# ON THE CHOICE NUMBER OF COMPLETE MULTIPARTITE GRAPHS WITH PART SIZE FOUR 

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#### Abstract

Let $\operatorname{ch}(G)$ denote the choice number of a graph $G$, and let $K_{s * k}$ be the complete $k$-partite graph with $s$ vertices in each part. Erdős, Rubin, and Taylor showed that $\operatorname{ch}\left(K_{2 * k}\right)=k$, and suggested the problem of determining the choice number of $K_{s * k}$. The first author established $\operatorname{ch}\left(K_{3 * k}\right)=\left\lceil\frac{4 k-1}{3}\right\rceil$. Here we prove $\operatorname{ch}\left(K_{4 * k}\right)=\left\lceil\frac{3 k-1}{2}\right\rceil$.


## 1. Introduction

Let $G=(V, E)$ be a graph. A list assignment $L$ for $G$ is a function $L: V \rightarrow 2^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers and $2^{\mathbb{N}}$ is the power set of $\mathbb{N}$. If $|L(v)|=k$ for all vertices $v \in V$, then $L$ is a $k$-list assignment for $G$. An $L$-coloring $f$ from a list assignment $L$ is a function $f: V \rightarrow \mathbb{N}$ such that $f(v) \in L(v)$ for all vertices $v \in V$ and $f(x) \neq f(y)$ whenever $x y \in E . G$ is $L$-colorable if there exists an $L$-coloring of $G$; it is $k$-choosable if it is $L$-choosable for all $k$-list assignments $L$. The list chromatic number or choice number of $G$, denoted $\operatorname{ch}(G)$, is the smallest integer $k$ such that $G$ is $k$-choosable. The general list coloring problem may consider list assignments with uneven list sizes.

The study of list coloring was initiated by Vizing [13] and by Erdős, Rubin and Taylor [2]. It is a generalization of two well studied areas of combinatorics - graph coloring and transversal theory. Restricting the list assignment to a constant function, yields ordinary graph coloring; restricting the graph to a clique yields the problem of finding a system of distinct representatives (SDR) for the family of lists. Both restrictions play a role in this paper. Given the general nature of this parameter, it is hardly surprising that there are not many graphs whose exact choice number is known. However, there are some amazingly elegant results that add to the subject's charm. For example, Thomassen 12 proved that planar graphs have choice number at most 5, Voight [14 proved that this is tight, and Galvin $[3]$ proved that line graphs of bipartite graphs have choice number equal to their clique number.

Erdős et al. [2] suggested determining the choice number of uniform complete multipartite graphs. More generally, let $K_{1 * k_{1}, 2 * k_{2} \ldots}$ denote the complete multipartite graph with $k_{i}$ parts of size $i$, where zero terms in the subscript are deleted. Since $K_{1 * k}$ is a clique and $K_{s * 1}$ is an independent set, these cases are trivial. Alon [1] proved the general bounds $c_{1} k \log s \leq \operatorname{ch}\left(K_{s * k}\right) \leq c_{2} k \log s$ for some constants $c_{1}, c_{2}>0$. This was tightened by Gazit and Krivelevich [4].
Theorem 1 (Gazit and Krivelevich [4]). $\operatorname{ch}\left(K_{s * k}\right)=(1+o(1)) \frac{\log s}{\log (1+1 / k)}$.
The next well-known example provides the best lower bounds for small values of $s$.

[^0]Example 2. $\operatorname{ch}\left(K_{s * k}\right) \geq\left\lceil\frac{2(s-1) k-s+2}{s}\right\rceil$ : Let $G=K_{s * k}$ have parts $\left\{X_{1}, \ldots, X_{k}\right\}$ with $X_{i}=\left\{v_{i, 1}, \ldots, v_{i, s}\right\}$. We will construct an $(l-1)$-list assignment $L$ from which $G$ cannot be colored. Equitably partition $C:=[2 k-1]$ into $s$ parts $C_{1}, \ldots, C_{s}$. Define a list assignment $L$ for $G$ by $L\left(v_{i, j}\right)=C \backslash C_{j}$. Then each list has size at least

$$
2 k-1-\left\lceil\frac{2 k-1}{s}\right\rceil=\left\lfloor\frac{2 k s-s-2 k+1}{s}\right\rfloor=\left\lceil\frac{2(s-1) k-2 s+2}{s}\right\rceil=l-1 .
$$

Consider any color $\alpha \in C$. Then $\alpha \in C_{i}$ for some $i \in[s]$. So $\alpha \notin L\left(x_{i, j}\right)$ for every $j \in[k]$. Thus any $L$-coloring of $G$ uses at least two colors for every part $X_{j}$. Since vertices in distinct parts are adjacent, they require distinct colors. As there are $k$ parts this would require $2 k>|C|$ colors, which is impossible.

Restricting the question of Erdös et al., we ask for those integers $s$ such that:

$$
\begin{equation*}
\left(\forall k \in \mathbb{Z}^{+}\right)\left[\operatorname{ch}\left(K_{s * k}\right)=l(s, k):=\left\lceil\frac{2(s-1) k-s+2}{s}\right\rceil\right] . \tag{1.1}
\end{equation*}
$$

The first two cases $s=2$ and $s=3$ have been solved:
Theorem 3 (Erdôs, Rubin and Taylor [2]). All positive integers $k$ satisfy $\operatorname{ch}\left(K_{2 * k}\right)=k$.
Theorem 4 (Kierstead [5|). All positive integers $k$ satisfy $\operatorname{ch}\left(K_{3 * k}\right)=\left\lceil\frac{4 k-1}{3}\right\rceil$.
Recently, Kozik, Micek, and Zhu [6] gave a very different proof of Theorem 4. The following more general result appears in [8].
Theorem 5 (Ohba |8|). $\operatorname{ch}\left(K_{1 * k_{1}, 3 * k_{3}}\right)=\max \left\{k,\left\lceil\frac{n+k-1}{3}\right\rceil\right\}$, where $k=k_{1}+k_{3}$ and $n=$ $k_{1}+3 k_{3}$.

The next example shows that the largest $s$ satisfying (1.1) is at most 14.
Example 6. If $k$ is even then $\operatorname{ch}\left(K_{15 * k}\right) \geq l:=2 k$ : Let $G=K_{s * k}$ have parts $\left\{X_{1}, \ldots, X_{k}\right\}$ with $X_{i}=\left\{v_{i, 1}, \ldots, v_{i, s}\right\}$. We will construct an $(l-1)$-list assignment $L$ from which $G$ cannot be colored. Equitably partition $C:=[3 k-1]$ into 6 parts $C_{1}, \ldots, C_{6}$, and fix a bijection $f:[15] \rightarrow\binom{[6]}{2}$. Define a list assignment $L$ for $G$ by

$$
L\left(v_{i, j}\right)=C \backslash \bigcup\left\{C_{h}: h \in f(i)\right\} .
$$

Then each list has size at least

$$
3 k-1-2\left\lceil\frac{3 k-1}{6}\right\rceil=2 k-1=l-1 .
$$

Consider any two colors $\alpha, \beta \in C$. Then $\alpha, \beta \in \bigcup\left\{C_{h}: h \in f(i)\right\}$ for some $i \in[15]$. So $\alpha, \beta \notin L\left(x_{i, j}\right)$ for every $j \in[k]$. Thus any $L$-coloring of $G$ uses at least three colors for every part $X_{j}$. Since $3 k>|C|$, this is impossible.

Yang 15] proved $\left\lceil\frac{3 k}{2}\right\rceil \leq \operatorname{ch}\left(K_{4 * k}\right) \leq\left\lceil\frac{7 k}{4}\right\rceil$, and Noel et al. 7 improved the upper bound to $\left\lceil\frac{5 k-1}{3}\right\rceil$. The main result of this paper is that (1.1) holds for $s=4$. To prove this theorem we first extract a simple proof of Theorem 4 from [7], and then elaborate on it.

Theorem 7. $\operatorname{ch}\left(K_{4 * k}\right)=l(4, k):=\left\lceil\frac{3 k-1}{2}\right\rceil$.
Some of the recent development of list coloring of complete multipartite graphs has been motivated by paintability, or on-line choosability. Introduced by Schauz [11], paintability is a coloring game played between two players Alice and Bob on a graph $\bar{G}=(V, E)$ and a function $f: V \rightarrow \mathbb{N}$. Let $V_{i}$ denote the vertex set at the start of round $i$; so $V_{1}=V$. At
round $i$, Alice selects a nonempty set of vertices $A_{i} \subseteq V_{i}$, and Bob selects an independent set $B_{i} \subseteq A_{i}$. Then $B_{i}$ is deleted from the graph so that $V_{i+1}=V_{i} \backslash B_{i}$, and the rounds are continued until $V_{n}=\emptyset$. Alice's goal is to present some vertex $v$ more than $f(v)$ times, while Bob's goal is to choose every vertex before it has been presented $f(v)+1$ times. We say that $G$ is on-line $f$-choosable if player $B$ has a strategy such that any vertex $v \in V$ is in at most $f(v)$ sets $A_{i}$, and on-line $k$ choosable if $G$ is on-line $f$-choosable when $f(v)=k$ for all $v \in V$. The on-line choice number, denoted $\operatorname{ch}^{O L}(G)$, is the least $k$ such that $G$ is on-line $k$-choosable.

This game formulation hides the on-line nature of the problem. Another way of thinking about it is that Alice has secretly assigned lists of colors to all the vertices. At round $i$ she reveals all vertices whose list contains color $i$, and Bob colors an independent set of them with color $i$. In this formulation it is clear that $\operatorname{ch}(G) \leq \operatorname{ch}^{O L}(G)$.

Surprisingly, Schauz [11] proved that many results on choice number, including Brooks' theorem, Thomassen's theorem, and the Bondy-Boppana kernel lemma carry over to online choice number. It is unknown whether $\operatorname{ch}^{O L}(G)-\operatorname{ch}(G)$ is bounded by a constant. Indeed, no graphs are known for which $\operatorname{ch}^{O L}(G)-\operatorname{ch}(G) \geq 2$. It is known that

$$
\operatorname{ch}\left(K_{2,2,3}\right)=3<4=\operatorname{ch}^{O L}\left(K_{2,2,3}\right) .
$$

The explicit value of $\operatorname{ch}\left(K_{4 * k}\right)$ provided by Theorem 7 may be useful for establishing larger gaps. In Section 4 we show that $\operatorname{ch}\left(K_{4 * 3}\right)<\operatorname{ch}^{O L}\left(K_{4 * 3}\right)$.

## 2. SET-UP

Fix $s, k \in \mathbb{Z}^{+}$. Let $G=(V, E)=K_{s * k}$, and $\mathcal{P}$ be the partition of $V$ into $k$ independent $s$-sets. Let $l=l(k, s)=\left\lceil\frac{(s-1) 2 k-s+2}{s}\right\rceil$, and consider any $l$-list assignment $L$ for $G$. Put $C^{*}=\bigcup_{x \in V} L(x)$. Let $L \neg \alpha$ be the result of deleting $\alpha$ from every list of $L$.
We may write $x_{1} \ldots x_{t}$ for the subpart $S=\left\{x_{1}, \ldots, x_{t}\right\} \subseteq X \in \mathcal{P}$; when we use this notation we implicitly assume the $x_{i}$ are distinct. Also set $\bar{S}=X \backslash S$. For a set of verties $S \subseteq V$ let $\mathcal{L}(S)=\{L(x): x \in S\}, L(S)=\bigcap \mathcal{L}(S), W(S)=\bigcup \mathcal{L}(S)$, and $l(S)=|L(S)|$. The operation of replacing the vertices in $S$ by a new vertex $v_{S}$ with the same neighborhood as $S$ is called merging. The new vertex $v_{S}$ is said to be merged; vertices that are not merged are called original. When merging a set $S$ we also create a list $L\left(v_{S}\right)=L(S)$.

For a color $\alpha \in C^{*}$, let $|X, \alpha|=|\{x \in X: \alpha \in L(x)\}|$ be the number of times $\alpha$ appears in the lists of vertices of $X, N_{i}(X)=\left\{\alpha \in C^{*}:|X, \alpha|=i\right\}$ be the set of colors that appear exactly $i$ times in the lists of vertices in $X, n_{i}(X)=\left|N_{i}(X)\right|$, and $N(X)=N_{2}(X) \cup N_{3}(X)$. Let $\sigma_{i}(X)=\sum\{l(I): I \subseteq X \wedge|I|=i\}$ and $\mu_{i}(X)=\max \{l(I): I \subseteq X \wedge|I|=i\}$.
For a set $S$ and element $x$ we use the notation $S+x=S \cup\{x\}$ and $S-x=S \backslash\{x\}$.
The following lemma was proved independently by Kierstead [5], and by Reed and Sudakov [9], [10], and named by Rabern.
Lemma 8 (Small Pot Lemma). If $\operatorname{ch}(G)>r$ then there exists a list assignment $L$ such that $G$ has no L-coloring, all lists have size $r$, and their union has size less than $|V(G)|$.
If $s$ does not satisfy (1.1) then there is a minimal counterexample $k$ with $\operatorname{ch}\left(K_{s, k}\right)>$ $l(s, k)$. By the Small Pot Lemma, this is witnessed by a list assignment $L$ with $\mid \bigcup\{L(x)$ : $x \in V(G)\}|<|V|$. We always assume $L$ has this property.
Lemma 9. Every part $X$ of $G$ satisfies $L(X)=\emptyset$.
Proof. Otherwise there exists a list assignment $L$, a color $\alpha$, and a part $X$ such that $\alpha \in L(X)$. Color each vertex in $X$ with $\alpha$, set $G^{\prime}=G-X$, and put $L^{\prime}=L \neg \alpha$. Then $L^{\prime}$ witnesses that $k-1$ is a smaller counterexample, a contradiction.

By Lemma 9, $n_{s}(X)=0$ for each part $X \in \mathcal{P}$. So by the Small Pot Lemma, $|W(X)|=$ $\sum_{i=1}^{s-1} n_{i}(X)<s k$. Also $\sum_{i=1}^{s-1} i n_{i}(X)=s l$ is the number of occurrences of colors in the lists of vertices of $X$. Thus

$$
\begin{equation*}
\sum_{i=2}^{s-1}(i-1) n_{i}(X) \geq s l-|W(X)| \geq s(l-k)+1 \tag{2.1}
\end{equation*}
$$

Now we warm-up by giving a short proof extracted from [7] of Theorem 4.
Proof of Theorem \& Let $s=3, l=l(3, k)$, and assume $G$ is a counterexample with $k$ minimal. Then $k>1$. By Lemma $9, n_{3}(X)=0$ for all $X \in \mathcal{P}$. We obtain a contradiction by $L$-coloring $G$. First we use the following steps to partition $V$ into sets of vertices that will receive the same color. Then we merge each set $I$ into a single vertex $v_{I}$, and assign $v_{I}$ the set of colors in $L(I)$. Finally we apply Hall's Theorem to chose a system of distinct representatives (SDR) for these new lists; this induces an $L$-coloring of $G$.
Step 1. Partition $\mathcal{P}$ into a set $\mathcal{R}$ of $l-k$ reserved parts together with a set $\mathcal{U}=\mathcal{P} \backslash \mathcal{R}$ of $2 k-l$ unreserved parts.
Step 2. Choose $\mathcal{U}_{1} \subseteq \mathcal{U}$ maximum subject to $\left|\mathcal{U}_{1}\right| \leq \mu_{2}(X)$ for all $X \in \mathcal{U}_{1}$, and subject to this, $\nu=\sum_{X \in \mathcal{U}_{1}} \mu_{2}(X)$ is maximum. Set $u_{1}=\left|\mathcal{U}_{1}\right|$. For each $X \in \mathcal{U}_{1}$ choose a pair $I_{X} \subseteq X$ with $l\left(I_{X}\right) \geq u_{1}$ maximum. Put $\mathcal{U}_{2}=\mathcal{U} \backslash \mathcal{U}_{1}$ and $u_{2}=\left|\mathcal{U}_{2}\right|$. So

$$
\begin{equation*}
\text { if } u_{1}<2 k-l \text { then } \mu_{2}(X) \leq u_{1} \text { for all } X \in \mathcal{U}_{2}, \tag{2.2}
\end{equation*}
$$

since otherwise we could increase $\nu$ by adding $X$ to $\mathcal{U}_{1}$, and deleting one part $Y \in \mathcal{U}_{1}$ with $\mu_{2}(Y)=u_{1}$, if such a part $Y$ exists.
Step 3. Using (2.1), each part $X \in \mathcal{P}$ satisfies

$$
n_{2}(X) \geq 3(l-k)+1 \geq 3\left\lceil\frac{k-1}{3}\right\rceil+1 \geq k-1+1=k .
$$

Form an $\operatorname{SDR} f$ for $\left\{L\left(v_{I_{X}}\right): X \in \mathcal{U}_{1}\right\} \cup\{N(X): X \in \mathcal{R}\}$ by greedily choosing representatives for the first family and then for the second family. For each $X \in \mathcal{R}$ choose a pair $I_{X} \subseteq X$ so that $f(x) \in L\left(I_{X}\right)$.
Step 4. For each $X \in \mathcal{U}_{1} \cup \mathcal{R}$, merge $I_{X}$ to a new vertex $v_{I_{X}}$, let $z_{X} \in X \backslash I_{X}$, and set $X^{\prime}=\left\{v_{I_{X}}, z_{X}\right\}$. If $X \in \mathcal{U}_{2}$, set $X^{\prime}=X$. This yields a graph $G^{\prime}$ with parts $\mathcal{P}^{\prime}=\left\{X^{\prime}\right.$ : $X \in \mathcal{P}\}$, and list assignment $L$.
Next we use Hall's Theorem to prove that $\left\{L(x): x \in V\left(G^{\prime}\right)\right\}$ has an SDR. For this it suffices to prove:

$$
\begin{equation*}
|S| \leq|\bigcup\{L(x): x \in S\}| \text { for every } S \subseteq V\left(G^{\prime}\right) \tag{2.3}
\end{equation*}
$$

To prove (2.3), let $S \subseteq V\left(G^{\prime}\right)$ be arbitrary, and set $W=W(S):=\bigcup\{L(x): x \in S\}$. We consider several cases in order, always assuming all previous cases fail.
Case 1: There exists $X \in \mathcal{P}$ with $\left|S \cap X^{\prime}\right|=3$. Then $|S| \leq 2 k+u_{2}, X^{\prime}=X \in \mathcal{U}_{2}$ and $u_{2} \geq 1$. Thus $u_{1} \leq 2 k-l-u_{2}<2 k-l$, and so by (2.2), $u_{1} \geq \mu_{2}(X) \geq \sigma_{2}(X) / 3$. Using inclusion-exclusion, and Lemma 9,

$$
\begin{aligned}
|W| & \geq|W(X)| \geq \sigma_{1}(X)-\sigma_{2}(X)+\sigma_{3}(X) \geq 3 l-3 u_{1}=3 l-3\left(2 k-l-u_{2}\right) \\
& \geq 6(l-k)+3 u_{2} \geq(2 k-2)+\left(2+u_{2}\right) \geq 2 k+u_{2} \geq|S|
\end{aligned}
$$

Case 2: There is $X \in \mathcal{U}_{2}$ with $\left|S \cap X^{\prime}\right|=2$. Then $X=X^{\prime}$ and $|S| \leq 2 k$. Since $u_{1}=2 k-l-u_{2}<2 k-l$, (2.2) yields
$|W| \geq|W(S \cap X)| \geq 2 l-l(S \cap X) \geq 2 l-u_{1} \geq 2 l-\left(2 k-l-u_{2}\right) \geq 3 l+1-2 k=2 k \geq|S|$.


Figure 3.1. The partition $\mathcal{P}^{\prime}$ of $K_{4 * k}$.
Case 3: There is $X \in \mathcal{U}_{1}$ with $\left|S \cap X^{\prime}\right|=2$. As $|S| \leq 2 k-u_{2}=l+u_{1}$ and $L\left(v_{I_{X}} z_{X}\right)=$ $L(X)=\emptyset$,

$$
|W| \geq\left|W\left(S \cap X^{\prime}\right)\right| \geq l\left(v_{I_{X}}\right)+l\left(z_{X}\right)-l\left(v_{I_{X}} z_{X}\right) \geq u_{1}+l \geq|S| .
$$

Case 4: $S$ has an original vertex. Then $|S| \leq l \leq|W|$.
Case 5: All vertices of $S$ have been merged. Then $|S| \leq|f(S)| \leq|W|$.

## 3. The main theorem

In this section we prove our main result, Theorem 7. The case when $k$ is odd is considerably more technical. Casual or first time readers may wish to avoid these additional details; the proof is organized so that this is possible. In particular, in the even case Step 11 and Lemmas 13 and 14 are not involved. We often use the partition $k=(2 k-l)+(l-k)$ of the integer $k$, and note that $2 k-l=l-k+b$, where $b=k \bmod 2$.
Proof of Theorem 7. Our set-up is the same as in the proof of Theorem 4. Let $s=4$, $l=l(4, k)$, and $G=K_{4 * k}$. The theorem is trivial if $k=1$. Let $k>1$ be a minimal counterexample, and let $L$ be an $l$-list assignment for $G$ with $|W(V)| \leq 4 k-1$ and $L(X)=\emptyset$ for all parts $X \in \mathcal{P}$. Again we partition $V$ into sets of vertices that will receive the same color, and then find an SDR for the induced list assignment that in turn induces an $L$-coloring of $G$. See Figure 3.1.
Step 1. Partition $V$ as $\mathcal{P}=\mathcal{U} \cup \mathcal{R}$, where $|\mathcal{R}|=l-k,|\mathcal{U}|=2 k-l, \mathcal{R}=\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}$ and $\mathcal{U}=\mathcal{U}_{1} \cup \mathcal{U}_{2} \cup \mathcal{U}_{3} \cup \mathcal{U}_{4}$ as follows.
Step 2. Choose $\mathcal{U}_{1} \subseteq \mathcal{P}$ maximum subject to $\left|\mathcal{U}_{1}\right| \leq 2 k-l$ and for every $X \in \mathcal{U}_{1}$ there is a pair $I_{X} \subseteq X$ with $l\left(I_{X}\right), l\left(\bar{I}_{X}\right) \geq k$. Put $\mathcal{U}_{1} \subseteq \mathcal{U}$, and let $u_{1}:=\left|\mathcal{U}_{1}\right|$. Then:
(3.1) If $u_{1}<2 k-l$ then $\left(\forall X \in \mathcal{P} \backslash \mathcal{U}_{1}\right)(\forall I \subseteq X)[|I|=2 \rightarrow \min \{l(I), l(\bar{I})\} \leq k-1]$.

Step 3. Choose $\mathcal{U}_{2} \subseteq \mathcal{P} \backslash \mathcal{U}_{1}$ maximum subject to $\left|\mathcal{U}_{2}\right| \leq 2 k-l-u_{1}$ and $\left|\mathcal{U}_{2}\right| \leq \mu_{3}(X)$ for all $X \in \mathcal{U}_{2}$; subject to this let $\nu=\sum_{X \in \mathcal{U}_{2}} \mu_{3}(X)$ be maximum. Put $\mathcal{U}_{2} \subseteq \mathcal{U}$, and let $u_{2}=\left|\mathcal{U}_{2}\right|$. If $\mathcal{U}_{2} \neq \emptyset$ then let $\dot{Z} \in \mathcal{U}_{2}$; else $\dot{Z}=\emptyset$. For each $X \in \mathcal{U}_{2}$ choose a triple $I_{X} \subseteq X$ with $l\left(I_{X}\right) \geq u_{2}$ maximum. Since $\nu$ cannot be increased:

$$
\begin{equation*}
\text { If } u_{1}+u_{2}<2 k-l \text { then }\left(\forall X \in \mathcal{U}_{3} \cup \mathcal{U}_{4} \cup \mathcal{R}\right)\left[\mu_{3}(X) \leq u_{2}\right] \text {. } \tag{3.2}
\end{equation*}
$$

Step 4. Choose $\mathcal{R}_{1} \subseteq \mathcal{P} \backslash\left(\mathcal{U}_{1} \cup \mathcal{U}_{2}\right)$ maximum subject to $\left|\mathcal{R}_{1}\right| \leq l-k$ and for all $X \in \mathcal{R}_{1}$ there exists $I_{X} \subseteq X$ with $\left|I_{X}\right|=3$ such that there is an SDR $f_{1}$ of $\mathcal{L}\left(M_{1}\right)$, where $M_{1}:=\left\{v_{I_{X}}: X \in \mathcal{U}_{2} \cup \mathcal{R}_{1}\right\} ;$ let $C_{1}=\operatorname{ran}\left(f_{1}\right)$. Put $\mathcal{R}_{1} \subseteq \mathcal{R}$, and let $r_{1}:=\left|\mathcal{R}_{1}\right|$. Then:

$$
\begin{equation*}
\text { If } r_{1}<l-k \text { then }\left(\forall X \in \mathcal{U}_{3} \cup \mathcal{U}_{4} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right)\left[N_{3}(X) \subseteq C_{1}\right] . \tag{3.3}
\end{equation*}
$$

Step 5. Choose $\mathcal{U}_{3} \subseteq \mathcal{P} \backslash\left(\mathcal{U}_{1} \cup \mathcal{U}_{2} \cup \mathcal{R}_{1}\right)$ maximum subject to $\left|\mathcal{U}_{3}\right| \leq 2 k-l-u_{1}-u_{2}$ and $l-k+u_{2}+\left|\mathcal{U}_{3}\right| \leq \mu_{2}(X)$ for all $X \in \mathcal{U}_{3}$; subject to this let $\nu=\sum_{X \in \mathcal{U}_{3}} \mu_{2}(X)$ be maximum. Put $\mathcal{U}_{3} \subseteq \mathcal{U}$, and $u_{3}=\left|\mathcal{U}_{3}\right|$. Since $\nu$ cannot be increased:

$$
\begin{equation*}
\text { If } u_{1}+u_{2}+u_{3}<2 k-l \text { then }\left(\forall X \in \mathcal{U}_{4} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right)\left[\mu_{2}(X) \leq l-k+u_{2}+u_{3}\right] . \tag{3.4}
\end{equation*}
$$

For all $X \in \mathcal{U}_{3}$ choose a pair $I_{X}=x y \subseteq X$ with $l\left(I_{X}\right) \geq l-k+u_{2}+u_{3}$ maximum; subject to this choose $I_{X}$ so that $\Delta_{1}\left(I_{X}\right):=l\left(I_{X}\right)-l\left(\bar{I}_{X}\right)$ is maximum. Set $\Delta_{2}\left(I_{X}\right):=$ $2 u_{2}-l(x y z)-l(x y w)$, where $z w=\bar{I}_{X}$. Using $r_{1} \leq l-k$, extend $f_{1}$ to an SDR $f_{2}$ of $\mathcal{L}\left(M_{2}\right)$, where $M_{2}:=M_{1} \cup\left\{v_{I_{X}}: X \in \mathcal{U}_{3}\right\}$; set $C_{2}=\operatorname{ran}\left(f_{2}\right)$.
Step 6. Choose $\mathcal{R}_{2} \subseteq \mathcal{P} \backslash\left(\mathcal{U}_{1} \cup \mathcal{U}_{2} \cup \mathcal{U}_{3} \cup \mathcal{R}_{1}\right)$ maximum subject to $\left|\mathcal{R}_{2}\right| \leq l-k-r_{1}$ and $\sigma_{2}(X)-\sigma_{3}(X) \geq 5(l-k)+2 u_{1}+2 u_{2}+u_{3}+r_{1}+\left|\mathcal{R}_{2}\right|$ for all $X \in \mathcal{R}_{2}$; subject to this let $\sum_{X \in \mathcal{R}_{2}} \sigma_{2}(X)-\sigma_{3}(X)$ be maximum. Put $\mathcal{R}_{2} \subseteq \mathcal{R}$, and set $r_{2}=\left|\mathcal{R}_{2}\right|$. Then:

$$
\begin{align*}
& \text { If } r_{1}+r_{2}<l-k \text { then }\left(\forall X \in \mathcal{U}_{4} \cup \mathcal{R}_{3}\right) \\
& \quad\left[\sigma_{2}(X)-\sigma_{3}(X) \leq 5(l-k)+2 u_{1}+2 u_{2}+u_{3}+r_{1}+r_{2}\right] . \tag{3.5}
\end{align*}
$$

Step 7. Choose $\mathcal{R}_{3} \subseteq \mathcal{P} \backslash\left(\mathcal{U}_{1} \cup \mathcal{U}_{2} \cup \mathcal{U}_{3} \cup \mathcal{R}_{1} \cup \mathcal{R}_{2}\right)$ with $\left|\mathcal{R}_{3}\right|=l-k-r_{1}-r_{2}$, and set $\mathcal{R}=\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}$. Let $r_{3}=\left|\mathcal{R}_{3}\right|$. For $I \subseteq X$, put $L^{\prime}(I)=L(I) \backslash C_{2}$ and $l^{\prime}(I)=\left|L^{\prime}(I)\right|$. Using Lemma 11, for all $X \in \mathcal{R}_{3}$ there exists a pair $I_{X} \subseteq X$ with $l^{\prime}\left(\bar{I}_{X}\right) \leq l^{\prime}\left(I_{X}\right)$ such that $f_{2}$ can be extended to an SDR $f_{3}$ of $\mathcal{L}\left(M_{3}\right)$, where $M_{3}:=M_{2} \cup\left\{v_{I_{X}}: \bar{X} \in \mathcal{R}_{3}\right\}$. Let $C_{3}=\operatorname{ran}\left(f_{3}\right)$.
Step 8. Put $\mathcal{U}=\mathcal{P} \backslash \mathcal{R}, \mathcal{U}_{4}:=\mathcal{U} \backslash\left(\mathcal{U}_{1} \cup \mathcal{U}_{2} \cup \mathcal{U}_{3}\right)$, and $u_{4}:=\left|\mathcal{U}_{4}\right|$.
Step 9. Using Lemma 12, choose a pair $I_{X} \subseteq X$ for all $X \in \mathcal{R}_{2}$ so that $\mathcal{L}\left(M_{4}\right)$ has an $\operatorname{SDR} f_{4}$ extending $f_{3}$, where $M_{4}:=M_{3} \cup\left\{v_{I_{X}}, v_{\bar{I}_{X}}: X \in \mathcal{U}_{1} \cup \mathcal{R}_{2}\right\}$.
Step 10. Let $G^{\prime}:=\left(V^{\prime}, E^{\prime}\right)$ be the graph obtained from $G$ by merging each $I_{X}$ with $X \in \mathcal{U}_{2} \cup \mathcal{U}_{3} \cup \mathcal{U}_{1} \cup \mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}$ and each $\bar{I}_{X}$ with $X \in \mathcal{U}_{1} \cup \mathcal{R}_{2}$. For a part $X$, let $X^{\prime}$ be the corresponding part in $G^{\prime}$, and set $\mathcal{P}^{\prime}=\left\{X^{\prime}: X \in \mathcal{P}\right\}$.
Step 11. Set $0=\dot{u}=\dot{r}=\ddot{u}$. If $k$ is odd $(b=1)$ then we merge one more pair of vertices under any of the following special circumstances:
(a) there exists $X \in \mathcal{U}_{4}$ with $|W(X)|<\left|G^{\prime}\right|$. Fix such an $X=\dot{X}$. By Lemma 13 , $r_{3}=0$ and there is a pair $\dot{I} \subseteq \dot{X}$ such that (i) $f_{4}$ can be extended to an $S D R f$ of $\mathcal{L}(M)$, where $M:=M_{4}+v_{\dot{I}}$; (ii) $\left|W\left(\left\{v_{\dot{I}}, v\right\}\right)\right| \geq 2 k-1$, and if equality holds then $\left|W\left(\left\{v_{\dot{I}}, v\right\} \cup \dot{Z}^{\prime}\right) \cup C_{4}\right| \geq 2 k$ for both $v \in \bar{I}$; and (iii) $W\left(\overline{\bar{I}}+v_{\dot{I}}\right) \geq\left|G^{\prime}\right|-1$. Merge $\dot{I}$ and set $\dot{u}=1$.
(b) $u_{1}=r_{2}=0$ and there is $Y \in \mathcal{R}_{3}$ with $|W(Y)| \leq 3 k-1-u_{2}-r_{1}$. Then (a) fails since $r_{3} \geq 1$. Fix such a $Y=\dot{Y}$. As $u_{1}=0=r_{2}, M_{4}=M_{3}$. Since $r_{3} \neq 0$, (a) is not executed. By Lemma 11, $f_{3}$ can be chosen so that it is an SDR of $\mathcal{L}(M)$, where $M:=M_{4}+v_{\bar{I}_{\dot{\gamma}}}$. Merge $\bar{I}_{\dot{Y}}$ and set $\dot{r}=1$.
(c) condition (a) fails and there exist $X \in \mathcal{U}_{4}$ and $x y z \subseteq X$ with

$$
\left|W\left(x y z \cup \dot{Z}^{\prime}\right)\right| \leq 2 k+u_{4}-1<|W(X)| .
$$

Fix such an $X=x y z w=\ddot{X}$. By Lemma 14 there is a pair $\ddot{I} \subseteq x y z$ such that (i) $f_{4}$ can be extended to an SDR $f$ of $\mathcal{L}(M)$, where $M:=M_{4}+v_{\ddot{I}} ;(\mathrm{ii})\left|W\left(\left\{v_{\tilde{I}}, v\right\}\right)\right| \geq 2 k$
for $v \in x y z \backslash \ddot{I}$ and $\left|W\left(\left\{v_{\ddot{I}}, w\right\}\right)\right| \geq 2 k-1$; and (iii) $\left|W\left(\overline{\bar{I}}+v_{\bar{I}}\right)\right| \geq 2 k+u_{4}$. Merge $I_{\ddot{X}}:=\ddot{I}$ and set $\ddot{u}=1$.
Step 12. Recall that $G^{\prime}$ is the graph obtained after the first ten steps. Let $H$ be the final graph obtained by this merging procedure. (If $b=0$, and possibly otherwise, $\left.H=G^{\prime}\right)$. Also let $M$ be the final set of merged vertices, $f$ be the final $\operatorname{SDR}$ of $\mathcal{L}(M)$, and $C=\operatorname{ran}(f)$.

Our next task is to state and prove the four lemmas on which the algorithm is based. We will need the following easy claim.

Claim 10. Let $\mathcal{P}_{1}, \mathcal{P}_{2}, \mathcal{P}_{3}$ be the three partitions of a 4 -set $X$ into pairs. For all $I_{1} \in$ $\mathcal{P}_{1}, I_{2} \in \mathcal{P}_{2}, I_{3} \in \mathcal{P}_{3}$ there exists $v \in X$ such that either (i) $v \in I_{1} \cap I_{2} \cap I_{3}$ or (ii) $v \notin I_{1} \cup I_{2} \cup I_{3}$.

Lemma 11. There is a family $\mathcal{I}=\left\{I_{X}: X \in \mathcal{R}_{3}\right\}$ such that $I_{X} \subseteq X,\left|I_{X}\right|=2, l^{\prime}\left(I_{X}\right) \geq$ $l^{\prime}\left(\bar{I}_{X}\right)$, and $\mathcal{L}\left(M_{2} \cup\left\{v_{I_{X}}: X \in \mathcal{R}_{3}\right\}\right)$ has an $S D R f_{3}$ extending $f_{2}$.

Furthermore, if $u_{1}=0=r_{2}$ and there is $\dot{Y} \in \mathcal{R}_{3}$ with $|W(\dot{Y})| \leq 3 k-1-u_{2}-r_{1}$, then $I_{\dot{Y}}$ can be chosen so that there is an $S D R f$ of $\mathcal{L}(M)$ extending $f_{2}$, where $M=M_{3}+v_{\bar{I}_{\dot{Y}}}$.
Proof. Consider any $X \in \mathcal{R}_{3}$, and let $A(X)=N_{2}(X) \backslash C_{2}$ be the set of colors available for coloring a pair of vertices from $X$. Then $L^{\prime}(I)=L(I) \cap A(X)$ for all pairs $I \subseteq X$. For each color $\alpha \in A$, set $I(\alpha)=\{x \in X: \alpha \in L(x)\}$. As $A(X) \subseteq N_{2}(X),|I(\alpha)|=2$. Let $B(X)=\left\{\alpha \in A(X): l^{\prime}(I(\alpha)) \geq l^{\prime}(\bar{I}(\alpha))\right\}$. For the first part, it suffices to show that $\mathcal{B}=\left\{B(Z): Z \in \mathcal{R}_{3}\right\}$ has an SDR $g$ : for each $X \in \mathcal{R}_{3}$ set $I_{X}=I(\alpha)$, and $f\left(v_{I_{X}}\right)=\alpha$, where $\alpha=g(B(X))$.

By (3.3), $N_{3}(X) \subseteq C_{1} \subseteq C_{2}$; so $n_{3}(X) \leq u_{2}+r_{1}$. By (2.1)

$$
\begin{equation*}
n_{2}(X)+2 n_{3}(X) \geq 4 l-|W(X)| \geq 4(l-k)+1 \geq 2 k-1 . \tag{3.6}
\end{equation*}
$$

Thus

$$
\begin{align*}
|A(X)| & =n_{2}(X)+n_{3}(X)-\left|C_{2}\right| \geq n_{2}(X)+2 n_{3}(X)-n_{3}(X)-\left|C_{2}\right|  \tag{3.7}\\
& \geq 2 k-1-\left(2 u_{2}+u_{3}+2 r_{1}\right) \geq 2 r_{3}-1
\end{align*}
$$

If $\alpha \in A(X) \backslash B(X)$ then $A(X) \cap L(\bar{I}(\alpha)) \subseteq B(X)$. So $|B(X)| \geq\lceil|A(X)| / 2\rceil \geq r_{3}$. Hence $\mathcal{B}$ has an SDR $g$.

Now suppose $\dot{Y}$ is defined in Step 11(b). Then $b=1, u_{1}=r_{2}=0$, and $|W(\dot{Y})| \leq$ $3 k-1-u_{2}-r_{1}$. As $b=1, k$ is odd; so $k \geq 3$. If $r_{3} \geq 2$ then fix $Z \in \mathcal{R}_{3}-\dot{Y}$. A partition $\mathcal{Q}=\{I, \bar{I}\}$ of $\dot{Y}$ into pairs is bad if $l^{\prime}(I)=0$ or $l^{\prime}(\bar{I})=0$; else it is good. It is weak if $r_{3} \geq 2, L^{\prime}(I) \cup L^{\prime}(\bar{I}) \subseteq B(Z)$ and $|B(Z)|=r_{3}$; else it is strong.

For the second part, it suffices to show that $\dot{Y}$ has a good, strong partition: If $\{\dot{I}, \bar{I}\}$ is a good, strong partition then choose $\alpha, \beta \in L^{\prime}(I) \cup L^{\prime}(\bar{I})$ with $|B(Z)-\alpha| \geq r_{3}$ and $\alpha \in L^{\prime}(I)$ iff $\beta \in L^{\prime}(\bar{I})$. Then $\alpha$ and $\beta$ are the representatives for $L^{\prime}(I)$ and $L^{\prime}(\bar{I})$, or vice versa. We are done if $r_{3}=1$. If $r_{3} \geq 2$ then continue by greedily choosing an SDR of $\mathcal{B}-B(\dot{Y})-B(Z)+(B(Z)-\alpha)+L^{\prime}(I)+L^{\prime}(\bar{I})$ by picking representatives for $\mathcal{B}-B(\dot{Y})-B(Z)$, and finally picking a representative for $B(Z)-\alpha$.

Using the first half of (3.6),

$$
n_{2}(\dot{Y})+2 n_{3}(\dot{Y}) \geq 4 l-|W(\dot{Y})| \geq 2 l+u_{2}+r_{1}
$$

So by (3.7),

$$
|A(\dot{Y})| \geq 2 l+u_{2}+r_{1}-\left(2 u_{2}+u_{3}+\underset{7}{2 r_{1}}\right) \geq 2 l-u_{2}-u_{3}-r_{1} \geq 2 k+r_{3}-1
$$

First suppose for a contradiction that $\dot{Y}$ has no good partition. For each partition $\mathcal{P}$ of $X$ into pairs, choose $I \in \mathcal{P}$ with $L(I) \cap A(\dot{Y})=\emptyset$. Using Claim 10, there exists $w \in \dot{Y}$ such that either (i) $L(w x) \cap A(\dot{Y})=\emptyset$ for all $x \in \dot{Y} \backslash w$ or (ii) $L(x y) \cap A(\dot{Y})=\emptyset$ for all $x y \subseteq \dot{Y} \backslash w$. If (i) holds then $L(w) \cap A(\dot{Y})=\emptyset$. This yields the contradiction

$$
l+2 k+r_{3}-1 \leq l(w)+|A(\dot{Y})| \leq|W(\dot{Y})| \leq 3 k-1-u_{2}-r_{1}<l+2 k-1
$$

If (ii) holds then $A(\dot{Y}) \subseteq L(w)$, and so $l<|A(\dot{Y})| \leq l(w)$, another contradiction.
So $\dot{Y}$ has a good partition (say) $\mathcal{Q}_{1}=\{x y, z w\}$. Suppose $\mathcal{Q}_{1}$ is weak. Then $r_{3} \geq 2$ and $\left|A_{0}\right| \geq 2 k-1$, where $A_{0}:=A(\dot{Y}) \backslash B(Z) \subseteq A(\dot{Y}) \backslash\left(L^{\prime}(x y) \cup L^{\prime}(z w)\right)$. The former implies $2 \leq r_{3} \leq l-k \leq 2 k-l$; so $\left(^{*}\right) l \leq 2 k-2$. If the other two partitions of $\dot{Y}$ are both bad then there is $v \in \dot{Y}$ with $A_{0} \subseteq L(v)$. So $2 k-1 \leq\left|A_{0}\right| \leq l$ contradicting ( $\left.{ }^{*}\right)$. Say $\mathcal{Q}_{2}=\{x w, y z\}$ is good. If $\mathcal{Q}_{2}$ is weak then $A_{0} \subseteq A(\dot{Y}) \backslash\left(L^{\prime}(x y) \cup L^{\prime}(z w) \cup L^{\prime}(x w) \cup L^{\prime}(y z)\right)$. Then $\left|L^{\prime}(x z) \cup L^{\prime}(y w)\right| \geq 2 k-1$. So $\mathcal{Q}_{3}=\{x z, y w\}$ is strong. By $\left({ }^{*}\right), l^{\prime}(x z), l^{\prime}(y w) \leq l<2 k-1$. Thus $l^{\prime}(x z), l^{\prime}(y w) \geq 1$, and so $\mathcal{Q}_{3}$ is also good.

Lemma 12. For each $X \in \mathcal{R}_{2}$ there is a pair $I_{X} \subseteq X$ such that $\left\{L\left(I_{X}\right): X \in \mathcal{P} \backslash \mathcal{U}_{4}\right\} \cup$ $\left\{L\left(\bar{I}_{X}\right): X \in \mathcal{U}_{1} \cup \mathcal{R}_{2}\right\}$ has an $S D R f_{4}$ that extends $f_{3}$.
Proof. Each $X \in \mathcal{U}_{1}$ satisfies $L\left(I_{X}\right), L\left(\bar{I}_{X}\right) \geq k$. Thus $\left|L\left(I_{X}\right) \backslash C_{3}\right|,\left|L\left(\bar{I}_{X}\right) \backslash C_{3}\right| \geq$ $k-u_{2}-u_{3}-r_{1}-r_{3} \geq u_{1}$. By Theorem 3, $\left\{L\left(I_{X}\right) \backslash C_{3}, L\left(\bar{I}_{X}\right) \backslash C_{3}: X \in \mathcal{U}_{1}\right\}$ has an SDR, and so $f_{3}$ can be extended to an $\operatorname{SDR} g$ for $\mathcal{L}\left(M_{3}^{\prime}\right)$, where $M_{3}^{\prime}:=M_{3} \cup\left\{I_{X}, \bar{I}_{X}: X \in \mathcal{U}_{1}\right\}$. Let $C^{g}=\operatorname{ran}(g)$. Then $\left|C^{g}\right|=2 u_{1}+u_{2}+u_{3}+r_{1}+r_{3}$. Consider any $X=x y z w \in \mathcal{R}_{2}$. Let $A(X)=N_{2}(X) \backslash C^{g}$. Again by Theorem 3 it suffices to show:

$$
\begin{equation*}
\left(\exists I_{X} \subseteq X\right)\left[\left|I_{X}\right|=2 \wedge\left|L\left(I_{X}\right) \cap A(X)\right| \geq r_{2} \wedge\left|L\left(\bar{I}_{X}\right) \cap A(X)\right| \geq r_{2}\right] \tag{3.8}
\end{equation*}
$$

Observe $\sigma_{2}(X)=n_{2}(X)+3 n_{3}(X)$ and $\sigma_{3}(X)=n_{3}(X)$. So $n(X)=n_{2}(X)+n_{3}(X)=$ $\sigma_{2}(X)-2 \sigma_{3}(X)$. By (3.3), $N_{3}(X) \subseteq C^{g}$, and by (3.4) $\sigma_{3}(X) \leq u_{2}+r_{1}$. So

$$
\begin{align*}
n(X) & =\sigma_{2}(X)-2 \sigma_{3}(X) \geq 5(l-k)+2 u_{1}+2 u_{2}+u_{3}+r_{1}+r_{2}-\left(u_{2}+r_{1}\right) \\
& \geq 5(l-k)+2 u_{1}+u_{2}+u_{3}+r_{2} \text { and }  \tag{3.9}\\
|A(X)| & =\left|N_{2}(X) \backslash C^{g}\right|=\left|N_{2}(X) \cup N_{3}(X) \backslash C^{g}\right|=n(X)-\left|C^{g}\right| \\
& \geq 5(l-k)+2 u_{1}+u_{2}+u_{3}+r_{2}-\left(2 u_{1}+u_{2}+u_{3}+r_{1}+r_{3}\right) \\
& \geq 5(l-k)-r_{1}+r_{2}-r_{3} \geq 4(l-k)+2 r_{2} . \tag{3.10}
\end{align*}
$$

Suppose (3.8) fails. Then for each of the three partitions of $X$ into pairs, there is a pair $u v$ with $|L(u v) \cap A(X)| \leq r_{2}-1$. Using Claim 10, there exists $v \in X$ such that either (i) $|L(v w) \cap A(X)| \leq r_{2}-1$ for all $w \in X-v$ or (ii) $|L(v w) \cap A(X)| \leq r_{2}-1$ for all $w \in X-v$.

If (i) holds then

$$
|L(v) \cap N(X)| \leq\left|C^{g}\right|+\sum_{w \in X-v}|L(v w) \cap A(X)| \leq\left|C^{g}\right|+3 r_{2}-3 .
$$

Since $|L(w) \cap N(X)| \leq l$ for all $w \in X-v$,

$$
2 n(X) \leq \sum_{v \in X}|L(v) \cap N(X)| \leq 3 l+\left(\left|C^{g}\right|+3 r_{2}-3\right)
$$

Using $\left|C^{g}\right|=2 u_{1}+u_{2}+u_{3}+r_{1}+r_{3}$ and (3.9) implies

$$
\text { 11) } \begin{align*}
10(l-k)+4 u_{1}+2 u_{2}+2 u_{3}+2 r_{2} & \leq 3 l-2 u_{1}+u_{2}+u_{3}+r_{1}+r_{3}+3 r_{2}-3  \tag{3.11}\\
4 l-k+(6 l-9 k+3)+2 u_{1}+u_{2}+u_{3} & \leq 3 l+r_{1}+r_{2}+r_{3} \leq 4 l-k .
\end{align*}
$$

Since $6 l-9 k=-3 b$, both $b=1$ and $0=u_{1}=u_{2}=u_{3}$. Now, by (3.4), $\mu_{2}(X) \leq l-k$. So $|L(w) \cap N(X)| \leq 3(l-k)$ for all $w \in X$. Strengthening the estimate in (3.11) yields the contradiction:

$$
\begin{aligned}
10(l-k)+2 r_{2} & \leq 9(l-k)+\left(\left|C^{g}\right|+3 r_{2}-3\right) \\
l-k & \leq r_{1}+r_{2}+r_{3}-3<l-k .
\end{aligned}
$$

Thus (ii) holds. So

$$
\begin{equation*}
|A(X)| \leq l(v)+\sum_{w x \subseteq X-v}|L(u v) \cap A(X)| \leq l+3\left(r_{2}-1\right) . \tag{3.12}
\end{equation*}
$$

Using (3.10), (3.12) and $2 l-3 k=-b$, this yields the contradiction

$$
\begin{aligned}
4(l-k)+2 r_{2} \leq|A(X)| & \leq l+3\left(r_{2}-1\right) \\
l-k+2 \leq 3 l-4 k+3 & \leq r_{2} \leq l-k
\end{aligned}
$$

Lemma 13. Suppose $X=x y z w \in \mathcal{U}_{4}$ and $|W(X)|<\left|G^{\prime}\right|$. Then $b=1, u_{1}=0=r_{3}$, $u_{2}+u_{3} \geq 1$, and there exists a pair $J \subseteq X$ such that:
(1) $L(J) \nsubseteq C_{4}$;
(2) $\left|W\left(\left\{v_{J}, v\right\}\right)\right| \geq 2 k-1$ and if $\left|W\left(\left\{v_{J}, v\right\}\right)\right|=2 k-1$ then $\left|W\left(\left\{v_{J}, v\right\} \cup \dot{Z}\right) \cup C_{4}\right| \geq 2 k$ for both $v \in \bar{J}$;
(3) $\left|W\left(\bar{J}+v_{J}\right)\right| \geq\left|G^{\prime}\right|-1$; in particular $|W(X)| \geq\left|G^{\prime}\right|-1$.

Proof. Now $\left|G^{\prime}\right|=3 k-u_{1}-u_{2}+u_{4}-r_{1}-r_{2}$. Observe that

$$
\begin{equation*}
\sigma_{2}(X)-\sigma_{3}(X) \geq 5(l-k)+2 u_{1}+2 u_{2}+u_{3}+r_{1}+r_{2}+1, \tag{3.13}
\end{equation*}
$$

since otherwise inclusion-exclusion yields the contradiction:

$$
\begin{aligned}
|W(X)| & =\sigma_{1}(X)-\sigma_{2}(X)+\sigma_{3}(X) \\
& \geq 4 l-5(l-k)-2 u_{1}-2 u_{2}-u_{3}-r_{1}-r_{2} \\
& \geq 3 k+\left(2 k-l-u_{1}-u_{2}-u_{3}\right)-u_{1}-u_{2}-r_{1}-r_{2} \\
& \geq 3 k-u_{1}-u_{2}+u_{4}-r_{1}-r_{2}=\left|G^{\prime}\right|>|W(X)| .
\end{aligned}
$$

By (3.13) and (3.5), $r_{1}+r_{2}=l-k$ and $r_{3}=0$. Consider any pair $I=x y \subseteq X$. Then

$$
\begin{align*}
\left|W\left(\bar{I}+v_{I}\right)\right| & \geq l(x y)+l(z)+l(w)-l(x y z)-l(x y w)-l(z w)  \tag{3.14}\\
& \geq 2 l-2 u_{2}+\Delta_{1}(I)+\Delta_{2}(I) \\
\left|G^{\prime}\right|-\left|W\left(\bar{I}+v_{I}\right)\right| & \leq b-2 u_{1}+\left(u_{1}+u_{2}+u_{4}-l+k\right)-\Delta_{1}(I)-\Delta_{2}(I)  \tag{3.15}\\
1 & \leq 2 b-2 u_{1}-u_{3}-\Delta_{1}(I)-\Delta_{2}(I) \tag{3.16}
\end{align*}
$$

By (3.16), $\Delta_{1}(I)+\Delta_{2}(I) \leq 1$. As $\Delta_{1}(I)=-\Delta_{1}(\bar{I})$ and $\Delta_{2}(I), \Delta_{2}(\bar{I}) \geq 0$, we could choose $I$ with $\Delta_{1}(I)+\Delta_{2}(I) \geq 0$. So $b=1, u_{1}=0, u_{3} \leq 1$, and

$$
\begin{equation*}
1 \leq|G|-\left|W\left(\bar{I}+v_{I}\right)\right| \leq 2-u_{3}-\Delta_{1}(I)-\Delta_{2}(I) \leq 2 . \tag{3.17}
\end{equation*}
$$

Furthermore, using $\Delta_{1}(I)=-\Delta_{1}(\bar{I})$ again,

$$
\begin{equation*}
0 \leq 4 u_{2}-\sigma_{3}(X)=\Delta_{2}(I)+\Delta_{2}(\bar{I})=\Delta_{1}(I)+\Delta_{2}(I)+\Delta_{1}(\bar{I})+\Delta_{2}(\bar{I}) \leq 2 . \tag{3.18}
\end{equation*}
$$

By (3.13), $r_{1}+r_{2}=l-k, \sigma_{2}(X) \leq 6 \mu_{2}(X)$, 3.4), and $\sigma_{3}=4 u_{2}-\Delta_{2}(I)-\Delta_{2}(\bar{I})$,

$$
\begin{align*}
1+6(l-k)+2 u_{2}+u_{3}+\sigma_{3}(X) & \leq \sigma_{2}(X) \leq 6\left(l-k+u_{2}+u_{3}\right)  \tag{3.19}\\
1+u_{3}+6\left(l-k+u_{2}\right)-\Delta_{2}(I)-\Delta_{2}(\bar{I}) & \leq \sigma_{2}(X) \leq 6\left(l-k+u_{2}+u_{3}\right) .
\end{align*}
$$

By (3.19) $u_{2}+u_{3} \geq 1$. So the first three assertions of the lemma have been proved. It remains to find a pair $J \subseteq X$ satisfying (1-3).

First suppose $u_{3}=1$. By (3.17), $\Delta_{1}(I)+\Delta_{2}(I)=0$ for all pairs $I \subseteq X$. So $\Delta_{1}(I) \leq 0$ and $\Delta_{1}(\bar{I}) \leq 0$. As $\Delta_{1}(I)=-\Delta_{1}(\bar{I})$, this implies $\Delta_{1}(I)=0=\Delta_{1}(\bar{I})$. So $\Delta_{2}(I)=0=$ $\Delta_{2}(\bar{I})$. By (3.20), there exists a pair $I \subseteq X$ with $l(I)=l-k+u_{2}+u_{3}$. As $\Delta_{1}(I)=0$, $l(\bar{I})=l-k+u_{2}+u_{3}$. Thus

$$
\left|W\left(\left\{v_{I}, v_{\bar{I}}\right\}\right)\right|=l(I)+l(\bar{I})=2\left(l-k+u_{2}+u_{3}\right)>2(l-k)+u_{2}+u_{3} \geq\left|C_{4}\right| .
$$

Pick $J \in\{I, \bar{I}\}$ such that $L(J) \nsubseteq C_{4}$. Then (1) holds. For (2), let $v^{\prime} \in \bar{J}$, and observe

$$
\left|W\left(\left\{v_{J}, v^{\prime}\right\}\right)\right|=l(J)+l\left(v^{\prime}\right)-l\left(J+v^{\prime}\right) \geq 2 l-k+u_{2}+u_{3}-u_{2}=2 k .
$$

Thus (2) holds. As $u_{3}=1$, (3.17) implies (3).
Otherwise $u_{3}=0$. Then $u_{2} \geq 1$, and so $\dot{Z}$ is defined in Step 3. Put $C_{0}:=C_{4} \cup W\left(\dot{Z}^{\prime}\right)$. By Step $3,\left|C_{0}\right| \geq\left|W\left(\dot{Z}^{\prime}\right)\right| \geq l+u_{2}$. Call a vertex $x \in X$ bad if $\left|L(x) \cup C_{0}\right| \leq 2 k-1$; otherwise $x$ is good. If $x$ is bad then $\left|C_{0} \backslash L(x)\right| \leq 2 k-1-l \leq l-k$. If another vertex $y$ is also bad, then using (3.4) and (3.17),

$$
\begin{aligned}
l-k+u_{2} \geq l(x y) & \geq\left|L(x y) \cap C_{0}\right| \geq\left|C_{0}\right|-\left|C_{0} \backslash L(x)\right|-\left|C_{0} \backslash L(y)\right| \\
& \geq l+u_{2}-2(l-k) \geq l-k+u_{2}+1,
\end{aligned}
$$

a contradiction. So at most one vertex of $X$ is bad.
Call a pair $I \subseteq X$ bad if $L(I) \subseteq C_{4}$; otherwise $I$ is good. Note that if $I$ is good then $I$ satisfies (1). By (3.18), (3.20), and $u_{3}=0,6\left(l-k+u_{2}\right)-1 \leq \sigma_{2} \leq 6\left(l-k+u_{2}\right)$; and so by (3.4), every pair $I \subseteq X$ satisfies

$$
l-k+u_{2}-1 \leq l(I) \leq l-k+u_{2}
$$

If the upper bound is sharp then call I normal; otherwise call $I$ abnormal. Then there is at most one abnormal pair. If $I$ is normal then $l(\bar{I}) \leq l(I)$; so $\Delta_{1}(I) \geq 0$.

By (3.2), every triple $T \subseteq X$ satisfies $l(T) \leq u_{2}$. If equality holds then call $T$ normal; otherwise call $T$ abnormal; if $\left|L(T) \cap C_{0}\right| \leq u_{2}-2$ then call T very abnormal. Suppose two pairs $I, J \subseteq T$ are both bad. At least one, say $I$, is normal. Then

$$
\begin{align*}
2(l-k)+u_{2} & \geq\left|C_{4}\right| \geq|L(I) \cup L(J)| \geq l-k+u_{2}+l(J)-l(I \cup J)  \tag{3.21}\\
l(I \cup J) & \geq l(J)-l+k= \begin{cases}u_{2} & \text { if } J \text { is normal } \\
u_{2}-1 & \text { if } J \text { is abnormal }\end{cases}
\end{align*}
$$

So an abnormal triple contains at most one bad, normal pair, and a very abnormal triple contains at most one bad pair. A pair $I$ contained in an abnormal triple satisfies $\Delta_{2}(I) \geq 1$.

Let $J$ be a good, normal pair contained in a abnormal triple $T$ with $w \in X \backslash T$. Then $\Delta_{1}(J)+\Delta_{2}(J) \geq 1$. So $J$ satisfies (3) by (3.17). Also,

$$
\left|W\left(v_{J}, v\right)\right|=l(J)+l(v)-l(J+v) \geq \begin{cases}2 l-k+u_{2}-\left(u_{2}-1\right)=2 k & \text { if } v \in T \backslash J \\ 2 l-k+u_{2}-u_{2}=2 k-1 & \text { if } v=w\end{cases}
$$

So $\left(^{*}\right) J$ satisfies (2), provided $\left|W\left(v_{j}, w\right) \cup C_{0}\right| \geq 2 k$. In particular, (2) holds if $w$ is good.
By (3.20) and (3.18), $1 \leq \Delta_{2}(I)+\Delta_{2}(\bar{I}) \leq 2$. As $\sigma_{3}=4 u_{2}-\Delta_{2}(I)-\Delta_{2}(\bar{I})$, we have $4 u_{2}-2 \leq \sigma_{3}(X)=4 u_{2}-1$. In the first case there is one abnormal triple. In the second case, either there is a very abnormal triple or there are two abnormal triples.

First suppose there are two abnormal triples. Choose an abnormal triple $T$ so that if there is a bad vertex then it is in $T$. As $T$ contains three pairs and at most one is bad
and at most one is abnormal, $T$ contains a good, normal pair $J$. Say $J=y z, T=x y z$, and $w \in X \backslash T$. Then $w$ is good, and thus $J$ satisfies (2) by (*).

Otherwise, let $T=x y z$ be the only abnormal triple and $w \in X \backslash T$. There is at most one abnormal pair, and only if $T$ is very abnormal. So $T$ contains at most one bad pair. Now suppose $T$ has two good, normal pairs $x y$ and $y z$. By ( ${ }^{*}$ ), some $J \in P:=\{x y, y z\}$ satisfies (2), unless $C_{0} \subseteq L(J) \cup L(w)$ for both $J \in P$. Then, using $u_{1}=u_{3}=r_{3}=0$,

$$
l+u_{2}=\left|C_{0}\right| \leq|L(x y) \cup L(w)|+|L(y z) \cup L(w)|-|L(x y) \cup L(y z) \cup L(w)| .
$$

As $T$ is abnormal, and both $x y$ and $y z$ are normal,

$$
\begin{aligned}
|L(x y) \cup L(y z) \cup L(w)| & =l(x y)+l(y z)+l(w)-l(x y w)-l(y z w)-l(x y z) \\
& \geq 3 l-2 k+2 u_{2}-\left(3 u_{2}-1\right)=k+l-u_{2} .
\end{aligned}
$$

Combining the last two expressions yields the contradiction,

$$
l+u_{2} \leq\left|C_{0}\right| \leq 2(2 k-1)-\left(k+l-u_{2}\right)=3 k-1-l+u_{2}-1=l+u_{2}-1 .
$$

Otherwise, $T$ is very abnormal, and (say) both $x z$ is bad and $J=y z$ is normal. As $T$ contains at most one bad pair, $y z$ is also good. Since $x z$ is bad, $x z \subseteq C_{0}$. Now

$$
\left|C_{0} \backslash W\left(\left\{v_{J}, w\right\}\right)\right| \geq|L(x z) \backslash(L(w) \cup L(J))| \geq l-k+u_{2}-\left(u_{2}-2\right) \geq 1
$$

and $(2)$ holds by $\left({ }^{*}\right)$, since $\emptyset \neq L(x z) \backslash(L(w) \cup L(J)) \subseteq C_{0}$ implies

$$
\left|W\left(\left\{v_{J}, w\right\} \cup C_{0}\right)\right| \geq\left|W\left(\left\{v_{J}, v\right\}\right)\right|+1 \geq 2 k .
$$

Lemma 14. Suppose $b=1$ and $X=x y z w \in \mathcal{U}_{4}$. If

$$
|W(x y z)| \leq 2 k+u_{4}-1<|W(X)|
$$

then $u_{1}=0$ and there exists a pair $J \subseteq X$ such that:
(1) $L(J) \nsubseteq C_{4}$;
(2) $\left|W\left(\left\{v_{J}, v\right\}\right)\right| \geq 2 k$ for $v \in x y z \backslash J$ and $\left|W\left(\left\{v_{J}, w\right\}\right)\right| \geq 2 k-1+u_{3}$; and
(3) $\left|W\left(\bar{J}+v_{J}\right)\right| \geq 2 k+u_{4}$.

Proof. Consider a pair $v v^{\prime} \subseteq x y z$. Then

$$
\begin{aligned}
2 k+u_{4}-1 & \geq|W(x y z)| \geq\left|W\left(v v^{\prime}\right)\right| \geq l(v)+l\left(v^{\prime}\right)-l\left(v v^{\prime}\right) \\
& \geq 2 l-\left(l-k+u_{2}+u_{3}\right) \geq 3 k-1-k+u_{1}+u_{4} \\
& \geq 2 k+u_{1}+u_{4}-1 \geq 2 k
\end{aligned}
$$

So $u_{1}=0, l\left(v v^{\prime}\right)=l-k+u_{2}+u_{3}$, and $W(x y z)=W\left(v v^{\prime}\right)$. Since $v v^{\prime}$ is arbitrary, every color in $W(x y z)$ appears in at least two of the lists $L(x), L(y), L(z)$. So $W\left(\left\{v_{J}, v\right\}\right)=$ $W(x y z)$ and $\left|W\left(\left\{v_{J}, v\right\}\right)\right| \geq 2 k$ for every pair $J \subseteq x y z$ and vertex $v \in x y z \backslash J$. As $\left|C_{4}\right|<2 k \leq|W(x y z)|$, there is a pair $J \subseteq x y z$ with $L(J) \nsubseteq C_{4}$. Furthermore,

$$
\left|W\left(\left\{v_{J}, w\right\}\right)\right| \geq l(J)+l(w)-l(J+w) \geq l-k+u_{2}+u_{3}+l-u_{2}=2 k-1+u_{3} .
$$

Finally, as $W\left(\left\{v_{J}, v\right\}\right)=W(x y z)$ for $v \in x y z \backslash J$,

$$
\left|W\left(\bar{J}+v_{J}\right)\right|=\left|W\left(\left\{v_{J}, v\right\}\right) \cup W(w)\right|=|W(x y z w)| \geq 2 k+u_{4} .
$$

Lemma 15. $G^{\prime}$ is L-choosable.

Proof. First observe that if $k$ is even then $b=\dot{u}=\ddot{u}=\dot{r}=0$ and $H=G^{\prime}$. In this case the following argument is much simpler.

Using Hall's Theorem it suffices to show $|S| \leq|W|:=\left|\bigcup_{x \in S} L(x)\right|$ for every $S \subseteq V(H)$. Suppose for a contradiction that $|S|>|W|$ for some $S \subseteq V(H)$. We consider several cases. Each case assumes the previous cases fail.
Case 1: There is $X \in \mathcal{U}_{4}$ with $|S \cap X|=4$. Then $|W|<|S| \leq\left|G^{\prime}\right|$. By Lemma 13 , $b=1$ and $\left|G^{\prime}\right|-1=|W(X)|<|S|=\left|G^{\prime}\right|$. So Step 11(a) is executed, and $S=V\left(G^{\prime}\right)$. In particular, $\dot{X} \subseteq S$. Thus

$$
|S| \leq|H|=\left|G^{\prime}\right|-1 \leq\left|W\left(\bar{J}+v_{J}\right)\right| \leq|W(\dot{X})| \leq|W| .
$$

Case 2: There exists $Z=x y z w \in \mathcal{U}_{3}$ with $\left|S \cap Z^{\prime}\right|=3$. Now $|S| \leq 3 k-u_{1}-u_{2}-r_{1}-r_{2}-\dot{r}$, since Case 1 fails. Say $I_{Z}=x y$. By Step $4, \Delta_{1}(x y) \geq 0$ and $l(x y z)+l(x y w)=$ $2 u_{2}-\Delta_{2}(x y)$. By Step $3, l(x y z)+l(x y w) \leq u_{2}+r_{1}$. So

$$
\begin{align*}
|W| & \geq\left|W\left(Z^{\prime}\right)\right| \geq l(x y)+l(z)+l(w)-l(x y z)-l(x y w)-l(z w)  \tag{3.22}\\
& =2 l+\Delta_{1}(x y)-2 u_{2}+\Delta_{2}(x y)=3 k-b+\Delta_{1}(x y)-2 u_{2}+\Delta_{2}(x y) \\
& \geq 3 k-b+\Delta_{1}(x y)-u_{2}-r_{1} \geq|S|-b
\end{align*}
$$

As $|S|>|W|$ equality holds throughout. Thus $b=1, u_{1}=r_{2}=\dot{r}=\Delta_{1}(x y)=0, r_{1} \leq u_{2}$, and $\left({ }^{*}\right) Y^{\prime} \subseteq S$ for all $Y \in \mathcal{R}_{3}$. If $u_{4}=0$ then

$$
k=l-k+u_{2}+u_{3} \leq l(x y)=l(\overline{x y})+\Delta_{1}(x y)=l(\overline{x y}) .
$$

By (3.1) this contradicts $u_{1}=0$. So $u_{4} \geq 1, u_{3}+u_{4} \geq 2$, and

$$
r_{1}+r_{2} \leq u_{2}+0=2 k-l-u_{3}-u_{4} \leq l-k-1 .
$$

Thus $r_{3} \geq 1$. Say $Y \in \mathcal{R}_{3}$. By $\left(^{*}\right), Y^{\prime} \subseteq S$; by (3.22), $\left|W\left(Y^{\prime}\right)\right| \leq|W|=3 k-1-u_{2}-r_{1}$. So, using $b=1$ and $u_{1}=0=r_{2}$, Step 11(b) is executed, and $\dot{r}=1$, a contradiction.
Case 3: There exists $X=w x y z \in \mathcal{R}_{3}$ with $\left|S \cap X^{\prime}\right|=3$. Say $I_{X}=x y$. Now $|S| \leq$ $3 k-u_{1}-u_{2}-u_{3}-r_{1}-r_{2}-\dot{r}$. By Step $7, l^{\prime}(x y) \geq l^{\prime}(w z)$. By (3.3), $N_{3}(X) \subseteq C_{1} \subseteq C_{2}$. So $l^{\prime}(x y z)=0=l^{\prime}(x y w)$. Set $t=\left|C_{2} \cap W\right|$. Then $t \leq u_{2}+u_{3}+r_{1}$. So

$$
\begin{aligned}
|W| & =\left|W \backslash C_{2}\right|+\left|C_{2} \cap W\right| \geq l^{\prime}(x y)+l^{\prime}(z)+l^{\prime}(w)-l^{\prime}(x y z)-l^{\prime}(x y w)-l^{\prime}(z w)+t \\
& \geq l^{\prime}(x y)+l(z)-t+l(w)-t-l^{\prime}(z w)+t \\
& \geq 3 k-b-\left(u_{2}+u_{3}+r_{1}\right) \geq|S|-b .
\end{aligned}
$$

Thus $b=1,0=r_{2}=u_{1}=\dot{r}$, and $|W(X)| \leq|W| \leq 3 k-1-u_{2}-r_{1}$. So Step 11(b) is executed, and $\dot{r}=1$, a contradiction.
Case 4: There exists $X \in \mathcal{U}_{4}$ with $\left|S \cap X^{\prime}\right|=3$. As the previous cases fail, $|S| \leq 2 k+u_{4}$. Let $x y \subseteq S \cap X^{\prime} \backslash M$. By (3.4),

$$
\begin{aligned}
|W| & \geq l(x)+l(y)-l(x y) \geq 3 k-b-\left(l-k+u_{2}+u_{3}\right) \\
& \geq 2 k+(2 k-l)-\left(u_{2}+u_{3}\right)-b \geq 2 k+u_{1}+u_{4}-b \geq|S|-b .
\end{aligned}
$$

So $b=1, u_{1}=0,|W|=2 k+u_{4}-1$, and $|S|=2 k+u_{4}$. Thus $S$ has exactly two vertices in every class of $\mathcal{P}^{\prime} \backslash \mathcal{U}_{4}^{\prime}$ and exactly three vertices in every class of $\mathcal{U}_{4}^{\prime}$. In particular, $\dot{Z}^{\prime} \subseteq S$. If $\dot{u}=1$, then $\dot{X}^{\prime} \subseteq S$ and $\left|W\left(\dot{X}^{\prime}\right)\right| \geq\left|G^{\prime}\right|-1 \geq 2 k+u_{4} \geq|S|$ by Lemma 13 , else $|W(X)| \geq 2 k+u_{4}$. If $\ddot{u}=1$ then $\ddot{X}^{\prime} \subseteq S$ and $|W| \geq\left|W\left(\ddot{X}^{\prime}\right) \geq|S|\right.$ by Lemma 14 . else $X=X^{\prime}$. As Step 11(c) is not executed,

$$
|W| \geq|W((S \cap X) \cup \dot{Z})| \geq 2 k+u_{4} \geq|S|
$$

Case 5: There exists $X \in \mathcal{U}_{1}$ with $\left|S \cap X^{\prime}\right|=2$. Say $S \cap X^{\prime}=\left\{v_{I}, v_{\bar{I}}\right\}$. As the previous cases fail, $|S| \leq 2 k$. Now

$$
|W| \geq L\left(v_{I}\right)+L\left(v_{\bar{I}}\right) \geq 2 k \geq|S|
$$

Case 6: There exists $X \in \mathcal{U}_{3}$ with $\left|S \cap X^{\prime}\right|=2$. Say $S \cap X^{\prime}=v v^{\prime}$. As the previous cases fail, $|S| \leq 2 k-u_{1}$. If $v, v^{\prime} \notin M$ then $\bar{I}_{X}=v v^{\prime}$. By (3.1), $l\left(\bar{I}_{X}\right) \leq k-1$. So

$$
\left|W\left(v v^{\prime}\right)\right| \geq l(v)+l\left(v^{\prime}\right)-l\left(v v^{\prime}\right) \geq 2 l-(k-1) \geq 2 k \geq|S| .
$$

Otherwise $v=v_{x y}$, where $I_{X}=x y$, and $v^{\prime}=z \notin M$. Then

$$
\begin{aligned}
\left|W\left(v v^{\prime}\right)\right| & \geq l\left(v_{x y}\right)+l(z)-l(x y+z) \\
& \geq l-k+u_{2}+u_{3}+l-u_{2} \geq 2 k-b+u_{3} \geq 2 k \geq|S| .
\end{aligned}
$$

Case 7: There exists $X \in \mathcal{U}_{4}$ with $\left|S \cap X^{\prime}\right|=2$. Say $S \cap X^{\prime}=v v^{\prime}$. If possible, choose $X$ so that $S \cap X^{\prime} \cap M=\emptyset$. As the previous cases fail, $|S| \leq 2 k-u_{1}-u_{3}$. If $v, v^{\prime} \notin M$ then

$$
\begin{align*}
\left|W\left(v v^{\prime}\right)\right| & =l(v)+l\left(v^{\prime}\right)-l\left(v v^{\prime}\right) \geq 2 l-\left(l-k+u_{2}+u_{3}\right) \\
& \geq 2 k-b+u_{1}+u_{4} \geq 2 k \geq|S| . \tag{3.23}
\end{align*}
$$

Else $b=1$, and (say) $v \in M$. By Step $11, v=v_{i}$ or $v=v_{\tilde{I}}$, and $u_{1}=0$.
If $v=v_{I}$ then Step 11(a) was executed. So (i) $r_{3}=0$, (ii) $\left|W\left(v v^{\prime}\right)\right| \geq 2 k-1$, and (iii) if $\left|W\left(v v^{\prime}\right)\right|=2 k-1$ then $u_{2} \geq 1$ and $\left|W\left(v v^{\prime} \cup \dot{Z}^{\prime}\right) \cup C_{4}\right| \geq 2 k$. Since

$$
2 k \geq|S|>|W| \geq\left|W\left(v v^{\prime}\right)\right| \geq 2 k-1
$$

$|S|=2 k$. Thus $S$ contains exactly two vertices of each part $Y^{\prime} \in \mathcal{P}^{\prime}$. In particular, $\dot{Z}^{\prime} \subseteq S$. The choice of $X$ implies $u_{3}=0$ and $u_{4}=1$; thus $u_{2}=l-k \geq 1$. Since $u_{3}=0=r_{3}, M_{4} \subseteq S$. So $|W| \geq\left|W\left(v v^{\prime} \cup \dot{Z}^{\prime}\right) \cup C_{4}\right| \geq 2 k$, a contradiction.

Otherwise $x=v_{\ddot{I}}$. Then Step 11(c) was executed. So there is a part $\ddot{X}=x y z w \in \mathcal{U}_{4}$ with $\ddot{I}=x y$ such that

$$
|W(x y z \cup \dot{Z})| \leq 2 k+u_{4}-1<|W(\ddot{X})|,
$$

$\left|W\left(\left\{v_{x y}, w\right\}\right)\right| \geq 2 k-1+u_{3}$, and $\left|W\left(\left\{v_{x y}, z\right\}\right)\right| \geq 2 k$. So we are done, unless $v^{\prime}=w$ and

$$
2 k \geq|S|>\left|W\left(\left\{v_{x y}, w\right\}\right)\right| \geq 2 k-1+u_{3} .
$$

Thus $u_{3}=0$ and $|S|=2 k$. So $S$ contains exactly two vertices of each class $Y^{\prime} \in \mathcal{P}^{\prime}$. In particular, $\dot{Z}^{\prime} \subseteq S$. As $|W(\ddot{X})|>|W(x y z \cup \dot{Z})|$, we have $\left|L(w) \backslash W\left(x y z \cup \dot{Z}^{\prime}\right)\right| \geq 1$. So

$$
|W| \geq\left|W\left(\left\{v_{x y}, w\right\} \cup \dot{Z}^{\prime}\right)\right| \geq W\left(\dot{Z}^{\prime}\right)+1=l+u_{2}+1=2 l-k+1=2 k
$$

Case 8: There exists $X=x y z w \in \mathcal{U}_{2}$ with $\left|S \cap X^{\prime}\right|=2$. Say $S \cap X^{\prime}=\left\{v_{I}, w\right\}$. As the previous cases fail, $|S| \leq 2 k-u_{1}-u_{3}-u_{4}=l+u_{2}$. Since $L(x y z) \cap L(w)=\emptyset$, we have

$$
|W| \geq\left|W\left(X^{\prime}\right)\right| \geq l(x y z)+l(w) \geq u_{2}+l \geq|S|
$$

Case 9: Otherwise. As the previous cases fail,

$$
|S| \leq u_{1}+u_{2}+u_{3}+u_{4}+2|\mathcal{R}|=l .
$$

As $\mathcal{L}(M)$ has an SDR, there is a vertex $x \in S \backslash M$. Thus $|W| \geq l(x)=l \geq|S|$.


Figure 4.1. Strategy for Alice demonstrating $\operatorname{ch}^{O L}\left(K_{4 * 3}\right) \geq 5$.

## 4. On-line Choosability

By Theorem 7, $\operatorname{ch}\left(K_{4 * 3}\right)=4$. Using a computer we have checked that $\operatorname{ch}^{O L}\left(K_{4 * 3}\right)=5$, but do not have a readable argument to verify the upper bound. Here we prove the lower bound.

Theorem 16. $\mathrm{ch}^{O L}\left(K_{4 * 3}\right) \geq 5$.
Proof. Figure 4.1 describes a strategy for Alice. The top left matrix depicts the initial game position, and Alice's first move. The positions in the matrix correspond to the vertices of $K_{4 * 3}$ arranged so that vertices in the same part correspond to positions in the same column. The order of vertices within a column is irrelevant, as is the order of the columns. The numbers represent the size of the list of each corresponding vertex. The sequence of numbers represents a function $f$. The shaded positions represent the vertices that Alice presents on here first move.

As play progresses Bob chooses certain vertices presented by Alice and passes over others. When a vertex is chosen its position is removed from the next matrix (and the positions in its column of the remaining vertices and the order of the columns may be rearranged). When he passes over a vertex its list size is decreased by one (and its position in its column and the order of the columns may change). The arrows between the matrices point to the possible new game positions that arise from Bob's choice, not counting equivalent positions and omitting clearly inferior positions for Bob. In particular we assume Bob always chooses a maximal independent set.

For example, after Bob's first move there is only one possible game position, provided Bob chooses a maximal independent set. It is shown in the second column of the first row, along with Alice's second move. Now Bob has two possible responses that are pointed to by two arrows. Also consider the matrix in the third row and third column. There are three nonequivalent responses for Bob, but choosing the offered vertex in the second
column of the matrix results in a position that is inferior to choosing the offered vertex in the first column. So this option is not shown.

Eventually, Alice forces one of five positions $(G, f)$ such that $G$ is not $f$-choosable, and Bob, being a gentleman, resigns.

## References

1. Noga Alon, Choice numbers of graphs: a probabilistic approach., Combinatorics, Probability and Computing 1 (1992), 107-114.
2. Paul Erdős, Arthur L. Rubin, and Herbert Taylor, Choosability in graphs, Congress. Numer., XXVI (1980), 125-157. MR 593902 (82f:05038)
3. Fred Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995), no. 1, 153-158. MR 1309363 (95m:05101)
4. Nurit Gazit and Michael Krivelevich, On the asymptotic value of the choice number of complete multi-partite graphs, J. Graph Theory 52 (2006), no. 2, 123-134. MR 2218737 (2007a:05047)
5. H.A. Kierstead, On the choosability of complete multipartite graphs with part size three, Discrete Mathematics 211 (2000), 255-259.
6. Jakub Kozik, Piotr Micek, and Xuding Zhu, Towards an on-line version of Ohba's conjecture, European Journal of Combinatorics 36 (2014), no. 0, 110-121.
7. J. A. Noel, D. B. West, H. Wu, and X. Zhu, Beyond Ohba's Conjecture: A bound on the choice number of $k$-chromatic graphs with $n$ vertices, ArXiv e-prints (2013).
8. Kyoji Ohba, Choice number of complete multipartite graphs with part size at most three., Ars Comb. 72 (2004).
9. Bruce A. Reed and Benny Sudakov, List colouring of graphs with at most $(2-o(1)) \chi$ vertices, Proceedings of the International Congress of Mathematicians Vol. III (2002), 587-603.
10._, List colouring when the chromatic number is close to the order of the graph., Combinatorica 25 (2004), no. 1, 117-123.
10. Uwe Schauz, Mr. Paint and Mrs. Correct, Electron. J. Combin. 16 (2009), no. 1, Research Paper 77, 18. MR 2515754 (2010i:91064)
11. Carsten Thomassen, Every planar graph is 5 -choosable, J. Combin. Theory Ser. B 62 (1994), no. 1, 180-181. MR 1290638 (95f:05045)
12. V. G. Vizing, Coloring the vertices of a graph in prescribed colors, Diskret. Analiz (1976), no. 29 Metody Diskret. Anal. v Teorii Kodov i Shem, 3-10, 101. MR 0498216 (58 \#16371)
13. Margit Voigt, List colourings of planar graphs, Discrete Math. 120 (1993), no. 1-3, 215-219. MR 1235909 (94d:05061)
14. D. Yang, Extension of the game coloring number and some results on the choosability of complete multipartite graphs, Arizona State University, 2003.

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