# Cycles in graphs of fixed girth with large size 

József Solymosi*, Ching Wong ${ }^{\dagger}$


#### Abstract

Consider a family of graphs having a fixed girth and a large size. We give an optimal lower asymptotic bound on the number of even cycles of any constant length, as the order of the graphs tends to infinity.


## 1 Introduction

All graphs we consider in this article are simple graphs. We denote the size of $G$ by $e(G)$ and the order of $G$ by $v(G)$. A $j$-path in $G$ is a path of length $j$ in $G$. A $j$-cycle in $G$ is a cycle of length $j$ in $G$, and it is called an even cycle if $j$ is even. The girth of a graph $G$ is the length of the shortest cycles in $G$. For $x \in V(G)$, let $\Gamma_{G}^{k}(x)=\Gamma^{k}(x)$ denote the set of vertices of $G$ having distance exactly $k$ from the vertex $x$.

In the following, the big- $O$ notations $f(n)=O(g(n))$ are understood as $f(n)=O(g(n))$ as $n \rightarrow \infty$, where $n$ denotes the order of a graph. The same applies to $\Theta$.

It is easy to see that a graph having large girth cannot have too many edges. The famous Erdős girth conjecture asserts the existence of graphs of any given girth with a size of maximum possible order.

Conjecture 1 (Erdős girth conjecture). For any positive integer $m$, there exist a constant $c>0$ depending only on $m$ and a family of graphs $\left\{G_{n}\right\}$ such that $v\left(G_{n}\right)=n, e\left(G_{n}\right) \geq c n^{1+1 / m}$ and $\operatorname{girth}\left(G_{n}\right)>2 m$.

[^0]Indeed, such size is maximum by the result of Bondy and Simonovits [2], in which an explicit constant is given. They showed that a graph $G_{n}$ of order $n$ with girth $\left(G_{n}\right)>2 m$ has a size less than $100 m n^{1+1 / m}$.

This conjecture has been proved true for $m=1,2,3,5$. See [5], [3], [1] and [7]. For a general $m$, Sudakov and Verstraëte [6] showed that if such graphs exist, then they contain at least one cycle of any even length between and including $2 m+2$ and $C n$, for some constant $C>0$.

For $\ell>m$, by counting the number of $(2 \ell-2 m)$-paths, one can show that the number of $2 \ell$-cycles in such graphs has an order not greater than $O\left(n^{2 \ell / m}\right)$, provided that $\operatorname{deg}_{G_{n}}(x)=\Theta\left(n^{1 / m}\right)$ for any vertex $x$ in $G_{n}$. We will see in Section 2 that in the asymptotic case, one can assume this without loss of generality. This suggests the definition of almost regularity given in Section 2.

In this article, we give a lower bound on the number of $2 \ell$-cycles when $\ell=O(1)$, and conclude that the number of $2 \ell$-cycles is $\Theta\left(n^{2 \ell / m}\right)$. The precise statement is the following.

Theorem 2. For any real number $c>0$ and integers $M$, $m$ with $M>m \geq 2$, there exist a constant $\alpha>0$ and an integer $N$ such that if $\left\{G_{n}\right\}$ is a family of graphs satisfying $v\left(G_{n}\right)=n, e\left(G_{n}\right) \geq c n^{1+1 / m}$ and $\operatorname{girth}\left(G_{n}\right)>2 m$, then for $n \geq N$ and $m+1 \leq \ell \leq M$, the number of $2 \ell$-cycles in $G_{n}$ is at least $\alpha n^{2 \ell / m}$.

We proceed as follows. In Section 2, we show that by adjusting the threshold $N$ in our theorem, we can further assume that the graphs have some nice properties, namely bipartite and almost regular. In Section 3, we count the number of short even cycles in $G_{n}$ up to length $4 m$. Finally, in Section 4, we entend the argument to longer cycles, completing the proof of the main theorem.

## 2 Reduction to a simpler case

In this section, we show that it suffices to consider only bipartite graphs which are almost regular, defined as follows.

Definition 3. Suppose $\left\{G_{n}\right\}$ is a family of graphs with $v\left(G_{n}\right)=n$ and $\operatorname{girth}\left(G_{n}\right)>2 m$, we say that $\left\{G_{n}\right\}$ is almost regular if there exist $c_{1}, c_{2}>0$ such that $c_{1} n^{1 / m} \leq \operatorname{deg}(x) \leq c_{2} n^{1 / m}$ for any vertex $x \in V\left(G_{n}\right)$.

It is a well-known fact that any graph has a bipartite subgraph with at least half of its edges. It remains to construct subgraphs whose maximum and minimum degree is of order $n^{1 / m}$. To achieve this, we repeatedly apply a theorem of Bondy and Simonovits [2], which states that if an $n$-vertex graph $G_{n}$ has girth larger than $2 m$, then $G_{n}$ has less than $100 m n^{1+1 / m}$ edges.

First we delete vertices of small degree.
Lemma 4. For any real number $c>0$ and integer $m \geq 2$, there exists a constant $\beta>0$ such that any bipartite graph $G$ of order $n$, size at least cn ${ }^{1+1 / m}$ and girth larger than $2 m$ has a subgraph $H$ having at least $\beta$ n vertices, at least $\frac{9 c}{10} n^{1+1 / m}$ edges and minimum degree at least $\frac{c}{10} n^{1 / m}$.

Proof. Let $G=H_{0}$. For $i \geq 1$, inductively define $H_{i}$ to be the subgraph of $H_{i-1}$ induced by all vertices having degree at least $\frac{c}{10} n^{1 / m}$ in $H_{i-1}$. Then for all $i$ we have $e\left(H_{i}\right) \geq e\left(H_{i+1}\right)$ and

$$
\begin{aligned}
e\left(H_{i}\right) & \geq e\left(H_{0}\right)-\left(v\left(H_{0}\right)-v\left(H_{i}\right)\right) \frac{c}{10} n^{1 / m} \\
& \geq c n^{1+1 / m}-n \frac{c}{10} n^{1 / m} \\
& =\frac{9 c}{10} n^{1+1 / m}
\end{aligned}
$$

and so there exists some $j \geq 0$ such that $H_{i}=H_{j}$ for all $i \geq j$.
Set $H=H_{j}$, which has girth larger than $2 m$ and size at least $\frac{9 c}{10} n^{1+1 / m}$. Then,

$$
\frac{9 c}{10} n^{1+1 / m}<100 m \cdot v(H)^{1+1 / m}
$$

or $v(H) \geq \beta n$, where

$$
\beta=\left(\frac{9 c}{1000 m}\right)^{m /(m+1)}
$$

Lemma 5. For every real number $c>0$ and integer $m \geq 2$, there exists a constant $\gamma>0$ such that any bipartite graph $G$ of order $n$, size at least $c n^{1+1 / m}$ and girth larger than $2 m$ has a subgraph $H$ having at least $\frac{n}{2}$ vertices, at least $\frac{c}{4} n^{1+1 / m}$ edges and maximum degree at most $\gamma n^{1 / m}$.

Proof. For any $\gamma>0$, let $S_{\gamma}$ be the set of vertices of $G$ having degree at least $\gamma n^{1 / m}$, and let $T_{\gamma}$ be the remaining vertices of $G$. We want to find a $\gamma$ so
large that $H$ can be chosen as the subgraph $G\left[T_{\gamma}\right]$ of $G$ induced by $T_{\gamma}$. It suffices to find $\gamma$ large enough so that $e\left(G\left[S_{\gamma}\right]\right)<e(G) / 4, e\left(T_{\gamma}, S_{\gamma}\right)<e(G) / 2$ and $v\left(G\left[T_{\gamma}\right]\right) \geq n / 2$.

Since both $G$ and $G\left[S_{\gamma}\right]$ have girth larger than $2 m$, we can apply the result of [2] twice to obtain

$$
\begin{aligned}
e\left(G\left[S_{\gamma}\right]\right) & <100 m\left|S_{\gamma}\right|^{1+1 / m} \\
& \leq 100 m\left(\frac{2 e(G)}{\gamma n^{1 / m}}\right)^{1+1 / m} \\
& \leq 100 m\left(\frac{2 \cdot 100 m n^{1+1 / m}}{\gamma n^{1 / m}}\right)^{1+1 / m} \\
& =\left(\frac{2}{\gamma}\right)^{1+1 / m}(100 m)^{2+1 / m} n^{1+1 / m} .
\end{aligned}
$$

To satisfy the first condition, we choose $\gamma$ large enough so that $e\left(G\left[S_{\gamma}\right]\right)<$ $\frac{e(G)}{4}<\frac{100 m}{4} n^{1+1 / m}$, or

$$
\begin{equation*}
\gamma>2 \cdot 100 m \cdot 4^{m /(m+1)}>400 m \tag{1}
\end{equation*}
$$

The second condition can be obtained via its contrapositive. Suppose $e\left(S_{\gamma}, T_{\gamma}\right) \geq \frac{c}{2} n^{1+1 / m}$. Let $G_{\gamma}$ be the subgraph of $G$ induced by $E\left(S_{\gamma}, T_{\gamma}\right)$. Then apply Lemma 4 to $G_{\gamma}$, we get a subgraph $H_{\gamma}$ of $G_{\gamma}$ having $\nu \geq$ $\left(\frac{9 c}{2000 m}\right)^{m /(m+1)} n$ vertices and minimum degree at least $\frac{c}{20} n^{1 / m}$. Now, we consider the subgraph $G_{\gamma}^{\prime}$ of $G$ deleting the edges in $E\left(T_{\gamma}\right)$. Then, in $G_{\gamma}^{\prime}$, every vertex in $S_{\gamma} \cap V\left(H_{\gamma}\right)$ still has degree at least $\gamma n^{1 / m}$ and every vertex in $T_{\gamma} \cap V\left(H_{\gamma}\right)$ has degree at least $\frac{c}{20} n^{1 / m}$. Note that any $m$-path in $G_{\gamma}^{\prime}$ has at most $\lfloor m / 2\rfloor$ internal vertices in $T_{\gamma}$, therefore the number of $m$-paths in $G_{\gamma}^{\prime}$ is at least

$$
\frac{1}{2} \nu\left(\frac{c}{20} n^{1 / m}\right)^{\lfloor m / 2\rfloor}\left(\gamma n^{1 / m}\right)^{\lceil m / 2\rceil} \geq \frac{1}{2}\left(\frac{9 c}{2000 m}\right)^{m /(m+1)}\left(\frac{c \gamma}{20}\right)^{\lfloor m / 2\rfloor} n^{2}
$$

But the number of $m$-paths in $G$ cannot be larger than $n^{2}$, since otherwise there is a pair of vertices being the endpoints of two $m$-paths, contradicting the girth of $G$ is larger than $2 m$. Hence,

$$
\frac{1}{2}\left(\frac{9 c}{2000}\right)^{m /(m+1)}\left(\frac{c \gamma}{20}\right)^{\lfloor m / 2\rfloor}<1
$$

or

$$
\gamma<\frac{20}{c}\left(2\left(\frac{2000}{9 c}\right)^{m /(m+1)}\right)^{1 /\lfloor m / 2\rfloor}<\frac{40}{c}\left(\frac{2000}{9 c}\right)^{3 /(m+1)} .
$$

Therefore, if

$$
\gamma>\frac{40}{c}\left(\frac{2000}{9 c}\right)^{3 /(m+1)},
$$

then $e\left(S_{\gamma}, T_{\gamma}\right)<\frac{c}{2} n^{1+1 / m} \leq \frac{e(G)}{2}$.
Finally, since $\left|S_{\gamma}\right| \leq \frac{200 m}{\gamma} n$, the third condition is fulfilled by (11). This finishes the proof.

## 3 Counting short cycles

From now on, we suppose that $G$ is a bipartite $n$-vertex graph having at least $c n^{1+1 / m}$ edges with girth larger than $2 m$, such that for some constants $c_{1}, c_{2}>0$, there holds $c_{1} n^{1 / m} \leq \operatorname{deg}_{G}(x) \leq c_{2} n^{1 / m}$ for any vertex $x \in V(G)$.

In this section, we give a lower bound on the number of the $2 \ell$-cycles in $G$, for each $m+1 \leq \ell \leq 2 m$.

We first sketch the idea. Let $x$ be a vertex in $G$. Suppose we have a path of odd length $k$ in $\Gamma_{G}^{m}(x) \cup \Gamma_{G}^{m+1}(x)$ with endpoints $w_{0} \in \Gamma_{G}^{m}(x)$ and $w_{k} \in \Gamma_{G}^{m+1}(x)$. For each neighbor $y$ in $\Gamma_{G}^{m}(x)$ of $w_{k}$, the four paths joining $x$ to $w_{0}, w_{0}$ to $w_{k}, w_{k}$ to $y$, and $y$ to $x$ form a closed walk, which contains a cycle, as shown in Figure 1. We show in Section 3.1 that generically these paths are internally disjoint, i.e. the length of the cycle is $2 m+k+1$. Then we count in Section 3.2 the number of such paths and the number of neighbours of $w_{k}$. Finally, we obtain the desired lower bound in Section 3.3.

### 3.1 Internally disjoint closed walk

Note that for distinct $i, j \leq m+1$, we know that $\Gamma_{G}^{i}(x) \cap \Gamma_{G}^{j}(x)$ is empty because $G$ is bipartite and has girth larger than $2 m$. In particular, the subgraph $G_{x}$ of $G$ induced by the vertices $\Gamma_{G}^{m}(x) \cup \Gamma_{G}^{m+1}(x)$ is bipartite with bipartition $\left\{\Gamma_{G}^{m}(x), \Gamma_{G}^{m+1}(x)\right\}$. Hence, any path of odd length in $G_{x}$ has one endpoint in $\Gamma_{G}^{m}(x)$ and the other endpoint in $\Gamma_{G}^{m+1}(x)$.

For $1 \leq i \leq m$ and any vertex $w \in \Gamma_{G}^{i}(x)$, there is a unique $(x, w)$ path $P_{w}^{x}$ of length $i$ in $G$. Note that for any two vertices $y_{1}, y_{2} \in \Gamma_{G}^{m}(x)$, the intersection of the paths $P_{y_{1}}^{x}$ and $P_{y_{2}}^{x}$ must be a path, of which $x$ is an


Figure 1: The four paths form a closed walk, which contains a cycle
endpoint. The following lemma guarantees that if $w_{0} \in \Gamma_{G}^{m}(x)$ and $w_{k} \in$ $\Gamma_{G}^{m+1}(x)$, there is at most one neighbour $u \in \Gamma_{G_{x}}^{1}\left(w_{k}\right)$ so that $P_{w_{0}}^{x}$ and $P_{u}^{x}$ intersect internally, see Figure 2 .


Figure 2: Other neighbours of $w_{k}$ give internally disjoint paths

Lemma 6. Suppose two vertices $y_{1}, y_{2} \in \Gamma_{G}^{m}(x)$ share a common neighbour $w \in \Gamma_{G}^{m+1}(x)$, then the paths $P_{y_{1}}$ and $P_{y_{2}}$ are internally disjoint.

Proof. Suppose the paths $P_{y_{1}}^{x}$ and $P_{y_{2}}^{x}$ intersects internally, then their intersection must be a path of length $L \geq 1$, with endpoints $x$ and $v$, for some


Figure 3: A cycle of length at most $2 m$ formed
$v \in \Gamma_{G}^{L}(x)$. Thus, the union of the paths $P_{y_{1}} \backslash P_{v}, P_{y_{2}} \backslash P_{v}$ and the edges $\left(y_{2}, w\right),\left(w, y_{1}\right)$ is a cycle of length $2(m-L)+2 \leq 2 m$ in $G$, as in Figure 3, contradiction.

Now, given a path $P=\left(w_{0}, w_{1}, \ldots, w_{k}\right)$ of odd length $k \leq 2 m-1$ in $G_{x}$ with $w_{0} \in \Gamma_{G}^{m}(x)$ and $w_{k} \in \Gamma_{G}^{m+1}(x)$. Note that $V(P) \cap \Gamma_{G_{\sim}}^{1}\left(w_{k}\right)=\left\{w_{k-1}\right\}$ as $\operatorname{girth}\left(G_{n}\right)>2 m$. Let $y \in \Gamma_{G_{x}}^{1}\left(w_{k}\right)$. As shown in Figure 4, the four paths $P_{w_{0}}^{x}, P,\left(w_{k}, y\right)$ and $P_{y}^{x}$ contain a cycle of length $2 m+k+1$, with at most two exceptions, namely $y=w_{k-1}$ and $y=u$.


Figure 4: Most neighbours of $w_{k}$ give cycles of length $2 m+k+1$

### 3.2 Number of paths in $G_{x}$

Note that in $G_{x}$, the minimum degree can be as small as 1 . Instead of counting the number of paths of a given length in $G_{x}$, we work with a subgraph of $G_{x}$ having large minimum degree. We adopt the result from Section 2 .

It is easy to see that $v\left(G_{x}\right) \leq n$ and

$$
e\left(G_{x}\right) \geq\left|\Gamma_{G}^{m}(x)\right| c_{1} n^{1 / m} \geq\left(c_{1} n^{1 / m}\right)^{m+1}=c_{1}^{m+1} n^{1+1 / m}
$$

Using Lemma 4, we obtain a bipartite subgraph $H_{x}$ of $G_{x}$ having order at least $\left(\frac{9 c_{1}^{m+1}}{1000 m}\right)^{1 /(1+m)} n$, size at least $\frac{9 c_{1}^{m+1}}{10} n^{1+1 / m}$, and $\frac{c_{1}^{m+1}}{10} n^{1 / m} \leq \operatorname{deg}_{H_{x}}(u) \leq$ $c_{2} n^{1 / m}$, for any vertex $u \in V\left(H_{x}\right)$, with bipartition $\left\{A_{x}, B_{x}\right\}$, where $A_{x}=$ $\Gamma_{G}^{m}(x) \cap H_{x}$, and $B_{x}=\Gamma_{G}^{m+1}(x) \cap H_{x}$.

Lemma 7. Let $k$ be an odd number satisfying $1 \leq k \leq 2 m-1$. The number of $k$-paths in $G_{x}$ is at least

$$
\frac{9 c_{1}^{(m+1)(k+1)}}{10^{k+1} c_{2}} n^{1+k / m}
$$

Proof. The result follows from

$$
\left|B_{x}\right| \geq \frac{e\left(H_{x}\right)}{c_{2} n^{1 / m}} \geq \frac{9 c_{1}^{m+1}}{10 c_{2}} n
$$

and that the number of $k$-paths in $G_{x}$ is at least

$$
\left|B_{x}\right|\left(\frac{c_{1}^{m+1}}{10} n^{1 / m}\right)^{k}
$$

### 3.3 Lower bound on the number of short cycles

The work in the preceding sections allows us to find a lot cycles in $G$. It is clear that a $2 \ell$-cycle can be counted by at most $2 \ell$ times as each vertex of the cycle can play the role of $x$ once.

We are ready to give a lower bound on the number of short even cycles, up to length $4 m$ in $G$.

Proposition 8. Let $m$ be a positive integer. Let $G$ be a bipartite $n$-vertex graph having girth larger than $2 m$ and $c_{1} n^{1 / m} \leq \operatorname{deg}_{G}(v) \leq c_{2} n^{1 / m}$ for any vertex $v \in V(G)$, for some $c_{1}, c_{2}>0$. Then for $m+1 \leq \ell \leq 2 m$, the number of $2 \ell$-cycles in $G$ is at least $\alpha_{\ell} n^{2 \ell / m}$, where

$$
\alpha_{\ell}=\frac{9}{2 \ell c_{2}}\left(\frac{c_{1}^{m+1}}{10}\right)^{2 \ell-2 m+1}>0
$$

Proof. Using Lemma 7 with $k=2 \ell-2 m-1$ and the observation above, the number of $2 \ell$-cycles in $G$ is at least

$$
\begin{aligned}
& \frac{v(G)}{2 \ell}\left(\min _{x \in V(G)} \operatorname{deg}\left(H_{x}\right)\right)\left(\min _{x \in V(G)} \text { number of }(2 \ell-2 m-1) \text {-paths in } H_{x}\right) \\
\geq & \frac{n}{2 \ell}\left(\frac{c_{1}^{m+1}}{10} n^{1 / m}\right)\left(\frac{9 c_{1}^{(2 \ell-2 m)(m+1)}}{10^{2 \ell-2 m} c_{2}} n^{(2 \ell-m-1) / m}\right) \\
= & \frac{9}{2 \ell c_{2}}\left(\frac{c_{1}^{m+1}}{10}\right)^{2 \ell-2 m+1} n^{2 \ell / m} .
\end{aligned}
$$

## 4 Proof of main theorem

To count the number of longer cycles, we observe that $H_{x}$ has all the nice properties we wanted, namely bipartite, almost regular and large girth, we can apply Proposition 8 to $H_{x}$ and get many short cycles in $H_{x}$. From them, we obtain a lot of paths in $H_{x}$, and each of them corresponds to many longer cycles in $G$ as in Section 3. These longer cycles give many longer paths in $H_{x}$ and again, each of these paths corresponds to many even longer cycles in $G_{x}$. Eventually, we have Theorem 2.

For simplicity, we will assume that $m$ is even from now on. For odd $m$, one can proceed similarly.

Changing the parameters in Proposition 8, the number of $2 \ell$-cycles in $H_{x}$ is at least

$$
\frac{9}{2 \ell c_{2}}\left(\frac{c_{1}^{(m+1)^{2}}}{10^{m+2}}\right)^{2 \ell-2 m+1} n^{2 \ell / m}
$$

for $2 \ell \in L_{0}:=\{3 m, 3 m+2,3 m+4, \ldots, 4 m\}$, and so the number of paths of length $2 \ell-m \in\{2 m, 2 m+2,2 m+4, \ldots, 3 m\}$ in $H_{x}$ is at least

$$
\frac{9}{c_{2}}\left(\frac{c_{1}^{(m+1)^{2}}}{10^{m+2}}\right)^{2 \ell-2 m+1} n^{2 \ell / m}
$$

Then, for $2 \ell \in L_{0}$, the number of $((2 \ell-m)+2 m+2)$-cycles in $G$ is at least

$$
\begin{aligned}
& \frac{n}{2 \ell+m+2} \frac{c_{1}^{2(m+1)}}{100} n^{2 / m} \frac{9}{c_{2}}\left(\frac{c_{1}^{(m+1)^{2}}}{10^{m+2}}\right)^{2 \ell-2 m+1} n^{2 \ell / m} \\
= & \frac{9}{(2 \ell+m+2) c_{2}} \frac{c_{1}^{2(m+1)}}{100}\left(\frac{c_{1}^{(m+1)^{2}}}{10^{m+2}}\right)^{2 \ell-2 m+1} n^{(2 \ell+m+2) / m},
\end{aligned}
$$

or for $2 \ell \in L_{1}:=\{4 m+2,4 m+4, \ldots, 5 m+2\}$, the number of $2 \ell$-cycles in $G$ is at least $\alpha_{\ell} n^{2 \ell / m}$, where $\alpha_{\ell}$ is a positive constant depending on $m, c_{1}, c_{2}, \ell$ only. Repeating the same argument with the sets $L_{j}:=\{3 m+j(m+2), 3 m+$ $j(m+2)+2, \ldots, 4 m+j(m+2)\}$, the proof of Theorem 2 is completed.

## References

[1] C. T. Benson, Minimal regular graphs of girths eight and twelve, Canad. J. Math. 181966 1091-1094.
[2] J. A. Bondy and M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974), 97-105.
[3] W. G. Brown, On graphs that do not contain a Thomsen graph, Canad. Math. Bull. 91966 281-285.
[4] P. Erdős, Extremal problems in graph theory In Proc. Symp. Theory of Graphs and its Applications, page 2936, 1963.
[5] I. Reiman, Über ein Problem von K. Zarankiewicz, Acta. Math. Acad. Sci. Hungar. 91958 269-273.
[6] B. Sudakoov and J. Verstraëte, Cycle lengths in sparse graphs, Combinatorica 28 (2008), no. 3, 357372.
[7] R. Wenger, Extremal graphs with no C4's, C6's, or C10's, J. Combin. Theory Ser. B 52 (1991), no. 1, 113-116.


[^0]:    *Research was supported by NSERC, ERC-AdG. 321104, and OTKA NK 104183 grants.
    ${ }^{\dagger}$ Research was supported by FYF (UBC).

