MULTIPLY UNION FAMILIES IN $\mathbb{N}^n$

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Abstract. Let $A \subset \mathbb{N}^n$ be an r-wise s-union family, that is, a family of sequences with n components of non-negative integers such that for any r sequences in A the total sum of the maximum of each component in those sequences is at most s. We determine the maximum size of A and its unique extremal configuration provided (i) n is sufficiently large for fixed r and s, or (ii) $n = r + 1$.

1. Introduction

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of non-negative integers, and let $[n] := \{1, 2, \ldots, n\}$. Intersecting families in $2^{[n]}$ or $\{0, 1\}^n$ are one of the main objects in extremal set theory. The equivalent dual form of an intersecting family is a union family, which is the subject of this paper. In [2] Frankl and Tokushige proposed to consider such problems not only in $\{0, 1\}^n$ but also in $[q]^n$. They determined the maximum size of 2-wise s-union families (i) in $[q]^n$ for $n > n_0(q, s)$, and (ii) in $\mathbb{N}^3$ for all s (the definitions will be given shortly). In this paper we extend their results and determine the maximum size and structure of r-wise s-union families in $\mathbb{N}^n$ for the following two cases: (i) $n \geq n_0(r, s)$, and (ii) $n = r + 1$.

For a vector $\mathbf{x} \in \mathbb{R}^n$, we write x_i or $(\mathbf{x})_i$ for the ith component, so $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Define the *weight* of $\mathbf{a} \in \mathbb{N}^n$ by

$$|\mathbf{a}| := \sum_{i=1}^{n} a_i.$$

For a finite number of vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{z} \in \mathbb{N}^n$ define the join $\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z}$ by

$$(\mathbf{a} \vee \mathbf{b} \vee \cdots \vee \mathbf{z})_i := \max\{a_i, b_i, \ldots, z_i\},$$

and we say that $A \subset \mathbb{N}^n$ is r-wise *s-union* if

$$|\mathbf{a}_1 \vee \mathbf{a}_2 \vee \cdots \vee \mathbf{a}_r| \leq s \text{ for all } \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r \in A.$$

The *width* of $A \subset \mathbb{N}^n$ is defined to be the maximum s such that A is s-union. In this paper we address the following problem.

Problem. *For given n, r and s, determine the maximum size $|A|$ of r-wise s-union families $A \subset \mathbb{N}^n$.*

To describe candidates A that give the maximum size to the above problem, we need some more definitions. Let us introduce a partial order $\prec$ in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we let $\mathbf{a} \prec \mathbf{b}$ iff $a_i \leq b_i$ for all $1 \leq i \leq n$. Then we define a *down set* for $\mathbf{a} \in \mathbb{N}^n$ by

$$\mathcal{D}(\mathbf{a}) := \{\mathbf{c} \in \mathbb{N}^n : \mathbf{c} \prec \mathbf{a}\},$$

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and for $A \subset \mathbb{N}^n$ let

$$\mathcal{D}(A) := \bigcup_{\mathbf{a} \in A} \mathcal{D}(\mathbf{a}).$$

Similarly, we define an *up set* at distance d from $\mathbf{a} \in \mathbb{N}^n$ by

$$\mathcal{U}(\mathbf{a}, d) := \{\mathbf{a} + \boldsymbol{\epsilon} \in \mathbb{N}^n : \boldsymbol{\epsilon} \in \mathbb{N}^n, \, |\boldsymbol{\epsilon}| = d\}.$$

We say that $\mathbf{a} \in \mathbb{N}^n$ is an *equitable partition*, if all a_i's are as close to each other as possible, more precisely, $|a_i - a_j| \leq 1$ for all i, j. Let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$.

For $r, n \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{N}^n$ define a family K by

$$K = K(r, n, \mathbf{a}, d) := \bigcup_{i=0}^{\lfloor \frac{d}{u} \rfloor} \mathcal{D}(\mathcal{U}(\mathbf{a} + i\mathbf{1}, d - ui)),$$

where $u = n - r + 1$. We will show that this is an r-wise s-union family, see Claim 3 in the next section.

Conjecture. *If $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \leq \max_{0 \leq d \leq \lfloor \frac{s}{r} \rfloor} |K(r, n, \mathbf{a}, d)|,$$

where $\mathbf{a} \in \mathbb{N}^n$ is an equitable partition with $|\mathbf{a}| = s - rd$. Moreover if equality holds, then $A = K(r, n, \mathbf{a}, d)$ for some $0 \leq d \leq \lfloor \frac{s}{r} \rfloor$.

We first verify the conjecture when n is sufficiently large for fixed r, s. Let $\mathbf{e}_i$ be the i-th standard base of $\mathbb{R}^n$, that is, $(\mathbf{e}_i)_j = \delta_{ij}$. Let $\tilde{\mathbf{e}}_0 = \mathbf{0}$, and $\tilde{\mathbf{e}}_i = \sum_{j=1}^{i} \mathbf{e}_i$ for $1 \leq i \leq n$, e.g., $\tilde{\mathbf{e}}_n = \mathbf{1}$.

Theorem 1. *Let r and s be fixed positive integers. Write $s = dr + p$ where d and p are non-negative integers with $0 \leq p < r$. Then there exists $n_0(r, s)$ such that if $n > n_0(r, s)$ and $A \subset \mathbb{N}^n$ is r-wise s-union, then*

$$|A| \leq |\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d))|.$$

Moreover if equality holds, then A is isomorphic to $\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d)) = K(r, n, \tilde{\mathbf{e}}_p, d)$.

We mention that the case $A \subset \{0, 1\}^n$ of Theorem 1 is settled in [**?**], and the case $r = 2$ of Theorem 1 is proved in [2] in slightly stronger form. We also notice that if $A \subset \{0, 1\}^n$ is 2-wise $(2d + p)$-union, then the Katona's t-intersection theorem [3] states that $|A| \leq |\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d) \cap \{0, 1\}^n)|$ for all $n \geq s$.

Next we show that the conjecture is true if $n = r + 1$. We also verify the conjecture or general n if A satisfies some additional properties described below.

Let $A \subset \mathbb{N}^n$ be r-wise s-union. For $1 \leq i \leq n$ let

$$m_i := \max\{x_i : \mathbf{x} \in A\}.$$

If $n - r$ divides $|\mathbf{m}| - s$, then we define

$$d := \frac{|\mathbf{m}| - s}{n - r} \geq 0,$$

and for $1 \leq i \leq n$ let

$$a_i := m_i - d,$$

and we assume that $a_i \geq 0$. In this case we have $|\mathbf{a}| = s - rd$. Since $|\mathbf{a}| \geq 0$ it follows that $d \leq \lfloor \frac{s}{r} \rfloor$. For $1 \leq i \leq n$ define $P_i \in \mathbb{N}^n$ by

$$P_i := \mathbf{a} + d\mathbf{e}_i,$$

where $\mathbf{e}_i$ denotes the ith standard base, for example, $P_2 = (a_1, a_2 + d, a_3, \ldots, a_n)$.

Theorem 2. *Let $A \subset \mathbb{N}^n$ be r-wise s-union. Assume that P_i's are well-defined and*

$$\{P_1, \ldots, P_n\} \subset A. \tag{1}$$

Then it follows that

$$|A| \leq \max_{0 \leq d' \leq \lfloor \frac{s}{r} \rfloor} |K(r, n, \mathbf{a}', d')|,$$

where $\mathbf{a}' \in \mathbb{N}^n$ is an equitable partition with $|\mathbf{a}'| = s - rd'$. Moreover if equality holds, then $A = K(r, n, \mathbf{a}', d')$ for some $0 \leq d' \leq \lfloor \frac{s}{r} \rfloor$.

We will show that the assumption (1) is automatically satisfied when $n = r + 1$.

Corollary. *If $n = r + 1$, then Conjecture is true.*

Notation: For $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ we define $\mathbf{a} \setminus \mathbf{b} \in \mathbb{N}^n$ by $(\mathbf{a} \vee \mathbf{b}) - \mathbf{b}$, in other words, $(\mathbf{a} \setminus \mathbf{b})_i := \max\{a_i - b_i, 0\}$. The support of $\mathbf{a}$ is defined by $\mathrm{supp}(\mathbf{a}) := \{j : a_j > 0\}$.

2. Proof of Theorem 1 — the case when n is large

Let r, s be given, and let $s = dr + p$, $0 \leq p < r$.

Claim 1. $|\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d))| = 2^p \binom{n+d}{d}$.

Proof. By definition we have

$$\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d)) = \{\mathbf{x} + \mathbf{y} \in \mathbb{N}^n : |\mathbf{x}| \leq d,\ \mathbf{y} \prec \mathbf{e}_p\}.$$

The number of $\mathbf{x} \in \mathbb{N}^n$ with $|\mathbf{x}| \leq d$ is equal to the number of non-negative integer solutions of $x_1 + \cdots + x_n \leq d$, which is $\binom{n+d}{d}$. It is 2^p that the number of $\mathbf{y} \in \mathbb{N}^n$ satisfying $\mathbf{y} \prec \tilde{\mathbf{e}}_p$. □

Let $A \subset \mathbb{N}^n$ be r-wise s-union with maximal size. So A is a downset. We will show that $|A| \leq 2^p \binom{n+d}{d}$. Notice that this RHS is $\Theta(n^d)$ for fixed r, s.

First suppose that there is t with $2 \leq t \leq r$ such that A is t-wise $(dt + p)$-union, but not $(t-1)$-wise $(d(t-1) + p)$-union. In this case, by the latter condition, there are $\mathbf{b}_1, \ldots, \mathbf{b}_{t-1} \in A$ such that $|\mathbf{b}| \geq d(t-1) + p + 1$, where $\mathbf{b} = \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_{t-1}$. Then, by the former condition, for every $\mathbf{a} \in A$ it follows that $|\mathbf{a} \vee \mathbf{b}| \leq dt + p$, so $|\mathbf{a} \setminus \mathbf{b}| \leq d - 1$. This gives us

$$A = \{\mathbf{x} + \mathbf{y} \in \mathbb{N}^n : |\mathbf{x}| \leq d - 1,\ \mathbf{y} \prec \mathbf{b}\}.$$

There are $\binom{n+(d-1)}{d-1}$ choices for $\mathbf{x}$ satisfying $|\mathbf{x}| \leq d - 1$. On the other hand, the number of $\mathbf{y}$ with $\mathbf{y} \prec \mathbf{b}$ is independent of n (so it is a constant depending on r and s only). In fact $|\mathbf{b}| \leq (t-1)s < rs$, and there are less than 2^{rs} choices for $\mathbf{y}$. Thus we get $|A| < \binom{n+(d-1)}{d-1} 2^{rs} = O(n^{d-1})$ and we are done.

Next we suppose that

$$A \text{ is } t\text{-wise } (dt + p)\text{-union for all } 1 \leq t \leq r. \tag{2}$$

The case $t = 1$ gives us $|\mathbf{a}| \le d + p$ for every $\mathbf{a} \in A$. If $p = 0$, then this means that $A \subset \mathcal{D}(\mathcal{U}(\mathbf{0}, d))$, which finishes the proof for this case. So, from now on, we assume that $1 \le p < r$. Then there is u with $u \ge 1$ such that there exist $\mathbf{b}_1, \dots, \mathbf{b}_u \in A$ satisfying

$$|\mathbf{b}| = u(d + 1), \tag{3}$$

where $\mathbf{b} := \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_u$. In fact we have (3) for $u = 1$, if otherwise $A \subset \mathcal{D}(\mathcal{U}(\mathbf{0}, d))$. If $u = p+1$ then (3) fails. In fact setting $t = p+1$ in (2) we see that A is $(p+1)$-wise $((p + 1)(d + 1) - 1)$-union. We choose maximal u with $1 \le u \le p$ satisfying (3), and fix $\mathbf{b} = \mathbf{b}_1 \vee \cdots \vee \mathbf{b}_u$. By this maximality, for every $\mathbf{a} \in A$, it follows that $|\mathbf{a} \vee \mathbf{b}| \le (u + 1)(d + 1) - 1$, and

$$|\mathbf{a} \setminus \mathbf{b}| \le d. \tag{4}$$

Using (4) we partition A into $\bigsqcup_{i=0}^{d} A_i$, where

$$A_i := \{\mathbf{x} + \mathbf{y} \in A : |\mathbf{x}| = i,\ \mathbf{y} \prec \mathbf{b}\}.$$

Then we have $|A_i| \le \binom{n+i}{i} 2^{|\mathbf{b}|}$. Noting that $|\mathbf{b}| \le (d + p)u = O(1)$ it follows $\sum_{i=0}^{d-1} |A_i| = O(n^{d-1})$. So the size of A_d is essential as we will see below.

We naturally identify $\mathbf{a} \in A_d$ with a subset of $[n] \times \{1, \dots, d + p\}$. Formally let

$$\phi(\mathbf{a}) := \{(i, j) : 1 \le i \le n,\ 1 \le j \le a_i\}.$$

We say that $\mathbf{b}' \prec \mathbf{b}$ is rich if there exist vectors $\mathbf{c}_1, \dots, \mathbf{c}_{dr}$ of weight d such that $\mathbf{b}' \vee \mathbf{c}_j \in A$ for every j, and the $dr + 1$ subsets $\phi(\mathbf{c}_1), \dots, \phi(\mathbf{c}_{dr}), \phi(\mathbf{b})$ are pairwise disjoint. Informally, $\mathbf{b}'$ is rich if it can be extended to a $(|\mathbf{b}'| + d)$-element subset of A in dr ways disjointly outside $\mathbf{b}$. We are comparing our family A with the reference family $\mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p), d)$, and we define $\tilde{\mathbf{b}}$ which plays a role of $\tilde{\mathbf{e}}_p$ in our family, namely, let us define

$$\tilde{\mathbf{b}} := \bigvee \{\mathbf{b}' \prec \mathbf{b} : \mathbf{b}' \text{ is rich}\}.$$

Claim 2. $|\tilde{\mathbf{b}}| \le p$.

Proof. Suppose the contrary, then there are distinct rich $\mathbf{b}'_1, \dots, \mathbf{b}'_{p+1}$. Let $\mathbf{c}^{(i)}_1, \dots, \mathbf{c}^{(i)}_{dr}$ support the richness of $\mathbf{b}'_i$. Let $\mathbf{a}_1 := \mathbf{b}'_1 \vee \mathbf{c}^{(1)}_{j_1} \in A$, say, $j_1 = 1$. Then choose $\mathbf{a}_2 := \mathbf{b}'_2 \vee \mathbf{c}^{(2)}_{j_2}$ so that $\phi(\mathbf{a}_1)$ and $\phi(\mathbf{a}_2)$ are disjoint. If $i \le p$, then having $\mathbf{a}_1, \dots, \mathbf{a}_i$ chosen, we only used id elements as $\bigcup_{l=1}^{i} \phi(\mathbf{c}^{(l)}_{j_l})$, which intersect at most id of $\mathbf{c}^{(i+1)}_1, \dots, \mathbf{c}^{(i+1)}_{dr}$, and since $id \le pd < rd$ we still have some $\mathbf{c}^{(i+1)}_{j_{i+1}}$ disjoint from any already chosen vectors. So we can continue this procedure until we get $\mathbf{a}_{p+1} := \mathbf{b}'_{p+1} \vee \mathbf{c}^{(p+1)}_{j_{p+1}} \in A$ such that all $\phi(\mathbf{a}_1), \dots, \phi(\mathbf{a}_{p+1})$ are disjoint. However, these vectors yield $|\mathbf{a}_1 \vee \cdots \vee \mathbf{a}_{p+1}| \ge (p + 1)(d + 1)$, which contradicts (2) at $t = p + 1$. $\square$

If $\mathbf{y} \prec \mathbf{b}$ is not rich, then

$$\{\phi(\mathbf{x} + \mathbf{y}) \setminus \phi(\mathbf{b}) : \mathbf{x} + \mathbf{y} \in A_d, |\mathbf{x}| = d\}$$

is a family of d-element subsets on $(d+p)n$ vertices, which has no dr pairwise disjoint subsets (so the matching number is $dr - 1$ or less). Thus, by the Erdős matching theorem [1], the size of this family is $O(n^{d-1})$. There are at most $2^{|\mathbf{b}|} = O(1)$ choices for non-rich $\mathbf{y} \prec \mathbf{b}$, and we can conclude that the number of vectors in A_d coming from non-rich $\mathbf{y}$ is $O(n^{d-1})$. Then the remaining vectors in A_d comes from rich $\mathbf{y} \prec \tilde{\mathbf{b}}$, and the number of such vectors is at most $2^{|\tilde{\mathbf{b}}|}\binom{n+d}{d}$. Consequently we get

$$|A| \le 2^{|\tilde{\mathbf{b}}|}\binom{n+d}{d} + O(n^{d-1}).$$

Recall that the reference family is of size $2^p\binom{n+d}{d}$, and $|\tilde{\mathbf{b}}| \le p$ from Claim 2. So we only need to deal with the case when there are exactly 2^p rich sets, in other words, $\tilde{\mathbf{b}} = \tilde{\mathbf{e}}_p$ (by renaming coordinates if necessary). We show that $A \subset \mathcal{D}(\mathcal{U}(\tilde{\mathbf{e}}_p, d))$. Suppose the contrary, then there is $\mathbf{a} \in A$ such that $|\mathbf{a} \setminus \tilde{\mathbf{e}}_p| \ge d+1$. Since $\tilde{\mathbf{e}}_p$ is rich there are pairwise disjoint vectors $\mathbf{c}_1, \dots, \mathbf{c}_{r-1}$ of weight d, outside $\mathbf{b}$. Let $\mathbf{a}_i := \tilde{\mathbf{e}}_p \vee \mathbf{c}_i \in A_d$. Then we get

$$|\mathbf{a} \vee (\mathbf{a}_1 \vee \cdots \vee \mathbf{a}_{r-1})| \ge (d+1) + p + (r-1)d = dr + p + 1 = s + 1,$$

which contradicts that A is r-wise s-union. This completes the proof of Theorem 1.

3. The polytope P and proof of Theorem 2

We introduce a convex polytope $\mathbf{P} \subset \mathbb{R}^n$, which will play a key role in our proof. This polytope is defined by the following $n + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-r+1}$ inequalities:

$$x_i \ge 0 \qquad \text{if } 1 \le i \le n, \tag{5}$$

$$\sum_{i\in I} x_i \le \sum_{i \in I} a_i + d \qquad \text{if } 1 \le |I| \le n-r+1,\ I \subset [n]. \tag{6}$$

Namely,

$$\mathbf{P} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ satisfies (5) and (6)}\}.$$

Let L denotes the integer lattice points in $\mathbf{P}$:

$$L = L(r, n, \mathbf{a}, d) := \{\mathbf{x} \in \mathbb{N}^n : \mathbf{x} \in \mathbf{P}\}.$$

Lemma 1. *The two sets K and L are the same, and r-wise s-union.*

Proof. This lemma is a consequence of the following three claims.

Claim 3. *The set K is r-wise s-union.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r \in K$. We show that $|\mathbf{x}_1 \vee \mathbf{x}_2 \vee \cdots \vee \mathbf{x}_r| \le s$. We may assume that $\mathbf{x}_j \in \mathcal{U}(\mathbf{a} + i_j\mathbf{1}, d - ui_j)$, where $u = n - r + 1$. We may also assume that $i_1 \ge i_2 \ge \cdots \ge i_r$. Let $\mathbf{b} := \mathbf{a} + i_1\mathbf{1}$. Then, informally, $|\mathbf{b} \vee \mathbf{x} - \mathbf{b}|$ counts the excess of $\mathbf{x}$ above $\mathbf{b}$, more precisely, it is $\sum_{j\in[n]}\max\{0,x_j-b_j\}$. Thus we have

$$\begin{aligned}
|\mathbf{x}_1\vee\mathbf{x}_2\vee\cdots\vee\mathbf{x}_r| &\le |\mathbf{b}|+\sum_{j=1}^{r}|\mathbf{b}\vee\mathbf{x}_j-\mathbf{b}|\\
&\le |\mathbf{a}|+ni_1+\sum_{j=1}^{r}\big((d-ui_j)-(i_1-i_j)\big)\\
&= a+dr+(n-r)i_1-\sum_{j=1}^{r}(u-1)i_j\\
&= s-\sum_{j=2}^{r}j_j\le s,
\end{aligned}$$

as required. □

Claim 4. $K\subset L$.

Proof. Let $\mathbf{x}\in K$. We show that $\mathbf{x}\in L$, that is, $\mathbf{x}$ satisfies (5) and (6). Since (5) is clear by definition of K, we show that (6). To this end we may assume that $\mathbf{x}\in\mathcal{U}(\mathbf{a}+i\mathbf{1},d-ui)$, where $u=n-r+1$ and $i\le\lfloor\frac{d}{u}\rfloor$. Let $I\subset[n]$ with $1\le|I|\le u$. Then $i|I|\le ui$. Thus it follows

$$\sum_{j\in I}x_j\le\sum_{j\in I}a_j+i|I|+(d-ui)\le\sum_{j\in I}a_j+d,$$

which confirms (6). □

Claim 5. $K\supset L$.

Proof. Let $\mathbf{x}\in L$. We show that $\mathbf{x}\in K$, that is, there exists some i' such that $0\le i'\le\lfloor\frac{d}{n-r+1}\rfloor$ and

$$|\mathbf{x}\setminus(\mathbf{a}+i'\mathbf{1})|\le d-(n-r+1)i'.$$

We write $\mathbf{x}$ as

$$\mathbf{x}=(a_1+i_1,a_2+i_2,\dots,a_n+i_n),$$

where we may assume that $d\ge i_1\ge i_2\ge\cdots\ge i_n$. We notice that some i_j can be negative. Since $\mathbf{x}\in L$ it follows from (6) (a part of the definition of L) that if $1\le|I|\le n-r+1$ and $I\subset[n]$, then

$$\sum_{j\in I}i_j\le d.$$

Let $J:=\{j:x_j\ge a_j\}$ and we argue separately by the size of $|J|$.

If $|J|\le n-r+1$, then we may choose $i'=0$. In fact,

$$\begin{aligned}
|\mathbf{x}\setminus\mathbf{a}| &= \max\{0,i_1\}+\max\{0,i_2\}+\cdots+\max\{0,i_{n-r+1}\}\\
&= \max\left\{\sum_{j\in I}i_j : I\subset 2^{[n-r+1]}\right\}\le d.
\end{aligned}$$

If $|J| \geq n-r+2$, then we may choose $i' = i_{n-r+2}$. In fact, by letting $i' := i_{n-r+2}$, we have

$$\begin{aligned}|\mathbf{x} \setminus (\mathbf{a}+i'\mathbf{1})| &= (i_1 - i') + (i_2 - i') + \cdots + (i_{n-r+1} - i')\\ &\leq d - (n-r+1)i'.\end{aligned}$$

We need to check $0 \leq i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. It follows from $|J| \geq n-r+2$ that $i' \geq 0$. Also $d \geq i_1 \geq i_2 \geq \cdots \geq i_{n-r+2}$ and $i_1 + i_2 + \cdots + i_{n+r-1} \leq d$ yield $i' \leq \lfloor \frac{d}{n-r+1} \rfloor$. □

This completes the proof of Lemma 1. □

Let

$$\sigma_k(\mathbf{a}) := \sum_{K \in \binom{[n]}{k}} \prod_{i \in K} a_i$$

be the kth elementary symmetric polynomial of $a_1, \ldots, a_n$.

Lemma 2. *The size of $K(r,n,\mathbf{a},d)$ is given by*

$$\begin{aligned}|K(r,n,\mathbf{a},d)| &= \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(\mathbf{a})\\ &\quad + \sum_{i=1}^{\lfloor \frac{d}{u} \rfloor} \sum_{j=u+1}^{n} \left(\binom{d-ui+j}{j} - \binom{d-ui+u}{j} \right) \sigma_{n-j}(\mathbf{a}+i\mathbf{1}),\end{aligned}$$

where $u = n-r+1$. Moreover, for fixed n, r, d and $|\mathbf{a}|$, this size is maximized if and only if $\mathbf{a}$ is an equitable partition.

Proof. For $J \subset [n]$ let $\mathbf{x}|_J$ be the restriction of $\mathbf{x}$ to J, that is, $(\mathbf{x}|_J)_i$ is a_i if $i \in J$ and 0 otherwise.

First we count the vectors in the base layer $\mathcal{D}(\mathcal{U}(\mathbf{a},d))$. To this end we partition this set into $\bigsqcup_{J \subset [n]} A_0(J)$, where

$$A_0(J) = \{\mathbf{a}|_J + \mathbf{e} + \mathbf{b} : \mathrm{supp}(\mathbf{e}) \subset J,\ |\mathbf{e}| \leq d,\ \mathrm{supp}(\mathbf{b}) \subset [n] \setminus J,\ b_i < a_i \text{ for } i \notin J\}.$$

The number of vectors $\mathbf{e}$ with the above property is equal to the number of non-negative integer solutions of the inequality $x_1 + x_2 + \cdots + x_{|J|} \leq d$, which is $\binom{d+|J|}{|J|}$. The number of vectors $\mathbf{b}$ is clearly $\prod_{l \in [n] \setminus J} a_l$. Thus we get

$$\sum_{J \in \binom{[n]}{j}} |A_0(J)| = \sum_{J \in \binom{[n]}{j}} \binom{d+|J|}{|J|} \prod_{l \in [n] \setminus J} a_l = \binom{d+j}{j} \sigma_{n-j}(\mathbf{a}),$$

and $|\mathcal{D}(\mathcal{U}(\mathbf{a},d))| = \sum_{j=0}^{n} \binom{d+j}{j} \sigma_{n-j}(\mathbf{a})$.

Next we count the vectors in the ith layer:

$$\mathcal{D}(\mathcal{U}(\mathbf{a}+i\mathbf{1}, d-ui)) \setminus \left(\bigcup_{j=0}^{i-1} \mathcal{D}(\mathcal{U}(\mathbf{a}+j\mathbf{1}, d-uj)) \right).$$

For this we partition the above set into $\bigsqcup_{J\subset[n]} A_i(J)$, where

$$A_i(J) = \{(\mathbf{a}+i\mathbf{1})|_J + \mathbf{e} + \mathbf{b} : \mathrm{supp}(\mathbf{e}) \subset J,\ d-u(i-1)-|J| < |\mathbf{e}| \le d-ui,$$
$$\mathrm{supp}(\mathbf{b}) \subset [n]\setminus J,\ b_l < a_l + i \text{ for } l \notin J\}.$$

In this case we need $d-u(i-1) < |J|+|\mathbf{e}|$ because the vectors satisfying the opposite inequality are already counted in the lower layers $\bigcup_{j<i} A_j(J)$. We also notice that $d-u(i-1)-|J| < d-ui$ implies that $|J| > u$. So $A_i(J) = \emptyset$ for $|J| \le u$. Now we count the number of vectors $\mathbf{e}$ in $A_i(J)$, or equivalently, the number of non-negative integer solutions of

$$d-u(i-1)-|J| < x_1 + x_2 + \cdots + x_{|J|} \le d-ui.$$

This number is $\binom{d-ui+j}{j} - \binom{d-ui+u}{j}$, where $j = |J|$. On the other hand, the number of vectors $\mathbf{b}$ in $A_i(J)$ is $\prod_{l\in[n]\setminus J}(a_l+i)$. Consequently we get

$$\sum_{J\subset[n]} |A_i(J)| = \sum_{j=u+1}^{n} \left(\binom{d-ui+j}{j} - \binom{d-ui+u}{j} \right) \sigma_{n-j}(\mathbf{a}+i\mathbf{1}).$$

Summing this term over $1 \le i \le \lfloor \frac{d}{u} \rfloor$ we finally obtain the second term of the RHS of $|K|$ in the statement of this lemma. Then, for fixed $|\mathbf{a}|$, the size of K is maximized when $\sigma_{n-1}(\mathbf{a})$ and $\sigma_{n-1}(\mathbf{a}+i\mathbf{1})$ are maximized. By the property of symmetric polynomials, this happens if and only if $\mathbf{a}$ is an equitable partition. □

Proof of Theorem 2. Let $A \subset \mathbb{N}^n$ be an r-wise s-union with (1). For $I \subset [n]$ let

$$m_I := \max\left\{ \sum_{i\in I} x_i : \mathbf{x} \in A \right\}.$$

Claim 6. *If $I \subset [n]$ and $1 \le |I| \le n-r+1$, then*

$$m_I = \sum_{i\in I} a_i + d.$$

Proof. Choose $j \in I$. By (1) we have $P_j \in A$ and

$$m_I \ge \sum_{i\in I} (P_j)_i = \sum_{i\in I} a_i + d. \tag{7}$$

We need to show that this inequality is actually an equality. Let $[n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$ be a partition of $[n]$. Then it follows that

$$s \ge m_{I_1} + m_{I_2} + \cdots + m_{I_r} \ge \sum_{i\in[n]} a_i + rd = s,$$

where the first inequality follows from the r-wise s-union property of A, and the second inequality follows from (7). Since the left-most and the right-most sides are the same s, we see that all inequalities are equalities. This means that (7) is equality, as needed. □

By this claim if $\mathbf{x} \in A$ and $1 \leq |I| \leq n - r + 1$, then we have

$$\sum_{i \in I} x_i \leq m_I = \sum_{i \in I} a_i + d.$$

This means that $A \subset L$. Finally the theorem follows from Lemmas 1 and 2. $\square$

Proof of Corollary. Let $n = r + 1$ and we show that (1) is satisfied. Let $A \subset \mathbb{N}^{r+1}$ be r-wise s-union with maximum size.

We first check that P_i's are well-defined. For this, we need (i) $(n-r)|(|\mathbf{m}| - s)$, and (ii) $a_i \geq 0$ for all i. Since $n - r = 1$ we have (i). To verify (ii) we may assume that $m_1 \geq m_2 \geq \cdots \geq m_{r+1}$. Then $a_i \geq a_{r+1} = m_{r+1} - d$, so it suffices to show $m_{r+1} \geq d$. Since A is r-wise s-union it follows that $m_1 + m_2 + \cdots + m_r \leq s$. This together with the definition of d implies $d = |\mathbf{m}| - s \leq m_{r+1}$, as needed.

Next we check that $\mathbf{x} \in A$ satisfies (5) and (6). By definition we have $x_i \leq m_i = a_i + d$, so we have (5). Since A is r-wise s-union, we have

$$(x_1 + x_2) + m_3 + \cdots + m_{r+1} \leq s,$$

or equivalently,

$$(x_1 + x_2) + (a_3 + d) + \cdots + (a_{r+1} + d) \leq s = |\mathbf{a}| + rd.$$

Rearranging we get $x_1 + x_2 \leq a_1 + a_2 + d$, and we get the other cases similarly, so we obtain (6). Thus $A \subset L$. But by the maximality of $|A|$ we have $A = L$. Now noting that every P_i satisfies (5) and (6), namely, P_i is in L, and thus (1) is satisfied. $\square$

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