

GL_n(F_q)-ANALOGUES OF FACTORIZATION PROBLEMS IN THE SYMMETRIC GROUP

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ABSTRACT. We consider GL_n(F_q)-analogues of certain factorization problems in the symmetric group \mathfrak{S}_n : rather than counting factorizations of the long cycle $(1, 2, \dots, n)$ given the number of cycles of each factor, we count factorizations of a regular elliptic element given the fixed space dimension of each factor. We show that, as in \mathfrak{S}_n , the generating function counting these factorizations has attractive coefficients after an appropriate change of basis. Our work generalizes several recent results on factorizations in GL_n(F_q) and also uses a character-based approach.

As an application of our results, we compute the asymptotic growth rate of the number of factorizations of fixed genus of a regular elliptic element in GL_n(F_q) into two factors as $n \rightarrow \infty$. We end with a number of open questions.

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1. INTRODUCTION

There is a rich vein in combinatorics of problems related to factorizations in the symmetric group \mathfrak{S}_n . Frequently, the size of a certain family of factorizations is unwieldy but has an attractive generating function, possibly after an appropriate change of basis. As a prototypical example, one might seek to count factorizations $c = u \cdot v$ of the **long cycle** $c = (1, 2, \dots, n)$ in \mathfrak{S}_n as a product of two permutations, keeping track of the *number of cycles* or even the *cycle types* of the two factors. Such results have been given by Harer–Zagier [HZ86, §5] when one of the factors is a fixed point-free involution, and in the general setting by Jackson [Jac87, §4], [Jac88].

Theorem 1.1 (Jackson [Jac88]; Morales–Vassilieva [MV13]). *Let $a_{r,s}$ be the number of pairs (u, v) of elements of \mathfrak{S}_n such that u has r cycles, v has s cycles, and $c = u \cdot v$. Then*

$$(1.1) \quad \frac{1}{n!} \sum_{r,s \geq 0} a_{r,s} \cdot x^r y^s = \sum_{t,u \geq 1} \binom{n-1}{t-1; u-1; n-t-u+1} \binom{x}{t} \binom{y}{u}.$$

Moreover, for λ, μ partitions of n , let $a_{\lambda,\mu}$ be the number of pairs (u, v) of elements of \mathfrak{S}_n such that u has cycle type λ , v has cycle type μ , and $c = u \cdot v$. Then

$$(1.2) \quad \frac{1}{n!} \sum_{\lambda, \mu \vdash n} a_{\lambda,\mu} \cdot p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) = \sum_{\alpha, \beta} \frac{(n - \ell(\alpha))! (n - \ell(\beta))!}{(n-1)! (n+1 - \ell(\alpha) - \ell(\beta))!} \mathbf{x}^\alpha \mathbf{y}^\beta,$$

where p_λ denotes the usual power-sum symmetric function and the sum on the right is over all weak compositions α, β of n .

Recently, there has been interest in q -analogues of such problems, replacing \mathfrak{S}_n with the finite general linear group $\mathrm{GL}_n(\mathbf{F}_q)$, the long cycle with a *Singer cycle* (or, more generally, *regular elliptic element*) c , and the number of cycles with the *fixed space dimension* [LRS14, HLR15]; or in more general geometric settings [HLRV11]. In the present paper, we extend this approach to give the following q -analogue of Theorem 1.1. Our theorem statement uses the standard notations

$$(a; q)_m = (1-a)(1-aq) \cdots (1-aq^{m-1})$$

and

$$[m]!_q = \frac{(q; q)_m}{(1-q)^m} = 1 \cdot (1+q) \cdots (1+q+\dots+q^{m-1}).$$

Theorem 1.2. *Fix a regular elliptic element c in $G = \mathrm{GL}_n(\mathbf{F}_q)$. Let $a_{r,s}(q)$ be the number of pairs (u, v) of elements of G such that u has fixed space dimension r , v has fixed space dimension s , and $c = u \cdot v$. Then*

$$(1.3) \quad \frac{1}{|G|} \sum_{r,s \geq 0} a_{r,s}(q) \cdot x^r y^s = \frac{(x; q^{-1})_n}{(q; q)_n} + \frac{(y; q^{-1})_n}{(q; q)_n} + \sum_{\substack{0 \leq t, u \leq n-1 \\ t+u \leq n}} q^{tu-t-u} \frac{[n-t-1]!_q \cdot [n-u-1]!_q (q^n - q^t - q^u + 1)}{[n-1]!_q \cdot [n-t-u]!_q (q-1)} \cdot \frac{(x; q^{-1})_t (y; q^{-1})_u}{(q; q)_t (q; q)_u}.$$

More generally, in either \mathfrak{S}_n or $\mathrm{GL}_n(\mathbf{F}_q)$ one may consider factorizations into more than two factors. In \mathfrak{S}_n , this gives the following result.

Theorem 1.3 (Jackson [Jac88]; Bernardi–Morales [BM13]). *Let a_{r_1, r_2, \dots, r_k} be the number of k -tuples (u_1, u_2, \dots, u_k) of permutations in \mathfrak{S}_n such that u_i has r_i cycles and $u_1 u_2 \cdots u_k = c$. Then*

$$(1.4) \quad \frac{1}{(n!)^{k-1}} \sum_{1 \leq r_1, r_2, \dots, r_k \leq n} a_{r_1, \dots, r_k} \cdot x_1^{r_1} \cdots x_k^{r_k} = \sum_{1 \leq p_1, \dots, p_k \leq n} M_{p_1-1, \dots, p_k-1}^{n-1} \binom{x_1}{p_1} \cdots \binom{x_k}{p_k},$$

where

$$(1.5) \quad M_{r_1, \dots, r_k}^m := \sum_{d=0}^{\min(r_i)} (-1)^d \binom{m}{d} \prod_{i=1}^k \binom{m-d}{r_i-d}.$$

Moreover, let $a_{\lambda^{(1)}, \dots, \lambda^{(k)}}$ be the number of k -tuples (u_1, u_2, \dots, u_k) of permutations in \mathfrak{S}_n such that u_i has cycle type $\lambda^{(i)}$ and $u_1 u_2 \cdots u_k = c$. Then

$$\frac{1}{(n!)^{k-1}} \sum_{\lambda^{(1)}, \dots, \lambda^{(k)} \vdash n} a_{\lambda^{(1)}, \dots, \lambda^{(k)}} \cdot p_{\lambda^{(1)}}(\mathbf{x}_1) \cdots p_{\lambda^{(k)}}(\mathbf{x}_k) = \sum_{\alpha^{(1)}, \dots, \alpha^{(k)}} \frac{M_{\ell(\alpha^{(1)})-1, \dots, \ell(\alpha^{(k)})-1}^{n-1}}{\prod_{i=1}^k \binom{n-1}{\ell(\alpha^{(i)})-1}} (\mathbf{x}_1)^{\alpha^{(1)}} \cdots (\mathbf{x}_k)^{\alpha^{(k)}},$$

where the sum on the right is over all weak compositions $\alpha^{(1)}, \dots, \alpha^{(k)}$ of n .

In the present paper, we prove the following q -analogue of this result. The statement uses the standard q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]!_q / ([k]!_q \cdot [n-k]!_q)$.

Theorem 1.4. *Fix a regular elliptic element c in $G = \text{GL}_n(\mathbf{F}_q)$. Let $a_{r_1, \dots, r_k}(q)$ be the number of tuples (u_1, \dots, u_k) of elements of G such that u_i has fixed space dimension r_i and $u_1 \cdots u_k = c$. Then*

$$(1.6) \quad \frac{1}{|G|^{k-1}} \sum_{r_1, \dots, r_k} a_{r_1, \dots, r_k}(q) \cdot x_1^{r_1} \cdots x_k^{r_k} = \sum_{\substack{\mathbf{p}=(p_1, \dots, p_k): \\ 0 \leq p_i \leq n}} \frac{M_{\tilde{\mathbf{p}}}^{n-1}(q)}{\prod_{p \in \tilde{\mathbf{p}}} \begin{bmatrix} n-1 \\ p \end{bmatrix}_q} \cdot \frac{(x_1; q^{-1})_{p_1}}{(q; q)_{p_1}} \cdots \frac{(x_k; q^{-1})_{p_k}}{(q; q)_{p_k}},$$

where $\tilde{\mathbf{p}}$ is the result of deleting all copies of n from \mathbf{p} ,

$$(1.7) \quad M_{r_1, \dots, r_k}^m(q) := \sum_{d=0}^{\min(r_i)} (-1)^d q^{\binom{d+1}{2} - kd} \begin{bmatrix} m \\ d \end{bmatrix}_q \prod_{i=1}^k \begin{bmatrix} m-d \\ r_i-d \end{bmatrix}_q$$

for $k > 0$, and $M_{\emptyset}^m(q) := 0$.

Remark 1.5. In viewing Theorems 1.2 and 1.4 as q -analogues of Theorems 1.1 and 1.3, it is helpful to first observe that if $x = q^N$ is a positive integer power of q then $\frac{(x; q^{-1})_m}{(q; q)_m} = \begin{bmatrix} N \\ m \end{bmatrix}_q$.

Further, we have the equality

$$\lim_{q \rightarrow 1} \frac{[n-t-1]!_q \cdot [n-u-1]!_q (q^n - q^t - q^u + 1)}{[n-1]!_q \cdot [n-t-u]!_q (q-1)} = \frac{(n-(t+1))!(n-(u+1))!}{(n-1)!(n+1-(t+1)-(u+1))!}$$

between the limit of a coefficient in (1.3) and a coefficient on the right side of (1.2), and more generally the equality $\lim_{q \rightarrow 1} M_{r_1, \dots, r_k}^m(q) = M_{r_1, \dots, r_k}^m$.

Note that the generating function (1.6) is (in its definition) analogous to the less-refined generating function (1.4), while the coefficient

$$M_{\tilde{\mathbf{P}}}^{n-1}(q) / \prod_{p \in \tilde{\mathbf{P}}} \left[\begin{matrix} n-1 \\ p \end{matrix} \right]_q$$

is analogous (in the $q \rightarrow 1$ sense) to a coefficient in the more refined half of Theorem 1.3. This phenomenon is mysterious. A similar phenomenon was observed in the discussion following Theorem 4.2 in [HLR15], namely, that the counting formula $q^{e(\alpha)}(q^n - 1)^{k-1}$ for factorizations of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ into k factors with fixed space codimensions given by the composition α of n is a q -analogue of the counting formula n^{k-1} for factorizations of an n -cycle as a genus-0 product of k *cycles* of specified lengths.

On the other hand, we *can* give a heuristic explanation for the fact that the lower indices in the M -coefficients in Theorem 1.4 are shifted by 1 compared with those in Theorem 1.3: the matrix group \mathfrak{S}_n does not act irreducibly in its standard representation, as every permutation fixes the all-ones vector. Thus, morally, the subtraction of 1 should correct for the irrelevant dimension of fixed space.

Our approach is to follow a well-worn path, based on character-theoretic techniques that go back to Frobenius. In the case of the symmetric group, this approach has been extensively developed in the '80s and '90s, notably in work of Stanley [Sta81], Jackson [Jac87, Jac88], Hanlon–Stanley–Stembridge [HSS92], and Goupil–Schaeffer [GS98]; see also the survey [GJ14]. In $\mathrm{GL}_n(\mathbf{F}_q)$, the necessary character theory was worked out by Green [Gre55], building on work of Steinberg [Ste51]. This approach has been used recently by the first-named author and coauthors to count factorizations of Singer cycles into reflections [LRS14] and to count *genus-0* factorizations (that is, those in which the codimensions of the fixed spaces of the factors sum to the codimension of the fixed space of the product) of regular elliptic elements [HLR15]. The current work subsumes these previous results while requiring no new character values. (See Remarks 4.2 and 4.3 for derivations of these earlier results from Theorem 1.4.)

The plan of the paper is as follows. In Section 2, we provide background, including an overview of the character-theoretic approach to problems of this sort and a quick introduction to the character theory of $\mathrm{GL}_n(\mathbf{F}_q)$ necessary for our purposes. Theorems 1.2 and 1.4 are proved in Sections 3 and 4, respectively. The **genus** of a factorization counted in $a_{r,s}(q)$ is $n - r - s$. In Section 5, we give an application of Theorem 1.2 to asymptotic enumeration, giving the precise growth rate $\Theta(q^{(n+g)^2/2}/|\mathrm{GL}_g(\mathbf{F}_q)|)$ of the number of factorizations of fixed genus g of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ as a product of two factors, as $n \rightarrow \infty$. Finally, in Section 6 we give a few additional remarks and open questions. In particular, in Section 6.1 we briefly discuss the history of *combinatorial* approaches to Theorems 1.1 and 1.3, and discuss whether this story can be given a q -analogy.

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2. REGULAR ELLIPTICS, CHARACTER THEORY, AND THE SYMMETRIC GROUP APPROACH

2.1. Singer cycles and regular elliptic elements. The field \mathbf{F}_{q^n} is an n -dimensional vector space over \mathbf{F}_q , and multiplication by a fixed element in the larger field is a linear transformation.

Thus, any choice of basis for \mathbf{F}_{q^n} over \mathbf{F}_q gives a natural inclusion $\mathbf{F}_{q^n}^\times \hookrightarrow G_n := \mathrm{GL}_n(\mathbf{F}_q)$. The image of any cyclic generator c for $\mathbf{F}_{q^n}^\times$ under this inclusion is called a **Singer cycle**. A strong analogy between Singer cycles in G_n and n -cycles in \mathfrak{S}_n has been established over the past decade or so, notably in work of Reiner, Stanton, and collaborators [RSW04, RSW06, LRS14, HLR15]. As one elementary example of this analogy, the Singer cycles act transitively on the lines of \mathbf{F}_q^n , just as the n -cycles act transitively on the points $\{1, \dots, n\}$.

A more general class of elements of G_n , containing the Singer cycles, is the set of images (under the same inclusion) of field generators σ for \mathbf{F}_{q^n} over \mathbf{F}_q . (That is, one should have $\mathbf{F}_q[\sigma] = \mathbf{F}_{q^n}$ but not necessarily $\{\sigma^m \mid m \in \mathbb{Z}\} = \mathbf{F}_{q^n}^\times$.) Such elements are called **regular elliptic elements**. They may be characterized in several other ways; see [LRS14, Prop. 4.4].

2.2. The character-theoretic approach to factorization problems. Given a finite group G , let $\mathrm{Irr}(G)$ be the collection of its irreducible (finite-dimensional, complex) representations V . For each V in $\mathrm{Irr}(G)$, denote by $\deg(V) := \dim_{\mathbb{C}} V$ its **degree**, by $\chi^V(g) := \mathrm{Tr}(g : V \rightarrow V)$ its **character value** at g , and by $\tilde{\chi}^V(g) := \chi^V(g) / \deg(V)$ its **normalized character value**. The functions $\chi^V(-)$ and $\tilde{\chi}^V(-)$ on G extend by \mathbb{C} -linearity to functionals on the group algebra $\mathbb{C}[G]$.

The following result allows one to express every factorization problem of the form we consider as a computation in terms of group characters.

Proposition 2.1 (Frobenius [Fro68]). *Let G be a finite group, and $A_1, \dots, A_\ell \subseteq G$ unions of conjugacy classes in G . Then for g in G , the number of ordered factorizations (t_1, \dots, t_ℓ) with $g = t_1 \cdots t_\ell$ and t_i in A_i for $i = 1, 2, \dots, \ell$ is*

$$(2.1) \quad \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(g^{-1}) \cdot \tilde{\chi}^V(z_1) \cdots \tilde{\chi}^V(z_\ell),$$

where $z_i := \sum_{t \in A_i} t$ in $\mathbb{C}[G]$.

In practice, it is often the case that one does not need the full set of character values that appear in (2.1) in order to evaluate the sum. As an example of this phenomenon, we show how to derive Theorem 1.1 without needing access to the full character table for the symmetric group \mathfrak{S}_n . This argument also provides a template for our work in $\mathrm{GL}_n(\mathbf{F}_q)$.

Let c be the long cycle $c = (1, 2, \dots, n)$ in \mathfrak{S}_n . Consider the generating function

$$(2.2) \quad F(x, y) = \sum_{1 \leq r, s \leq n} a_{r,s} \cdot x^r y^s$$

for the number $a_{r,s}$ of factorizations $c = u \cdot v$ in which u, v have r, s cycles, respectively. By Proposition 2.1, we have that

$$a_{r,s} = \frac{1}{n!} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot \tilde{\chi}^V(z_r) \tilde{\chi}^V(z_s),$$

where z_i is the formal sum in $\mathbb{C}[\mathfrak{S}_n]$ of all elements with i cycles. Substituting this in (2.2) gives

$$\begin{aligned} F(x, y) &= \frac{1}{n!} \sum_{1 \leq r, s \leq n} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot \tilde{\chi}^V(z_r) \tilde{\chi}^V(z_s) \cdot x^r y^s \\ &= \frac{1}{n!} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y), \end{aligned}$$

where $f_V(x) := \sum_{r=1}^n \tilde{\chi}^V(z_r)x^r$. The irreducible representations of \mathfrak{S}_n are indexed by partitions λ of n , and we write $\chi^V = \chi^\lambda$ if V is indexed by λ . The degree of a character is given by the *hook-length formula*. It follows from the *Murnaghan–Nakayama rule* that the character value $\chi^\lambda(c^{-1})$ on the n -cycle c^{-1} is equal to 0 unless $\lambda = \langle n-d, 1^d \rangle$ is a **hook shape**, in which case $\chi^{\langle n-d, 1^d \rangle}(c^{-1}) = (-1)^d$. Thus it suffices to understand $f_\lambda(x)$ for hooks λ . One can show that

$$f_{\langle n-d, 1^d \rangle}(x) = (x-d)_n := (x-d) \cdot (x-d+1) \cdot (x-d+2) \cdots (x-d+n-1) = n! \cdot \sum_{k=d+1}^n \binom{n-1-d}{k-1-d} \cdot \binom{x}{k},$$

and the result follows by identities for binomial coefficients after extracting the coefficient of $\binom{x}{t} \binom{y}{u}$.

2.3. Character theory of the finite general linear group. In this section, we give a (very) brief overview of the character theory of $G_n = \mathrm{GL}_n(\mathbf{F}_q)$, including the specific character values necessary to prove the main results in this paper. For a proper treatment, see [Zel81, Ch. 3] or [GR15, §4]. Throughout this section, we freely conflate the (complex, finite-dimensional) representation V for $\mathrm{GL}_n(\mathbf{F}_q)$ with its character χ^V .

The basic building-block of the character theory of G_n is **parabolic** (or **Harish-Chandra induction**), defined as follows. For nonnegative integers a, b , let $P_{a,b}$ be the parabolic subgroup

$$P_{a,b} = \left\{ \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : A \in G_a, B \in G_b, \text{ and } C \in \mathbf{F}_q^{a \times b} \right\}$$

of G_{a+b} . Given characters χ_1 and χ_2 for G_a and G_b , respectively, one obtains a character $\chi_1 * \chi_2$ for G_{a+b} by the formula

$$(\chi_1 * \chi_2)(g) = \frac{1}{|P_{a,b}|} \sum_{\substack{h \in G_{a+b}: \\ hgh^{-1} \in P_{a,b}}} \chi_1(A) \chi_2(B),$$

where A and B are the diagonal blocks of hgh^{-1} as above.

Many irreducible characters for G_n may be obtained as irreducible components of induction products of characters on smaller general linear groups. A character \mathcal{C} for G_n that cannot be so-obtained is called **cuspidal**, of **weight** $\mathrm{wt}(\mathcal{C}) = n$. The set of cuspidals for G_n is denoted Cusp_n , and the set of all cuspidals for all general linear groups is denoted $\mathrm{Cusp} = \sqcup_{n \geq 1} \mathrm{Cusp}_n$. (Though we will not need this, we note that cuspidals may be indexed by irreducible polynomials over \mathbf{F}_q , or equivalently by primitive q -colored necklaces.)

Let Par denote the set of all integer partitions. The set of *all* irreducible characters for G_n is indexed by functions $\underline{\lambda}: \mathrm{Cusp} \rightarrow \mathrm{Par}$ such that

$$\sum_{\mathcal{C} \in \mathrm{Cusp}} \mathrm{wt}(\mathcal{C}) \cdot |\underline{\lambda}(\mathcal{C})| = n.$$

(In particular, $\underline{\lambda}(\mathcal{C})$ must be equal to the empty partition for all but finitely many choices of \mathcal{C} .) A particular representation of interest is the trivial representation $\mathbf{1}$ for $\mathrm{GL}_1(\mathbf{F}_q)$. (The trivial representation for $\mathrm{GL}_n(\mathbf{F}_q)$ is indexed by the function associating to $\mathbf{1}$ the partition $\langle n \rangle$.) If V is indexed by $\underline{\lambda}$ having support on a single cuspidal representation \mathcal{C} , we call V **primary** and denote it by the pair (\mathcal{C}, λ) where $\lambda = \underline{\lambda}(\mathcal{C})$.

A priori, in order to use Proposition 2.1 in our setting, we require the degrees and certain other values of all irreducible characters for G_n . In fact, however, we will only need a very

small selection of them. The character degrees were worked out by Steinberg [Ste51] and Green [Gre55], and the special case relevant to our work is

$$(2.3) \quad \deg \left(\chi^{\mathbf{1}, \langle n-d, 1^d \rangle} \right) = q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q.$$

The relevant character values on regular elliptic elements were computed by Lewis–Reiner–Stanton.

Proposition 2.2 ([LRS14, Prop. 4.7]). *Suppose c is a regular elliptic element and χ^λ an irreducible character of G_n .*

- (i) *One has $\chi^\lambda(c) = 0$ unless χ^λ is a primary irreducible character $\chi^{U, \lambda}$ for some s dividing n and some cuspidal character U in Cusp_s , and $\lambda = \langle \frac{n}{s} - d, 1^d \rangle$ is a hook-shaped partition of n/s .*
- (ii) *If $U = \mathbf{1}$ is the trivial character then*

$$\chi^{\mathbf{1}, \langle n-d, 1^d \rangle}(c) = (-1)^d.$$

Finally, denote by z_k the formal sum (in $\mathbb{C}[G_n]$) of all elements of G_n having fixed space dimension¹ equal to k . Huang–Lewis–Reiner computed the relevant character values on the z_k .

Proposition 2.3 ([HLR15, Prop. 4.10]). (i) *For any s dividing n , any cuspidal representation U in Cusp_s other than $\mathbf{1}$, and any partition λ of $\frac{n}{s}$, we have*

$$\tilde{\chi}^{U, \lambda}(z_r) = (-1)^{n-r} q^{\binom{n-r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

- (ii) *For $U = \mathbf{1}$ and $\lambda = \langle n-d, 1^d \rangle$ a hook, we have*

$$\begin{aligned} \tilde{\chi}^{\mathbf{1}, \langle n-d, 1^d \rangle}(z_r) = & (-1)^{n-r} q^{\binom{n-r}{2}} \left(\begin{bmatrix} n \\ r \end{bmatrix}_q + \right. \\ & \left. \frac{(1-q)[n]_q}{[r]!_q} \cdot \sum_{j=1}^{n-\max(r,d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \right). \end{aligned}$$

3. FACTORING REGULAR ELLIPTIC ELEMENTS INTO TWO FACTORS

3.1. Proof of Theorem 1.2. In this section, we prove Theorem 1.2 by following the approach for \mathfrak{S}_n sketched in Section 2.2. Let c be a regular elliptic element in $G = \mathrm{GL}_n(\mathbf{F}_q)$, and let $a_{r,s}(q)$ be the number of pairs (u, v) of elements of G such that $u \cdot v = c$ and u, v have fixed space dimensions r, s , respectively. Define the generating function

$$F(x, y) := \sum_{r,s \geq 0} a_{r,s}(q) x^r y^s.$$

Our goal is to rewrite this generating function in the basis $\frac{(x;q^{-1})_t}{(q;q)_t} \frac{(y;q^{-1})_u}{(q;q)_u}$ of polynomials q -analogous to the binomial coefficients.

By Proposition 2.1, we may write

$$(3.1) \quad a_{r,s}(q) = \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot \tilde{\chi}^V(z_r) \cdot \tilde{\chi}^V(z_s)$$

¹Caution: our indexing here differs from that of [HLR15], where the symbol z_k is used to represent the sum of elements with fixed space codimension k .

where z_k is defined (as above) to be the element of the group algebra $\mathbb{C}[G]$ equal to the sum of all elements of fixed space dimension k . Thus, our generating function is given by

$$\begin{aligned}
 F(x, y) &= \sum_{r, s \geq 0} x^r y^s \frac{1}{|G|} \sum_{V \in \text{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot \tilde{\chi}^V(z_r) \cdot \tilde{\chi}^V(z_s) \\
 &= \frac{1}{|G|} \sum_{V \in \text{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot \left(\sum_r \tilde{\chi}^V(z_r) x^r \right) \cdot \left(\sum_s \tilde{\chi}^V(z_s) y^s \right) \\
 (3.2) \quad &= \frac{1}{|G|} \sum_{V \in \text{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) \cdot f_V(y),
 \end{aligned}$$

where

$$f_V(x) := \sum_{r=0}^n \tilde{\chi}^V(z_r) \cdot x^r.$$

By Proposition 2.2, the character value $\chi^V(c^{-1})$ is typically 0, and so in order to prove Theorem 1.2 it suffices to compute f_V for only a few select choices of V . We do this now.

Proposition 3.1. *If $V = (U, \lambda)$ for $U \neq \mathbf{1}$ we have*

$$(3.3) \quad f_{U, \lambda}(x) = |G| \cdot \frac{(x; q^{-1})_n}{(q; q)_n},$$

while if $V = (\mathbf{1}, \langle n-d, 1^d \rangle)$ we have

$$(3.4) \quad f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x) = |G| \cdot \left(\frac{(x; q^{-1})_n}{(q; q)_n} + q^{-d} \cdot \sum_{m=d}^{n-1} \frac{[m]!_q \cdot [n-d-1]!_q}{[m-d]!_q \cdot [n-1]!_q} \cdot \frac{(x; q^{-1})_m}{(q; q)_m} \right).$$

The proof is a reasonably straightforward computation using Proposition 2.3, the q -binomial theorem (essentially [Sta12, (1.87)])

$$(3.5) \quad \frac{(x; q^{-1})_m}{(q; q)_m} = \frac{1}{(q; q)_m q^{\binom{m}{2}}} \sum_{k=0}^m (-1)^k q^{\binom{m-k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q \cdot x^k,$$

and its inverse

$$(3.6) \quad x^k = \sum_{m=0}^k (-1)^m q^{\binom{m}{2}} (q^k; q^{-1})_m \cdot \frac{(x; q^{-1})_m}{(q; q)_m}.$$

It is hidden away in Appendix A.

We continue studying the factorization generating function $F(x, y)$. We split the sum (3.2) according to whether V is a primary irreducible over the cuspidal $\mathbf{1}$ of hook shape:

$$\begin{aligned}
 (3.7) \quad |G| \cdot F(x, y) &= \sum_{V \neq (\mathbf{1}, \langle n-d, 1^d \rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) \cdot f_V(y) + \\
 &\quad \sum_{d=0}^{n-1} \deg(\mathbf{1}, \langle n-d, 1^d \rangle) \chi^{\mathbf{1}, \langle n-d, 1^d \rangle}(c^{-1}) \cdot f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x) \cdot f_{\mathbf{1}, \langle n-d, 1^d \rangle}(y).
 \end{aligned}$$

We use Proposition 2.2(i) and (3.3) to rewrite the first sum on the right side of (3.7) as

$$\begin{aligned}
\sum_{V \neq (1, \langle n-d, 1^d \rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y) &= \sum_{V=(U, \lambda), U \neq 1} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y) \\
&= \sum_{V=(U, \lambda), U \neq 1} \deg(V) \chi^V(c^{-1}) \cdot |G|^2 \cdot \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n} \\
&= |G|^2 \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n} \sum_{V \neq (1, \langle n-d, 1^d \rangle)} \deg(V) \chi^V(c^{-1}).
\end{aligned}$$

Observe (following the same idea as in [HLR15, §4.3]) that $\sum_{V \in \text{Irr}(G)} \deg(V) \chi^V$ is the character of the regular representation for G . It follows that $\sum_{V \in \text{Irr}(G)} \deg(V) \chi^V(c^{-1}) = 0$ and so that

$$\begin{aligned}
\sum_{V \neq (1, \langle n-d, 1^d \rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y) &= -|G|^2 \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n} \sum_{V=(1, \langle n-d, 1^d \rangle)} \deg(V) \chi^V(c^{-1}) \\
&= -|G|^2 \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n} \cdot \sum_{d=0}^{n-1} q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \cdot (-1)^d \\
&= -(q; q)_{n-1} \cdot |G|^2 \cdot \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n}.
\end{aligned}$$

Substituting from (2.3), (3.8) and Proposition 2.2 into (3.7) yields

$$\begin{aligned}
(3.9) \quad \frac{F(x, y)}{|G|} &= -(q; q)_{n-1} \frac{(x; q^{-1})_n}{(q; q)_n} \frac{(y; q^{-1})_n}{(q; q)_n} + \\
&\quad \frac{1}{|G|^2} \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \cdot f_{1, \langle n-d, 1^d \rangle}(x) \cdot f_{1, \langle n-d, 1^d \rangle}(y).
\end{aligned}$$

In order to finish the proof of Theorem 1.2, we must extract the coefficient of $\frac{(x; q^{-1})_t}{(q; q)_t} \frac{(y; q^{-1})_u}{(q; q)_u}$ from the right side of this equation. Call this coefficient $b_{t,u}(q)$, so that

$$F(x, y) = |G| \cdot \sum_{t,u} b_{t,u}(q) \cdot \frac{(x; q^{-1})_t}{(q; q)_t} \frac{(y; q^{-1})_u}{(q; q)_u}.$$

By (3.4), this extraction reduces to a computation involving q -series. We begin with a few small simplifications in order to make the computation more reasonable.

First, observe that the subadditivity of fixed space codimensions [HLR15, Prop. 2.9] implies that $a_{r,s}(q) = 0$ if $r + s > n$. By the triangularity of the change-of-basis formulas (3.5), (3.6), it follows immediately that $b_{t,u}(q) = 0$ if $t + u > n$. When $r + s = n$, the value of $a_{r,s}(q)$ was computed (by the same methods) in [HLR15], yielding $a_{n,0}(q) = a_{0,n}(q) = 1$ and $a_{r,s}(q) = q^{2rs-n}(q^n - 1)$ if r, s are both positive. Again by the triangularity of (3.6) we have

$$b_{r,s}(q) = a_{r,s}(q) \cdot (-1)^r q^{\binom{r}{2}} (q^r; q^{-1})_r \cdot (-1)^s q^{\binom{s}{2}} (q^s; q^{-1})_s / |G|$$

when $r + s = n$. This yields $b_{n,0}(q) = b_{0,n}(q) = 1$ and $b_{r,s}(q) = q^{rs-r-s}(q^n - 1) \frac{[r]!_q [s]!_q}{[n]!_q}$ when $r + s = n$, and this latter formula may be seen by a short computation to be equal to the desired value. Thus, it remains to compute $b_{t,u}(q)$ when $0 \leq t, u$ and $t + u < n$.

In this case, it follows from (3.9) and (3.4) that

$$(3.10) \quad b_{t,u}(q) = \sum_{d=0}^{\min(t,u)} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \cdot q^{-d} \frac{[t]!_q \cdot [n-d-1]!_q}{[t-d]!_q \cdot [n-1]!_q} \cdot q^{-d} \frac{[u]!_q \cdot [n-d-1]!_q}{[u-d]!_q \cdot [n-1]!_q}.$$

Collecting powers of q , cancelling common factors and rewriting with the easy identity

$$(3.11) \quad \frac{[m]!_q}{[m-d]!_q} = \frac{q^{md - \binom{d}{2}}}{(q-1)^d} \cdot (q^{-m}; q)_d$$

shows that $b_{t,u}(q)$ is equal to the q -hypergeometric function²

$$b_{t,u}(q) = {}_2\phi_1(q^{-t}, q^{-u}; q^{1-n}; q^{t+u-n}).$$

This function is almost summable by the q -Chu–Vandermonde identity (see [GR04, (II.7)]), but the power of q in the final parameter is off by one; instead, we use the ${}_2\phi_1$ -to- ${}_3\phi_2$ identity [GR04, (III.7)]

$${}_2\phi_1(q^{-k}, B; C; z) = \frac{(C/B; q)_k}{(C; q)_k} \cdot {}_3\phi_2(q^{-k}, B, Bzq^{-k}/C; Bq^{1-k}/C, 0; q)$$

to rewrite

$$b_{t,u}(q) = \frac{(q^{1+u-n}; q)_t}{(q^{1-n}; q)_t} {}_3\phi_2(q^{-t}, q^{-u}, q^{-1}; q^{n-t-u}, 0; q).$$

Since one of the numerator parameters is q^{-1} , this ${}_3\phi_2$ summation truncates after two terms:

$$\begin{aligned} b_{t,u}(q) &= \frac{(q^{1+u-n}; q)_t}{(q^{1-n}; q)_t} \sum_{d=0}^{\infty} \frac{(q^{-t}; q)_d (q^{-u}; q)_d (q^{-1}; q)_d}{(q; q)_d (q^{n-t-u}; q)_d} q^d \\ &= \frac{(q^{1-n+u}; q)_t}{(q^{1-n}; q)_t} \cdot \left(1 + \frac{(1-q^{-t}) \cdot (1-q^{-u}) \cdot (1-q^{-1})}{(1-q) \cdot (1-q^{n-t-u})} \cdot q \right) \\ &= q^{tu} \frac{[n-t-1]!_q \cdot [n-u-1]!_q}{[n-t-u-1]!_q \cdot [n-1]!_q} \cdot \left(1 - \frac{(1-q^{-t}) \cdot (1-q^{-u})}{1-q^{n-t-u}} \right) \\ &= q^{tu-t-u} \frac{[n-t-1]!_q \cdot [n-u-1]!_q}{[n-t-u]!_q \cdot [n-1]!_q} \cdot \frac{q^n - q^t - q^u + 1}{q-1}. \end{aligned}$$

This concludes the proof of Theorem 1.2.

3.2. Additional remarks on Theorem 1.2. We note here two special cases of Theorem 1.2; compare with the remarks following [SV08, Thm. 1.1].

Remark 3.2. If we set $x = y = 1$ in (1.3), all terms on the right vanish except $t = u = 0$, leaving

$$F(1, 1) = |\mathrm{GL}_n(\mathbf{F}_q)| \cdot \frac{[n-1]!_q \cdot [n-1]!_q}{[n-1]!_q \cdot [n]!_q} \cdot \frac{q^n - 1}{q - 1} = |\mathrm{GL}_n(\mathbf{F}_q)|.$$

Indeed, for each element u of $\mathrm{GL}_n(\mathbf{F}_q)$ there is exactly one element v such that $u \cdot v = c$.

² Recall the definition

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; z) := \sum_{d=0}^{\infty} \frac{(a_1; q)_d \cdot (a_2; q)_d \cdots (a_r; q)_d}{(q; q)_d \cdot (b_1; q)_d \cdots (b_s; q)_d} \left((-1)^d q^{\binom{d}{2}} \right)^{1+s-r} z^d.$$

Remark 3.3. More generally, if we set $y = 1$ in (1.3), we get the generating function $F(x, 1)$ for GL_n(F_q) by dimension of fixed space. On the right side, only terms with $u = 0$ survive, and so

$$\begin{aligned} F(x, 1) &= |\mathrm{GL}_n(\mathbf{F}_q)| \cdot \left(\frac{(x; q^{-1})_n}{(q; q)_n} + \sum_{t=0}^{n-1} q^{-t} \frac{[n-t-1]!_q \cdot [n-1]!_q}{[n-1]!_q \cdot [n-t]!_q} \cdot \frac{q^n - q^t}{q-1} \frac{(x; q^{-1})_t}{(q; q)_t} \right) \\ &= |\mathrm{GL}_n(\mathbf{F}_q)| \cdot \sum_{t=0}^n \frac{(x; q^{-1})_t}{(q; q)_t}. \end{aligned}$$

This formula is analogous to the generating function for \mathfrak{S}_n by number of cycles; see also [HLR15, §3]. Formulas for the coefficients of this generating function were obtained by Fulman [Ful99] (attributed there to unpublished work of Rudvalis and Shinoda).

4. FACTORING REGULAR ELLIPTIC ELEMENTS INTO MORE THAN TWO FACTORS

4.1. Proof of Theorem 1.4. We follow the same framework as in Section 3. Let $a_{r_1, \dots, r_k}(q)$ be the number of tuples (g_1, \dots, g_k) of elements of $G = \mathrm{GL}_n(\mathbf{F}_q)$ such that g_i has fixed space dimension r_i for all i and $g_1 \cdots g_k = c$. Define the generating function

$$F(x_1, \dots, x_k) = \sum_{r_1, \dots, r_k} a_{r_1, \dots, r_k}(q) x_1^{r_1} \cdots x_k^{r_k}.$$

The statement we wish to prove asserts a formula for this generating function when expressed in another basis. Applying Proposition 2.1 and making calculations analogous to those in (3.2) gives

$$|G| \cdot F(\mathbf{x}) = \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) f_V(x_1) \cdots f_V(x_k).$$

The same regular representation trick that leads from (3.7) to (3.9) works with more variables; it yields

$$\begin{aligned} (4.1) \quad |G| \cdot F(\mathbf{x}) &= -(q; q)_{n-1} |G|^k \cdot \frac{(x_1; q^{-1})_n}{(q; q)_n} \cdots \frac{(x_k; q^{-1})_n}{(q; q)_n} + \\ &\quad \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x_1) \cdots f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x_k). \end{aligned}$$

To finish the proof of Theorem 1.4, we must extract from this expression the coefficient of $\prod_i \frac{(x_i; q^{-1})_{p_i}}{(q; q)_{p_i}}$. For convenience, we denote by $b_{p_1, \dots, p_k}(q)$ the coefficient of $\prod_i \frac{(x_i; q^{-1})_{p_i}}{(q; q)_{p_i}}$ in the normalized generating function $F(\mathbf{x})/|G|^{k-1}$.

Because of the form (3.4) of the polynomial $f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x)$, it is convenient to introduce a new parameter j , marking the number of indices p_i not equal to n . Up to permuting variables, we may assume without loss of generality that $p_i < n$ if $1 \leq i \leq j$ and $p_i = n$ if $j < i \leq k$. Then substituting from (3.4) into (4.1) and extracting coefficients yields

$$\begin{aligned} (4.2) \quad b_{p_1, \dots, p_k}(q) &= \sum_{d=0}^{\min(p_i)} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q q^{-jd} \left(\frac{[n-d-1]!_q}{[n-1]!_q} \right)^j \prod_{i=1}^j \frac{[p_i]!_q}{[p_i-d]!_q} \\ &= \sum_{d=0}^{\min(p_i)} (-1)^d q^{\binom{d+1}{2} - jd} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \prod_{i=1}^j \begin{bmatrix} n-1-d \\ p_i-d \end{bmatrix}_q / \prod_{i=1}^j \begin{bmatrix} n-1 \\ p_i \end{bmatrix}_q, \end{aligned}$$

as desired. This completes the proof of Theorem 1.4.

Remark 4.1. When $k = 2$ the expression in (4.2) equals (3.10). In the case of two factors we applied an additional q -identity to arrive at the expression in Theorem 1.2. In the case of k factors, using a trick of Stanton (personal communication), one can split (4.2) into several terms and apply similar identities (like [GR04, (3.2.2)] for the case $k = 3$). However, the resulting “less alternating” expressions have several terms. Because of this, and like in the \mathfrak{S}_n case (1.4), we opted to stop at (4.2).

4.2. Additional remarks on Theorem 1.4. The following remarks show how to recover the main theorems of [LRS14, HLR15] as special cases of Theorem 1.4. Incidentally, the first remark also settles a conjecture of Lewis–Reiner–Stanton.

Remark 4.2. In [LRS14, Thm. 1.2], Lewis–Reiner–Stanton gave the following formula for the number $t_q(n, \ell) := a_{n-1, \dots, n-1}(q)$ of factorizations of a Singer cycle c as a product of ℓ reflections (that is, elements with fixed space dimension $n - 1$):

$$t_q(n, \ell) = \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left((-1)^{n-1}(q; q)_{n-1} + \sum_{k=0}^{n-1} (-1)^{k+n} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (1 + q^{n-k-1} - q^{n-k})^\ell \right).$$

Here, we show how to derive this formula from Theorem 1.4. It was conjectured [LRS14, Conj. 6.3] that this formula should count factorizations of any regular elliptic element, not just a Singer cycle; since the derivation here is valid for all regular elliptic elements, it settles the conjecture.

Using square brackets to denote coefficient extraction, we have by definition that

$$(4.3) \quad t_q(n, \ell) = [x_1^{n-1} \cdots x_\ell^{n-1}] F(x_1, \dots, x_\ell),$$

where $F(x_1, \dots, x_\ell)$ is the generating function appearing in (1.6). By (3.5) we have

$$[x^{n-1}] \frac{(x; q^{-1})_p}{(q; q)_p} = \begin{cases} (-1)^{n-1} q^{-\binom{n-1}{2}} / (q; q)_{n-1} & \text{if } p = n-1, \\ (-1)^{n-1} q^{-\binom{n}{2}} [n]_q / (q; q)_n & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, carrying out the coefficient extraction (4.3) from $F(x_1, \dots, x_\ell)$ yields

$$\begin{aligned} t_q(n, \ell) &= |G|^{\ell-1} \sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^{(n-1)\ell} \left(\frac{1}{(q; q)_{n-1} q^{\binom{n-1}{2}}} \right)^i \left(\frac{[n]_q}{(q; q)_n q^{\binom{n}{2}}} \right)^{\ell-i} M_{(n-1)^i}^{n-1}(q) \\ &= (-1)^n \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left(\sum_{i=0}^{\ell} \binom{\ell}{i} (1-q)^i q^{(n-1)i} M_{(n-1)^i}^{n-1}(q) \right). \end{aligned}$$

By the definition of $M_{\mathbf{p}}^m(q)$ we have $M_{\emptyset}^{n-1} = 0$ and $M_{(n-1)^i}^{n-1}(q) = \sum_{k=0}^{n-1} (-1)^k q^{\binom{k+1}{2}-ik} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q$ for $i > 0$. Thus

$$\begin{aligned} t_q(n, \ell) &= (-1)^n \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left(\sum_{i=1}^{\ell} \binom{\ell}{i} (1-q)^i q^{(n-1)i} \left(\sum_{k=0}^{n-1} (-1)^k q^{\binom{k+1}{2}-ik} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q \right) \right) \\ &= (-1)^n \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left(\sum_{k=0}^{n-1} (-1)^k q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q \left(\sum_{i=1}^{\ell} \binom{\ell}{i} q^{(n-k-1)i} (1-q)^i \right) \right) \\ &= \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left(\sum_{k=0}^{n-1} (-1)^{k+n} q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q ((1 + q^{n-k-1} - q^{n-k})^\ell - 1) \right), \end{aligned}$$

where the last step follows by an application of the binomial theorem. Finally, by (3.5) we have

$$\sum_{k=0}^{n-1} (-1)^k q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q = (q; q)_{n-1}$$

and so

$$\begin{aligned} t_q(n, \ell) &= \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left(\sum_{k=0}^{n-1} (-1)^{k+n} q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q ((1 + q^{n-k-1} - q^{n-k})^\ell - 1) \right) \\ &= \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q; q)_n} \left((-1)^{n-1} (q; q)_{n-1} + \sum_{k=0}^{n-1} (-1)^{k+n} q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q (1 + q^{n-k-1} - q^{n-k})^\ell \right), \end{aligned}$$

as desired.

Remark 4.3. In the genus-0 case $r_1 + \dots + r_k = (k-1)n$, there is a simple formula [HLR15, Thm. 4.2]

$$(4.4) \quad a_{r_1, \dots, r_k}(q) = q^{\sum_{i=1}^k (n-r_i-1)r_i} (q^n - 1)^{k-1}$$

for $a_{r_1, \dots, r_k}(q)$ with $0 \leq r_i < n$. Here, we show how to derive this formula from Theorem 1.4.

In the genus-0 case, it follows from the triangularity of the basis change (3.5) that $a_{r_1, \dots, r_k}(q)$ is equal to a predictable factor times $b_{r_1, \dots, r_k}(q)$. Thus, we focus on computing this latter coefficient. Our approach is to evaluate a certain q -difference in two different ways: the first will give a formula involving $M_{\mathbf{r}}^m(q)$, while the second will give an explicit product formula.

Let Δ_q be the operator that acts on a function f by

$$\Delta_q(f)(x) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(qx) - f(x)}{(q-1)x}.$$

Then it is not hard to show using the q -Pascal recurrence [Sta12, (1.67)] that

$$(4.5) \quad \Delta_q^N(f)(x) = q^{-\binom{N}{2}} (q-1)^{-N} \sum_{d=0}^N (-1)^d q^{\binom{d}{2}} \left[\begin{matrix} N \\ d \end{matrix} \right]_q \frac{f(q^{N-d}x)}{x^N}.$$

On the other hand, Δ_q acts in a predictable way on polynomials: if f is a polynomial of degree n and leading coefficient a then $\Delta_q(f)$ is a polynomial of degree $n-1$ and leading coefficient $[n]_q \cdot a$.

If one defines for $0 \leq r \leq m$ the polynomial

$$P_r^m(x) := x \cdot \frac{(xq^{r+1-m}; q)_{m-r}}{(q; q)_{m-r}} = x \cdot \frac{(x; q^{-1})_{m-r}}{(q; q)_{m-r}} = x \cdot (1-x)(1-xq^{-1}) \cdots (1-xq^{m-r-1})/(q; q)_{m-r}$$

and for a tuple $\mathbf{r} = (r_1, \dots, r_k)$

$$\mathcal{P}_{\mathbf{r}}^m(x) := \frac{1}{x} \prod_{\substack{1 \leq i \leq k: \\ r_i \leq m}} P_{r_i}^m(x),$$

then one has by Equation (4.5) that

$$\begin{aligned} \Delta_q^m(\mathcal{P}_{\mathbf{r}}^m)(x) &= q^{-\binom{m}{2}}(q-1)^{-m} \sum_{d=0}^m (-1)^d q^{\binom{d}{2}} \begin{bmatrix} m \\ d \end{bmatrix}_q \frac{\mathcal{P}_{\mathbf{r}}^m(q^{m-d}x)}{x^m} \\ &= q^{-\binom{m}{2}}(q-1)^{-m} \sum_{d=0}^m (-1)^d q^{\binom{d}{2}} \begin{bmatrix} m \\ d \end{bmatrix}_q \frac{(q^{m-d}x)^{k-1}}{x^m} \prod_{i=1}^k \frac{(xq^{m-d}; q^{-1})_{m-r_i}}{(q; q)_{m-r_i}} \\ &= q^{-\binom{m}{2}+m(k-1)}(q-1)^{-m} \sum_{d=0}^m (-1)^d q^{\binom{d}{2}-d(k-1)} \begin{bmatrix} m \\ d \end{bmatrix}_q x^{k-m-1} \prod_{i=1}^k \frac{(xq^{m-d}; q^{-1})_{m-r_i}}{(q; q)_{m-r_i}}. \end{aligned}$$

Plugging in $x = 1$ gives

$$\begin{aligned} \Delta_q^m(\mathcal{P}_{\mathbf{r}}^m)(1) &= q^{-\binom{m}{2}+m(k-1)}(q-1)^{-m} \sum_{d=0}^m (-1)^d q^{\binom{d}{2}-d(k-1)} \begin{bmatrix} m \\ d \end{bmatrix}_q \prod_{i=1}^k \frac{(q^{m-d}; q^{-1})_{m-r_i}}{(q; q)_{m-r_i}} \\ &= q^{-\binom{m}{2}+m(k-1)}(q-1)^{-m} \sum_{d=0}^m (-1)^d q^{\binom{d}{2}-d(k-1)} \begin{bmatrix} m \\ d \end{bmatrix}_q \prod_{i=1}^k \begin{bmatrix} m-d \\ m-r_i \end{bmatrix}_q \\ (4.6) \quad &= q^{-\binom{m}{2}+m(k-1)}(q-1)^{-m} M_{\mathbf{r}}^m(q). \end{aligned}$$

On the other hand, when $m = n - 1$ we have that $\mathcal{P}_{\mathbf{r}}^m$ is a polynomial of degree $(k-1) + \sum_i (n-1-r_i) = kn-1 - \sum_i r_i$. Thus, in the genus-0 case $\sum_i r_i = (k-1)n$, we have that $\deg \mathcal{P}_{\mathbf{r}}^m = n-1$. It follows immediately that $\Delta_q^{n-1}(\mathcal{P}_{\mathbf{r}}^{n-1})$ is equal to $[n-1]!_q$ times the leading coefficient of $\mathcal{P}_{\mathbf{r}}^{n-1}$. It is easy to see from the definition that $\mathcal{P}_{\mathbf{r}}^{n-1}$ has leading coefficient

$$\prod_{r \in \mathbf{r}: r < n} \frac{(-1)^{m-r} q^{-\binom{m-r}{2}}}{(q; q)_{n-1-r}}.$$

Combining this with (4.6) gives a simple product formula³ for $b_{r_1, \dots, r_k}(q)$. Putting everything together gives (4.4), as desired.

Remark 4.4. We again suppose that exactly j of the p_i are less than n , and without loss of generality take them to be p_1, \dots, p_j . Applying (3.11), one may rewrite

$$b_{p_1, \dots, p_k}(q) = b_{p_1, \dots, p_j}(q)$$

as a q -hypergeometric function of a particularly simple form:

$$b_{p_1, \dots, p_k}(q) = {}_j\phi_{j-1} \left(\begin{matrix} q^{-p_1}, \dots, q^{-p_j} & ; & q^{1-n}, \dots, q^{1-n} & ; & q^{n(1-j)+\sum p_i} \end{matrix} \right).$$

³ More generally, one could obtain by the same techniques a formula for $b_{r_1, \dots, r_k}(q)$ in the genus $g := (k-1)n - \sum_i r_i$ case as a sum of $g+1$ terms, depending on the top $g+1$ coefficients of \mathcal{P} .

Then the calculation in Remark 4.3 in the genus-0 case is equivalent to a q -analogue of the Karlsson–Minton formulas; see [GR04, §1.9].

5. AN APPLICATION TO ASYMPTOTIC ENUMERATION OF FACTORIZATIONS BY GENUS

In the spirit of (e.g.) [FS15] and [GS98, §4.2], one may ask to study the *asymptotic* enumeration of factorizations. Here, we compute the asymptotic growth of the number of fixed-genus factorizations of a regular elliptic element in $G = \text{GL}_n(\mathbf{F}_q)$ as $n \rightarrow \infty$.

Theorem 5.1. *Let $g \geq 0$ and q be fixed. As $n \rightarrow \infty$, the number of genus- g factorizations of a regular elliptic element in $\text{GL}_n(\mathbf{F}_q)$ into two factors has growth rate*

$$\Theta \left(\frac{q^{(n+g)^2/2}}{|\text{GL}_g(\mathbf{F}_q)|} \right),$$

where the implicit constants depend on q but not g .

Remark 5.2. For the sake of comparison, we include the analogous asymptotics in \mathfrak{S}_n . Goupil and Schaeffer showed [GS98, Cor. 4.3] that for fixed $g \geq 0$, as $n \rightarrow \infty$ the number of genus- g factorizations of a fixed long cycle in \mathfrak{S}_n into two factors is asymptotic to

$$\frac{n^{3(g-\frac{1}{2})} 4^n}{g! 48^g \sqrt{\pi}}.$$

It is interesting to observe that in both results we see that the constant depends on the size ($|\mathfrak{S}_g|$ or $|\text{GL}_g(\mathbf{F}_q)|$) of a related group. Is there an explanation for this phenomenon?

The rest of this section is a sketch of the proof of Theorem 5.1. To begin, we use Theorem 1.2 to give an explicit formula for $a_{r,s}(q)$. For every positive integer g , let $P_g(x, y, z, q)$ be the following Laurent polynomial of four variables:

$$(5.1) \quad P_g(x, y, z, q) := (-1)^g q^{-g} \left(y^{-g} z^g \prod_{i=1}^g (yq^i - 1) + y^g z^{-g} \prod_{i=1}^g (zq^i - 1) \right) + \sum_{\substack{0 \leq t', u' \leq g-1 \\ 0 \leq t' + u' \leq g}} (-1)^{t' + u'} \times \\ \left[t'; u'; g - t' - u' \right]_q \cdot y^{u-t'} z^{t'-u'} q^{t'u'-t'-u'} (x - yq^{t'} - zq^{u'} + 1) \prod_{i=1}^{g-t'-1} (zq^i - 1) \prod_{i=1}^{g-u'-1} (yq^i - 1).$$

Proposition 5.3. *If $g, r, s > 0$ satisfy $r + s = n - g$ then*

$$a_{r,s}(q) = \frac{q^{2rs+(g-1)n-\binom{g}{2}}(q^n - 1)}{(q - 1)^g [g]_q!} \cdot P_g(q^n, q^r, q^s, q).$$

Proof sketch. Extracting the coefficient of $x^r y^s$ from (1.3) using (3.5) gives

$$(5.2) \quad \frac{a_{r,s}(q)}{|G|} = (-1)^{n-g} q^{\binom{r+1}{2} + \binom{s+1}{2}} \times \\ \sum_{\substack{r \leq t, s \leq u, \\ t+u \leq n}} \frac{1}{(q; q)_t (q; q)_u} \begin{bmatrix} t \\ r \end{bmatrix}_q \begin{bmatrix} u \\ s \end{bmatrix}_q \cdot q^{tu-t-u-rt-su} \frac{[n-t-1]_q! \cdot [n-u-1]_q! (q^n - q^t - q^u + 1)}{[n-1]_q! \cdot [n-t-u]_q! (q-1)}.$$

The rest of the proof is a long but totally unenlightening calculation: expanding the q -binomials, making the change of variables $t = r + t'$, $u = s + u'$ with $0 \leq t' + u' \leq g$, separating the $(t', u') = (g, 0)$ and $(0, g)$ terms, rearranging various factors, and doing some basic arithmetic. In particular, no nontrivial q -identities are required. \square

Proposition 5.4. *For constants a, b, c, g and $q > 1$, we have*

$$\sum_{\substack{r, s \geq 1 \\ r+s=n-g}} q^{2rs+an+br+cs} = \Theta \left(q^{(b-c)^2/8+g(g-b-c)/2} \cdot q^{n^2/2} \cdot q^{(2a+b+c-2g)n/2} \right)$$

as $n \rightarrow \infty$, where the implicit constants depend only on q and $b - c$.

Proof. In the sum on the left side, set $m := n - g$ and rearrange to get

$$\sum_{\substack{r, s \geq 1 \\ r+s=n-g}} q^{2rs+an+br+cs} = q^{an+cm+(m+(b-c)/2)^2/2} \sum_{r=1}^{m-1} q^{-\frac{1}{2}((m-2r)+(b-c)/2)^2}.$$

The exponent $an+cm+(m+(b-c)/2)^2/2$ is equal to $\frac{n^2}{2} + (2a+b+c-2g)n/2 + \frac{g(g-b-c)}{2} + \frac{(b-c)^2}{8}$. For m large, the sum on the right is easily seen to be $\Theta(1)$: it oscillates with the parity of m between two evaluations of the convergent Jacobi theta function $\vartheta(w, t) := \sum_{r=-\infty}^{\infty} t^{r^2} \cdot w^{2r}$, depending on the value $b - c$. This completes the proof. \square

It follows from the preceding proposition that for any fixed g and any Laurent polynomial $Q(x, y, z)$ with coefficients in $\mathbb{Q}(q)$, the asymptotic growth of $\sum_{r=1}^{n-g-1} q^{2r(n-g-r)} Q(q^n, q^r, q^{n-g-r})$ as $n \rightarrow \infty$ is determined entirely by the monomials in Q of maximal **weight**, where the weight of $x^a y^b z^c$ is defined to be $2a + b + c$. This inspires the following result.

Proposition 5.5. *Let $g > 0$ and $P_g(x, y, z, q)$ be as in (5.1). Then P_g has a unique (x, y, z) -monomial $x^1 y^{g-1} z^{g-1}$ of maximal weight.*

Proof. By inspection, the first two summands in P_g contribute maximum-weight monomials $y^{-g} \cdot z^g \cdot y^g = x^0 y^0 z^g$ and $y^g \cdot z^{-g} \cdot z^g = x^0 y^g z^0$, both of weight $g/2$. In the summation, the t', u' -summand contributes a unique maximum-weight monomial $z^{g-t'-1} \cdot y^{g-u'-1} \cdot y^{u'-t'} \cdot z^{t'-u'} \cdot x = x^1 \cdot y^{g-t'-1} \cdot z^{g-u'-1}$, of weight $(2g - t' - u' - 2)/2 + 1$. This expression is uniquely maximized over the region of summation at $t' = u' = 0$, with value g . This is also strictly larger than the maximum-weight contribution from the first two summands. Thus, this maximum-weight monomial in the $t' = u' = 0$ summand is the unique maximum-weight monomial in P_g , as desired. \square

Finally, we put the preceding results together to get asymptotics for factorizations of fixed genus.

Proof sketch of Theorem 5.1. The case $g = 0$ is straightforward from the formula (4.4) already available in [HLR15]. For $g > 0$, Proposition 5.3 provides an explicit polynomial formula for $a_{r,s}(q)$. Notably, the number of terms in this formula does not depend on n . Proposition 5.4 provides the asymptotics for the sum (over r, s such that $r + s = n - g$) of a single monomial from this polynomial. Finally, Proposition 5.5 detects the unique monomial whose asymptotic

growth dominates the others. To finish, one needs to extract the coefficient of the $xy^{g-1}z^{g-1}$ monomial in P_g and do the arithmetic.⁴ \square

6. CLOSING REMARKS

6.1. Combinatorial proofs. Theorems 1.1 and 1.3 were originally proved in [Jac88] by character methods (as in the outline in Section 2.2), but these are not the only known proofs. The first result (the case of two factors in \mathfrak{S}_n) has several combinatorial proofs, by Schaeffer–Vassilieva [SV08], Bernardi [Ber12], and Chapuy–Féray–Fusy [CFF13]. The second result (the case of k factors) has an intricate combinatorial proof [BM13, BM15]. It would be of interest to find combinatorial proofs of our q -analogous Theorems 1.2 and 1.4

Most of the combinatorial proofs in \mathfrak{S}_n for two and more factors include (following [Ber12]) the following elements:

1. For a permutation σ in \mathfrak{S}_n , denote by $\text{cycles}(\sigma)$ the set of cycles of the permutation. When x, y are both positive integers, the term $a_{r,s}x^r y^s$ counts triples consisting of a factorization $(1, 2, \dots, n) = u \cdot v$ and *cycle colorings*: functions $\text{cycles}(u) \rightarrow \{1, \dots, x\}$ and $\text{cycles}(v) \rightarrow \{1, \dots, y\}$.
2. When we do the change of bases $x^k = \sum_{j=0}^k S(k, j)(x)_j$ implicit in (1.1), where $S(k, j)$ is a Stirling number of the first kind and $(x)_j$ is a falling factorial, the term $n! \cdot b_{r,s}$ counts tuples (u, v, A', B') consisting of
 - a factorization $(1, 2, \dots, n) = u \cdot v$,
 - a surjective function $A' : \text{cycles}(u) \rightarrow \{1, \dots, r\}$, and
 - a surjective function $B' : \text{cycles}(v) \rightarrow \{1, \dots, s\}$,

Call such tuples **colored factorizations**.⁵

3. Using the classical correspondence (e.g., see [LZ04, Ch. 1]) between factorizations of a long cycle and *unicellular bipartite maps*, certain bipartite graphs with a one-cell embedding on a locally orientable surface. Under this correspondence, the cycles of u and v correspond to vertices and colored factorizations correspond to unicellular bipartite maps with a coloring on the vertices of each part.
4. Doing a bijective construction on the colored bipartite map. (See [Ber12, §3] for the bijective construction proving (1.1).)

The next remark gives a combinatorial interpretation of the coefficients $|G| \cdot b_{t,u}(q)$ that appear in Theorem 1.2, analogous to the first two steps above. However, we do not know if there is a map interpretation of the factorizations in $\text{GL}_n(\mathbf{F}_q)$ that we consider.

Remark 6.1. The second change of basis formula (3.6) may be understood combinatorially in the following way: x^k is the number of linear maps from a k -dimensional vector space V to an x -element vector space X over \mathbf{F}_q . The term $\frac{(x; q^{-1})_m}{(q; q)_m}$ gives the number of m -dimensional subspaces of X , while $(-1)^m q^{\binom{m}{2}} (q^k; q^{-1})_m = (q^k - 1)(q^k - q) \cdots (q^k - q^{m-1})$ is the number of surjective maps from V to the selected m -dimensional subspace of X . Thus, the right side refines the left side by dimension of the image.

For an element u of $\text{GL}_n(\mathbf{F}_q) \cong \text{GL}(V)$, denote by V^u the fixed space $\ker(u - 1)$ of u . When x, y are both powers of q , the term $a_{r,s}(q)x^r y^s$ counts triples consisting of a factorization $c = u \cdot v$

⁴We have swept under the rug the issue of the contribution from the terms $a_{0,n-g}(q) = a_{n-g,0}(q)$. Using the same approach as in Proposition 5.3, it is another long-but-not-difficult computation to show that this term is low order and so does not contribute to the asymptotic growth rate.

⁵The first explicit appearance of cycle coloring of factorizations that we are aware of is in [GJ95, Thm. 2.1].

with $\dim V^u = r$ and $\dim V^v = s$, a linear map $V^u \rightarrow X$, and a linear map $V^v \rightarrow Y$. When we change bases, the term $|G| \cdot b_{r,s}(q) \frac{(x;q^{-1})_r}{(q;q)_r} \frac{(y;q^{-1})_s}{(q;q)_s}$ counts triples consisting of a factorization $c = u \cdot v$, a linear map $A : V^u \rightarrow X$, and a linear map $B : V^v \rightarrow Y$ such that $\dim \operatorname{Im}(A) = r$ and $\dim \operatorname{Im}(B) = s$. Equivalently, $|G| \cdot b_{r,s}(q)$ counts tuples (u, v, A', B') consisting of

- a factorization $c = u \cdot v$,
- a surjective linear map $A' : V^u \rightarrow \mathbf{F}_q^r$, and
- a surjective linear map $B' : V^v \rightarrow \mathbf{F}_q^s$.

Remark 6.2. One might hope to exploit special properties of the coefficients M that appear in (1.5), (1.7) to find combinatorial connections. For example, by inclusion-exclusion one has that M_{r_1, \dots, r_k}^m counts tuples (T_1, T_2, \dots, T_k) of subsets $T_i \subseteq [m]$ such that $T_1 \cap T_2 \cap \dots \cap T_k = \emptyset$ and $|T_i| = r_i$. The obvious analogy is to count tuples (W_1, \dots, W_k) of subspaces $W_i \subseteq \mathbf{F}_q^m$ such that $W_1 \cap \dots \cap W_k = \{0\}$ and $\dim W_i = r_i$. The number of such tuples of subspaces may be computed by Möbius inversion, but unfortunately it is

$$\sum_{a=0}^{\min(r_i)} (-1)^a q^{\binom{a}{2}} \begin{bmatrix} m \\ a \end{bmatrix}_q \prod_{i=1}^k \begin{bmatrix} m-a \\ r_i-a \end{bmatrix}_q,$$

which is off by a power of q in the summand from $M_{r_1, \dots, r_k}^m(q)$. Thus, we currently lack a more concrete connection than the obvious $\lim_{q \rightarrow 1} M_{r_1, \dots, r_k}^m(q) = M_{r_1, \dots, r_k}^m$.

Remark 6.3. Note that the derivation in Remark 4.3 is very involved, while the corresponding derivation in the symmetric group using the combinatorial interpretation of M_{r_1, \dots, r_k}^n is straightforward. One expected benefit of a combinatorial interpretation of $M_{r_1, \dots, r_k}^n(q)$ would be a simpler derivation of (4.4).

6.2. Connection with random matrices. If A is a square matrix, define (in an abuse of notation) the power-sum symmetric function p_k on A by $p_k(A) := \operatorname{Tr}(A^k)$. Hanlon–Stanley–Stembridge showed [HSS92, §2] that for $x \leq y$ positive integers, the left side of (1.1) can be written as

$$(6.1) \quad \sum_{r,s \geq 0} a_{r,s} \cdot x^r y^s = \mathbb{E}_U(p_n(ABU^*)) = \mathbb{E}_V(p_n(VV^*)),$$

where the first expectation is over $y \times y$ random matrices U with independent standard normal complex entries, $A = I_x \oplus \mathbf{0}_{y-x}$ and $B = I_y$, and the second expectation is over $x \times y$ random matrices V with independent standard normal complex entries. Similarly, the left side of (1.2) can be written as

$$(6.2) \quad \sum_{\lambda, \mu \vdash n} a_{\lambda, \mu} \cdot p_\lambda(a_1, \dots, a_m) p_\mu(b_1, \dots, b_m) = \mathbb{E}_U(p_n(ABU^*)),$$

where the expectation is over random $m \times m$ matrices U with independent standard normal complex entries and A, B are arbitrary fixed $m \times m$ Hermitian complex matrices with eigenvalues a_1, \dots, a_m and b_1, \dots, b_m , respectively. In [GJ95], Goulden and Jackson gave a combinatorial proof of these equations in terms of factorizations and coloring of cycles. Is there an analogue of (6.1) for $\operatorname{GL}_n(\mathbf{F}_q)$ expressing the left side of (1.3) as an expectation over random matrices?

6.3. Other asymptotic questions. As usual, define $F(x, y) = \sum_{r,s \geq 0} a_{r,s}(q) x^r y^s$. In light of Remark 3.2, we may view $F(x, x)/|G| = \frac{1}{|G|} \sum_{r,s \geq 0} a_{r,s}(q) x^{r+s}$ as encoding the probability

distribution of the genus of a factorization chosen uniformly at random. In particular, we have that

$$\left. \frac{d}{dx} \left(\frac{F(x, x)}{|G|} \right) \right|_{x=1} = \frac{1}{|G|} \sum_{r,s \geq 0} (r+s) \cdot a_{r,s}(q)$$

is exactly equal to n minus the expected genus of a random factorization. Since $(x; q^{-1})_t|_{x=1} = 0$ for $t > 0$, we have by Theorem 1.2 that

$$\begin{aligned} \left. \frac{d}{dx} \left(\frac{F(x, x)}{|G|} \right) \right|_{x=1} &= 2 \sum_{t=1}^n b_{t,0}(q) \cdot \left. \frac{d}{dx} \frac{(x; q^{-1})_t}{(q; q)_t} \right|_{x=1} \\ &= 2 \sum_{t=1}^n -\frac{(q^{-1}; q^{-1})_{t-1}}{(q; q)_t} \\ (6.3) \quad &= 2 \sum_{t=1}^n \frac{(-1)^t}{q^{\binom{t}{2}}(1-q^t)}. \end{aligned}$$

Thus, the expected genus of a random factorization of a regular elliptic element in GL_n(F_q) into two factors is exactly

$$n - 2 \sum_{t=1}^n \frac{(-1)^t}{q^{\binom{t}{2}}(1-q^t)}.$$

(Similar calculations could be made for the case of more factors.)

Since the sum (6.3) converges as $n \rightarrow \infty$, the vast majority of factorizations of a regular elliptic element $c \in \text{GL}_n(\mathbf{F}_q)$ into two factors must have large genus. Unfortunately, the techniques used to prove Theorem 5.1 are not sufficient to compute asymptotics for the number of genus- g factorizations if g grows with n . This leads to several natural questions.

Question 6.4. What is the asymptotic growth rate of the number of genus- g factorizations of a regular elliptic element in GL_n(F_q) into two factors if g grows with n ? For example, if $g = \alpha n$ for $\alpha \in (0, 1)$?

Question 6.5. Can one compute the limiting distribution of the genus of a random factorization of a regular elliptic element $c \in \text{GL}_n(\mathbf{F}_q)$ into two factors when n is large? That is, choose u uniformly at random in GL_n(F_q) and let $v = u^{-1}c$; what is the distribution of the genus of the factorization $c = u \cdot v$? As a first step, can one compute any higher moments of this distribution?

6.4. Connection with supercharacters. Proposition 2.3 shows that certain characters have very simple values on the sum of a large number of elements. This behavior is one characteristic of supercharacter theories (see, e.g., [DI08, DT09]).

Question 6.6. Does the grouping of elements by fixed space dimension, as in Proposition 2.3, correspond to a (very coarse) supercharacter theory for GL_n(F_q)?

APPENDIX A. CALCULATION OF GENERATING FUNCTIONS FOR CHARACTERS

In this section we derive formulas for the character generating functions

$$f_V(x) = \sum_{r=0}^n \tilde{\chi}^V(z_r) \cdot x^r$$

required for the proof of our main theorems.

Proposition 3.1. *If $V = (U, \lambda)$ for $U \neq \mathbf{1}$ we have*

$$(3.3) \quad f_{U, \lambda}(x) = |G| \cdot \frac{(x; q^{-1})_n}{(q; q)_n},$$

while if $V = (\mathbf{1}, \langle n - d, 1^d \rangle)$ we have

$$(3.4) \quad f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x) = |G| \cdot \left(\frac{(x; q^{-1})_n}{(q; q)_n} + q^{-d} \cdot \sum_{m=d}^{n-1} \frac{[m]!_q \cdot [n-d-1]!_q}{[m-d]!_q \cdot [n-1]!_q} \cdot \frac{(x; q^{-1})_m}{(q; q)_m} \right).$$

Proof. To prove (3.3), we substitute the value of $\tilde{\chi}^{U, \lambda}(z_r)$ from Proposition 2.3(i) into the definition of f_V and apply the q -binomial theorem (3.5) directly to get

$$f_{U, \lambda}(x) = \sum_{r=0}^n (-1)^{n-r} q^{\binom{n-r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q x^r = |G| \cdot \frac{(x; q^{-1})_n}{(q; q)_n},$$

as desired.

To prove (3.4), we substitute from Proposition 2.3(ii) to get the monstrous equation

$$f_{\mathbf{1}, \langle n-d, 1^d \rangle}(x) = \sum_{r=0}^n x^r \cdot \left((-1)^{n-r} q^{\binom{n-r}{2}} \left(\begin{bmatrix} n \\ r \end{bmatrix}_q + \frac{(1-q)[n]_q}{[r]!_q} \cdot \sum_{j=1}^{n-\max(r, d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \right) \right).$$

By (3.6), the coefficient of $\frac{(x; q^{-1})_m}{(q; q)_m}$ in the right side after changing bases is

$$(A.1) \quad c_m := \sum_{r=m}^n (-1)^m q^{\binom{m}{2}} (q^r; q^{-1})_m \cdot (-1)^{n-r} q^{\binom{n-r}{2}} \left(\begin{bmatrix} n \\ r \end{bmatrix}_q + \frac{1-q^n}{[r]!_q} \cdot \sum_{j=1}^{n-\max(r, d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \right).$$

We manipulate this expression, using the following special case of the q -binomial theorem (3.5):

$$\sum_{i=0}^k (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q = \begin{cases} 1 & k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$\begin{aligned}
c_m &= (-1)^m q^{\binom{m}{2}} \left(\sum_{r=m}^n (-1)^{n-r} q^{\binom{n-r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q (q^r; q^{-1})_m + \right. \\
&\quad \left. (1 - q^n) \cdot \sum_{r=m}^{n-1} \frac{(-1)^{n-r} q^{\binom{n-r}{2}} (q^r; q^{-1})_m}{[r]!_q} \sum_{j=1}^{n-\max(r,d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \right) \\
&= (-1)^m q^{\binom{m}{2}} \left((1-q)^m \sum_{r=m}^n (-1)^{n-r} q^{\binom{n-r}{2}} \frac{[n]!_q}{[n-r]!_q [r-m]!_q} + \right. \\
&\quad \left. (1-q^n)(1-q)^m \cdot \sum_{r=m}^{n-1} \frac{(-1)^{n-r} q^{\binom{n-r}{2}}}{[r-m]!_q} \sum_{j=1}^{n-\max(r,d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \right) \\
&= (q-1)^m q^{\binom{m}{2}} \left(\frac{[n]!_q}{[n-m]!_q} \sum_{r=m}^n (-1)^{n-r} q^{\binom{n-r}{2}} \begin{bmatrix} n-m \\ n-r \end{bmatrix}_q + \right. \\
&\quad \left. (1-q^n) \sum_{j=1}^{n-\max(m,d)} q^{jn-d} [n-j]!_q (q^{n-d-j+1}; q)_{j-1} \sum_{r=m}^{n-j} \frac{(-1)^{n-r} q^{\binom{n-r}{2} - (n-r)j}}{[n-r-j]!_q [r-m]!_q} \right) \\
&= (q-1)^m q^{\binom{m}{2}} \left(\frac{[n]!_q}{[n-m]!_q} \delta_{m,n} + \right. \\
&\quad \left. (1-q^n) \sum_{j=1}^{n-\max(m,d)} q^{jn-d - \binom{j+1}{2}} \frac{[n-j]!_q (q^{n-d-j+1}; q)_{j-1}}{[n-m-j]!_q} \sum_{r=m}^{n-j} (-1)^{n-r} q^{\binom{n-r-j}{2}} \begin{bmatrix} n-m-j \\ n-r-j \end{bmatrix}_q \right) \\
&= (q-1)^m q^{\binom{m}{2}} \left([n]!_q \delta_{m,n} + (1-q^n) \sum_{j=1}^{n-\max(m,d)} (-1)^j q^{jn-d - \binom{j+1}{2}} \frac{[n-j]!_q (q^{n-d-j+1}; q)_{j-1}}{[n-m-j]!_q} \delta_{n-j,m} \right).
\end{aligned}$$

We now consider cases: if $d > m$ then this evaluates to 0 (we have $d < n$ so $\delta_{m,n} = 0$, while every term in the second sum satisfies $j \leq n-d < n-m$ and so $\delta_{n-j,m} = 0$). If $m = n$ then it evaluates to $(q-1)^n q^{\binom{n}{2}} [n]!_q = |G|$ (the second sum is empty). Finally, if $d \leq m < n$ it simplifies to

$$\begin{aligned}
&(q-1)^m q^{\binom{m}{2}} (1-q^n) (-1)^{n-m} q^{(n-m)n-d - \binom{n-m+1}{2}} \frac{[m]!_q (q^{m-d+1}; q)_{n-m-1}}{[0]!_q} \\
&= (q-1)^n q^{\binom{n}{2}-d} \frac{[n]_q \cdot [m]!_q \cdot [n-d-1]!_q}{[m-d]!_q}.
\end{aligned}$$

Factoring out $|G|$ gives the desired result. \square

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