{R^-}, y'/_{R^-}, z'/_{R^-}) \\ &= \phi(M^+(\phi^{-1}(x'/_{R^-}), \phi^{-1}(y'/_{R^-}), \phi^{-1}(z'/_{R^-}))) \\ &= \phi(M^+(x/_{R^+}, y/_{R^+}, z/_{R^+})). \end{aligned}$$

Now observe that actually $M(x, y, z)^+ = \phi(M^+(x/_{R^+}, y/_{R^+}, z/_{R^+}))$. Putting the last two equalities together, we obtain $M(x, y, z)^+ = M(x', y', z')/_{R^-}$ which can only happen when we have $(M(x, y, z), M(x', y', z')) \in E(H)$, concluding the proof. $\square$

It is straightforward to translate Theorem 13 to the language of universal algebra varieties (see [3] for background and details):

Corollary 14. *If V is a variety generated by the algebra of all polymorphisms of some digraph G then V is congruence permutable iff V is arithmetic.*

Let us close this section with a description of the class of all Maltsev digraphs.

Lemma 15. *Let G be a rectangular digraph. Then G is Maltsev iff G^+ is Maltsev.*

Proof. We already know the "$\Rightarrow$" implication from Observation 12.

On the other hand, if m^+ is a Maltsev polymorphism of G^+, we can use a construction similar to the one from the proof of Theorem 13 to obtain a Maltsev polymorphism m of G. $\square$

From this lemma we see that if we start from a disjoint union of directed cycles G_0 and then in each step choose a rectangular digraph G_{i+1} so that $(G_{i+1})^+ = G_i$, all the graphs $G_0, G_1, \dots$ will be Maltsev. Moreover, every Maltsev digraph can be obtained in this way (with a suitable choice of the sequence $G_0, G_1, \dots, G_n$) because every Maltsev digraph becomes a disjoint union of directed cycles ofter applying the $^+$ operation sufficiently many times.

Let us state our findings in a more compact form:

Corollary 16. *The class of all Maltsev digraphs $\mathcal{M}$ is the smallest class of digraphs such that:*

1. *All digraphs in $\mathcal{M}$ are rectangular,*
2. *$\mathcal{M}$ contains all disjoint unions of directed cycles and all edgeless digraphs,*
3. *$\mathcal{M}$ is closed under taking the preimages under the map $G \mapsto G^+$ (i.e. if $H \in \mathcal{M}$ and G is rectangular such that $G^+ = H$ then $G \in \mathcal{M}$).*

Note: We explicitly mention edgeless digraphs in part (2) so that the corollary is true even if we disallow the null digraph.

4. Maltsev digraphs and the CSP

We conclude our paper with a note about the connections with the Constraint Satisfaction Problem. However, we must first introduce two new notions: adding constants and the Datalog language.

Observe that both majority and Maltsev polymorphisms preserve the unary constant relation $c_v = \{(v)\}$ for every $v \in V(G)$ (because $m(v,v,v) = M(v,v,v) = v$). Therefore, we can "enhance" any Maltsev digraph G by adding one constant relation for every $v \in V(G)$. Call the resulting relational structure G_c. Observe that $\mathrm{CSP}(G_c)$ is essentially the problem of determining whether a given partial mapping $V(H) \to V(G)$ can be extended to a digraph homomorphism $H \to G$. It is not difficult to observe that CSP(G_c) is at least as hard as CSP(G). To make our results more meaningful, we will now be talking about the complexity of CSP(G_c).

There is an important class of CSPs that can be solved using the Datalog language or some subset thereof (note that "can be solved" actually means that the *complements* of these problems lie in the DATALOG class, however that is just a technicality). Another name for such CSPs is problems of *bounded width* (see [1] for an overview).

Putting Theorem 13 together with known body of knowledge about Datalog, we obtain that if G is a Maltsev digraph then CSP(G_c) can be solved using a rather simple kind of consistency test in logarithmic space.

As shown in [5], if G admits a majority polymorphism, then CSP(G_c) can be solved using linear Datalog (in nondeterministic logarithmic space). On the other hand, Maltsev polymorphism in general relational structure G does *not* guarantee that there is a Datalog solution to CSP(G_c). However, if G is actually a digraph, then Maltsev implies majority by Theorem 13 and hence there exists a linear Datalog solution. We can improve this statement further, as [6] tells us that in this case it is enough to use the so-called symmetric Datalog, ensuring that CSP(G_c) is solvable in *deterministic* logarithmic space.

5. Conclusions and open problems

Digraphs are a rather versatile structures that can often "emulate" other structures in various ways (see eg. [9] or Section 5 of [7]). Our result, however, shows that sometimes digraphs are not general enough: In general relational structures, Maltsev and majority polymorphisms are independent of each other while in digraphs Maltsev implies majority.

Therefore, we would like to know what makes digraphs behave like this. And, perhaps more importantly, what other implications of this kind (i.e. "If G has a polymorphism s then G has a polymorphism t.") hold for digraphs but not for general relational structures?

Finally, two more direct future tasks spring to mind: First, to characterize all Maltsev digraphs in a more explicit way than Corollary 16 and, returning to combinatorics, to count the number of all Maltsev digraphs on n vertices (or at least give some asymptotics).

6. Acknowledgements

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