NORDHAUS-GADDUM FOR TREEWIDTH

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ABSTRACT. We prove that for every *n*-vertex graph G, the treewidth of G plus the treewidth of the complement of G is at least n-2. This bound is tight.

Nordhaus-Gaddum-type theorems establish bounds on $f(G) + f(\overline{G})$ for some graph parameter f, where \overline{G} is the complement of a graph G. The literature has numerous examples; see [1, 4, 5, 8, 10, 13, 14] for a few. Our main result is the following Nordhaus-Gaddum-type theorem for treewidth¹, which is a graph parameter of particular importance in structural and algorithmic graph theory. Let $\operatorname{tw}(G)$ denote the treewidth of a graph G.

Theorem 1. For every graph G with n vertices,

$$\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) \ge n - 2$$

The following lemma is the key to the proof of Theorem 1.

Lemma 2. Let G be a graph with n vertices, no induced 4-cycle, and no k-clique. Then $\operatorname{tw}(\overline{G}) \ge n-k$.

Proof. Let $\mathcal{B} := \{\{v, w\} : vw \in E(\overline{G})\}$. If $\{v, w\}$ and $\{x, y\}$ do not touch for some $vw, xy \in E(\overline{G})$, then the four endpoints are distinct and (v, x, w, y) is an induced 4-cycle in G, which is a contradiction. Thus \mathcal{B} is a bramble in \overline{G} . Let S be a hitting set for \mathcal{B} . Thus no edge in \overline{G} has both endpoints in $V(\overline{G}) \setminus S$. Hence $V(G) \setminus S$ is a clique in G. Therefore $n - |S| \leq k - 1$ and $|S| \geq n - k + 1$. That is, the order of \mathcal{B} is at least n - k + 1. By the Treewidth Duality Theorem, $\operatorname{tw}(\overline{G}) \geq n - k$, as desired.

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¹While treewidth is normally defined in terms of tree decompositions (see [3]), it can also be defined as follows. A graph G is a k-tree if $G \cong K_{k+1}$ or G - v is a k-tree for some vertex v whose neighbours induce a k-clique. Then the treewidth of a graph G is the minimum integer k such that G is a spanning subgraph of a k-tree. See [2, 11] for surveys on treewidth.

Let G be a graph. Two subsets of vertices A and B in G touch if $A \cap B \neq \emptyset$, or some edge of G has one endpoint in A and the other endpoint in B. A bramble in G is a set of subsets of V(G) that induce connected subgraphs and pairwise touch. A set S of vertices in G is a hitting set of a bramble \mathcal{B} if S intersects every element of \mathcal{B} . The order of \mathcal{B} is the minimum size of a hitting set. Seymour and Thomas [12] proved the Treewidth Duality Theorem, which says that a graph G has treewidth at least k if and only if G contains a bramble of order at least k + 1.

Proof of Theorem 1. Let $k := \operatorname{tw}(G)$. Let H be a k-tree that contains G has a spanning subgraph. Thus H has no induced 4-cycle (it is chordal) and has no (k+2)-clique. By Lemma 2 and since $\overline{G} \supseteq \overline{H}$, we have $\operatorname{tw}(\overline{G}) \ge \operatorname{tw}(\overline{H}) \ge n - k - 2$. That is, $\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) \ge n - 2$.

Lemma 2 immediately implies the following result of independent interest.

Theorem 3. For every graph G with girth at least 5, we have $tw(\overline{G}) \ge n-3$.

For k-trees we have the following precise result, which proves that the bound in Theorem 1 is tight. Let Q_n^k be the k-tree consisting of a k-clique C with n - k vertices adjacent only to C.

Theorem 4. For every k-tree G,

$$\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) = \begin{cases} n-1 & \text{if } G \cong Q_n^k \\ n-2 & \text{otherwise} \end{cases}.$$

Proof. First suppose that $G \cong Q_n^k$. Then \overline{G} consists of K_{n-k} and k isolated vertices. Thus $\operatorname{tw}(\overline{G}) = n - k - 1$, and $\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) = n - 1$.

Now assume that $G \not\cong Q_n^k$. By the definition of k-tree, V(G) can be labelled v_1, \ldots, v_n such that $\{v_1, \ldots, v_{k+1}\}$ is a clique, and for $j \in \{k+2, \ldots, n\}$, the neighbourhood of v_j in $G[\{v_1, \ldots, v_{j-1}\}]$ is a k-clique C_j . If C_{k+2}, \ldots, C_n are all equal then $G \cong Q_n^k$. Thus $C_j \neq C_{k+2}$ for some minimum integer j. Observe that each vertex in C_j has a neighbour outside of C_j . Arbitrarily label $C_j = \{x_1, \ldots, x_{k+1}\}$, and let y_i be a neighbour of each x_i outside of C_j .

We now describe an (n - k - 2)-tree H that contains \overline{G} . Let $A := V(G) \setminus C_j$ be the starting (n-k-1)-clique of H. Add each vertex x_i to H adjacent to $A \setminus \{y_i\}$. Observe that H is an (n-k-2)-tree and \overline{G} is a spanning subgraph of H. Thus $\operatorname{tw}(\overline{G}) \leq n-k-2$ and $\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) \leq n-2$, with equality by Theorem 1.

In view of Theorem 1, it is natural to also consider how large $\operatorname{tw}(G) + \operatorname{tw}(\overline{G})$ can be. Every *n*-vertex graph *G* satisfies $\operatorname{tw}(G) \leq n-1$, implying $\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) \leq 2n-2$. It turns out that this trivial upper bound is tight up to lower order terms. Indeed, Perarnau and Serra [9] proved that, if $G \in \mathcal{G}(n,p)$ is a random *n*-vertex graph with edge probability $p = \omega(\frac{1}{n})$ in the sense of Erdős and Rényi, then asymptotically almost surely $\operatorname{tw}(G) = n - o(n)$; see [6, 7] for related results. Setting $p = \frac{1}{2}$, it follows that asymptotically almost surely, $\operatorname{tw}(G) = n - o(n)$ and $\operatorname{tw}(\overline{G}) = n - o(n)$, and hence $\operatorname{tw}(G) + \operatorname{tw}(\overline{G}) = 2n - o(n)$.

Theorems 1 and 4 can be reinterpreted as follows.

Proposition 5. For all graphs G_1 and G_2 , the union $G_1 \cup G_2$ contains no clique on $tw(G_1) + tw(G_2) + 3$ vertices. Conversely, there exist graphs G_1 and G_2 such that $G_1 \cup G_2$ contains a clique on $tw(G_1) + tw(G_2) + 2$ vertices.

Proof. For the first claim, we may assume that $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. Let S be a clique in $G_1 \cup G_2$. Thus $G_1[S]$ and $G_2[S]$ are complementary. By Theorem 1, $\operatorname{tw}(G_1) + \operatorname{tw}(G_2) \ge$ $\operatorname{tw}(G_1[S]) + \operatorname{tw}(G_2[S]) \ge |S| - 2$. Thus $|S| \le \operatorname{tw}(G_1) + \operatorname{tw}(G_2) + 2$ as desired. The converse claim follows from Theorem 4. Proposition 5 suggests studying $G_1 \cup G_2$ further. For example, what is the maximum of $\chi(G_1 \cup G_2)$ taken over all graphs G_1 and G_2 with $\operatorname{tw}(G_1) \leq k$ and $\operatorname{tw}(G_2) \leq k$? By Proposition 5 the answer is at least 2k + 2. A minimum-degree greedy algorithm proves that $\chi(G_1 \cup G_2) \leq 4k$. This question is somewhat similar to Ringel's earth-moon problem which asks for the maximum chromatic number of the union of two planar graphs.

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