The Real-Rootedness and Log-concavities of Coordinator Polynomials of Weyl Group Lattices

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Abstract

It is well-known that the coordinator polynomials of the classical root lattice of type A_n and those of type C_n are real-rooted. They can be obtained, either by the Aissen-Schoenberg-Whitney theorem, or from their recurrence relations. In this paper, we develop a trigonometric substitution approach which can be used to establish the real-rootedness of coordinator polynomials of type D_n. We also find the coordinator polynomials of type B_n are not real-rooted in general. As a conclusion, we obtain that all coordinator polynomials of Weyl group lattices are log-concave.

Keywords: coordinator polynomial; log-concavity; real-rootedness; trigonometric substitution; Weyl group lattice

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1 Introduction

Let $f(x) = \sum_{i=1}^{n} a_i x^i$ be a polynomial of degree n with nonnegative coefficients. We say that $f(x)$ is *real-rooted* if all its zeros are real. Real-rooted polynomials have attracted much attention during the past decades. One of the most significant reasons is that for any polynomial, the real-rootedness implies the log-concavity of its coefficients, which in turn implies the unimodality of the coefficients. Indeed, unimodal and log-concave sequences occur naturally in combinatorics, algebra, analysis, geometry, computer science, probability and statistics. We refer the reader to the survey papers, Brenti [8] and Stanley [20], for various results on the unimodality and log-concavity.

There is a characterization of real-rooted polynomials in the theory of total positivity; see Karlin [14]. A matrix $(a_{ij})_{i,j\geq 0}$ is said to be *totally positive* if all its minors have nonnegative determinants. The sequence $\{a_k\}_{k=0}^{n}$ is called a *Pólya frequency sequence* if the lower triangular matrix $(a_{i-j})_{i,j=0}^{n}$ is totally positive, where a_k is set to be zero if $k < 0$. A basic link between Pólya frequency sequences and real-rooted polynomial was given by the Aissen-Schoenberg-Whitney theorem [1], which stated that the polynomial $f(x)$ is real-rooted if and only if the sequence $\{a_k\}_{k=0}^{n}$ is a Pólya frequency sequence. Another characterization from the probabilistic point of view can be found in Pitman [17], see also Schoenberg [18].

Polynomials arising from combinatorics are often real-rooted. Basic examples include the generating functions of binomial coefficients, of Stirling numbers of the first kind and of the second kind, of Eulerian numbers, and the matching polynomials; see, for example, Brenti [9, 10], Liu and Wang [16], Stanley [21] and Wang and Yeh [22].

This paper is concerned with the real-rootedness and the log-concavities of coordinator polynomials of Weyl group lattices.

Following Ardila et al. [2], we give an overview of the notions. Let $\mathcal{L}$ be a lattice, that is, a discrete subgroup of a finite-dimensional Euclidean vector space E. The dimension of the subspace spanned by $\mathcal{L}$ is called its rank. A lattice is said to be generated as a monoid if there exists a finite collection M of vectors such that every vector in the lattice is a nonnegative integer linear combination of the vectors in M. Suppose that $\mathcal{L}$ is a lattice of rank d, generated by M. For any vector v in $\mathcal{L}$, define the length of v with respect to M to be the minimum sum of the coefficients among all nonnegative integer linear combinations, denoted by $\ell(v)$. In other words,

$$\ell(v) = \min\left\{ \sum_{m\in M} c_m \;\middle|\; v = \sum_{m\in M} c_m m,\ c_m \geq 0 \right\}.$$

Let $S(k)$ be the number of vectors of length k in $\mathcal{L}$. Benson [5] proved that the generating function

$$\sum_{k\geq 0} S(k)x^k = \frac{h(x)}{(1-x)^d} \tag{1.1}$$

is rational, where $h(x)$ is a polynomial of degree at most d. Following Conway and Sloane [12], we call $h(x)$ the *coordinator polynomial* with respect to M.

We concern ourselves with the classical root lattices as $\mathcal{L}$. Let e_i denote the vector in E, having the ith entry one, and all other entries zero, where the space E is taken to be $\mathbb{R}^{n+1}$ for the root lattice A_n, and to be $\mathbb{R}^n$ for the root lattices B_n, C_n, and D_n. The root lattices can be defined to be generated as monoids respectively by

$$\begin{aligned}
M_{A_n} &= \left\{\pm(e_i - e_j) \,\middle|\, 1 \leq i < j \leq n+1\right\},\\
M_{B_n} &= \left\{\pm e_i \pm e_j \,\middle|\, 1 \leq i < j \leq n\right\} \cup \left\{\pm e_i \,\middle|\, 1 \leq i \leq n\right\},\\
M_{C_n} &= \left\{\pm e_i \pm e_j \,\middle|\, 1 \leq i < j \leq n\right\} \cup \left\{\pm 2e_i \,\middle|\, 1 \leq i \leq n\right\},\\
M_{D_n} &= \left\{\pm e_i \pm e_j \,\middle|\, 1 \leq i < j \leq n\right\}.
\end{aligned}$$

We denote the coordinator polynomial of type T by $h_T(x)$. Conway and Sloane [12] established the explicit expression

$$h_{A_n}(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k, \tag{1.2}$$

which were also called the Narayana polynomials of type B by Chen, Tang, Wang and Yang [11]. In fact, these polynomials appeared as the rank generating function of the lattice of noncrossing partitions of type B on the set $\{1, 2, \ldots, n\}$. With Colin Mallows's help, Conway and Sloane [12] conjectured that

$$h_{D_n}(x) = \frac{(1+\sqrt{x})^{2n} + (1-\sqrt{x})^{2n}}{2} - 2nx(1+x)^{n-2}. \tag{1.3}$$

Baake and Grimm [3] pointed out that the methods outlined in [12] can be used to deduce that the coordinator polynomials of type C have the expression

$$h_{C_n}(x) = \sum_{k=0}^{n} \binom{2n}{2k} x^k. \tag{1.4}$$

They also conjectured that

$$h_{B_n}(x) = \sum_{k=0}^{n} \binom{2n+1}{2k} x^k - 2nx(1+x)^{n-1}. \tag{1.5}$$

Bacher, de la Harpe and Venkov [4] rederived (1.2) and proved the formulas (1.3), (1.4) and (1.5). Recently, Ardila et al. [2] gave alternative proofs for (1.2), (1.3) and (1.4) by computing the f-vectors of a unimodular triangulation of the corresponding root polytope.

The real-rootedness of coordinator polynomials has received much attention. As pointed out by Conway and Sloane [12], coordinator polynomials of type A can be expressed as

$$h_{A_n}(x) = (1-x)^n L_n\left(\frac{1+x}{1-x}\right),$$

where $L_n(x)$ denotes the nth Legendre polynomial. Since Legendre polynomials are orthogonal, and thus real-rooted, we are led to the following result.

Theorem 1.1. *The coordinator polynomials of type A are real-rooted.*

In fact, Theorem 1.1 follows immediately from a classical result of Schur [19], see also Theorems 2.4.1 and 3.5.3 in Brenti [7]. For $h_{C_n}(x)$, one may easily deduce the real-rootedness by the Aissen-Schoenberg-Whitney theorem.

Theorem 1.2. *The coordinator polynomials of type C are real-rooted.*

Liang and Yang [15] reproved both Theorems 1.1 and 1.2 by establishing recurrences of the coefficients. Moreover, they verified the real-rootedness of coordinator polynomials of types E_6, E_7, F_4 and G_2. In contrast, $h_{E_8}(x)$ is not real-rooted. They also conjectured that $h_{D_n}(x)$ is real-rooted.

For coordinator polynomials of type B_n, we find $h_{B_{16}}(x)$ has 14 real roots and 2 non-real roots. So $h_{B_n}(x)$ are not real-rooted in general. In the next section, we develop a trigonometric substitution approach which enables us to confirm Liang-Yang's conjecture. In Section 3, we establish the log-concavities of all coordinator polynomials of Weyl group lattices.

2 The real-rootedness

In this section, we show the real-rootedness of $h_{D_n}(x)$.

Theorem 2.1. *The coordinator polynomials of type D are real-rooted.*

We shall adopt a technique of trigonometric transformation. To be precise, we transform the polynomial $h_{D_n}(x)$ into a trigonometric function, say, $g_n(\theta)$, and then consider the roots of $g_n(\theta)$. It turns out that the signs of $g_n(\theta)$ at a sequence of $n+1$ fixed values of θ are interlacing. Hence $g_n(\theta)$ has n distinct zeros in a certain domain, and so does $h_{D_n}(x)$.

Proof. We are going to show that the polynomial $h_{D_n}(x)$ has n distinct negative roots for $n \geq 2$. For this purpose, we let $y > 0$ and substitute $x = -y^2$ in the expression (1.3) of $h_{D_n}(x)$. Note that $\sqrt{-y^2}$ is two-valued, denoting $\pm yi$. However, taking $\sqrt{-y^2} = yi$ and taking $\sqrt{-y^2} = -yi$ yields the same expression of $h_{D_n}(-y^2)$, that is,

$$h_{D_n}(-y^2) = \frac{(1+yi)^{2n} + (1-yi)^{2n}}{2} + 2ny^2(1-y^2)^{n-2}. \tag{2.1}$$

Without loss of generality, we can suppose that

$$y = \tan\frac{\phi}{2},$$

where $\phi \in (0, \pi)$. Then $1 + yi = \sqrt{1+y^2}\, e^{i\phi/2}$, and thus

$$(1+yi)^{2n} + (1-yi)^{2n} = (1+y^2)^n e^{in\phi} + (1+y^2)^n e^{-in\phi} = 2(1+y^2)^n \cos n\phi.$$

It follows that

$$h_{D_n}(-y^2) = (1+y^2)^n \cos n\phi + 2ny^2(1-y^2)^{n-2} = \left(1+y^2\right)^n g_n(\phi),$$

where

$$g_n(\phi) = \cos n\phi + \frac{n}{2}\sin^2\phi\cos^{n-2}\phi. \tag{2.2}$$

Now it suffices to prove that the function $g_n(\phi)$ has n distinct roots ϕ in the interval $(0, \pi)$. Let

$$h_n(\phi) = \frac{n}{2}\sin^2\phi\cos^{n-2}\phi.$$

We claim that

$$|h_n(\phi)| < 1. \tag{2.3}$$

In fact, by the arithmetic-geometric mean inequality,

$$\begin{aligned}
h_n^2(\phi) &= \Big(\frac{n}{2}\sin^2\phi\Big)\Big(\frac{n}{2}\sin^2\phi\Big)\big(\cos^2\phi\big)^{n-2} \\
&\le \Big(\frac{n}{2}\sin^2\phi + \frac{n}{2}\sin^2\phi + (n-2)\cos^2\phi\Big)^n \Big/ n^n \qquad (2.4)\\
&= \Big(1 - \frac{2\cos^2\phi}{n}\Big)^n \le 1. \qquad (2.5)
\end{aligned}$$

Note that the equality in (2.4) holds if and only if

$$\frac{n}{2}\sin^2\phi = \cos^2\phi, \tag{2.6}$$

while the equality in (2.5) holds if and only if

$$\cos\phi = 0. \tag{2.7}$$

However, the conditions (2.6) and (2.7) contradict each other. So the equality in (2.5) does not hold. This confirms the claim (2.3).

Let j be an integer. From (2.2), we see that

$$g_n\Big(\frac{j\pi}{n}\Big) = (-1)^j + h_n\Big(\frac{j\pi}{n}\Big).$$

Since $|h(\phi)| < 1$ for any ϕ, we have

$$(-1)^j g_n\Big(\frac{j\pi}{n}\Big) = 1 + (-1)^j h_n\Big(\frac{j\pi}{n}\Big) > 0.$$

By its continuity, we obtain that $g_n(\phi)$ has roots $\phi_0, \phi_1, \ldots, \phi_{n-1}$ such that

$$0 < \phi_0 < \frac{\pi}{n} < \phi_1 < \frac{2\pi}{n} < \phi_2 < \frac{3\pi}{n} < \cdots < \frac{(n-1)\pi}{n} < \phi_{n-1} < \pi.$$

In conclusion, the polynomial $h_{D_n}(x)$ has n distinct negative roots

$$x_j = -\tan^2\frac{\phi_j}{2}, \qquad j = 0, 1, \ldots, n-1.$$

This completes the proof. ∎

3 The log-concavity

In this section, we consider the log-concavities of coordinator polynomials of Weyl group lattices. For basic notions on the Weyl group, see Humphreys [13]. By the definition (1.1), it is easy to see that the coordinator polynomial of any Weyl group lattice is the product of the coordinator polynomials of the Weyl group lattices determined by the irreducible components. This has been noticed by, for instance, Conway and Sloane [12, Page 2373]. By the Cauchy-Binet theorem, the product of log-concave polynomials with nonnegative coefficients and no internal zero coefficients are log-concave; see Stanley [20, Proposition 2]. Therefore, we are led to consider the coordinator polynomials of the Weyl group lattices which are determined by irreducible root systems. By the Cartan-Killing classification, irreducible root systems can be classified into types A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 and G_2. For historical notes, see Bourbaki [6]. To conclude, we have the following result.

Theorem 3.1. *All coordinator polynomials of Weyl group lattices are log-concave.*

Proof. It is straightforward to verify the log-concavity of the coordinator polynomials of types E_6, E_7, E_8, F_4 and G_2. By Theorems 1.1, 1.2 and 2.1, it suffices to show the log-concavity of $h_{B_n}(x)$. Let b_k be the coefficient of x^k in $h_{B_n}(x)$, and let $b'_k = b_k/\binom{n}{k}$. By (1.5), we have

$$b'_k = \frac{(2n+1)!!}{(2k-1)!!(2n-2k+1)!!} - 2k.$$

It is easy to verify that the sequence $\{b'_k\}_{k=0}^n$ is log-concave, which implies the log-concavity of $\{b_k\}_{k=0}^n$. This completes the proof. ∎

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