The Price of Connectivity for Cycle Transversals[⋆]

Tatiana R. Hartinger[1], Matthew Johnson[2],
Martin Milanič[1], and Daniël Paulusma[2]

[1] University of Primorska, UP IAM and UP FAMNIT, Koper, Slovenia
tatiana.hartinger@iam.upr.si,martin.milanic@upr.si
[2] School of Engineering and Computing Sciences, Durham University, UK
{matthew.johnson2,daniel.paulusma}@durham.ac.uk

Abstract. For a family of graphs $\mathcal{F}$, an $\mathcal{F}$-transversal of a graph G is a subset $S \subseteq V(G)$ that intersects every subset of $V(G)$ that induces a subgraph isomorphic to a graph in $\mathcal{F}$. Let $t_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G, and $ct_{\mathcal{F}}(G)$ be the minimum size of an $\mathcal{F}$-transversal of G that induces a connected graph. For a class of connected graphs $\mathcal{G}$, we say that the price of connectivity of $\mathcal{F}$-transversals is multiplicative if, for all $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant, and additive if $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant. The price of connectivity is identical if $t_{\mathcal{F}}(G)$ and $ct_{\mathcal{F}}(G)$ are always equal and unbounded if $ct_{\mathcal{F}}(G)$ cannot be bounded in terms of $t_{\mathcal{F}}(G)$. We study classes of graphs characterized by one forbidden induced subgraph H and $\mathcal{F}$-transversals where $\mathcal{F}$ contains an infinite number of cycles and, possibly, also one or more anticycles or short paths. We determine exactly those classes of connected H-free graphs where the price of connectivity of these $\mathcal{F}$-transversals is unbounded, multiplicative, additive, or identical. In particular, our tetrachotomies extend known results for the case when $\mathcal{F}$ is the family of all cycles.

⋆ The research in this paper was supported by a London Mathematical Society Scheme 4 Grant and by EPSRC Grant EP/K025090/1, and, in part, by the Slovenian Research Agency (I0-0035, research programs P1-0285, research projects N1-0032, J1-5433, J1-6720, J1-6743, J1-7051, and a Young Researchers Grant). An extended abstract of the paper appeared in the proceedings of MFCS 2015 [19].

1 Introduction

Let $\mathcal{F}$ be a family of graphs. A graph is *$\mathcal{F}$-free* if it contains no induced subgraph isomorphic to some graph in $\mathcal{F}$ (if $\mathcal{F} = \{F\}$ for some graph F then we write F-free instead). An *$\mathcal{F}$-transversal* of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that $G - S$ is $\mathcal{F}$-free; that is, S intersects every subset of V that induces a subgraph isomorphic to a graph in $\mathcal{F}$. In certain cases, $\mathcal{F}$-transversals are well studied. For example, a *vertex cover* is a $\{P_2\}$-transversal (here, P_k is the path on k vertices). Note that, for any $\{P_2\}$-transversal S of a graph G, the graph $G - S$ is an independent set. To give another example, a *feedback vertex set* is an $\mathcal{F}$-transversal for the infinite family $\mathcal{F} = \{C_3, C_4, C_5, \ldots\}$ (where C_k is the cycle on k vertices). In this case, for any $\mathcal{F}$-transversal S of a graph G, the graph $G - S$ is a forest. As the examples suggest, it is natural to study minimum size $\mathcal{F}$-transversals.

We can put an additional constraint on an $\mathcal{F}$-transversal S of a connected graph G by requiring that the subgraph of G induced by S is connected. Minimum size *connected* $\mathcal{F}$-transversals of a graph have also been investigated. In particular, minimum size connected vertex covers are well studied (see, for example, [4, 6, 8, 11, 14, 17, 21, 23]) and minimum size connected feedback vertex sets have also received attention (see, for example, [2, 10, 18, 20, 22]). We study the following question:

> *What is the effect of adding the connectivity constraint on the minimum size of an $\mathcal{F}$-transversal for a graph family $\mathcal{F}$?*

We first give two definitions: for a connected graph G, let $t_{\mathcal{F}}(G)$ denote the minimum size of an $\mathcal{F}$-transversal of G, and let $ct_{\mathcal{F}}(G)$ denote the minimum size of a connected $\mathcal{F}$-transversal of G. So our aim is to find relationships between $ct_{\mathcal{F}}(G)$ and $t_{\mathcal{F}}(G)$; more particularly, we ask for a class of connected graphs $\mathcal{G}$, whether we can find a bound for $ct_{\mathcal{F}}(G)$ in terms of $t_{\mathcal{F}}(G)$ that holds for all $G \in \mathcal{G}$.

We briefly survey existing work starting with a number of results on vertex cover, that is, for $\mathcal{F} = \{P_2\}$. Cardinal and Levy [8] proved that for every $\epsilon > 0$ there is a multiplicative bound of $2/(1 + \epsilon) + o(1)$ in the class of connected n-vertex graphs with average degree at least ϵn; that is, $ct_{\mathcal{F}}(G) \leq (2/(1 + \epsilon) + o(1))t_{\mathcal{F}}(G)$ for such graphs G. Camby et al. [6] proved that for the class of all connected graphs, there is a multiplicative bound of 2 and that this bound is asymptotically sharp for paths and cycles. They also gave forbidden induced subgraph characterizations of classes of graphs such that for every connected induced subgraph there is a multiplicative bound of t, for each $t \in \{1, 4/3, 3/2\}$.

Belmonte et al. [2, 3] studied feedback vertex sets, that is, $\mathcal{F}$-transversals where $\mathcal{F} = \{C_3, C_4, C_5 \ldots\}$. They determined all finite families of graphs $\mathcal{H}$ such that for all connected graphs G in the class of $\mathcal{H}$-free graphs, $ct_{\mathcal{F}}(G)/t_{\mathcal{F}}(G)$ is bounded by a constant [3]. They also determined exactly those graphs classes $\mathcal{G}$ of H-free graphs for which, for all connected $G \in \mathcal{G}$, $ct_{\mathcal{F}}(G) - t_{\mathcal{F}}(G)$ is bounded by a constant (and they found exactly when that constant is zero) [2].

We also give two other examples of graph properties where the effect of requiring connectivity has been studied. A result of Duchet and Meyniel [13] implies that for all connected graphs the minimum size of a connected dominating set is at most 3 times the size of a minimum size dominating set. A result of Zverovich [24] implies that for connected (P_5, C_5)-free graphs this bound is exactly 1. Camby and Schaudt [7] showed that the equivalent multiplicative bound for connected (P_8, C_8)-free graphs is 2 and for connected (P_9, C_9)-free graphs it is 3; both bounds were shown to be sharp. They also proved that the problem of deciding whether, for a given class of graphs this bound is at most r is $P^{\mathrm{NP}[\log]}$-complete for every fixed rational r with $1 < r < 3$. The same authors also found an example of an additive bound: they proved that for every connected (P_6, C_6)-free graph, a minimum size connected dominating set contains at most one more vertex than a minimum size dominating set. Grigoriev and Sitters [18] proved that for connected planar graphs of minimum degree at least 3, a minimum size connected face hitting set is at most 11 times larger than a minimum size face hitting set. Schweitzer and Schweitzer [22] reduced this bound to 5 and proved tightness.

In this paper we consider a number of families $\mathcal{F}$ that contain cycles, paths and complements of cycles. We study $\mathcal{F}$-transversals for graph classes characterized by one forbidden induced subgraph and ask whether the size of a minimum size *connected* $\mathcal{F}$-transversal can be bounded in terms of the size of a minimum size $\mathcal{F}$-transversal. Before we can present our results we need to introduce some additional terminology and notation.

1.1 Terminology

We start by giving the following definition.

Definition 1. *Let H be a graph and let $\mathcal{G}$ be the class of connected H-free graphs. Let $\mathcal{F}$ be a family of graphs. We say that $\mathcal{G}$ is:*

(a) $\mathcal{F}$-unbounded *if for every function $f : \mathbb{N} \to \mathbb{N}$ there exists a graph $G \in \mathcal{G}$ such that $ct_{\mathcal{F}}(G) > f(t_{\mathcal{F}}(G))$;*
(b) $\mathcal{F}$-multiplicative *if $ct_{\mathcal{F}}(G) \leq c_H t_{\mathcal{F}}(G)$ for some constant c_H and for every $G \in \mathcal{G}$;*
(c) $\mathcal{F}$-additive *if $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G) + d_H$ for some constant d_H and for every $G \in \mathcal{G}$; and*
(d) $\mathcal{F}$-identical *if $ct_{\mathcal{F}}(G) = t_{\mathcal{F}}(G)$ for every $G \in \mathcal{G}$.*

If a graph class $\mathcal{G}$ is $\mathcal{F}$-unbounded, $\mathcal{F}$-multiplicative, $\mathcal{F}$-additive or $\mathcal{F}$-identical, respectively, for a family of graphs $\mathcal{F}$, then we say that the *price of connectivity* of $\mathcal{F}$-transversals for $\mathcal{G}$ is *unbounded*, *multiplicative*, *additive*, or *identical*, respectively. Note that this definition can also be introduced for graph properties other than $\mathcal{F}$-transversals. We note that our definition is a refinement of the term *price of connectivity* as it was used when first introduced by Cardinal and Levy [8] in their study of vertex cover. They were concerned only with multiplicative bounds.

For graphs F and G, we write $F \subseteq_i G$ to denote that F is an induced subgraph of G. We let C_n, K_n and P_n denote the cycle, complete graph, and path on n vertices, respectively. The *disjoint union* of two vertex-disjoint graphs G and H is the graph $G+H$ that has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ where $V(G) \cap V(H) = \emptyset$. We denote the disjoint union of r copies of G by rG. A graph is a *linear forest* if it is the disjoint union of a set of paths.

The *complement* $\overline{G}$ of a graph G has the same vertex set as G and an edge between two distinct vertices if and only if these vertices are not adjacent in G. A *hole* is a cycle of length at least 4. An *antihole* is the complement of a hole. A cycle, hole or antihole is *even* if it contains an even number of vertices; otherwise it is *odd*. A hole is *long* if it is of length at least 5, and a *long antihole* is the complement of a long hole.

A graph is *odd-hole-free* or *odd-antihole-free* if it contains no induced odd holes or no induced odd antiholes, respectively. An *even-hole-free* graph is defined similarly. A graph is *chordal* if it has no induced hole, that is, if it has no induced cycles of length at least 4. A graph is *weakly chordal* if it has no induced long hole and no induced long antihole. A graph is *perfect* if the chromatic number of every induced subgraph equals the size of a largest clique in that subgraph. By the Strong Perfect Graph Theorem [9], a graph is perfect if and only if it is odd-hole-free and odd-antihole-free. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set. Split graphs coincide with the $(C_4, C_5, 2P_2)$-free graphs [16]. A graph is *threshold* if it is $(C_4, 2P_2, P_4)$-free, *trivially perfect* if it is (C_4, P_4)-free, *cotrivially perfect* if it is $(2P_2, P_4)$-free and a *cograph* if it is P_4-free.

1.2 Our Results

Table 1 summarizes our results together with related previous work. Results can be seen both according to the family $\mathcal{F}$ and the corresponding property of the graph $G - S$, where S is an $\mathcal{F}$-transversal of G. We note that when $\mathcal{F}$ is the family of even cycles or of holes there is an open case. In all other cases, the stated conditions in Table 1 are both necessary and sufficient for $\mathcal{F}$-multiplicativity ($\mathcal{F}$-boundedness), $\mathcal{F}$-additivity, and $\mathcal{F}$-identity, respectively, in the class of connected H-free graphs.

$\mathcal{F}$	Property of $G - S$	Condition for $\mathcal{F}$-multiplicativity (for $\mathcal{F}$-boundedness)	Condition for $\mathcal{F}$-additivity	Condition for $\mathcal{F}$-identity
cycles	forest	H is a linear forest [2]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ [2]	$H \subseteq_i P_3$ [2]
odd cycles	bipartite	H is a linear forest	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
even cycles† (equiv.: even holes)	even-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1{}^\dagger$	$H \subseteq_i P_3$
holes†	chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1{}^\dagger$	$H \subseteq_i P_3$
odd holes	odd-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
odd holes and odd antiholes	perfect	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes	long-hole-free	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
long holes and long antiholes	weakly chordal	H is a linear forest	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$
cycles and P_2 (equiv.: $\{P_2\}$)	edgeless	no restriction [6]	$H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$	$H \subseteq_i P_3$
holes and $2P_2$ (equiv.: $\{C_4, C_5, 2P_2\}$)	split	no restriction	$H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$	$H \subseteq_i P_3$
holes and $2P_2, P_4$ (equiv.: $\{C_4, 2P_2, P_4\}$)	threshold	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
holes and P_4 (equiv.: $\{C_4, P_4\}$)	trivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$
long holes and $2P_2$ (equiv.: $\{C_5, 2P_2\}$)	$(C_5, 2P_2)$-free	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ $H \subseteq_i P_2 + P_1$
long holes and $2P_2, P_4$ (equiv.: $\{2P_2, P_4\}$)	cotrivially perfect	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$
long holes and P_4 (equiv.: $\{P_4\}$)	cograph	no restriction	$H \subseteq_i P_4 + sP_1$	$H \subseteq_i P_4$

Table 1. Conditions on the graph H for the price of connectivity of $\mathcal{F}$-transversal for the class of H-free graphs to be multiplicative, additive or identical, respectively, when $\mathcal{F}$ is a family of graphs that contains the specified infinite family of cycles and possibly some other small graphs. The results on cycles in the first row are due to Belmonte et al. [2] and the multiplicativity result on cycles and P_2 in the ninth row is due to Camby et al. [6]. All other results are new and presented in this paper. All conditions are *necessary* and *sufficient* except for even cycles and holes, as in these two cases (marked by a † in the table) we do not know if H-free graphs are $\mathcal{F}$-additive for $H \subseteq_i P_3 + P_2 + sP_1$.

From Table 1 we can draw a number of conclusions. If a transversal that intersects (small) paths is wanted, we obtain multiplicative bounds for any class of H-free graphs. In all other cases, H may not contain a cycle or a claw (so is a linear forest). We also see that when we add a requirement that all triangles are intersected, there is always a jump from $H = P_4 + sP_1$ to $H = P_5 + sP_1$ for the additive bound. In general, it can be noticed that adding small graphs to $\mathcal{F}$ has differing effects. We say that a family of graphs $\mathcal{F}$ or a graph F *positively (negatively) influences* a family of graphs $\mathcal{F}'$ if the row in the table for their union contains more (fewer) bounded cases than the row for $\mathcal{F}'$. So, for example, $2P_2$ does not influence $\{C_4, C_5, C_6, \ldots\} \cup \{P_4\}$, and P_4 does not influence the family of long holes. Moreover, odd holes do not influence even holes, whereas even holes influence odd holes positively.

In the remainder of our paper, after presenting some known and new basic results in Section 2, we present a number of general theorems, from which the results in Table 1 directly follow. We emphasize that all proofs of these theorems are algorithmic in nature, that is, they can be translated directly into polynomial-time algorithms that modify an $\mathcal{F}$-transversal into a connected $\mathcal{F}$-transversal of appropriate cardinality.

We provide a brief guide to the proof of Table 1. Theorem 2 implies the second row. Theorem 3 implies the third and fourth row, and Theorem 4 implies the next four rows. The ninth row follows from Theorem 5 and the tenth from Theorem 6. Theorem 7 implies the eleventh and twelfth rows. The final three rows follow from Theorems 8, 9 and 10, respectively.

2 Initial Results

In this section we present a number of known results, along with some new ones, that we need as lemmas in order to prove our results. We also state some more terminology. Throughout the paper we consider finite undirected graphs with no multiple edges and no self-loops. We refer to the textbook of Diestel [12] for any undefined terms.

Let $G = (V, E)$ be a connected graph. For a subset $S \subseteq V$, we let $G[S]$ denote the subgraph of G induced by S (that is, the graph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$). Two vertex-disjoint subgraphs (or vertex subsets) F_1 and F_2 of a graph G are *adjacent* if there is at least one edge in G between a vertex in F_1 and a vertex in F_2. Similarly, a vertex u not in F_1 is *adjacent* to F_1 if $\{u\}$ and F_1 are adjacent. A set $D \subseteq V$ *dominates* G if every vertex $u \in V \setminus D$ is adjacent to D. We also say that $G[D]$ dominates G. If $D = \{u, v\}$ for two adjacent vertices u, v, then uv is called a *dominating edge* of G. A set $D \subseteq V$ *dominates* a set $S \subseteq V \setminus D$ if every vertex in S is adjacent to D.

2.1 Some Structural Results

We give four structural results (three known ones and one observation). The first result is well known (see, for example, [5]).

Lemma 1. *Every connected P_4-free graph on two or more vertices has a dominating edge.*

We will need the following result of Bacsó and Tuza [1] for the class of connected P_5-free graphs.

Lemma 2 (Bacsó and Tuza [1]). *Every connected P_5-free graph has a dominating P_3 or a dominating clique.*

We also need a lemma due to Duchet and Meyniel [13].

Lemma 3 (Duchet and Meyniel [13]). *Let G be a connected graph. Let β be the size of a minimum dominating set of G. Then G has a connected dominating set of size at most $3\beta - 2$.*

The *distance* between two vertices u and v in a graph G is the length of a shortest path between them. The maximum distance in G is called the *diameter* of G.

Lemma 4. *Let G be a connected graph with diameter d. Let A be a subgraph of G consisting of $r \geq 1$ components. Then G has a connected subgraph A' that contains A and that has less than $|V(A)| + (r-1)d$ vertices.*

Proof. Let the components of A be $D_1, \ldots, D_r$. We need to add less than d vertices to A in order to connect D_1 to each other D_i ($i \neq 1$). The resulting graph A' has size less than $|V(A)| + (r-1)d$. $\square$

For later use, we also state and prove five observations on linear forests.

Lemma 5. *Let H be a linear forest. Then, the following five statements hold:*

(i) If $H \not\subseteq_i P_4$, then $2P_2 \subseteq_i H$ or $3P_1 \subseteq_i H$.
(ii) If $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$, then $2P_2 \subseteq_i H$.
(iii) If $H \not\subseteq_i P_4 + sP_1$ and $H \not\subseteq_i P_3 + P_2 + sP_1$ for any $s \geq 0$, then $P_5 \subseteq_i H$, $P_4 + P_2 \subseteq_i H$, $2P_3 \subseteq_i H$, or $3P_2 \subseteq_i H$.
(iv) If $H \not\subseteq_i P_4 + sP_1$ and $H \not\subseteq_i P_3 + sP_2$ for any $s \geq 0$, then $P_5 \subseteq_i H$, $P_4 + P_2 \subseteq_i H$, or $2P_3 \subseteq_i H$.
(v) If $H \not\subseteq_i P_5 + sP_1$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$, then $P_6 \subseteq_i H$ or $P_4 + P_2 \subseteq_i H$.

Proof. Let P be a longest path in H.

(i) If $|V(P)| \geq 5$, then $2P_2 \subseteq_i P$ and thus $2P_2 \subseteq_i H$. Hence we may assume that $|V(P)| \in \{1, 2, 3, 4\}$. First suppose that $|V(P)| = 1$. Then every component of H consists of exactly one vertex. As $H \not\subseteq_i P_4$, this means that H has at least three components. Therefore, $3P_1 \subseteq_i H$. Now suppose that $|V(P)| = 2$. If H has only one further component and this is isomorphic to P_1, then $H \subseteq_i P_4$, a contradiction. Hence H contains either at least two further components, in which case $3P_1 \subseteq_i H$, or one further component which is isomorphic to P_2, in which case $2P_2 \subseteq_i H$. Finally, if $|V(P)| \in \{3, 4\}$, then H must have at least one more component (else $H \subseteq_i P_4$), and thus $3P_1 \subseteq_i H$.
(ii) Note first that $|V(P)| > 1$, or otherwise $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$. If $|V(P)| \geq 5$, then $2P_2 \subseteq_i H$. Therefore we may assume that $|V(P)| \in \{2, 3, 4\}$. In each case, there must be a component of H with at least two vertices other than P, or we would have $H \subseteq_i P_4 + sP_1$, and thus $2P_2 \subseteq_i H$.
(iii) Since $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$, we infer that $|V(P)| > 1$. If $|V(P)| \geq 5$, then $P_5 \subseteq_i H$. So we may assume that $|V(P)| \in \{2, 3, 4\}$. If $|V(P)| = 2$, there must exist at least two more components in H isomorphic to P_2 (since otherwise $H \subseteq_i P_3 + P_2 + sP_1$ for some $s \geq 0$) and therefore $3P_2 \subseteq_i H$. Suppose now that $|V(P)| = 3$. If, of the other components of H, zero or one is isomorphic to P_2, and the others are each isomorphic to P_1, then $H \subseteq_i P_3 + P_2 + sP_1$ for some $s \geq 0$. Thus, either H contains at least two components isomorphic to P_2, in which case $3P_2 \subseteq_i H$, or it contains at least one more component isomorphic to P_3, in which case $2P_3 \subseteq_i H$. Finally, if $|V(P)| = 4$, then H has a component with at least two vertices other than P, or else $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, and thus $P_4 + P_2 \subseteq_i H$.

(iv) Since $H \not\subseteq_i P_3 + sP_2$ for any $s \geq 0$, we infer that $|V(P)| > 2$. If $|V(P)| \geq 5$, then $P_5 \subseteq_i H$. Therefore we may assume that $|V(P)| \in \{3, 4\}$. Suppose that $|V(P)| = 3$. If all other components of H are isomorphic to P_1 or P_2, then $H \subseteq_i P_3 + sP_2$. Therefore there must exist another component of H isomorphic to P_3, and thus $2P_3 \subseteq_i H$. Now, if $|V(P)| = 4$, H must contain a component with at least two vertices other than P (else $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$) in which case $P_4 + P_2$ is an induced subgraph of H.

(v) Since $H \not\subseteq_i sP_3$ for any $s \geq 0$, we infer that $|V(P)| \geq 4$. If $|V(P)| \geq 6$, then $P_6 \subseteq_i H$. Suppose $|V(P)| \in \{4, 5\}$. Then, since $H \not\subseteq_i P_5 + sP_1$ for any $s \geq 0$, we find that H contains a component with at least two vertices other than P. Therefore $P_4 + P_2 \subseteq_i H$.

□

2.2 Some Results on the Price of Connectivity

We now give five results that are directly related to the concept of price of connectivity and that we will need in our later proofs. All results, except the first one, which follows from Lemma 1, can be found in the papers of Belmonte et al. [2,3] or follow from results in these papers after a straightforward generalization (which we need).

Lemma 6. *For every family $\mathcal{F}$ of graphs, the class of connected P_4-free graphs is $\mathcal{F}$-additive.*

Proof. Let G be a connected P_4-free graph with two or more vertices, with a minimum $\mathcal{F}$-transversal S. By Lemma 1, G has a dominating edge, say uv. So $S \cup \{u, v\}$ is a connected $\mathcal{F}$-transversal of G, implying that $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G) + 2$. Since the above inequality trivially holds for the one-vertex graph, we conclude that the class of connected P_4-free graphs is $\mathcal{F}$-additive, with $d_{P_4} \leq 2$. □

The second result has been proven by Belmonte et al. [2] for the special case when the family $\mathcal{F}$ consists of all cycles.

Lemma 7. *For any family of graphs $\mathcal{F}$ with $K_r \in \mathcal{F}$ for some integer $r \geq 1$, the class of connected P_5-free graphs is $\mathcal{F}$-additive.*

Proof. Let G be a connected P_5-free graph. Let S be a minimum $\mathcal{F}$-transversal of G. By Lemma 2, G has a dominating set D that induces a P_3 or a complete graph. In the first case, $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $|S| + 3$. In the second case, $|D \setminus S| \leq r - 1$. So in this case $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $|S| + r - 1$. □

We also need to generalize a result that was proved by Belmonte et al. [2] for the graph $H = P_5$. The proof for the general case is the same and we state it here for completeness.

Lemma 8. *For a family of graphs $\mathcal{F}$ and a graph H, if the class of connected H-free graphs is $\mathcal{F}$-additive, then so is the class of connected $(H + sP_1)$-free graphs for all $s \geq 1$.*

Proof. Let G be a connected $(H + sP_1)$-free graph for some $s \geq 0$. We prove that $ct_{\mathcal{F}}(G) \leq t_{\mathcal{F}}(G) + d_{H+sP_1}$ for some constant d_{H+sP_1} by induction on s. If $s = 0$ the statement holds by assumption. Now let $s \geq 1$. If G is $(H + (s-1)P_1)$-free, then the statement holds by the induction hypothesis. Suppose G is not $(H+(s-1)P_1)$-free. Let F be an induced subgraph of G isomorphic to $H+(s-1)P_1$. Because G is $(H+sP_1)$-free, F dominates G. By Lemma 3 we find that G has a connected dominating set D

of size at most $3|V(F)|-2$. Let S be a minimum $\mathcal{F}$-transversal of G. Then $S \cup D$ is a connected $\mathcal{F}$-transversal of G of size at most $t_{\mathcal{F}}(G)+3|V(F)|-2$. Hence, we can take $d_{H+sP_1} = 3|V(H)|+3s-5$. □

Belmonte et al. [2] proved that the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive if $\mathcal{F}$ is the class of all cycles. To prove this result they showed that the family $\{L_k : k \geq 1\}$ of connected (P_2+P_4, P_6)-free graphs displayed in Figure 1 is not $\mathcal{F}$-additive. Using the observation made in the caption of Figure 1 leads to the following more general result.

Lemma 9. *For any family of cycles $\mathcal{F}$ with $C_3 \in \mathcal{F}$, the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive.*

As a consequence of Lemma 9, any class of connected graphs that contains all connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive either. More generally, if $\mathcal{G}$ and $\mathcal{G}'$ are two classes of connected graphs such that $\mathcal{G} \subseteq \mathcal{G}'$ and $\mathcal{G}$ is not $\mathcal{F}$-additive, then neither is $\mathcal{G}'$. We will use this fact implicitly throughout the paper.

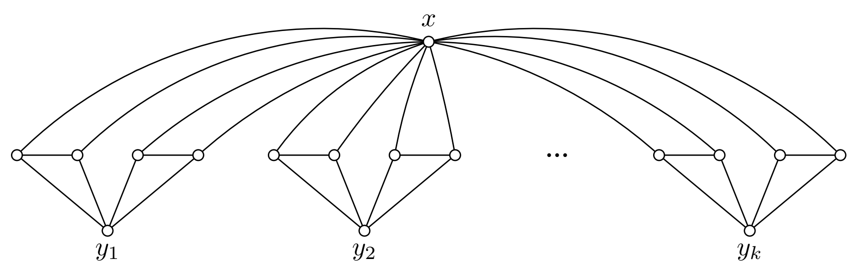


Fig. 1. The graph L_k, defined by Belmonte et al. [2] for every $k \geq 1$; note that $\{y_1, \ldots, y_k, x\}$ is the unique minimum $\mathcal{F}$-transversal whenever $\mathcal{F}$ is any family of cycles with $C_3 \in \mathcal{F}$ and that any minimum connected $\mathcal{F}$-transversal has size $2k+1$.

Finally, the following technical lemma of Belmonte et al. [2] will also be useful for proving our results.

Lemma 10 (Belmonte et al. [2]). *Let $s \geq 1$ be an integer and let G be a connected sP_3-free graph with a subset $S \subseteq V(G)$ and an independent set $U \subseteq V(G) \setminus S$. If there exists a component Z of $G[S]$ that contains an induced copy of $(s-1)P_3$, then there exists a set S' with $S \subseteq S'$ of size at most $|S|+2s-2$ such that*

(i) $G[S']$ has a component Z' containing all vertices of $V(Z) \cup (S' \setminus S)$;
(ii) every vertex of $U' = U \setminus S'$ is adjacent to at most one component of $G[S']$ that is not equal to Z';
(iii) every component of $G[S']$ not equal to Z' is adjacent to at most one vertex of U'.

2.3 A New General Theorem

For $r \geq 1$, $s \geq 1$, the *complete bipartite graph* $K_{r,s}$ is a bipartite graph whose vertex set can be partitioned into two sets of sizes r and s such that there is an edge joining each pair of vertices from distinct sets. The graph $K_{1,3}$ is also called a *claw*.

The following theorem is used in all our tetrachotomies. The third part was shown by Belmonte et al. [2] for the case when $\mathcal{F}$ is the family of all cycles, and our proof for that part is a modification of theirs.

Theorem 1. *Let $\mathcal{F}$ be a family of graphs and let H be a graph. Then, the following three statements hold:*

(i) If $\mathcal{F}$ contains a linear forest, then the class of all connected graphs is $\mathcal{F}$-multiplicative.

(ii) If H is a linear forest, then the class of connected H-free graphs is $\mathcal{F}$-multiplicative.

(iii) If $\mathcal{F}$ contains an infinite number of cycles and no linear forests and H is not a linear forest, then the class of connected H-free graphs is $\mathcal{F}$-unbounded.

Proof. We start with (i). First suppose that $\mathcal{F}$ contains a linear forest F; that is, it is, say, the disjoint union of p paths. Let G be a connected graph, and let S be a minimum $\mathcal{F}$-transversal of G with components $D_1, \ldots, D_r$ for some integer $r \geq 1$. Because G is connected, we can connect the components of S by $r-1$ paths using vertices of $G-S$ only. Let S' be the resulting connected $\mathcal{F}$-transversal. Because $G-S$ is $\mathcal{F}$-free, $G-S$ is F-free. Let q be the length of a longest path in F. As the path $P_{p(q+2)}$ contains F as an induced subgraph and $G-S$ is F-free, $G-S$ is $P_{p(q+2)}$-free. Hence, each of the $r-1$ paths contains less than $p(q+2)$ vertices. Thus we find that $|S'| \leq |S| + rp(q+2) \leq |S| + |S|(p(q+2)) = (p(q+2)+1)|S|$, and we can take $c_{\mathcal{F}} = (p(q+2)+1)$.

Now we prove (ii). Suppose that H is a linear forest; that is, it is, say, the union of k paths, each of length at most ℓ. Let $G = (V, E)$ be a connected H-free graph. Then, as G is H-free, we find that G has diameter less than $k(\ell+2)$. Let $S \subseteq V$ be a minimum $\mathcal{F}$-transversal of G. Let $D_1, D_2, \ldots, D_r$ $(r \geq 1)$ be the components of $G[S]$. In order to make S connected we need to add less than $(r-1)k(\ell+2) \leq (|S|-1)k(\ell+2)$ vertices by Lemma 4. Hence we can take $c_H = k(\ell+2)$.

Finally, we prove (iii). Suppose that $\mathcal{F}$ contains an infinite number of cycles and no linear forests and that H is not a linear forest.

Let p' be an integer greater than the maximum length of a cycle in H; if H has no cycle, let $p' = 5$. Let p be an integer such that $p \geq p'$ and $C_p \in \mathcal{F}$ (such an integer p exists because $\mathcal{F}$ contains infinitely many cycles).

First suppose that H is C_3-free. We construct the following graph. Take two cycles $C = u_1 \cdots u_{p+1} u_1$ and $C' = u'_1 \cdots u'_{p+1} u'_1$. Connect u_1 and u'_1 via a path $u_1 v_1 \ldots v_k u'_1$ for some $k \geq 1$. Add the edges $u_2 u_{p+1}$ and $u'_2 u'_{p+1}$. Denote the resulting graph by G_k; see Figure 2 for an example. Note that G_k is connected and $K_{1,3}$-free and that it has four induced cycles, two of which have length p and two of which have length 3.

As H is not a linear forest, H either contains an induced $K_{1,3}$ or an induced cycle, which has length between 4 and $p-1$ by our choice of p and our assumption that H is C_3-free. Hence, every G_k is H-free. Let $S = \{u_2, u'_2\}$. As $G_k - S$ is a path and $\mathcal{F}$ contains no linear forests, S is an $\mathcal{F}$-transversal. Because G_k has two induced copies of C_p at distance more than k and $C_p \in \mathcal{F}$, the family $\{G_k\}$ is $\mathcal{F}$-unbounded.

Now suppose that H contains an induced C_3. Take two cycles $C = u_1 \cdots u_p u_1$ and $C' = u'_1 \cdots u'_p u'_1$. Connect u_1 and u'_1 via a path $u_1 v_1 \ldots v_k u'_1$ for some $k \geq 1$. The resulting graph G^*_k is connected and H-free, as it is C_3-free. We repeat the above arguments and find that the family $\{G^*_k\}$ is $\mathcal{F}$-unbounded. $\square$

Parts *(ii)* and *(iii)* of Theorem 1 imply the following.

Corollary 1. *For any graph H and for any family of graphs $\mathcal{F}$ containing an infinite number of cycles and no linear forests, the class of connected H-free graphs is $\mathcal{F}$-multiplicative if and only if H is a linear forest.*

3 Cycle Families with Odd Cycles

In this section we assume we are given a family $\mathcal{F}$ of graphs that contains all odd cycles, although we will show more general results whenever possible. We start with

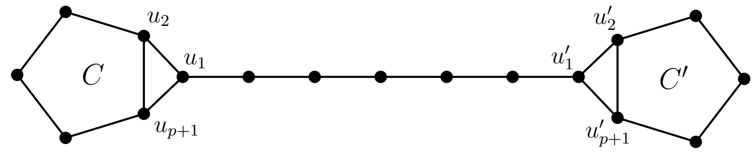


Fig. 2. An example of the construction in the proof of Theorem 1 (iii) in the case when H is C_3-free, only contains cycles of length at most 4 and $C_5 \in \mathcal{F}$.

the following lemma, which generalizes the corresponding result of Belmonte et al. [2] when $\mathcal{F}$ is the family of all cycles. We use a similar approach as used in their proof but our arguments (which are based on bipartiteness instead of cycle-freeness) are different and this proof demonstrates some techniques used several times in obtaining our results.

Lemma 11. *For any family of graphs $\mathcal{F}$ containing either all odd cycles or P_2 and for any fixed $s \geq 1$, the class of connected sP_3-free graphs is $\mathcal{F}$-additive.*

Proof. The proof is by induction on s. Let $s = 1$. Then every connected sP_3-free graph G is complete. Hence, every minimum $\mathcal{F}$-transversal of G is connected.

Now let $s \geq 2$. Let G be a connected sP_3-free graph. We may assume by induction that G contains an induced copy Γ_0 of an $(s-1)P_3$. Let S be a minimum $\mathcal{F}$-transversal of G. Let Γ be a minimum connected induced subgraph of G that contains Γ_0. Because G is sP_3-free, G has diameter less than $4s$. Then, by Lemma 4, we find that Γ has size less than $3(s-1) + (s-2)4s = 4s^2 - 5s - 3$. Let $S' = S \cup V(\Gamma)$. Then we have that $|S'| \leq |S| + 4s^2 - 5s - 3$.

If S' is connected then we take $d_{sP_3} = 4s^2 - 5s - 3$ as our desired constant and we are done. Suppose S' is not connected. Below we describe how to refine S'. During this process, we always use Z to denote the component of S' containing Γ, and we will never remove a vertex of Z from S'; in fact, one can think of the proof as "growing" Z and connecting it to the other vertices of S' until $Z = S'$.

Observe that the sP_3-freeness of G implies that every component of S' other than Z is complete. Throughout the proof, we let A denote the union of clique components of S', so $V(A) = S' \setminus V(Z) = S \setminus V(Z)$. We also note that the graph $G - S'$ is bipartite, as even its supergraph $G - S$ contains no odd cycles by the definition of S. Hence we can partition $G - S'$ into two (possibly empty) sets U_1 and U_2 so that U_1 and U_2 are independent sets.

We start with the following two claims, both of which follow from Lemma 10, which we apply twice, namely once with respect to U_1 and once with respect to U_2. By Lemma 10 this leads to a total increase in the size of S' by an additive factor of at most $2(2s-2) = 4s-4$.

Claim 1: Without loss of generality, we may assume that every vertex of $U_1 \cup U_2$ is adjacent to at most one component of A.

Claim 2: Without loss of generality, we may assume that every component of A is adjacent to at most one vertex of U_1 and to at most one vertex of U_2.

Using Claims 1 and 2 we prove the following crucial claim.

Claim 3: Without loss of generality, we may assume that every vertex of every component of A has exactly one neighbour in U_1 and exactly one neighbour in U_2.

We prove Claim 3 as follows. Let A^* be the union of components for which the statement of Claim 3 does not hold. Let D be a component of A^*. By Claim 2, D is adjacent to at most one vertex of U_1 and to at most one vertex of U_2. First suppose that D is non-adjacent to U_1 or to U_2, say D is not adjacent to U_1. Because G is connected, this means that D is adjacent to (exactly one) vertex $z \in U_2$, say $v \in D$ is adjacent to z. As D belongs to A^*, we find that D contains a vertex v' not adjacent to z. Hence, $vv'z$ is an induced P_3. Now suppose that D is adjacent to U_1 and to U_2, say D has vertices u, v (possibly $u = v$) so that u is adjacent to $x \in U_1$ and v is adjacent to $z \in U_2$. Then, as D is in A^*, there exists a vertex v' that is non-adjacent to at least one of x, z, say to z. Again, $vv'z$ is an induced P_3. As G is sP_3-free and no vertex in $U_1 \cup U_2$ is adjacent to more than one component of A by Claim 1, we deduce that A^* contains at most $s-1$ components. Moreover, each vertex $z \in U_1 \cup U_2$ included in an induced P_3 as described above must be adjacent to Z (due to sP_3-freeness of G and the fact that Z contains an induced $(s-1)P_3$). Hence, we can add these vertices to Z increasing the size of Z, and thus the size of S', by at most $s-1$. The remaining components of A have the desired property. Moreover, Claims 1 and 2 are still valid. This completes the proof of Claim 3.

Due to Claim 3 we may assume without loss of generality that each vertex v in each component D of A has exactly two neighbours in $G - S'$, namely one neighbour in U_1 and one neighbour in U_2. By Claim 2, these neighbours are the same for all vertices in D. Hence, we may denote these two neighbours by s_D and t_D, respectively,

Consider a component D of A. If one of its neighbours in $U_1 \cup U_2$, say s_D, is adjacent to Z, then replacing S' with $(S' \cup \{s_D\}) \setminus \{v\}$ and Z with the connected component of S' containing $Z \cup \{s_D\}$ does not result in an odd cycle in $G - S'$. Moreover, such a swap does not increase the size of S' either. It does, however, reduce the number of vertices of S' that are not in Z (which is our goal). Consequently, we perform these swaps until, in the end, both the neighbours s_D and t_D of each component of A are not adjacent to Z. In particular this implies that s_D and t_D are adjacent, so $V_D \cup \{s_D, t_D\}$ is a clique. Then, due to Claims 1–3, the components in A together with their neighbours in $U_1 \cup U_2$ induce a union of complete graphs. This union is a disjoint union, as otherwise G would contain an induced P_3 not adjacent to Z and, as Z has an induced $(s-1)P_3$, we would obtain an induced sP_3 in G. Note that the swaps did not change the size of S'.

Let U_1' and U_2' denote the subsets of U_1 and U_2, respectively, that consist of vertices adjacent to no components of A. Let W_1 consist of all vertices s_D adjacent to U_2' and let W_2 consist of all vertices t_D adjacent to U_1'. Note that $W_1 \subseteq U_1 \setminus U_1'$ and that $W_2 \subseteq U_2 \setminus U_2'$. Because G is connected and no s_D or t_D is adjacent to Z or to some other component of A not equal to D, we find that $W_1 \cup W_2$ contain at least one of s_D, t_D for each component D of A.

We choose smallest sets U_1'' and U_2'' in U_1' and U_2', respectively, that dominate W_2 and W_1, respectively. By minimality, each vertex $u \in U_1''$ must have a "private" neighbour t_D in W_2, and hence together with t_D and s_D, corresponds to a "private" P_3. Consequently, as G is sP_3-free and $U_1'' \subseteq U_1$ is an independent set, U_1'' has size at most $s-1$. Similarly, U_2'' has size at most $s-1$. Moreover, each vertex in $U_1'' \cup U_2''$ is adjacent to Z (again due to the sP_3-freeness of G).

Figure 3 shows an example in which the components of A consist on three cliques (the first two of size two and the last one of size one) to illustrate the situation.

We now do as follows. First, for each component D of A we pick one of its vertices v and swap v with s_D if $s_D \in W_1$ and otherwise we swap v with t_D (note that $t_D \in W_2$ in that case). We also add all vertices of $U_1'' \cup U_2''$ to Z and thus to S'. The results of these swaps are as follows. First, $G[S']$ has become connected. Second, S' has increased in size at most by $2(s-1)$, which is allowed. Third, $G - S'$ is still bipartite (as swapping a vertex of a component D of A with s_D or t_D does not create

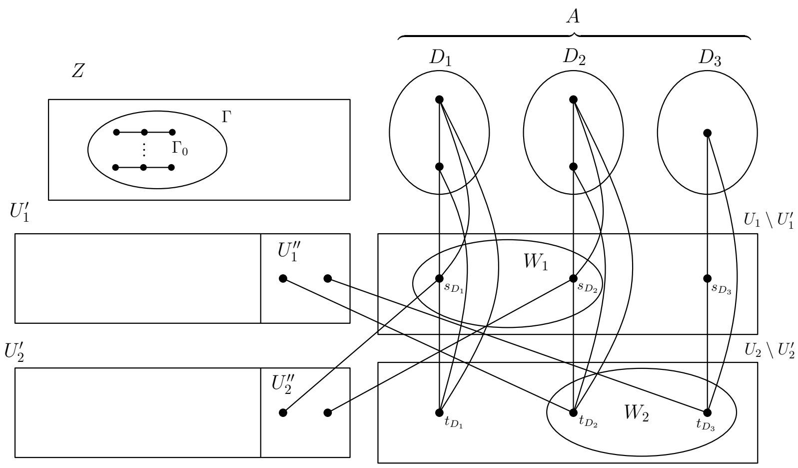


Fig. 3. The situation in the proof of Lemma 11.

any odd cycles). Consequently, we have found a connected $\mathcal{F}$-transversal of size at most $|S| + 4s^2 - 5s - 3 + 4s - 4 + (s-1) + 2(s-1) = |S| + 4s^2 + 2s - 10$, so we can take $d_{sP_3} = 4s^2 + 2s - 10$. □

We are now ready to prove the main result of this section.

Theorem 2. *For any graph H and for any family of cycles $\mathcal{F}$ containing all odd cycles, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Corollary 1. We now prove the second claim. First suppose $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 0$. If $H \subseteq_i P_5 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7 and 8. If $H \subseteq_i sP_3$ for some $s \geq 1$, the result follows from Lemma 11. Now suppose $H \nsubseteq_i P_5 + sP_1$ and $H \nsubseteq_i sP_3$ for any $s \geq 0$. By Theorem 1 (iii), we may assume that H is a linear forest. Then $P_6 \subseteq_i H$ or $P_2 + P_4 \subseteq_i H$, hence the class of connected H-free graphs is a superclass of the class of connected $(P_2 + P_4, P_6)$-free graphs and we can use Lemma 9.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. If $H \nsubseteq_i P_3$ then, by Theorem 1 (iii), we may assume that H is a linear forest. Hence, $3P_1 \subseteq_i H$ or $P_1 + P_2 \subseteq_i H$. Let $K_{2,2,2}$ be the graph on vertices $u_1, u_2, v_1, v_2, w_1, w_2$ and edges $u_i w_j$, $u_i v_j$ and $v_i w_j$ for $1 \leq i \leq j \leq 2$. Note that $K_{2,2,2}$ is $(3P_1, P_1 + P_2)$-free. Any minimum $\mathcal{F}$-transversal has size 2, whereas any minimum connected $\mathcal{F}$-transversal is of size 3. □

4 Cycle Families with 4-Cycles but no 3-Cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3 \notin \mathcal{F}$ but $C_4 \in \mathcal{F}$. We need the following lemma.

Lemma 12. *For any family $\mathcal{F}$ of cycles with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$,*

- *the class of connected P_5-free graphs is not $\mathcal{F}$-additive;*
- *the class of connected $P_2 + P_4$-free graphs is not $\mathcal{F}$-additive;*

- *the class of connected $2P_3$-free graphs is not $\mathcal{F}$-additive;*
- *the class of connected $3P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We consider the four parts one at a time.

First, we describe a family of connected P_5-free graphs that is not $\mathcal{F}$-additive. Each graph G is a clique on k vertices, $k \geq 4$, and k copies of C_4. Each vertex in the clique is adjacent to every vertex in a distinct copy of C_4. Figure 4 gives an example with $k = 4$. Note that G is P_5-free: any induced path on at least four vertices can contain at most one vertex from each C_4, and thus at most two such vertices in total, and can only contain two vertices from the clique.

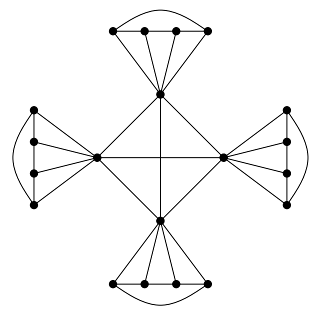


Fig. 4. A graph in a family of P_5-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$ since a set S containing one vertex from each copy of C_4 is an $\mathcal{F}$-transversal as $G - S$ is chordal. On the other hand, every connected $\mathcal{F}$-transversal of G contains, in addition to at least one vertex from each C_4, all the vertices of the clique. So $ct_{\mathcal{F}}(G) \geq 2k$.

Second, we describe a family of connected $P_2 + P_4$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of $k \geq 2$ copies of $K_{3,3}$, identified at a single vertex denoted v. Figure 5 shows the construction for $k = 4$.

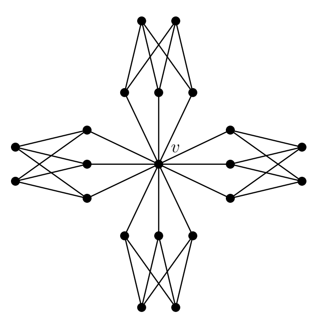


Fig. 5. A graph in a family of $P_4 + P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_4 \in \mathcal{F}$.

Note that G is $P_4 + P_2$-free: every induced P_4 contains v, and deleting the vertices in such a P_4 and their neighbours results in an edgeless graph. We have $t_{\mathcal{F}}(G) \leq k+1$ since a set S containing v and one vertex that is not adjacent to v from each $K_{3,3}$ is an $\mathcal{F}$-transversal as $G - S$ is a forest. On the other hand, every connected $\mathcal{F}$-transversal of G contains, in addition to v, at least two other vertices from each copy of $K_{3,3}$. So $ct_{\mathcal{F}}(G) \geq 2k+1$.

Third, we describe a family of connected $2P_3$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of a complete graph K_{4k} for $k \geq 2$ denoted K, and a set M of $2k$ additional vertices forming an induced matching and each joined to two other vertices in K. Figure 6 shows the construction for $k = 3$. Note that G is $2P_3$-free: any induced P_3 contains a vertex from K, and deleting this vertex and all its neighbours results in a disjoint union of cliques, a P_3-free graph.

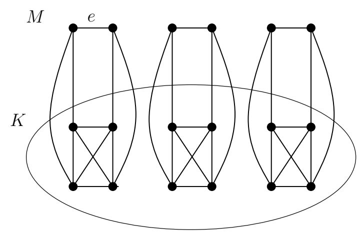


Fig. 6. A graph in a family of $2P_3$-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$, $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$, since a set S containing one vertex from each edge in M is an $\mathcal{F}$-transversal as $G - S$ is chordal. On the other hand, every connected $\mathcal{F}$-transversal of G contains at least two vertices from each subgraph consisting of an edge e in M and vertices in K adjacent to an endpoint of e. So $ct_{\mathcal{F}}(G) \geq 2k$.

Finally, we describe a family of connected $3P_2$-free graphs that is not $\mathcal{F}$-additive. Each graph G consists of three copies K, K' and K^* of a complete graph on $2k$ vertices for $k \geq 2$, and an independent set M of k vertices. Every vertex in K^* is joined to every vertex in K and K' and every vertex in M is joined to a distinct pair of vertices in K and K'. Figure 7 shows the construction for $k = 3$. Note that G is $3P_2$-free: when an induced P_2 and all its neighbours are deleted the resulting graph is either an independent set (if the P_2 is contained in K^*) or a graph in which every P_2 is incident with the same clique (if the P_2 intersects either K or K').

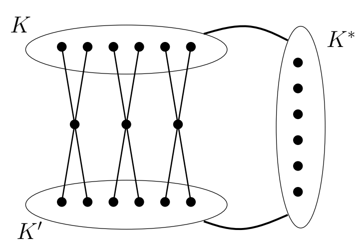


Fig. 7. A graph in a family of $3P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \notin \mathcal{F}$, $C_4 \in \mathcal{F}$.

We have $t_{\mathcal{F}}(G) \leq k$, since M is an $\mathcal{F}$-transversal as $G - M$ is chordal. On the other hand, a connected $\mathcal{F}$-transversal of G either contains K^* or, for each vertex v of M, either v and one of its neighbours, or, if it does not contain v, two of its neighbours. So $ct_{\mathcal{F}}(G) \geq 2k$. $\square$

We now state our result for infinite families of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$. It does not provide a complete characterization as we are unable to give necessary and sufficient conditions for the class of H-free graphs to be $\mathcal{F}$-additive. This would be possible if it could be shown that $(P_3 + P_2 + sP_1)$-free graphs are $\mathcal{F}$-additive for all $s \geq 0$. By Lemma 8, this is the case if and only if $(P_3 + P_2)$-free graphs are $\mathcal{F}$-additive, which we conjecture to be true.

Theorem 3. *For any graph H and for any infinite family of cycles $\mathcal{F}$ with $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, but not if $H \nsubseteq_i P_4 + sP_1$ nor $H \nsubseteq_i P_3 + P_2 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Corollary 1. We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from Lemmas 6 and 8. Now suppose $H \nsubseteq_i P_4 + sP_1$ and $H \nsubseteq_i P_3 + P_2 + sP_1$ for any $s \geq 0$. By Theorem 1 (iii), we may assume that H is a linear forest. Then, by Lemma 5 (iii), we find that $P_5 \subseteq_i H$, $P_2 + P_4 \subseteq_i H$, $2P_3 \subseteq_i H$, or $3P_2 \subseteq_i H$, and we can use Lemma 12.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. If $H \nsubseteq_i P_3$ then, by Theorem 1 (iii), we may assume that H is a linear forest. Hence, $3P_1 \subseteq_i H$ or $P_1 + P_2 \subseteq_i H$.

If $P_1 + P_2 \subseteq_i H$, then we have that the complete bipartite graph $G = K_{3,3}$ is a connected H-free graph (since it is $P_1 + P_2$-free). And $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$ so the class of connected H-free graphs is not $\mathcal{F}$-identical.

Finally, suppose that $3P_1 \subseteq_i H$, and let G be the complement of the graph shown in Figure 8. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph. As every $\mathcal{F}$-transversal of G must intersect every induced $2P_2$ in $\overline{G}$, the minimum $\mathcal{F}$-transversals of G are in bijective correspondence with the four edges of the 4-cycle in $\overline{G}$. So $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$, and the class of connected H-free graphs is also not $\mathcal{F}$-identical in this case. □

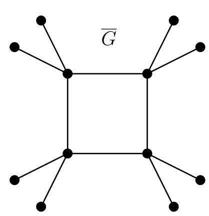


Fig. 8. The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $C_3 \notin \mathcal{F}$ and $C_4 \in \mathcal{F}$.

5 Cycle Families with 5-Cycles but no 3- or 4-Cycles

In this section we consider families of cycles $\mathcal{F}$ such that $C_3, C_4 \notin \mathcal{F}$ but $C_5 \in \mathcal{F}$. We first prove the following lemma; note that C_3 and C_4 are both induced subgraphs of $\overline{2P_4}$.

Lemma 13. *Let $\mathcal{F}$ be a family of graphs with $C_5 \in \mathcal{F}$ that contains no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$. Then the class of connected $2P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We describe a family of connected $2P_2$-free graphs that is not $\mathcal{F}$-additive, where $\mathcal{F}$ is any family of cycles as in the statement of the lemma. The graphs in the family are constructed from $k \geq 2$ copies $H_1, \ldots, H_k$ of the graph that is obtained from $2P_4$ by adding all possible edges between the vertices of one copy and the other one. For each H_i, there is a new vertex v_i adjacent to both endpoints of the two P_4s, and in addition there are all possible edges between vertices in different H_i's. Figure 9 shows an example for $k = 4$.

We first show that every graph G in this family is $2P_2$-free. Every edge e of G has at least one endpoint in some H_i, say in H_1. Deleting the closed neighbourhood of e results in the subgraph induced by a subset of $\{v_1, \ldots, v_k\}$ (if $e \in E(H_1)$), or in the subgraph induced by $\{u, v_2, \ldots, v_k\}$ for some $u \in V(H_1)$ (otherwise). In either case, the resulting graph is edgeless. Therefore, G is $2P_2$-free.

Let G be a graph in this family, and let k be the number of H_i's. We have $t_{\mathcal{F}}(G) \leq k$ since deleting the vertices $v_1, \ldots, v_k$ results in a graph that is isomorphic to $\overline{2kP_4}$ and thus $\mathcal{F}$-free. On the other hand, every connected $\mathcal{F}$-transversal S of G must contain at least two vertices from each subgraph induced by $V(H_i) \cup \{v_i\}$, for every i (otherwise it either misses an induced C_5 or contains only v_i, making it isolated in $G[S]$). Therefore, $ct_{\mathcal{F}}(G) \geq 2k$, which establishes the non-$\mathcal{F}$-additivity of the family. □

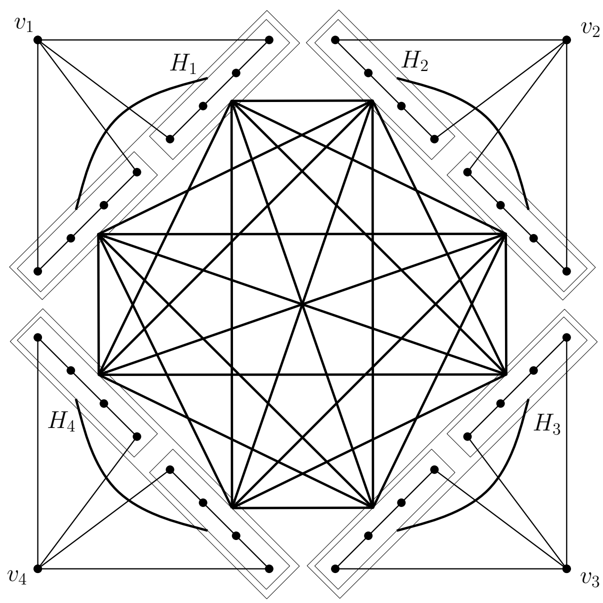


Fig. 9. A member of a family of connected $2P_2$-free graphs that is not $\mathcal{F}$-additive whenever $C_5 \in \mathcal{F}$ and $\mathcal{F}$ contains no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$. A thick edge between two sets of vertices inducing a P_4 means the presence of all possible edges between the two sets.

We also need the following lemma.

Lemma 14. *Let $\mathcal{F}$ be a family of graphs that contains C_5 but no induced subgraph of $\overline{4P_4}$. Then the class of connected $3P_1$-free graphs is not $\mathcal{F}$-identical.*

Proof. Let $\mathcal{F}$ be any family of cycles as in the statement of the lemma and let G be the complement of the graph depicted in Figure 10. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph.

Since $\overline{C_5} = C_5$, in the complement of G we need to cover all the C_5's. Therefore there is a unique minimum $\mathcal{F}$-transversal S of G, consisting of the two endpoints of the central edge of $\overline{G}$. Indeed $\overline{G} - S$ is isomorphic to $4P_4$, so the graph $G - S \cong \overline{4P_4}$ is $\mathcal{F}$-free. Since the graph $G[S]$ is not connected, we have $ct_{\mathcal{F}}(G) > t_{\mathcal{F}}(G)$. $\square$

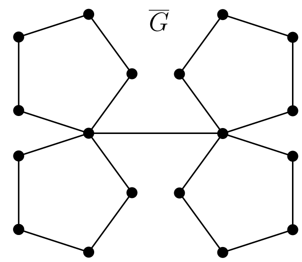


Fig. 10. The complement of a graph that shows that the class of connected $3P_1$-free graphs is not $\mathcal{F}$-identical whenever $C_5 \in \mathcal{F}$ and $\mathcal{F}$ contains no induced subgraphs of $\overline{4P_4}$.

Theorem 4. *For any graph H and for any graph family $\mathcal{F}$ which only contains graphs with an induced P_4, including C_5 and an infinite number of other cycles but no linear forests and no induced subgraphs of $\overline{sP_4}$ for any $s \geq 1$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative if and only if H is a linear forest;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_4$.*

Proof. The first claim follows immediately from Corollary 1. We now prove the second claim. First suppose that $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$. Then the class of connected H-free graphs is $\mathcal{F}$-additive due to Lemmas 6 and 8. Now suppose that $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. By Theorem 1 (iii), we may assume that H is a linear forest. Hence, by Lemma 5 (ii), $2P_2 \subseteq_i H$ and we can use Lemma 13. Finally, we show the third claim. Recall that if $H \subseteq_i P_4$ then any H-free graph is already $\mathcal{F}$-free. Suppose that $H \not\subseteq_i P_4$. By Lemma 5 (i), we find that $2P_2 \subseteq_i H$ or $3P_1 \subseteq_i H$. If $2P_2 \subseteq_i H$ we use Lemma 13 again. Hence $3P_1 \subseteq_i H$. In that case we use Lemma 14. This completes the proof of Theorem 4. $\square$

6 Families of Short Paths and Cycles

In Section 6.2 we prove our results for families $\mathcal{F}$ of graphs that contain P_2, $2P_2$ or P_4, in particular for families $\mathcal{F}$ for which the graph minus an $\mathcal{F}$-transversal is a split graph, a threshold graph, a trivially perfect graph, or a cograph, respectively. In order to show these results we need a number of lemmas, which we will prove in Section 6.1. As before, lemmas and theorems are often stated in a more general form than needed.

6.1 Lemmas

Lemma 15. *For $\mathcal{F} = \{C_4, C_5, 2P_2\}$ and any fixed $s \geq 0$, the class of connected $(P_3 + sP_2)$-free graphs is $\mathcal{F}$-additive.*

Proof. The proof is by induction on s. Let $s = 0$. Every connected P_3-free graph G is complete. Hence, every minimum $\mathcal{F}$-transversal of G is connected.

Now let $s \geq 1$. Let G be a connected $(P_3 + sP_2)$-free graph. We may assume by induction that G contains an induced copy Γ_0 of an $P_3+(s-1)P_2$. Let S be a minimum $\mathcal{F}$-transversal of G. Let Γ be a minimum connected induced subgraph of G that contains Γ_0. Because G is (P_3+sP_2)-free, G has diameter less than $3(s+1)-1 = 3s-2$. Then, by Lemma 4, we find that Γ has size less than $3(s-1) + (s-2)(3s-2) = 3s^2 - 3s + 1$. Let $S' = S \cup V(\Gamma)$. Then we have that $|S'| \leq |S| + 3s^2 - 3s + 1$.

If S' is connected then we take $d_{P_3+sP_2} = 3s^2 - 3s + 1$ as our desired constant and we are done. Suppose S' is not connected. Below we describe how to refine S'. During this process, we always use Z to denote the component of S' containing Γ, and we will never remove a vertex of Z from S'.

Observe that the (P_3+sP_2)-freeness of G implies that every component of S' other than Z consists of a single vertex. We let A denote the union of these single vertices, so $A = S' \setminus V(Z) = S \setminus V(Z)$. We also note that the graph $G - S'$ is split, as even its supergraph $G - S$ is $\{C_4, C_5, 2P_2\}$-free by the definition of S. Hence we can partition $G - S'$ into two (possibly empty) sets: a clique K and an independent set I.

We start with the following two claims, both of which follow from Lemma 10. By Lemma 10, this leads to a total increase of S' by an additive factor of at most $2s - 2$.

Claim 1: Without loss of generality, we may assume that every vertex of I is adjacent to at most one vertex of A.

Claim 2: Without loss of generality, we may assume that every vertex of A is adjacent to at most one vertex of I.

We proceed as follows. If A contains a vertex u not adjacent to a vertex in I then we move u from A to I. Hence, we may assume without loss of generality that A has no such vertices. Then, by Claim 2, every vertex in A is adjacent to exactly one vertex of I. Let $A = \{a_1, \ldots, a_q\}$ for some integer $q \geq 1$ and let $X = \{x_1, \ldots, x_q\}$ be the subset of I in which x_i is the unique neighbour of a_i for $i = 1, \ldots, q$. By Claim 1, $G[A \cup X]$ is isomorphic to qP_2. See Figure 11 for an example.

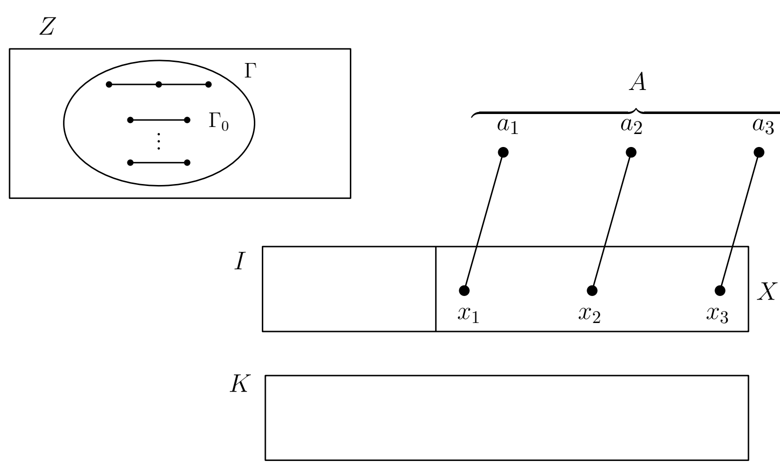


Fig. 11. The decomposition of the graph G in the proof of Lemma 15.

Due to the $(P_3 + sP_2)$-freeness of G and the fact that Z contains an induced $P_3 + (s-1)P_2$, each x_i is adjacent to Z. We swap a_i and x_i, that is, we put a_i into I

and x_i into A. Then, because a_i is not adjacent to any other vertex in I, we still have the property that $G-S'$ is split. However, we now also have that $Z = S'$, as desired. So we have found a connected $\mathcal{F}$-transversal S' of size at most $|S|+3s^2-3s+1+2s-2 = |S|+3s^2-s-1$ meaning we can take $d_{P_3+sP_2} = 3s^2-s-1$. This completes the proof of Lemma 15. □

Lemma 16. *Let $\mathcal{F}$ be a family of graphs with either $\mathcal{F} = \{P_2\}$, or $\mathcal{F} \cap \{P_4, 2P_2\} \neq \emptyset$ and $\mathcal{F} \setminus \{P_2, P_4, 2P_2\}$ a (possibly empty) set of holes. If H is not a linear forest then the class of connected H-free graphs is not $\mathcal{F}$-additive.*

Proof. Let H be a graph that is not a linear forest, so H contains a cycle or an induced $K_{1,3}$. Let us verify that the class of all paths is a class of H-free connected graphs that is not $\mathcal{F}$-additive.

If $\mathcal{F} = \{P_2\}$, then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$ (since taking every other vertex on the path results in an $\mathcal{F}$-transversal), while $ct_{\mathcal{F}}(P_n) \geq n-2$ (since any $\mathcal{F}$-transversal contains a vertex u from the first 2 vertices of P_n and also a vertex v from the last 2 vertices, and these two need to be made connected by taking all the vertices of the path that lie in between).

If $\mathcal{F} \cap \{P_4, 2P_2\} = \{P_4\}$ then, similarly, for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/4$ while $ct_{\mathcal{F}}(P_n) \geq n-6$. If $\mathcal{F} \cap \{P_4, 2P_2\} = \{2P_2\}$ then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$, while $ct_{\mathcal{F}}(P_n) \geq n-8$. Finally, if $\mathcal{F} \cap \{P_4, 2P_2\} = \{P_4, 2P_2\}$ then for large enough n we have $c_{\mathcal{F}}(P_n) \leq n/2$, while $ct_{\mathcal{F}}(P_n) \geq n-6$. □

Lemma 17. *Let $\mathcal{F}$ be a family of graphs that contains C_4 but no induced subgraph of $K_{1,3}$. Then the class of (P_2+P_1)-free graphs is not $\mathcal{F}$-identical.*

Proof. The complete bipartite graph $K_{3,3}$ is (P_2+P_1)-free. Removing a single vertex or two adjacent vertices does not make the graph C_4-free. If we remove two non-adjacent vertices then we obtain a claw, which is $\mathcal{F}$-free. Hence, a minimum $\mathcal{F}$-transversal has size 2 and a minimum connected $\mathcal{F}$-transversal has size at least 3. □

Lemma 18. *Let $\mathcal{F}$ be a family of graphs that contains P_4 but no complete graph. Then the class of $2P_2$-free graphs is not $\mathcal{F}$-additive.*

Proof. We construct a family of connected $2P_2$-free graphs $\{G_k\}$ as follows. Let G_k have a clique $K_k = \{u_1, \ldots, u_{2k}\}$ and an independent set $\{a_1, \ldots, a_k\}$. For $i = 1, \ldots, k$, add the edges a_iu_{2i-1} and a_iu_{2i}. (See Figure 12.)

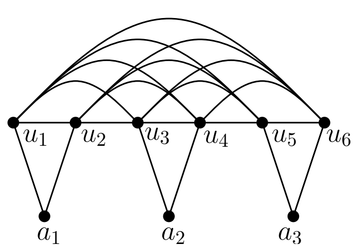


Fig. 12. The graph G_k for $k = 3$ used in the proof of Lemma 18.

Note that G_k is $2P_2$-free, for all $k \geq 1$. Note that each set $\{a_i, a_j, u_i, u_{2i}, u_j, u_{2j}\}$ induces four different P_4's. On the one hand, the set $\{a_1, \ldots, a_k\}$ forms an $\mathcal{F}$-transversal of G of size k. On the other hand, as any two distinct a_i and a_j are non-adjacent and have no common neighbour, any connected $\mathcal{F}$-transversal of G contains at least two vertices from at least $k-1$ of the k pairwise disjoint sets $\{a_i, u_{2i-1}, u_{2i}\}$ and therefore has size at least $2(k-1)$. □

Lemma 19. *Let $\mathcal{F}$ be a family of graphs that contains P_4 but no disjoint union of two complete graphs. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-identical.*

Proof. Construct the following 14-vertex graph G^*. Take a set A of seven vertices a, a', b, b', c, d, d', add the edges making each of $A_1 = \{a, a', b, b'\}$ and $A_2 = \{d, d'\}$ a clique, and add the edges $bc, b'c, cd, cd'$. Take a set B of seven vertices s, s', t, t', u, v, v', add the edges making each of $B_1 = \{s, s', t, t'\}$ and $B_2 = \{v, v'\}$ a clique, and add the edges $tu, t'u, uv, uv'$. Add every edge between a vertex of A_1 and a vertex of B_1 (thus making $A_1 \cup B_1$ a clique), every edge between a vertex of B_1 and a vertex of B_2 (thus making $A_2 \cup B_2$ a clique), add edges from c to every vertex of $B \setminus \{u\}$, and add edges from u to every vertex of $A \setminus \{c\}$. See Figure 13 for a picture of G^*. Note that G^* is $3P_1$-free and that $\{u, c\}$ is the unique minimum $\mathcal{F}$-transversal, hence every minimum connected $\mathcal{F}$-transversal has size (at least) 3. $\square$

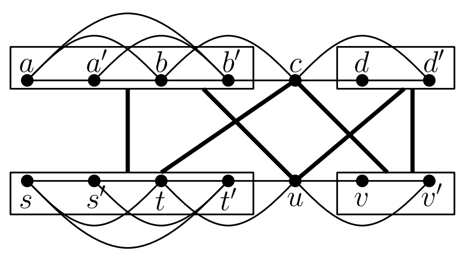


Fig. 13. The graph G^* used in the Proof of Lemma 19. A thick edge between two sets of vertices means the presence of all possible edges between the two sets.

Let K_6^+ be the graph that consists of a clique on six vertices and another vertex made adjacent to three vertices of the clique.

Lemma 20. *Let $\mathcal{F}$ be a family of graphs that contains $2P_2$ and P_4 but no induced subgraph of K_6^+. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-identical.*

Proof. We construct the following graph G with ten vertices $a_1, a_2, b_1, b_2, u_1, u_2, u_3, v_1, v_2, v_3$ so that $\{a_1, a_2, u_1, u_2, u_3\}$, $\{b_1, b_2, v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ are three cliques. See Figure 14 for a picture of G. Note that G is $3P_1$-free, as the first two cliques partition $V(G)$. Then every minimum $\mathcal{F}$-transversal consists of three vertices, namely one of $\{a_1, a_2\}$ and two of $\{b_1, b_2\}$, or vice versa (as otherwise either an induced $2P_2$ is left or an induced P_4). Consequently, the size of a minimum connected $\mathcal{F}$-transversal is 4. $\square$

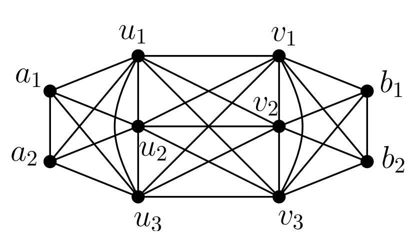


Fig. 14. The graph G used in the proof of Lemma 20.

Lemma 21. *Let $\mathcal{F}$ be a family of graphs that contains $2P_2$ but no induced subgraph of $\overline{4P_3}$. Then the class of $3P_1$-free graphs is not $\mathcal{F}$-identical.*

Proof. The proof mimics that of Lemma 14. Let G be the complement of the graph shown in Figure 15. Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free graph. Since $\overline{2P_2} = C_4$, in the complement of G we need to cover all the C_4's. Therefore there is a unique minimum $\mathcal{F}$-transversal S of G, consisting of the two endpoints of the central edge of $\overline{G}$. Indeed $\overline{G} - S$ is isomorphic to $4P_3$, so the graph $G - S \cong \overline{4P_3}$ is $\mathcal{F}$-free. Since the graph $G[S]$ is not connected, we have $ct_{\mathcal{F}}(G) > t_{\mathcal{F}}(G)$. □

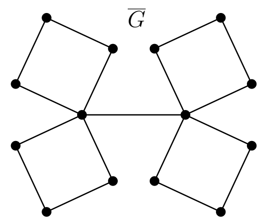


Fig. 15. The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $2P_2 \in \mathcal{F}$ and no induced subgraph of $\overline{4P_3}$ is in $\mathcal{F}$.

6.2 Theorems

We are now ready to prove the following six theorems.

Theorem 5. *For any graph H and for $\mathcal{F} = \{P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_5 + sP_1$ or $H \subseteq_i sP_3$ for some $s \geq 1$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. If $H \subseteq_i P_5 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 7 and 8. If $H \subseteq_i sP_3$ for some $s \geq 0$, the result follows from Lemma 11. Suppose that $H \not\subseteq_i P_5 + sP_1$ for any $s \geq 0$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then, since $H \not\subseteq_i P_5 + sP_1$ and $H \not\subseteq_i sP_3$ for any $s \geq 0$, it follows from Lemma 5 (v) that $P_4 + P_2 \subseteq_i H$ or $P_6 \subseteq_i H$. Consider the $(P_4 + P_2, P_6)$-free graph G_k obtained from k 4-cycles $a_ib_ic_id_ia_i$ for $i = 1, \ldots, k$ after identifying all $a_1, \ldots, a_k$ into a single vertex a (so G_k consists of disjoint P_3's, whose end-vertices are both adjacent to a). For every $k \geq 1$, a minimum $\mathcal{F}$-transversal has size $k+1$ and a minimum connected $\mathcal{F}$-transversal has size $2k+1$.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Suppose $H \not\subseteq_i P_3$. By the previous claim we may assume that H is a linear forest. Thus, $H \not\subseteq_i C_4$ and the graph $G = C_4$ is an H-free graph with $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$. □

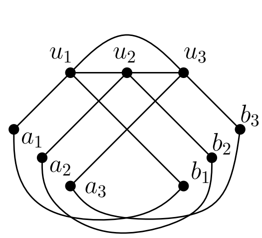
Theorem 6. *For any graph H and for $\mathcal{F} = \{C_4, C_5, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*

- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. First suppose $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 6 and 8. If $H \subseteq_i P_3 + sP_2$ for some $s \geq 0$, the result follows from Lemma 15. Now suppose $H \not\subseteq_i P_4 + sP_1$ and $H \not\subseteq_i P_3 + sP_2$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then by Lemma 5 (iv), we find that $P_5 \subseteq_i H$, $P_4 + P_2 \subseteq_i H$, or $2P_3 \subseteq_i H$.

First suppose that $P_5 \subseteq_i H$ or $2P_3 \subseteq_i H$. We construct a family of connected H-free graphs $\{G_k\}$ as follows. Let G_k have a clique $K_k = \{u_1, \ldots, u_k\}$ and two independent sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$. For $i = 1, \ldots, k$, add edges a_ib_i, a_iu_i and b_iu_i. See Figure 16 for an example.



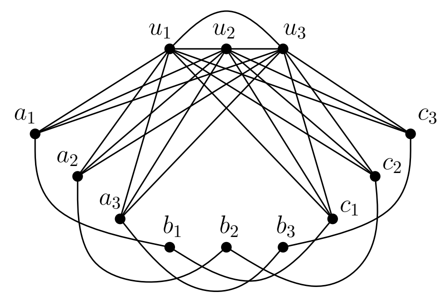


Fig. 16. The graphs G_k (left) and G_k^* (right) for $k = 3$ used in the proof of Theorem 6.

Note that G_k is $(2P_3, P_5)$-free, and thus H-free, for all $k \geq 1$. Every minimum $\mathcal{F}$-transversal consists of exactly one vertex of each pair $\{a_i, b_i\}$, as we need to remove at least one vertex from at least $k - 1$ pairs $\{a_i, b_i\}$ to remove induced $2P_2$'s and then another vertex from the remaining pair (which forms an induced $2P_2$ with a non-adjacent pair of clique vertices). On the other hand, every connected $\mathcal{F}$-transversal consists of at least $2k$ vertices.

Now suppose that $P_4 + P_2 \subseteq_i H$. We construct a family of connected H-free graphs $\{G_k^*\}$ as follows. Let G_k^* have a clique $K_k = \{u_1, \ldots, u_k\}$ and three independent sets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_k\}$. For $i = 1, \ldots, k$, add edges a_ib_i and b_ic_i. Also add an edge between each a_i and each u_j, and an edge between each c_i and each u_j. See Figure 16 for an example. As each u_j is adjacent to all vertices of G_k^* except the mutually non-adjacent vertices $b_1, \ldots, b_k$, we find that G_k^* is $(P_4 + P_2)$-free for all $k \geq 1$. By the same arguments as in the previous case, we find that $\{b_1, \ldots, b_k\}$ is the unique minimum $\mathcal{F}$-transversal. On the other hand, every connected $\mathcal{F}$-transversal contains at least $2k + 1$ vertices.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Now suppose $H \not\subseteq_i P_3$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ or $H \subseteq_i P_3 + sP_2$ for some integer $s \geq 0$.

Suppose that $3P_1 \subseteq_i H$, and let G be the complement of the graph shown in Figure 17.

Since $\overline{G}$ is triangle-free and every two vertices of $\overline{G}$ have a common non-neighbour, G is a connected $3P_1$-free (and hence H-free) graph. The set $S = \{v_1, v_2\}$ is an $\mathcal{F}$-transversal of G since $\overline{G} - S$ (and consequently $G - S$) is a split graph. On the other

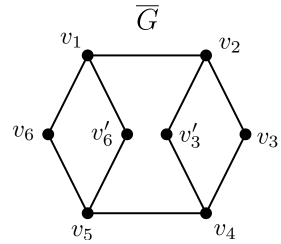


Fig. 17. The complement of a graph G with $t_{\mathcal{F}}(G) < ct_{\mathcal{F}}(G)$ whenever $\mathcal{F} = \{C_4, C_5, 2P_2\}$.

hand, deleting any pair of non-adjacent vertices from $\overline{G}$ leaves at least one subgraph isomorphic to $2P_2$ or C_4, which implies that $t_{\mathcal{F}}(G) = 2 < ct_{\mathcal{F}}(G)$.

Now suppose that $3P_1 \not\subseteq_i H$. If $P_2 + P_1 \subseteq_i H$ then we can apply Lemma 17. If H is $(3P_1, P_2 + P_1)$-free, then we conclude (since H is a linear forest) that $H \subseteq_i P_3$, a contradiction. $\square$

Theorem 7. *For any graph H and for $\mathcal{F} = \{C_4, P_4\}$ or $\mathcal{F} = \{C_4, P_4, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 6 and 8. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, by Lemma 5 (ii) we find that $2P_2 \subseteq_i H$ and we can use Lemma 18.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, so the result follows directly. Now suppose $H \not\subseteq_i P_3$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. Hence it holds that $3P_1 \subseteq_i H$ or $P_2 + P_1 \subseteq_i H$.

We start with the case where $3P_1 \subseteq_i H$. If $2P_2 \in \mathcal{F}$ then we use Lemma 20. Suppose that $2P_2 \notin \mathcal{F}$. Then $\mathcal{F} = \{C_4, P_4\}$ and we can use Lemma 19. We now consider the case $P_2 + P_1 \subseteq_i H$. As $C_4 \in \mathcal{F}$ we apply Lemma 17. This completes the proof of Theorem 7. $\square$

Theorem 8. *For any graph H and for $\mathcal{F} = \{C_5, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 6 and 8. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, by Lemma 5 (ii) we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 13.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, and if $H \subseteq_i P_1 + P_2$ then any connected H-free graph is $\mathcal{F}$-free. So in both cases the result follows directly. Now suppose that $H \not\subseteq_i P_3$ and $H \not\subseteq_i P_1 + P_2$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. If $3P_1 \subseteq_i H$, then we can apply Lemma 21. If $3P_1 \not\subseteq_i H$, then $H = P_4$ and we can consider the 7-vertex graph G consisting of 6 vertices forming a $3P_2$ and one more vertex adjacent to all the other vertices. Graph G is a connected P_4-free graph with $t_{\mathcal{F}}(G) = 2 < 3 = ct_{\mathcal{F}}(G)$. This completes the proof of Theorem 8. □

Theorem 9. *For any graph H and for $\mathcal{F} = \{P_4, 2P_2\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_3$ or $H \subseteq_i P_2 + P_1$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 6 and 8. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, by Lemma 5 (ii) we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 18.

We now prove the third claim. If $H \subseteq_i P_3$ then any connected H-free graph is complete, and if $H \subseteq_i P_1 + P_2$ then any connected H-free graph is $\mathcal{F}$-free. So in both cases the result follows directly. Now suppose that $H \not\subseteq_i P_3$ and $H \not\subseteq_i P_1 + P_2$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 0$. Hence it holds that $3P_1 \subseteq_i H$ and we can apply Lemma 20. This completes the proof of Theorem 9. □

Theorem 10. *For any graph H and for $\mathcal{F} = \{P_4\}$, the class of connected H-free graphs is*

- *$\mathcal{F}$-multiplicative;*
- *$\mathcal{F}$-additive if and only if $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$;*
- *$\mathcal{F}$-identical if and only if $H \subseteq_i P_4$.*

Proof. The first claim follows immediately from Theorem 1 (i). We now prove the second claim. If $H \subseteq_i P_4 + sP_1$ for some $s \geq 0$, the result follows from combining Lemmas 6 and 8. Now suppose $H \not\subseteq_i P_4 + sP_1$ for any $s \geq 0$. If H is not a linear forest then we can use Lemma 16. Hence we may assume that H is a linear forest. Then, as $H \not\subseteq_i P_4 + sP_1$, by Lemma 5 (ii) we find that $2P_2 \subseteq_i H$ and thus we can use Lemma 18.

We now prove the third claim. If $H \subseteq_i P_4$ then any connected H-free graph is $\mathcal{F}$-free, so the result follows directly. Now suppose $H \not\subseteq_i P_4$. By the previous claim, we may assume that $H \subseteq_i P_4 + sP_1$ for some integer $s \geq 1$. Hence, $3P_1 \subseteq_i H$ and we can use Lemma 19. □

7 Conclusions

We extended the tetrachotomy result of Belmonte et al. [2] for the family $\mathcal{F}$ of all cycles by giving tetrachotomy results for a number of natural families $\mathcal{F}$ containing cycles and anticycles (see Table 1). Let us recall that a tetrachotomy for the price of connectivity of $\mathcal{F}$-transversals when $\mathcal{F}$ is the family of even cycles or of all holes is still an open case. To settle it, it would suffice to show that the class of connected $(P_3 + P_2)$-free graphs is $\mathcal{F}$-additive, which we conjecture to be true.

Conjecture. *The class of connected $(P_3 + P_2)$-free graphs is $\mathcal{F}$-additive if $\mathcal{F}$ consists of all even cycles or all holes.*

We also have no tetrachotomy for infinite families $\mathcal{F}$ of cycles that contain C_3 but that miss some other odd cycle. The partial results below show that a more refined analysis is needed to obtain complete results in this direction.

We first summarize our current knowledge. By Corollary 1 we know that the class of H-free graphs is $\mathcal{F}$-multiplicative if and only if H is a linear forest. We also know, due to Lemma 9, that the class of connected (P_2+P_4, P_6)-free graphs is not $\mathcal{F}$-additive. Moreover, the class of connected H-free graphs is $\mathcal{F}$-identical if and only if $H \subseteq_i P_3$, as we can use the example of $G = K_{2,2,2}$ from Theorem 2. Hence, using Lemmas 6–8, we see that what remains is to check, for every $s \geq 2$, whether the class of H-free graphs is $\mathcal{F}$-additive if $H = sP_3$. We can show that already for $s = 2$ this is true for some families $\mathcal{F}$ and false for others.

In order to prove the first statement we need the following lemma.

Lemma 22. *Every connected $(C_3, C_5, 2P_3)$-free graph not isomorphic to C_7 is bipartite.*

Proof. Let G be a connected $(C_3, C_5, 2P_3)$-free graph not isomorphic to C_7. For contradiction, suppose that G is not bipartite. Then, as G is $(C_3, C_5, 2P_3)$-free, G must contain an induced subgraph F that is isomorphic to C_7. Let $F = v_1 v_2 \cdots v_7 v_1$. As G is connected and not isomorphic to C_7, there exists a vertex $u \in V(G) \setminus V(H)$ adjacent to a vertex of F, say u is adjacent to v_1. If u has no other neighbours in F, then $u, v_1, v_2, v_4, v_5, v_6$ form an induced $2P_3$, which is not possible. As G is (C_3, C_5)-free, u is not adjacent to v_2, v_4, v_5, v_7. If u is adjacent to both v_3 and v_6, then u, v_3, v_4, v_5, v_6 induce a C_5, which is not possible. This means that u is adjacent to exactly one of v_3, v_6, say to v_3. Then $u, v_1, v_2, v_4, v_5, v_6$ form an induced $2P_3$, which is not possible either. This completes the proof of the lemma. ◻

Using Lemma 22 we can now show the following result, the proof of which mimics the proof of Lemma 11 (although some changes are required).

Proposition 1. *For any family of cycles $\mathcal{F}$ containing C_3 and C_5, the class of connected $2P_3$-free graphs is $\mathcal{F}$-additive.*

Proof. Let $\mathcal{F}$ be a family of cycles containing C_3 and C_5. Let G be a $2P_3$-free graph. If G contains no induced P_3 then G is complete and we are done. Suppose that G contains an induced copy Γ of a P_3. Let S be a minimum $\mathcal{F}$-transversal of G. Let $S' = S \cup V(\Gamma)$. Note that $|S'| \leq |S| + 3$.

If S' is connected then we take $d_{2P_3} = 3$ and we are done. Suppose S' is not connected. Observe that the $2P_3$-freeness of G implies that every component of S' other than Z is complete. Moreover, as $G - S$ is $\mathcal{F}$-free and $2P_3$-free, the same holds for $G - S'$. As C_3 and C_5 both belong to $\mathcal{F}$, we find that each component of $G - S'$ is either bipartite or isomorphic to C_7. We place the vertices of any C_7 in $G - S'$ in S'. Because G is $2P_3$-free, $G - S'$ can have at most two components isomorphic to C_7, so this increases the size of S' by at most 14. Due to this operation, $G - S'$ becomes bipartite and the rest of the proof is a copy of the proof of Lemma 11. ◻

Proposition 2. *For any family $\mathcal{F}$ of cycles with $C_3 \in \mathcal{F}$ and $C_5 \notin \mathcal{F}$, the class of connected $2P_3$-free graphs is not $\mathcal{F}$-additive.*

Proof. We describe a family of connected $2P_3$-free graphs that is not $\mathcal{F}$-additive, where $\mathcal{F}$ is any family of cycles as in the statement of the lemma. The graphs in the family consist of $k \geq 3$ copies of the diamond (the K_4 minus an edge) with pairs of non-adjacent vertices denoted as $\{a_i, b_i\}$ in the i-th diamond. Moreover, for every $1 \leq i < j \leq k$, vertex a_i is adjacent to vertex b_j. Figure 18 gives an example of one of such graphs, for $k = 5$.

We first show that every graph G in this family is $2P_3$-free. Let P be an induced P_3 in G. Then P contains some a_i or some b_i, say it contains a_i. If it also contains b_i for some i, then $G - V(P)$ is a disjoint union of triangles, hence P_3-free. Otherwise, it must contain some b_j for $j > i$, and also in this case $G - V(P)$ is a disjoint union of cliques. Therefore, G is $2P_3$-free.

Let G be a graph in this family, and let k be the number of diamonds. We have $t_{\mathcal{F}}(G) \leq k$ since deleting a vertex of degree 3 from each diamond results in a graph in which every induced cycle is a C_5, hence in an $\mathcal{F}$-free graph. On the other hand, every connected $\mathcal{F}$-transversal S of G must contain at least two vertices from each diamond (otherwise it either misses an induced C_3 or contains only one vertex of degree 3 in some diamond, making it isolated in $G[S]$). Therefore, $ct_{\mathcal{F}}(G) \geq 2k$, which establishes the non-$\mathcal{F}$-additivity of the family. $\square$

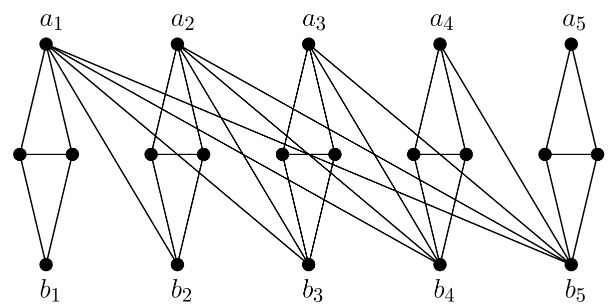


Fig. 18. A member of a family of $2P_3$-free graphs that is not $\mathcal{F}$-additive whenever $C_3 \in \mathcal{F}$ and $C_5 \notin \mathcal{F}$.

Propositions 1 and 2 suggest that we may want to distinguish between families $\mathcal{F}$ that contain C_3 and C_5 or that contain C_3 but not C_5. We leave this as future work.

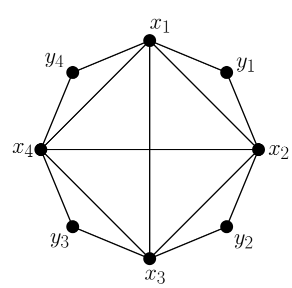


Fig. 19. A sun obtained from a cycle of length 8.

We finish our paper with the following open problem. A *chord* of a cycle C is an edge between two vertices $u, v \in V(C)$ with $uv \notin E(C)$; if the distance between u ad v in C is odd, then we speak of an *odd* chord. A graph is *strongly chordal* if it is chordal and every cycle of even length at least 6 in G has an odd chord. A *sun* is a cycle $x_1y_1x_2 \cdots x_\ell y_\ell x_1$ for some $\ell \geq 3$ to which all edges of the form x_ix_j are added; suns are sometimes called complete suns or (complete) trampolines. A graph G is strongly chordal if and only if it is chordal and contains no sun as an induced subgraph [15].

Is there a tetrachotomy for the price of connectivity for H-free graphs if $\mathcal{F}$ consists of holes and suns?

References

1. G. Bacsó and Zs. Tuza, Dominating cliques in P_5-free graphs, Periodica Mathematica Hungarica 21(1990) 303–308.
2. R. Belmonte, P. van 't Hof, M. Kamiński and D. Paulusma, The price of connectivity for feedback vertex set, Proc. EuroComb 2013, CRMS 16 (2013) 123–128 (full manuscript available as arXiv:1510.02639).
3. R. Belmonte, P. van 't Hof, M. Kamiński and D. Paulusma, Forbidden induced subgraphs and the price of connectivity for feedback vertex set, Proc. MFCS 2014, LNCS 8635 (2014) 57–68
4. D. Binkele-Raible and H. Fernau, Parameterized measure & conquer for problems with no small kernels, Algorithmica 64 (2012) 189–212.
5. A. Brandstädt, V.B. Le and J. Spinrad, Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), 1999.
6. E. Camby, J. Cardinal, S. Fiorini and O. Schaudt, The Price of Connectivity for Vertex Cover. Discrete Math. & Theoretical Computer Science 16 (2014) 207–224.
7. E. Camby and O. Schaudt, The price of connectivity for dominating set: Upper bounds and complexity, Discrete Applied Mathematics 17 (2014) 53–59.
8. J. Cardinal and E. Levy, Connected vertex covers in dense graphs, Theoretical Computer Science 411(2010) 2581–2590.
9. M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas, The strong perfect graph theorem, Annals of Mathematics 164 (2006) 51–229.
10. M. Cygan, M. Pilipczuk, M. Pilipczuk, and J. O. Wojtaszczyk, Kernelization hardness of connectivity problems in d-degenerate graphs, Discrete Applied Mathematics 160 (2012) 2131–2141.
11. F. Delbot, C. Laforest, and S. Rovedakis, Self-stabilizing algorithms for connected vertex cover and clique decomposition problems, In *Principles of distributed systems*, volume 8878 of *Lecture Notes in Comput. Sci.*, pages 307–322. Springer, Cham, 2014.
12. R. Diestel, Graph Theory, Springer-Verlag, Electronic Edition, 2005.
13. P. Duchet and H. Meyniel, On Hadwiger's number and the stability number, Annals of Discrete Mathematics 13 (1982) 71–74.
14. L. Fan, Z. Zhang, and W. Wang, PTAS for minimum weighted connected vertex cover problem with c-local condition in unit disk graphs, Journal of Combinatorial Optimization 22 (2011) 663–673.
15. M. Farber, Characterizations of strongly chordal graphs, Discrete Mathematics 43 (1983) 173–189.
16. S. Földes and P.L. Hammer, Split graphs, 8th South–Eastern Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium 19 (1977) 311–315.
17. T. Fujito, On approximability of the independent/connected edge dominating set problems, Information Processing Letters 79 (2001) 261–266.
18. A. Grigoriev and R. Sitters, Connected feedback vertex set in planar graphs, In: Proc. WG 2009, LNCS 5911 (2010) 143–153.
19. T.R. Hartinger, M. Johnson, M. Milanič, and D. Paulusma, The price of connectivity for cycle transversals, Proc. MFCS 2015, LNCS 9235 (2015) 395–406.
20. N. Misra, G. Philip, V. Raman, S. Saurabh and S. Sikdar, FPT algorithms for Connected Feedback Vertex Set.,J. Comb. Optim. 24(2012) 131–146.
21. G. Salamon, Vulnerability bounds on the number of spanning tree leaves, Ars Mathematica Contemporanea 2 (2009) 77–92.
22. P. Schweitzer and P. Schweitzer, Connecting face hitting sets in planar graphs, Information Processing Letters 111 (2010) 11–15.
23. F. Zou, X. Li, S. Gao, and W. Wu, Node-weighted Steiner tree approximation in unit disk graphs, Journal of Combinatorial Optimization 18 (2009) 342–349.
24. I.E. Zverovich, Perfect connected-dominant graphs, Discussiones Mathematicae Graph Theory 23 (2003)159–162.