Bounds on three- and higher-distance sets

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October 30, 2018

Abstract

A finite set X in a metric space M is called an s-distance set if the set of distances between any two distinct points of X has size s. The main problem for s-distance sets is to determine the maximum cardinality of s-distance sets for fixed s and M. In this paper, we improve the known upper bound for s-distance sets in the n-sphere for $s = 3, 4$. In particular, we determine the maximum cardinalities of three-distance sets for $n = 7$ and 21. We also give the maximum cardinalities of s-distance sets in the Hamming space and the Johnson space for several s and dimensions.

Key words: s-distance set, two-point-homogeneous space.

1 Introduction

A finite subset X of the Euclidean space $\mathbb{R}^n$ or the unit sphere S^{n-1} is called an s-distance set (or s-code) if there exist s Euclidean distances between two distinct vectors in X. The main problem for s-distance sets is to determine the maximum cardinality of s-distance sets for fixed s and n.

Bannai, Bannai and Stanton [2] proved that the size of s-distance sets in $\mathbb{R}^n$ is bounded above by $\binom{n+s}{s}$. When $s \geq 2$, we know only one example attaining this upper bound, namely, for $(n, s) = (8, 2)$ [17]. The maximum cardinality of s-distance sets in $\mathbb{R}^n$ are determined for the following n and s [6, 14, 17].

n	2	3	4	5	6	7	8
size	5	6	10	16	27	29	45

Table 1: Maximum cardinalities of two-distance sets in $\mathbb{R}^n$.

s	2	3	4	5
size	5	7	9	12

Table 2: Maximum cardinalities of s-distance sets in $\mathbb{R}^2$.

*Research supported in part by NSF grant DMS-0807640 and NSA grant MSPF-08G-201.
†Research supported by JSPS Research Fellowship. The second author stays at the University of Texas at Brownsville from August 24th, 2009 to August 23rd, 2010.
2010 Mathematics Subject Classification: 05E30 (94B65).

Moreover, Shinohara [24] proved the icosahedron is the unique maximum three-distance set in $\mathbb{R}^3$.

Delsarte, Goethals, and Seidel proved that the largest cardinality of s-distance sets in S^{n-1} is bounded above by $\binom{n+s-1}{s} + \binom{n+s-2}{s-1}$. In the circle, the regular $(2s+1)$-gons attain this upper bound. When $n \geq 3$, we have two examples attaining this upper bound, namely, for $(n,s) = (6,2), (22,2)$ [9]. We have the following results for the maximum cardinalities of two-distance sets in S^{n-1} [9, 19].

n	2	3	4	5	6	$7 \cdots 21$	22	$24 \cdots 39$
size	5	6	10	16	27	$\frac{n(n+1)}{2}$	275	$\frac{n(n+1)}{2}$

Table 3: Maximum cardinalities of two-distance sets in S^{n-1}.

When $s \geq 3$, we have only one result, namely, that of Shinohara [24] for $(n,s) = (3,3)$.

Recently, Musin [19] determined the maximum cardinalities of two-distance sets in S^{n-1} for $7 \leq n \leq 21$ and $24 \leq n \leq 39$ by a certain general method. This method needs three theorems, namely, Delsarte's linear programming bound, Larman-Rogers-Seidel's theorem and a certain useful bound. This bound in [19] is the following: for two-distance sets in S^{n-1} with inner products a_1 and a_2, if $a_1 + a_2 \geq 0$, then the size of two-distance set is at most $\binom{n+1}{2}$. Larman, Rogers, and Seidel proved that if the size of a two-distance set in $\mathbb{R}^n$ with distances b_1 and b_2 ($b_1 > b_2$) is greater than $2n+3$, then the ratio b_1^2/b_2^2 is equal to $k/(k-1)$ where k is a positive integer bounded above by some function of n [15]. This method in [19] is applicable to s-distance sets in a two-point-homogeneous space M with a certain assumption.

Nozaki extended the upper bound in [19] to spherical s-distance sets for any s [22]. This upper bound is applicable to M. By this generalized bound, Barg and Musin [4] gave the maximum s-distance sets in the Hamming space and the Johnson space for some s and small dimensions. Larman-Rogers-Seidel's theorem is also extended to s-distance sets for any s [21]. This theorem is also applicable to s-distance sets in M.

In the present paper, we improve the known upper bound for s-distance sets in S^{n-1} by the method in [19] with the generalized Larman-Rogers-Seidel's theorem and the Nozaki upper bound. In particular, we determine the maximum cardinalities of three-distance sets in S^7 and S^{21}. We also give the maximum cardinalities of s-distance sets in the Hamming space and the Johnson space for some $s \geq 3$ and more dimensions.

2 Few distance sets in two-point-homogeneous spaces

2.1 Basic definitions

In this subsection, we introduce the concept of two-point-homogeneous spaces M and our restrictive assumption [5, Chapter 9], [13, 16].

Let G be a finite group or a connected compact group. We call M a two-point-homogeneous G-space if M holds the following properties:

(1) M is a set on which G acts.

(2) M is a metric space with a distance function τ.

(3) τ is strongly invariant under G: for any $x, x', y, y' \in M$, $\tau(x,y) = \tau(x',y')$ if and only if there is an element $g \in G$ such that $g(x) = x'$ and $g(y) = y'$.

Let H be the subgroup of G that fixes a particular element $x_0 \in M$. Then M can be identified with the space G/H of left cosets gH. Throughout the present paper, we assume the following:

(1) If G is infinite, then M is a connected Riemannian manifold and τ is a constant times the natural distance on the manifold.

(2) If G is finite, and $d_0 = \min \tau(x,y)$ for $x, y \in M$, $x \neq y$, then M has the structure of a graph in which x is adjacent to y if and only if $\tau(x,y) = d_0$, and furthermore τ is a constant times the natural distance in the graph.

Under our assumptions, if G is infinite then Wang [26] proved that M is a sphere; real, complex or quaternionic projective space; or the Cayley projective plane. The finite two-point-homogeneous spaces have not yet been completely classified.

Let μ be the Haar measure, which is invariant under G. This induces a unique invariant measure on M, which will also be denoted by μ. We assume that μ is normalized so that $\mu(M) = 1$. Let $L^2(G)$ denote the vector space of complex-valued functions u on G, satisfying

$$\int_G |u(g)|^2 d\mu(g) < \infty$$

with inner product

$$(u_1, u_2) = \int_G u_1(g)\overline{u_2(g)} d\mu(g).$$

Those $u \in L^2(G)$ that are constant on left cosets of H can be regarded as belonging to $L^2(M)$, which is defined similarly and has the inner product

$$(u_1, u_2) = \int_M u_1(x)\overline{u_2(x)} d\mu(x).$$

The space $L^2(M)$ decomposes into a countable direct sum of mutually orthogonal subspaces $\{V_k\}_{k=0,1,\ldots}$ called (generalized) spherical harmonics. Let $\{\phi_{k,i}\}_{i=1}^{h_k}$ be an orthonormal basis for V_k, where $h_k = \dim V_k$. Since M is distance transitive, the function

$$\Phi_k(x,y) := \frac{1}{h_k} \sum_{i=1}^{h_k} \phi_{k,i}(x)\overline{\phi_{k,i}(y)}$$

depends only on $\tau(x,y)$. This expression is called the addition formula, and $\Phi_k(\tau)$ is called the zonal spherical function associated with V_k. It is immediate from the definition that Φ_k is positive definite, that is,

$$\sum_{x \in X} \sum_{y \in X} \Phi_k(\tau(x,y)) \geq 0$$

for any $X \subset M$. For all infinite M and for all currently known finite cases, $\{\Phi_i\}$ form families of classical orthogonal polynomials. We suppose that the degree of Φ_k is k. Note that $\Phi_k(\tau_0) = 1$.

We define

$$D(X) = \{\tau(x, y) \mid x, y \in X, x \neq y\}$$

for a finite set X in a two-point-homogeneous space M. The finite set X is called an s-distance set (or s-code) if $|D(X)| = s$. Let $A(M, s)$ be the maximum cardinality of s-distance sets in M.

2.2 Delsarte's linear programming bound

The following bound is known as Delsarte's linear programming bound, and give a good evaluation for some $D(X)$.

Theorem 2.1. *Let X be an s-distance set with $D(X) = \{d_1, d_2, \ldots, d_s\}$. Then*

$$|X| \leq \max\{1 + \alpha_1 + \cdots + \alpha_s \mid \sum_{i=1}^{s} \alpha_i \Phi_k(d_i) \geq -1, k \geq 0; \\ \alpha_i \geq 0, i = 1, 2, \ldots, s\}.$$

The following is corresponding to the dual problem of the above linear programming problem.

Theorem 2.2. *Let X be an s-distance set with $D(X) = \{d_1, d_2, \ldots, d_s\}$. Choose a natural number m. Then*

$$|X| \leq \min\{1 + f_1 + \cdots + f_m \mid \sum_{k=1}^{m} f_k \Phi_k(d_i) \leq -1, i = 1, 2, \ldots s; \\ f_i \geq 0, i = 1, 2, \ldots, s\}.$$

2.3 Harmonic absolute bound

The following upper bound was proved by Delsarte [7, 8, 16].

Theorem 2.3. *Let X be an s-distance set in M. Then*

$$|X| \leq \sum_{i=0}^{s} h_i.$$

Nozaki improved the above bound [22].

Theorem 2.4. *Let X be an s-distance set in M with $D(X) = \{d_1, d_2, \ldots, d_s\}$. Consider the polynomial $f(t) = \prod_{i=1}^{s}(d_i - t)/(d_i - \tau_0)$ and suppose that its expansion in the basis $\{\Phi_k\}$ has the form $f(t) = \sum_{i=0}^{s} f_i \Phi_i(t)$. Then*

$$|X| \leq \sum_{i : f_i > 0} h_i.$$

When the coefficients f_i are all positive, the bound coincides with the bound in Theorem 2.3.

2.4 LRS type theorem

Let

$$N(M,s) := h_0 + h_1 + \cdots + h_{s-1}.$$

For $d_1, d_2, \ldots, d_s$, we define the value

$$K_i := \prod_{j \neq i} \frac{d_j - \tau_0}{d_j - d_i}$$

for each $i \in \{1, 2, \ldots, s\}$. The following theorem is a good constraint to improve the upper bound [21].

Theorem 2.5. *Let X be an s-distance set in M with $D(X) = \{d_1, d_2, \ldots, d_s\}$. If $|X| \geq 2N(M,s)$, then K_i is an integer for each $i \in \{1, 2, \ldots, s\}$. Moreover, $|K_i| \leq \lfloor 1/2 + \sqrt{N(M,s)^2/(2N(M,s)-2) + 1/4} \rfloor$.*

The numbers K_i have the following properties.

Theorem 2.6. *For any $j \in \{0, 1, \ldots s-1\}$, we have $\sum_{i=1}^{s} d_i^j K_i = \tau_0^j$.*

Proof. For each $j \in \{1, 2, \ldots, s\}$, we define the polynomial

$$L_j(x) := \sum_{i=1}^{s} d_i^j \prod_{k \neq i} \frac{x - d_k}{d_i - d_k}$$

of degree at most $s-1$. Then the property $L_j(d_i) = d_i^j$ holds for any $i \in \{1, 2, \ldots, s\}$. The polynomial of degree at most $s-1$, that is interpolating distinct s points, is unique. Therefore we can determine $L_j(x) = x^j$. □

Corollary 2.7. (1) *When $s = 2$, we have*

$$d_1 = \frac{\tau_0 - d_2 K_2}{K_1}.$$

(2) *When $s = 3$, if $d_1 > d_2$, then*

$$d_1 = \frac{\tau_0 K_1 - d_3 K_1 K_3 - (d_3 - \tau_0)\sqrt{-K_1 K_2 K_3}}{K_1(K_1 + K_2)},$$

$$d_2 = \frac{\tau_0 K_2 - d_3 K_2 K_3 + (d_3 - \tau_0)\sqrt{-K_1 K_2 K_3}}{K_2(K_1 + K_2)}.$$

Proof. We solve the system of equations given by Theorem 2.6 □

Remark 2.8. For $s \geq 4$, there is no simple solution of the system of equations given by Theorem 2.6.

Corollary 2.9. *If $d_1 > d_2 > \cdots > d_s > \tau_0$ (i.e. $\tau(\rho)$ is a monotone increasing function) or $d_1 < d_2 < \cdots < d_s < \tau_0$ (i.e. $\tau(\rho)$ is a monotone decreasing function), then $|K_1| < |K_2|$.*

Proof. This is immediate because

$$\left| \frac{K_1}{K_2} \right| = \left| \frac{\tau_0 - d_2}{\tau_0 - d_1} \cdot \frac{d_3 - d_2}{d_3 - d_1} \cdot \cdots \cdot \frac{d_s - d_2}{d_s - d_1} \right| < 1.$$

□

2.5 New bounds

Let $\mathfrak{D}(M,s)$ be the set of all possible s distances $D(X) = \{d_1, d_2, \dots, d_s\}$ satisfying that K_i are integers. For each $D \in \mathfrak{D}(M,s)$, we have the two bounds, those are the harmonic absolute bound $H(D)$ in Theorem 2.4, and Delsarte's linear programming bound $L(D)$. Then the following immediately holds.

Theorem 2.10. *Let $B(D) := \min\{H(D), L(D)\}$ for $D \in \mathfrak{D}(M,s)$. Then*

$$A(M,s) \le \max_{D \in \mathfrak{D}(M,s)} \{B(D), 2N(M,s) - 1\}.$$

3 Bounds on sets with few distances

3.1 Hamming space

In this section, we deal with the Hamming space $\mathbb{F}_2^n$ with the Hamming distance $\tau(x,y) := |\{i \mid x_i \neq y_i\}|$ where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. Then Φ_k is the Krawtchouk polynomial of degree k:

$$\Phi_k(x) := \binom{n}{k}^{-1} \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}.$$

We have $h_i = q^{-n} \binom{n}{i} (q-1)^i$.

When $2s \le n$, we can construct an s-distance set in $\mathbb{F}_2^n$ with $\sum_{i=0}^{\lfloor s/2 \rfloor} \binom{n}{s-2i}$ points. Namely, the example consists of all vectors having k ones for all $k \equiv s$ mod 2. We obtain a lower bound

$$A(\mathbb{F}_2^n, s) \ge \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} \binom{n}{s-2i} \tag{3.1}$$

for $2s \le n$.

Maximum two-distance sets are studied in [4].

Theorem 3.1. *If $6 \le n \le 74$ with the exception of the values $n = 47, 53, 59, 65, 70, 71$, or if $n = 78$, then $A(\mathbb{F}_2^n, 2) \le (n^2 - n + 2)/2$.*

We determine the maximum cardinalities of three- or four-distance sets in $\mathbb{F}_2^n$ for some n.

Theorem 3.2. (1) *If $8 \le n \le 22$, $24 \le n \le 33$, or $n = 36, 37, 44$, then $A(\mathbb{F}_2^n, 3) = n + \binom{n}{3}$.*

(2) *If $10 \le n \le 47$, then $A(\mathbb{F}_2^n, 4) = 1 + \binom{n}{2} + \binom{n}{4}$.*

Proof. In [4] it is proved that (1) for $8 \le n \le 22$ and $n = 24$, and (2) for $10 \le n \le 24$. Since $\mathbb{F}_2^n$ is finite, we can obtain the finite set $\mathfrak{D}(\mathbb{F}_2^n, s)$. We apply Theorem 2.10 for $\mathfrak{D}(M,s)$. Then this theorem follows from (3.1). □

Remark 3.3. We also have $A(\mathbb{F}_2^{23}, 3) = 2048$, which is obtained from the even subcode of the Golay code $\mathcal{G}_{23}$ (*i.e.* the dual code $\mathcal{G}_{23}^{\perp}$ [4, 16]). Our method can be applied for other relatively small s. For $s \ge 3$, the authors know no example whose cardinality is greater than the value in the lower bound (3.1) except for $\mathcal{G}_{23}^{\perp}$.

3.2 Johnson space

The binary Johnson space $\mathbb{F}_2^{n,w}$ consists of n-dimensional binary vectors with w ones, where $2w \leq n$. The distance is $\tau(x,y) = |\{i \mid x_i \neq y_i\}|/2$. Then Φ_k is the Hahn polynomial of degree k:

$$\Phi_k(x) := \sum_{j=0}^{k} (-1)^j \frac{\binom{k}{j}\binom{n+1-k}{j}}{\binom{w}{j}\binom{n-w}{j}} \binom{x}{j}.$$

We have $h_i = \binom{n}{i} - \binom{n}{i-1}$.

When $s \leq n-w$, we can construct s-distance sets in $\mathbb{F}_2^{n,w}$ with $\binom{n-w+s}{s}$ points. The example consists of the all vectors with $w-s$ ones in the first coordinates and the remaining s ones anywhere outside them. Therefore we have a lower bound

$$A(\mathbb{F}_2^{n,w}, s) \geq \binom{n-w+s}{s} \tag{3.2}$$

for $s \leq n-w$.

The case $s=2$ was already considered in [4].

Theorem 3.4. *If n and w satisfy any of the following conditions:*

$$\begin{aligned} 6 \leq n \leq 8 \quad & \textit{and } w = 3, \\ 9 \leq n \leq 11 \quad & \textit{and } 3 \leq w \leq 4, \\ 12 \leq n \leq 14 \textit{ or } 25 \leq n \leq 34 \quad & \textit{and } 3 \leq w \leq 5, \\ 15 \leq n \leq 24 \textit{ or } 35 \leq n \leq 46 \quad & \textit{and } 3 \leq w \leq 6, \end{aligned}$$

then $A(\mathbb{F}_2^{n,w}, 2) = (n-w+1)(n-w+2)/2$.

We also have $A(\mathbb{F}_2^{23,7}, 2) = 253$, which is obtained from the 253 vectors of weight 7 in the binary Golay code of length 23 [4], [18, p. 69]. The code attains the upper bound in Theorem 2.3. Let X be the set of the 253 vectors. We can compute an upper bound $A(\mathbb{F}_2^{24,8}, 2) \leq 253$ by the method in Barg–Musin's paper [4]. Though they did not mention the tightness about this bound, an attaining example is easily constructed by

$$Y := \{(1,u) \mid u \in X\}.$$

Clearly Y is a two-distance set $\mathbb{F}_2^{24,8}$ with 253 points, and hence $A(\mathbb{F}_2^{24,8}, 2) = 253$.

We give the following maximum cardinalities of three- or four-distance sets in $\mathbb{F}_2^{n,w}$ for some n and w.

Theorem 3.5. (1) *For $11 \leq n \leq 45$ and $4 \leq w \leq n/2$, we have $A(\mathbb{F}_2^{n,w}, 3) \leq h_0 + h_1 + h_3 = \binom{n}{3} - \binom{n}{2} + n$.*

(2) *If n and w satisfy any of the following conditions:*

$$\begin{aligned} 11 \leq n \leq 12 \quad & \textit{and } w = 4, \\ 13 \leq n \leq 15 \quad & \textit{and } 4 \leq w \leq 5, \\ 16 \leq n \leq 19 \quad & \textit{and } 4 \leq w \leq 6, \\ 20 \leq n \leq 24 \quad & \textit{and } 4 \leq w \leq 7, \\ 25 \leq n \leq 50 \quad & \textit{and } 4 \leq w \leq 8, \end{aligned}$$

then $A(\mathbb{F}_2^{n,w}, 3) = \binom{n-w+3}{3}$.

Proof. We have the finite set $\mathfrak{D}(\mathbb{F}_2^{n,w}, s)$. This theorem is immediate from the bound in Theorem 2.10 and (3.2). $\square$

Theorem 3.6. (1) *For $14 \leq n \leq 58$ and $5 \leq w \leq n/2$, we have $A(\mathbb{F}_2^{n,w}, 4) \leq h_0 + h_1 + h_2 + h_4 = \binom{n}{4} - \binom{n}{3} + \binom{n}{2}$.*

(2) *If n and w satisfy any of the following conditions:*

$$\begin{array}{rl} 15 \leq n \leq 16 & \text{and } w = 5, \\ 17 \leq n \leq 19 & \text{and } 5 \leq w \leq 6, \\ 20 \leq n \leq 24 & \text{and } 5 \leq w \leq 7, \\ 25 \leq n \leq 29 & \text{and } 5 \leq w \leq 8, \\ 30 \leq n \leq 34 \text{ or } 41 \leq n \leq 47 & \text{and } 5 \leq w \leq 9, \\ 35 \leq n \leq 40 \text{ or } 48 \leq n \leq 59 & \text{and } 5 \leq w \leq 10, \\ 60 \leq n \leq 70 & \text{and } 5 \leq w \leq 11, \end{array}$$

then $A(\mathbb{F}_2^{n,w}, 4) = \binom{n-w+4}{4}$.

Proof. This proof is the same as that of Theorem 3.5 $\square$

Remark 3.7. For relatively small s, we can obtain similar results. For $s \geq 3$, the authors know no example whose cardinality is greater than the value in the lower bound (3.2). We can regard a bound for s-distance sets in $\mathbb{F}_2^{n,w}$ as that for w-uniform s-intersecting families [4, 1, 10, 25].

3.3 Spherical space

For the unit sphere S^{n-1}, we use the usual inner product as τ. Then Φ_k is the Gegenbauer polynomial of degree k. The Gegenbauer polynomials G_k are defined by the following manner:

$$xG_k(x) = \lambda_{k+1} G_{k+1}(x) + (1 - \lambda_{k-1}) G_{k-1}(x)$$

where $\lambda_k = k/(n + 2k - 2)$, $G_0(x) \equiv 1$, and $G_1(x) = nx$. We have $\Phi_k(x) = G_k(x)/h_k$ where $h_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$.

We can construct an s-distance set in S^{n-1} with $\binom{n+1}{s}$ points for $2s \leq n+1$. The example consists of all vectors those are of length $n+1$, and have exactly s entries of 1 and $n+1-s$ entries of 0. Since the finite set is on the hyper plane which is perpendicular to the vector of all ones, we can regard it as a subset of S^{n-1}. Thus we have a lower bound

$$A(S^{n-1}, s) \geq \binom{n+1}{s} \tag{3.3}$$

for $2s \leq n+1$.

The following are new bounds on three- or four-distance sets in S^{n-1} for some n.

Theorem 3.8. (1) *$A(S^7, 3) = 120$ and $A(S^{21}, 3) = 2025$.*

(2) *$A(S^3, 3) \leq 27$, $A(S^4, 3) \leq 39$ and $A(S^6, 3) \leq 91$.*

(3) *For* $n = 6$ *or* $9 \le n \le 19$*, we have* $A(S^{n-1}, 3) \le h_1 + h_3 = n(n+1)(n+2)/6$.

(4) *For* $20 \le n \le 30$*, we have* $A(S^{n-1}, 3) \le h_0 + h_1 + h_3 = (n+3)(n^2+2)/6$.

(5) *For* $31 \le n \le 50$*, we have* $A(S^{n-1}, 3) \le h_2 + h_3 = (n^2-1)(n+6)/6$.

Proof. Let $X \subset S^{n-1}$ be a three-distance set with $D(X) = \{d_1, d_2, d_3\}$ where $d_1 < d_2 < d_3 < \tau_0 = 1$. By Corollary 2.7, we write

$$d_1 = \frac{K_1 - d_3K_1K_3 - (d_3-1)\sqrt{-K_1K_2K_3}}{K_1(K_1+K_2)},$$
$$d_2 = \frac{K_2 - d_3K_2K_3 + (d_3-1)\sqrt{-K_1K_2K_3}}{K_2(K_1+K_2)}.$$

The maximum inner product d_3 should be greater than zero. Otherwise the cardinality is smaller than $2n+1$ by Rankin's third bound [23], [11, page 16]. Dividing the range $0 < d_3 < 1$ into sufficiently many parts, we obtain finitely many choices of d_3. For finitely many choices of three inner products from K_i and d_3, we apply Theorem 2.10. Then the upper bound of $A(S^{n-1}, 3)$ is obtained numerically.

For $n = 8$ and $n = 22$, we have examples attaining the upper bounds. For $n = 8$, the examples can be constructed from subsets of the E_8 root system. Let X be the E_8 root system normalized to have the norm 1. We have $D(X) = \{0, -1, \pm 1/2\}$ and $|X| = 240$. There exists $Y \subset X$ such that $Y \cup (-Y) = X$ and $|Y| = |X|/2$. Then, $D(Y) = \{0, \pm 1/2\}$, and hence Y is a three-distance set with 120 points in S^7. For $n = 22$, the example is a subset of the minimum vectors in the Leech lattice. Let $X \subset S^{23}$ be the minimum vectors normalized to have the norm 1. For fixed $x, y \in X$ such that $\tau(x, y) = -1/4$, we obtain

$$Y = \{z \in X \mid \tau(z, x) = 1/2, \tau(z, y) = 0\}.$$

Then, $Y \subset S^{21}$ has 2025 points and $D(Y) = \{7/22, -1/44, -4/11\}$. ☐

Remark 3.9. We have a lot of maximum three-distance sets in S^7 up to orthogonal transformations because there exist many choices of subsets Y in the above proof. Only one maximum three-distance set in S^{21} is known, and hence it might be unique.

Remark 3.10. For the case $s = 2$, giving polynomials in Theorem 2.2 concretely, we obtained a similar result (see details in [19]). We can use this approach also for $s = 3$.

Theorem 3.11. (1) $A(S^4, 4) \le 99$*,* $A(S^5, 4) \le 153$ *and* $A(S^6, 4) \le 223$.

(2) *For* $8 \le n \le 15$ *or* $n = 18$*, we have* $A(S^{n-1}, 4) \le h_0 + h_2 + h_4 = n(n+1)(n+2)(n+3)/24$.

(3) *For* $16 \le n \le 17$*, we have* $A(S^{n-1}, 4) \le h_0 + h_3 + h_4 = (n+3)(n^3 + 7n^2 - 10n + 8)/24$.

(4) *For* $19 \le n \le 21$*, we have* $A(S^{n-1}, 4) \le h_2 + h_3 = d(n+5)(n^2+n+6)/24$.

Proof. The proof of this theorem is the same as that of Theorem 3.8 except for the way to obtain d_i. For given K_i and d_4, we find the solutions of the system of equations given by Theorem 2.6 numerically. ☐

It is possible to calculate for $s \geq 5$ or large n, but it takes much time and needs more memory. The following table shows an example whose size is greater than the value in the lower bound (3.3) for $s \geq 3$, and except for $(n, s) = (8, 3), (22, 3)$.

n	s	$\lvert X \rvert$	inner products	absolute bound	new bound	bound (3.3)
23	3	2300	$0, \pm\frac{1}{3}$	2576	2301	2024
8	4	240	$-1, 0, \pm\frac{1}{2}$	450	330	126
24	5	98280	$0, \pm\frac{1}{4}, \pm\frac{1}{2}$	115830	?	53130
24	6	196560	$-1, 0, \pm\frac{1}{4}, \pm\frac{1}{2}$	573300	?	177100

The examples in the above table are obtained from tight spherical designs, or their subsets [9, 16]. The methods in Theorems 3.8 and 3.11 are applicable to other projective spaces.

Remark 3.12. Our method is applicable to a Q-polynomial association scheme defined in [7] (also see [3]). A Q-polynomial association scheme is not always a two-point-homogeneous space. There are two concepts which include the projective spaces and Q-polynomial association schemes, namely, Q-polynomial spaces [12] and Delsarte spaces [20]. The method in the present paper is applicable to both of the two concepts.

Acknowledgments. The authors thank Alexander Barg, Grigori Kabatianski, Eiichi Bannai, Masashi Shinohara and Sho Suda for useful discussions and comments.

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