

DENSE SETS OF INTEGERS WITH PRESCRIBED REPRESENTATION FUNCTIONS

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ABSTRACT. Let \mathcal{A} be a set of integers and let $h \geq 2$. For every integer n , let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form $n = a_1 + \cdots + a_h$, where $a_i \in \mathcal{A}$ for $1 \leq i \leq h$, and $a_1 \leq \cdots \leq a_h$. The function $r_{\mathcal{A},h} : \mathbb{Z} \rightarrow \mathbf{N}$, where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$, is the *representation function of order h for \mathcal{A}* .

We prove that every function $f : \mathbb{Z} \rightarrow \mathbf{N}$ satisfying $\liminf_{|n| \rightarrow \infty} f(n) \geq g$ is the representation function of order h for a sequence $\mathcal{A} = \{a_k\}$ of integers, and that \mathcal{A} can be constructed so that it increases “almost” as slowly as any given $B_h[g]$ sequence. In particular, for every $\varepsilon > 0$ and $g \geq g(h, \varepsilon)$, we can construct a sequence \mathcal{A} satisfying $r_{\mathcal{A},h} = f$ and $\mathcal{A}(x) \gg x^{(1/h)-\varepsilon}$.

1. INTRODUCTION

Let \mathcal{A} be a set of integers and let $h \geq 2$. For every integer n , let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form

$$n = a_1 + \cdots + a_h$$

where

$$a_1 \leq \cdots \leq a_h \quad \text{and} \quad a_i \in \mathcal{A} \quad \text{for} \quad 1 \leq i \leq h.$$

The function $r_{\mathcal{A},h} : \mathbb{Z} \rightarrow \mathbf{N}$ is the *representation function of order h for \mathcal{A}* , where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$.

Nathanson proved [8] that any function $f : \mathbb{Z} \rightarrow \mathbf{N}$ satisfying $\liminf_{|n| \rightarrow \infty} f(n) \geq 1$ is the *representation function of order h* of a set of integers \mathcal{A} such that¹

$$(1) \quad \mathcal{A}(x) \gg x^{1/(2h-1)},$$

where $\mathcal{A}(x)$ counts the number of elements $a \in \mathcal{A}$ with $|a| \leq x$. It is an open problem to determine how dense the sets \mathcal{A} can be.

In this paper we study the connection between this problem and the problem of finding dense $B_h[g]$ sequences. We recall that a set \mathcal{B} of nonnegative integers is called a $B_h[g]$ sequence if

$$r_{\mathcal{B},h}(n) \leq g$$

for every nonnegative integer n . It is usual to write B_h to denote $B_h[1]$ sequences.

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¹The notation $f(x) \gg g(x)$ means that there exists a constant $C > 0$ such that $f(x) \geq Cg(x)$ for x large enough

Luczak and Schoen proved that any B_h sequence satisfying an additional kind of Sidon property (see [7] for the definition of this property, which they call the S_h property) can be enlarged to obtain a sequence with any prescribed representation function given f satisfying that $\liminf_{|x| \rightarrow \infty} f(x) \geq 1$. In particular, since they prove that there exists a B_h sequence \mathcal{A} satisfying the S_h property with $\mathcal{A}(x) \gg x^{1/(2h-1)}$, they recover Nathanson's result.

In this paper we prove that any $B_h[g]$ sequence can be modified slightly to have any prescribed representation function f satisfying $\liminf_{|x| \rightarrow \infty} f(x) \geq g$. Our main theorem is the following.

Theorem 1. *Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ any function such that $\liminf_{|n| \rightarrow \infty} f(n) \geq g$ and let \mathcal{B} be any $B_h[g]$ sequence. Then, for any decreasing function $\epsilon(x) \rightarrow 0$ as $x \rightarrow \infty$, there exists a sequence \mathcal{A} of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$\mathcal{A}(x) \gg \mathcal{B}(x\epsilon(x)).$$

It is difficult problem to construct dense B_h sequences. A trivial counting argument gives

$$\mathcal{B}(x) \ll x^{1/h}$$

for these sequences. On the other hand, the greedy algorithm shows that there exists a B_h sequence \mathcal{B} such that

$$(2) \quad \mathcal{B}(x) \gg x^{1/(2h-1)}.$$

For B_2 sequences, also called Sidon sets, Ruzsa proved [11] that there exists a Sidon set \mathcal{B} such that

$$(3) \quad \mathcal{B}(x) \gg x^{\sqrt{2}-1+o(1)}$$

This result and Theorem 1 give the following corollary.

Corollary 1. *Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ any function such that $\liminf_{|n| \rightarrow \infty} f(n) \geq 1$. Then there exists a sequence of integers \mathcal{A} such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$\mathcal{A}(x) \gg x^{\sqrt{2}-1+o(1)}.$$

This result gives an affirmative answer to the third open problem in [1], which was also posed previously in [10]. Unfortunately, nothing better than (2) is known for B_h sequences for $h \geq 3$. Erdős and Renyi [3] proved that, for any $\epsilon > 0$, there exists a positive integer g and a $B_2[g]$ sequence \mathcal{B} such that $\mathcal{B}(x) \gg x^{(1/2)-\epsilon}$. They claimed that the same method could be extended to $B_h[g]$ sequences, but a serious problem with non-independent events appears when $h \geq 3$.

Vu [13], using ideas from Erdős and Tetali [4] to study a related problem for bases of order h , proved that for any $\epsilon > 0$, there exist an integer g and a $B_h[g]$ sequence \mathcal{B} such that

$$\mathcal{B}(x) \gg x^{1/h-\epsilon}.$$

This result and Theorem 1 imply the next corollary

Corollary 2. *Given $h \geq 2$, for any $\varepsilon > 0$, there exists $g = g(h, \varepsilon)$ such that, for any function $f : \mathbb{Z} \rightarrow \mathbb{N}$ satisfying $\liminf_{|n| \rightarrow \infty} f(n) \geq g$, there exists a sequence \mathcal{A} of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$\mathcal{A}(x) \gg x^{\frac{1}{h}-\varepsilon}.$$

The construction in [8] for the set \mathcal{A} satisfying the growth condition (1) was based on the greedy algorithm. In this paper we construct the set \mathcal{A} by adjoining a very sparse sequence $\mathcal{U} = \{u_k\}$ to a suitable $B_h[g]$ sequence \mathcal{B} . This idea was used in [2], but in a simpler way, to construct dense *perfect difference sets*, which are sets such that every nonzero integer has a unique representation as a difference of two elements of \mathcal{A} . The proof of the main theorem in [2] can be adapted easily to our problem in the simplest case $h = 2$.

Theorem 2. *Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be a function such that $\liminf_{|n| \rightarrow \infty} f(n) \geq g$, and let \mathcal{B} be a $B_2[g]$ sequence. Then there exists a sequence of integers \mathcal{A} such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all } n \in \mathbb{Z}$$

and

$$\mathcal{A}(x) \gg \mathcal{B}(x/3).$$

We omit the proof because it is very close to the proof of the main theorem in [2]. Unfortunately, that proof cannot be adapted to the case $h \geq 3$. We need another definition of a “suitable” B_h set. In section §2 we shall show how to modify a $B_h[g]$ sequence \mathcal{B} so that it becomes “suitable.” We do this by applying the “Inserting Zeros Transformation” to an arbitrary $B_h[g]$ set. This is the main ingredient in the proof of Theorem 1.

Chen [1] has proved that for any $\epsilon > 0$ there exists a unique representation basis \mathcal{A} (that is, a set \mathcal{A} with $r_{\mathcal{A},2}(k) = 1$ for all integers $k \neq 0$) such that $\limsup_{x \rightarrow \infty} \mathcal{A}(x)/x^{1/2-\epsilon} > 1$. J. Lee [6] has improved this result by proving that for any increasing function ω tending to infinity there exists a unique representation basis \mathcal{A} such that $\limsup_{x \rightarrow \infty} \mathcal{A}(x)\omega(x)/\sqrt{x} > 0$.

Theorem 2 and the classical constructions of Erdős [12] and Kruckeberg [5] of infinite Sidon sets \mathcal{B} such that $\limsup_{x \rightarrow \infty} \mathcal{B}(x)/\sqrt{x} > 0$ provide a unique representation basis \mathcal{A} such that $\limsup_{x \rightarrow \infty} \mathcal{A}(x)/\sqrt{x} > 0$. Indeed, we can easily adapt the proof of Theorem 1.3 in [2] to the case of the additive representation function $r(n)$ (instead of the subtractive representation function $d(n) = \#\{n = a - a', a, a' \in \mathcal{A}\}$).

Theorem 3. *There exists a unique representation basis \mathcal{A} such that*

$$\limsup_{x \rightarrow \infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

Again we omit the proof because it is very close to the proof of Theorem 1.3 in [2].

Theorem above answers affirmatively the first open problem in [1]. Note also that if \mathcal{A} is an infinite Sidon set of integers, then the set

$$\mathcal{A}' = \{4a : a \geq 0\} \cup \{-4a + 1 : a < 0\}$$

is also a Sidon set and, in this case, $\liminf |\mathcal{A} \cap (-x, x)|/\sqrt{x} = \liminf \mathcal{A}'(4x)/\sqrt{x}$. A well known result of Erdős states that $\liminf \mathcal{B}(x)/\sqrt{x} = 0$ for any Sidon set \mathcal{B} .

Then the above limit is zero, so it answers negatively the second open problem in [1].

We do not know if it is possible to obtain a similar result for $h \geq 3$, because it is open problem to determine if there exists an infinite B_h sequence \mathcal{B} with $\limsup_{x \rightarrow \infty} \mathcal{B}(x)/x^{1/h} > 0$. It is easy, however, to prove that for any function ω tending to infinity there exists a unique representation basis of order h such that $\limsup_{x \rightarrow \infty} \mathcal{B}(x)\omega(x)/x^{1/h} > 1$. We can construct the set \mathcal{B} as follows: Let x_1, \dots, x_k, \dots be a sequence of positive integers such that $\omega(x_k) > (hx_{k-1})^{1/h}$ and consider, for each k , a B_h sequence $\mathcal{B}_k \subset [1, x_k/(hx_{k-1})]$ with $|\mathcal{B}_k| \gg (x_k/(hx_{k-1}))^{1/h}$. The set $\mathcal{B} = \cup_k (hx_{k-1}) * \mathcal{B}_k$ satisfies the conditions, where we use the notation $t * \mathcal{A} = \{ta, a \in \mathcal{A}\}$.

The construction above and Theorem 1 yield the following Corollary, which extends Theorem 6 in [6] in several ways.

Corollary 3. *Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ any function such that $\liminf_{|n| \rightarrow \infty} f(n) \geq 1$. For any increasing function ω tending to infinity there exists a set \mathcal{A} such that $r_{\mathcal{A},h}(n) = f(n)$ for all integers n , and*

$$\limsup_{x \rightarrow \infty} \mathcal{A}(x)\omega(x)/x^{1/h} > 0.$$

2. THE INSERTING ZEROS TRANSFORMATION

Consider the binary expansion of the elements of a set \mathcal{B} of positive integers. We will modify these integers by inserting strings of zeros at fixed places. We will see that this transformation of the set \mathcal{B} preserves certain additive properties.

In this paper we denote by γ any strictly increasing function $\gamma : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $\gamma(0) = 0$. For every positive integer r , we define the ‘‘Inserting Zeros Transformation’’ T_γ^r by

$$(4) \quad T_\gamma^r \left(\sum_{i \geq 0} \varepsilon_i 2^i \right) = \sum_{k \geq 0} 2^{2rk} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i.$$

In other words, if the integer b has the binary expansion

$$b = \varepsilon_0 \cdots \varepsilon_{\gamma(1)-1} \varepsilon_{\gamma(1)} \cdots \varepsilon_{\gamma(2)-1} \varepsilon_{\gamma(2)} \cdots \varepsilon_{\gamma(k)-1} \varepsilon_{\gamma(k)} \cdots,$$

then

$$T_\gamma^r(b) = \varepsilon_0 \cdots \varepsilon_{\gamma(1)-1} \underbrace{0 \cdots 0}_{2r} \varepsilon_{\gamma(1)} \cdots \varepsilon_{\gamma(2)-1} \underbrace{0 \cdots 0}_{2r} \varepsilon_{\gamma(2)} \cdots \varepsilon_{\gamma(k)-1} \underbrace{0 \cdots 0}_{2r} \varepsilon_{\gamma(k)} \cdots$$

Note that if $b < b'$, then $T_\gamma^r(b) < T_\gamma^r(b')$. We define the set

$$(5) \quad T_\gamma^r(\mathcal{B}) = \{T_\gamma^r(b) : b \in \mathcal{B}\}.$$

The next proposition proves that the function T_γ^r preserves some Sidon properties.

Proposition 1. *Let $2r > \log_2 h$. If $b_1, \dots, b_h, b'_1, \dots, b'_h$ are positive integers such that*

$$T_\gamma^r(b_1) + \cdots + T_\gamma^r(b_h) = T_\gamma^r(b'_1) + \cdots + T_\gamma^r(b'_h),$$

then

$$b_1 + \cdots + b_h = b'_1 + \cdots + b'_h.$$

In particular, if \mathcal{B} is a $B_h[g]$ set and $2r \geq \log_2 h$, then $T_\gamma^r(\mathcal{B})$ is also a $B_h[g]$ set.

Proof. We write

$$(6) \quad t_k = \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i(b_1)2^i + \cdots + \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i(b_h)2^i.$$

For any $k \geq 1$ we define the integer

$$(7) \quad m_k = 2^{2rk+\gamma(k)}.$$

It follows from (4), (6) and (7) that

$$T_\gamma^r(b_1) + \cdots + T_\gamma^r(b_h) \equiv \sum_{j=0}^{k-1} 2^{2rj} t_j \pmod{m_k}.$$

Since $T_\gamma^r(b_1) + \cdots + T_\gamma^r(b_h) = T_\gamma^r(b'_1) + \cdots + T_\gamma^r(b'_h)$, we have

$$\sum_{j=0}^{k-1} 2^{2rj} t_j \equiv \sum_{j=0}^{k-1} 2^{2rj} t'_j \pmod{m_k}.$$

Notice that

$$0 \leq \sum_{j=0}^{k-1} 2^{2rj} t_j \leq 2^{2r(k-1)} \sum_{j=0}^{k-1} t_j \leq 2^{2r(k-1)} h \sum_{i=0}^{\gamma(k)-1} 2^i < 2^{2r(k-1)} 2^{2r} 2^{\gamma(k)} = m_k,$$

and the same inequality works for $\sum_{j=0}^{k-1} 2^{2rj} t'_j$. Then

$$\sum_{j=0}^{k-1} 2^{2rj} t_j = \sum_{j=0}^{k-1} 2^{2rj} t'_j.$$

It follows that $t_k = t'_k$ for all $k \geq 0$, and so

$$b_1 + \cdots + b_h = \sum_{k \geq 0} t_k = \sum_{k \geq 0} t'_k = b'_1 + \cdots + b'_h.$$

This completes the proof. \square

Definition 1. For all integers $m \geq 2$ and x , let

$$\|x\|_m = \min\{|y|, x \equiv y \pmod{m}\}.$$

Note that $\|x_1 + x_2\|_m \leq \|x_1\|_m + \|x_2\|_m$ for all integers x_1 and x_2 . Also, if $\|x\|_m \neq \|x'\|_m$ for some m , then $x \not\equiv x' \pmod{m}$ and so $x \neq x'$.

Proposition 2. For $k \geq 1$ and for any positive integer b

$$\|T_\gamma^r(b)\|_{m_k} < \frac{m_k}{2^{2r}},$$

where m_k is defined in (7).

Proof. Let $b = \varepsilon_0 \varepsilon_1 \varepsilon_2 \dots$ be the binary expansion of b . Then

$$T_\gamma^r(b) \equiv \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \pmod{m_k}$$

and

$$0 \leq \sum_{j=0}^{k-1} 2^{2rj} \sum_{i=\gamma(j)}^{\gamma(j+1)-1} \varepsilon_i 2^i \leq \sum_{l=0}^{2r(k-1)+\gamma(k)-1} 2^l < \frac{m_k}{2^{2r}}.$$

This completes the proof. \square

3. PROOF OF THEOREM 1

3.1. Two auxiliary sequences. Consider the sequence $\{z_j\}_{j=1}^\infty$ defined by

$$(8) \quad z_j = j - [\sqrt{j}]([\sqrt{j}] + 1).$$

For every positive integer j there is a unique positive integer s such that $s^2 \leq j < (s+1)^2$. Then $j = s^2 + s + i$ for some $i \in [-s, s]$ and $z_j = i$. It follows that for every integer i there are infinitely many positive integers j such that $z_j = i$. Moreover, $|z_j| \leq s \leq \sqrt{j}$ for all $j \geq 1$.

Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ any function such that $\liminf_{|n| \rightarrow \infty} f(n) \geq g$. Let n_0 be the least positive integer such that $f(n) \geq g$ for all $|n| \geq n_0$. Choose an integer $r > 1 + \log_2(h^2 + n_0)$. Then

$$(9) \quad h^2 < 2^{r-1} \quad \text{and} \quad n_0 < 2^{r-1}.$$

Let $\gamma : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be a strictly increasing function such that $\gamma(0) = 0$.

Consider the sequence $\mathcal{U} = \{u_i\}_{i=1}^\infty$ defined by

$$(10) \quad \begin{cases} u_{2k-1} &= -m_k 2^{-r}, \\ u_{2k} &= (h-1)m_k 2^{-r} + z_k \end{cases}$$

where $m_k = 2^{2rk + \gamma(k)}$. We write

$$(11) \quad \mathcal{U}_k = \{u_{2k-1}, u_{2k}\} \quad \text{and} \quad \mathcal{U}_{<k} = \bigcup_{s < k} \mathcal{U}_s.$$

Note that for all $j \leq k$ we have

$$(12) \quad |z_j| \leq \sqrt{k} < 2^k \leq 2^{\gamma(k)} < 2^{2r(k-1) + \gamma(k)} = m_k 2^{-2r}.$$

3.2. The recursive construction. For any $B_h[g]$ -sequence \mathcal{B} we consider the set $T_\gamma^r(\mathcal{B})$ defined in (5). Let $f : \mathbb{Z} \rightarrow \mathbb{N}$ be a function such that $f(n) \geq g$ for $|n| \geq n_0$. We construct an increasing sequence $\{\mathcal{A}_k\}_{k=0}^\infty$ of sets of integers as follows:

$$(13) \quad \mathcal{A}_0 = \{a \in T_\gamma^r(\mathcal{B}) : a \geq n_0\}$$

and, for $k \geq 1$,

$$\mathcal{A}_k = \begin{cases} \mathcal{A}_{k-1} \cup \mathcal{U}_k & \text{if } r_{\mathcal{A}_{k-1}, h}(z_k) < f(z_k) \\ \mathcal{A}_{k-1} & \text{otherwise} \end{cases}$$

where z_k and \mathcal{U}_k are defined in (8) and (11).

We shall prove that the set

$$\mathcal{A} = \bigcup_{k=0}^\infty \mathcal{A}_k$$

satisfies $r_{\mathcal{A}, h}(n) = f(n)$ for all integers n .

Lemma 1. *Let $k \geq 1$. For nonnegative integers s and t with $s + t \leq h$, let*

$$\mathcal{A}_k^{(s,t)} = (h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}.$$

The sets $\mathcal{A}_k^{(s,t)}$ are pairwise disjoint, except possibly the sets $\mathcal{A}^{(0,0)}$ and $\mathcal{A}^{(h-1,1)}$.

Proof. If $n \in \mathcal{A}_k^{(s,t)}$ then

$$\begin{aligned} n &= a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} \\ &= a_1 + \cdots + a_{h-s-t} + (t(h-1) - s)m_k 2^{-r} + tz_k. \end{aligned}$$

If $a_i \in \mathcal{A}_0$, then $\|a_i\|_{m_k} \leq m_k 2^{-2r}$ by Proposition 2. If $a_i \in \mathcal{U}_{<k}$ then we use (10) and (12) to obtain

$$\|a_i\|_{m_k} \leq |a_i| \leq (h-1)m_{k-1}2^{-r} + m_{k-1}2^{-2r} < hm_k 2^{-2r}.$$

Therefore,

$$\begin{aligned} \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} &\leq \|a_1\|_{m_k} + \cdots + \|a_{h-s-t}\|_{m_k} + \|tz_k\|_{m_k} \\ &\leq (h-s-t)m_k h 2^{-2r} + tm_k 2^{-2r} \\ &\leq h^2 m_k 2^{-2r}. \end{aligned}$$

Now suppose that $n \in \mathcal{A}_k^{(s',t')}$ for some $(s',t') \neq (s,t)$. If $\{(s,t), (s',t')\} \neq \{(0,0), (h-1,1)\}$, then

$$t(h-1) - s \neq t'(h-1) - s'$$

and

$$\begin{aligned} m_k 2^{-r} &\leq \|((t(h-1) - s) - (t'(h-1) - s'))m_k 2^{-r}\|_{m_k} \\ &= \|(t(h-1) - s)m_k 2^{-r} - (t'(h-1) - s')m_k 2^{-r}\|_{m_k} \\ &= \|(n - (t(h-1) - s)m_k 2^{-r}) - (n - (t'(h-1) - s')m_k 2^{-r})\|_{m_k} \\ &\leq \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} + \|a'_1 + \cdots + a'_{h-s'-t'} + t'z_k\|_{m_k} \\ &\leq 2h^2 m_k 2^{-2r}. \end{aligned}$$

It follows that $h^2 \geq 2^{r-1}$, which contradicts (9). This completes the proof. \square

Lemma 2. If $n \in \mathcal{A}_k^{(s,t)}$ for some $k \geq 1$ and $(s,t) \notin \{(0,0), (h-1,1)\}$, then $|n| > n_0$.

Proof. If $n \in \mathcal{A}_k^{(s,t)}$, then

$$n = a_1 + \cdots + a_{h-s-t} + (t(h-1) - s)m_k 2^{-r} + tz_k$$

and

$$\begin{aligned} |n| &\geq \|n\|_{m_k} \\ &= \|a_1 + \cdots + a_{h-s-t} + tz_k + ((h-1)t - s)m_k 2^{-r}\|_{m_k} \\ &\geq \|((h-1)t - s)m_k 2^{-r}\|_{m_k} - \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k} \\ &\geq |((h-1)t - s)m_k 2^{-r}| - h^2 m_k 2^{-2r} \\ &\geq m_k 2^{-r} - h^2 m_k 2^{-2r} \geq m_k 2^{-r-1} \geq 2^{2r} 2^{-r-1} \\ &\geq 2^{r-1} > n_0, \end{aligned}$$

We have used that if $|((h-1)t - s)m_k 2^{-r}| < m_k/2$, then

$$\|((h-1)t - s)m_k 2^{-r}\|_{m_k} = |((h-1)t - s)m_k 2^{-r}| \geq m_k 2^{-r}.$$

Also we have used $(h-1)t - s \neq 0$ and (9) in the last inequalities. \square

Lemma 3. *For any $k \geq 0$, for any $h' < h$ and for any integer m we have that*

$$r_{\mathcal{A}_k, h'}(m) \leq g$$

Proof. By induction on k . Proposition 1 implies that $T_\gamma^r(\mathcal{B})$ and consequently \mathcal{A}_0 are $B_h[g]$ -sequences. In particular, \mathcal{A}_0 is a $B_{h'}[g]$ sequence. Then $r_{\mathcal{A}_0, h'}(m) \leq g$ for any integer m .

Suppose that it is true that for any $h' < h$, and for any integer m we have that $r_{\mathcal{A}_{k-1}, h'}(m) \leq g$.

Consider $m \in h'\mathcal{A}_k$.

- Suppose $m \notin (h' - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s, t) \neq (0, 0)$. Then $r_{\mathcal{A}_k, h'}(m) = r_{\mathcal{A}_{k-1}, h'}(m) \leq g$ by induction hypothesis.
- Suppose that $m \in (h' - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s, t) \neq (0, 0)$. Consider an element $a \in \mathcal{A}_0$. Then

$$m + (h - h')a \in \mathcal{A}_k^{(s, t)} = (h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}.$$

We apply lemma 1 and since $(s, t) \neq (h - 1, 1)$ (because $h' < h$) we have that

$$r_{\mathcal{A}_k, h'}(m) \leq r_{\mathcal{A}_k, h}(m + (h - h')a) = r_{\mathcal{A}_{k-1}, h-s-t}(m + (h - h')a - su_{2k-1} - tu_{2k}),$$

and we can apply induction hypothesis because $h - s - t < h$. \square

Proposition 3. *The sequence \mathcal{A} defined above satisfies $r_{\mathcal{A}, h}(n) = f(n)$ for all integers n .*

Proof. Since

$$\underbrace{u_{2k-1} + \cdots + u_{2k-1}}_{h-1} + u_{2k} = z_k$$

it follows that if $r_{\mathcal{A}_{k-1}, h}(z_k) < f(z_k)$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k$ and

$$r_{\mathcal{A}_k, h}(z_k) \geq r_{\mathcal{A}_{k-1}, h}(z_k) + 1.$$

For every integer n there are infinitely many integers k such that $z_k = n$ and so $r_{\mathcal{A}_k, h}(n) \geq f(n)$ for some k .

Next we show that, for every integer k , the sequence \mathcal{A}_k satisfies $r_{\mathcal{A}_k, h}(n) \leq f(n)$ for all n . The proof is by induction on k .

Let $k = 0$. Since \mathcal{A}_0 is a $B_h[g]$ -sequences, we have $r_{\mathcal{A}_0, h}(n) \leq g \leq f(n)$ for $n \geq n_0$. If $n < n_0$, then $r_{\mathcal{A}_0, h}(n) = 0 \leq f(n)$.

Now, suppose that it is true for $k - 1$. In particular $r_{\mathcal{A}_{k-1}, h}(z_k) \leq f(z_k)$. If $r_{\mathcal{A}_{k-1}, h}(z_k) = f(z_k)$ there is nothing to prove because in that case $\mathcal{A}_k = \mathcal{A}_{k-1}$. But if $r_{\mathcal{A}_{k-1}, h}(z_k) \leq f(z_k) - 1$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \mathcal{U}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}\} \cup \{u_{2k}\}$. We will assume that until the end of the proof.

If $n \notin h\mathcal{A}_k$ then $r_{\mathcal{A}_k, h}(n) = 0 \leq f(n)$.

If $n \in h\mathcal{A}_k$, since $\mathcal{A}_k = \mathcal{A}_{k-1} \cap \mathcal{U}_k$ we can write

$$h\mathcal{A}_k = \bigcup_{\substack{s, t=0 \\ s+t \leq h}}^h ((h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}).$$

Then

$$(14) \quad n = a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k}$$

for some s, t , satisfying $0 \leq s, t$, $s + t \leq h$ and for some $a_1, \dots, a_{h-s-t} \in \mathcal{A}_{k-1}$.

For short we write $r_{s,t}(n)$ for the number of solutions of (14).

- If $n \in (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s, t) \neq (0, 0)$, $(s, t) \neq (h-1, 1)$ then, due to lemma 1, we have that $r_{\mathcal{A}_k, h}(n) = r_{s,t}(n)$.
 - For $0 \leq n \leq n_0$ we have that $r_{s,t}(n) = 0 \leq f(n)$ (due to lemma 2).
 - For $n > n_0$ we apply lemma 3 in the first inequality below with $h' = h - s - t$ and $m = n - su_{2k-1} - tu_{2k}$,

$$r_{s,t}(n) = r_{\mathcal{A}_{k-1}, h-s-t}(n - su_{2k-1} - tu_{2k}) \leq g \leq f(n)$$

- If $n \notin (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s, t) \neq (0, 0)$, $(s, t) \neq (h-1, 1)$, then $r_{\mathcal{A}_k, h}(n) = r_{0,0}(n) + r_{h-1,1}(n)$. Notice that $r_{0,0}(n) = r_{\mathcal{A}_{k-1}, h}(n)$ and that $r_{h-1,1}(n) = 1$ if $n = z_k$ and $r_{h-1,1}(n) = 0$ otherwise.
 - If $n \neq z_k$, then $r_{\mathcal{A}_k, h}(n) = r_{\mathcal{A}_{k-1}, h}(n) \leq f(n)$ by induction hypothesis.
 - If $n = z_k$, then $r_{\mathcal{A}_k, h}(n) = r_{\mathcal{A}_{k-1}, h}(z_k) + r_{h-1,1}(z_k) \leq (f(z_k) - 1) + 1 = f(n)$.

□

3.3. The density of \mathcal{A} . Recall that $\gamma : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is a strictly increasing function with $\gamma(0) = 0$. Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. We extend γ to a strictly increasing function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. (For example, define $\gamma(x) = \gamma(k+1)(x-k) + \gamma(k)(k+1-x)$ for $k \leq x \leq k+1$.)

We have

$$\mathcal{A}(x) \geq \mathcal{A}_0(x) \geq T_\gamma^r(\mathcal{B})(x) - n_0.$$

Thus, to find a lower bound for $\mathcal{A}(x)$ it suffices to find a lower bound for the density of $T_\gamma^r(\mathcal{B})$.

Lemma 4. $T_\gamma^r(\mathcal{B})(x) > \mathcal{B}(x2^{-2r\gamma^{-1}(\log_2 x)})$.

Proof. Let b be a positive integer such that

$$b \leq x2^{-2r\gamma^{-1}(\log_2 x)}.$$

Let ℓ be such that $2^{\gamma(\ell)} \leq b < 2^{\gamma(\ell+1)}$. Then we can write

$$(15) \quad b = \sum_{k=0}^{\ell} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i.$$

It follows from the definition (4) of the Zeros Inserting Transformation that

$$\begin{aligned} T_\gamma^r(b) &= \sum_{k=0}^{\ell} 2^{rk} \sum_{i=\gamma(k)}^{\gamma(k+1)-1} \varepsilon_i 2^i \\ &\leq 2^{2r\ell} b \\ &\leq 2^{2r\gamma^{-1}(\log_2 b)} b \\ &\leq 2^{2r(\gamma^{-1}(\log_2 b) - \gamma^{-1}(\log_2 x))} x \\ &\leq x. \end{aligned}$$

□

Recall that ϵ is a decreasing positive function defined on $[1, \infty)$ such that $\lim_{x \rightarrow \infty} \epsilon(x) = 0$. We complete the proof of Theorem 1 by choosing a function γ that satisfies the inequality

$$2^{-2r\gamma^{-1}(\log_2 x)} \geq \epsilon(x).$$

It suffices to take $\gamma(x) > \log_2(\epsilon^{-1}(2^{-2rx}))$.

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