Cycles in graphs of fixed girth with large size

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Abstract

Consider a family of graphs having a fixed girth and a large size. We give an optimal lower asymptotic bound on the number of even cycles of any constant length, as the order of the graphs tends to infinity.

1 Introduction

All graphs we consider in this article are simple graphs. We denote the size of G by e(G) and the order of G by v(G). A *j*-path in G is a path of length *j* in G. A *j*-cycle in G is a cycle of length *j* in G, and it is called an even cycle if *j* is even. The *girth* of a graph G is the length of the shortest cycles in G. For $x \in V(G)$, let $\Gamma_G^k(x) = \Gamma^k(x)$ denote the set of vertices of G having distance exactly *k* from the vertex *x*.

In the following, the big-O notations f(n) = O(g(n)) are understood as f(n) = O(g(n)) as $n \to \infty$, where n denotes the order of a graph. The same applies to Θ .

It is easy to see that a graph having large girth cannot have too many edges. The famous Erdős girth conjecture asserts the existence of graphs of any given girth with a size of maximum possible order.

Conjecture 1 (Erdős girth conjecture). For any positive integer m, there exist a constant c > 0 depending only on m and a family of graphs $\{G_n\}$ such that $v(G_n) = n$, $e(G_n) \ge cn^{1+1/m}$ and $girth(G_n) > 2m$.

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Indeed, such size is maximum by the result of Bondy and Simonovits [2], in which an explicit constant is given. They showed that a graph G_n of order n with girth $(G_n) > 2m$ has a size less than $100mn^{1+1/m}$.

This conjecture has been proved true for m = 1, 2, 3, 5. See [5], [3], [1] and [7]. For a general m, Sudakov and Verstraëte [6] showed that if such graphs exist, then they contain at least one cycle of any even length between and including 2m + 2 and Cn, for some constant C > 0.

For $\ell > m$, by counting the number of $(2\ell - 2m)$ -paths, one can show that the number of 2ℓ -cycles in such graphs has an order not greater than $O(n^{2\ell/m})$, provided that $\deg_{G_n}(x) = \Theta(n^{1/m})$ for any vertex x in G_n . We will see in Section 2 that in the asymptotic case, one can assume this without loss of generality. This suggests the definition of *almost regularity* given in Section 2.

In this article, we give a lower bound on the number of 2ℓ -cycles when $\ell = O(1)$, and conclude that the number of 2ℓ -cycles is $\Theta(n^{2\ell/m})$. The precise statement is the following.

Theorem 2. For any real number c > 0 and integers M, m with $M > m \ge 2$, there exist a constant $\alpha > 0$ and an integer N such that if $\{G_n\}$ is a family of graphs satisfying $v(G_n) = n$, $e(G_n) \ge cn^{1+1/m}$ and $girth(G_n) > 2m$, then for $n \ge N$ and $m + 1 \le \ell \le M$, the number of 2ℓ -cycles in G_n is at least $\alpha n^{2\ell/m}$.

We proceed as follows. In Section 2, we show that by adjusting the threshold N in our theorem, we can further assume that the graphs have some nice properties, namely bipartite and almost regular. In Section 3, we count the number of short even cycles in G_n up to length 4m. Finally, in Section 4, we entend the argument to longer cycles, completing the proof of the main theorem.

2 Reduction to a simpler case

In this section, we show that it suffices to consider only bipartite graphs which are almost regular, defined as follows.

Definition 3. Suppose $\{G_n\}$ is a family of graphs with $v(G_n) = n$ and $girth(G_n) > 2m$, we say that $\{G_n\}$ is almost regular if there exist $c_1, c_2 > 0$ such that $c_1 n^{1/m} \leq \deg(x) \leq c_2 n^{1/m}$ for any vertex $x \in V(G_n)$.

It is a well-known fact that any graph has a bipartite subgraph with at least half of its edges. It remains to construct subgraphs whose maximum and minimum degree is of order $n^{1/m}$. To achieve this, we repeatedly apply a theorem of Bondy and Simonovits [2], which states that if an *n*-vertex graph G_n has girth larger than 2m, then G_n has less than $100mn^{1+1/m}$ edges.

First we delete vertices of small degree.

Lemma 4. For any real number c > 0 and integer $m \ge 2$, there exists a constant $\beta > 0$ such that any bipartite graph G of order n, size at least $cn^{1+1/m}$ and girth larger than 2m has a subgraph H having at least βn vertices, at least $\frac{9c}{10}n^{1+1/m}$ edges and minimum degree at least $\frac{c}{10}n^{1/m}$.

Proof. Let $G = H_0$. For $i \ge 1$, inductively define H_i to be the subgraph of H_{i-1} induced by all vertices having degree at least $\frac{c}{10}n^{1/m}$ in H_{i-1} . Then for all i we have $e(H_i) \ge e(H_{i+1})$ and

$$e(H_i) \ge e(H_0) - (v(H_0) - v(H_i))\frac{c}{10}n^{1/m}$$

$$\ge cn^{1+1/m} - n\frac{c}{10}n^{1/m}$$

$$= \frac{9c}{10}n^{1+1/m},$$

and so there exists some $j \ge 0$ such that $H_i = H_j$ for all $i \ge j$.

Set $H = H_j$, which has girth larger than 2m and size at least $\frac{9c}{10}n^{1+1/m}$. Then,

$$\frac{9c}{10}n^{1+1/m} < 100m \cdot v(H)^{1+1/m},$$

or $v(H) \geq \beta n$, where



Lemma 5. For every real number c > 0 and integer $m \ge 2$, there exists a constant $\gamma > 0$ such that any bipartite graph G of order n, size at least $cn^{1+1/m}$ and girth larger than 2m has a subgraph H having at least $\frac{n}{2}$ vertices, at least $\frac{c}{4}n^{1+1/m}$ edges and maximum degree at most $\gamma n^{1/m}$.

Proof. For any $\gamma > 0$, let S_{γ} be the set of vertices of G having degree at least $\gamma n^{1/m}$, and let T_{γ} be the remaining vertices of G. We want to find a γ so

large that H can be chosen as the subgraph $G[T_{\gamma}]$ of G induced by T_{γ} . It suffices to find γ large enough so that $e(G[S_{\gamma}]) < e(G)/4$, $e(T_{\gamma}, S_{\gamma}) < e(G)/2$ and $v(G[T_{\gamma}]) \geq n/2$.

Since both G and $G[S_{\gamma}]$ have girth larger than 2m, we can apply the result of [2] twice to obtain

$$e(G[S_{\gamma}]) < 100m |S_{\gamma}|^{1+1/m}$$

$$\leq 100m \left(\frac{2e(G)}{\gamma n^{1/m}}\right)^{1+1/m}$$

$$\leq 100m \left(\frac{2 \cdot 100m n^{1+1/m}}{\gamma n^{1/m}}\right)^{1+1/m}$$

$$= \left(\frac{2}{\gamma}\right)^{1+1/m} (100m)^{2+1/m} n^{1+1/m}.$$

To satisfy the first condition, we choose γ large enough so that $e(G[S_{\gamma}]) < \frac{e(G)}{4} < \frac{100m}{4}n^{1+1/m}$, or

$$\gamma > 2 \cdot 100m \cdot 4^{m/(m+1)} > 400m. \tag{1}$$

The second condition can be obtained via its contrapositive. Suppose $e(S_{\gamma}, T_{\gamma}) \geq \frac{c}{2}n^{1+1/m}$. Let G_{γ} be the subgraph of G induced by $E(S_{\gamma}, T_{\gamma})$. Then apply Lemma 4 to G_{γ} , we get a subgraph H_{γ} of G_{γ} having $\nu \geq \left(\frac{9c}{2000m}\right)^{m/(m+1)}n$ vertices and minimum degree at least $\frac{c}{20}n^{1/m}$. Now, we consider the subgraph G'_{γ} of G deleting the edges in $E(T_{\gamma})$. Then, in G'_{γ} , every vertex in $S_{\gamma} \cap V(H_{\gamma})$ still has degree at least $\gamma n^{1/m}$ and every vertex in $T_{\gamma} \cap V(H_{\gamma})$ has degree at least $\frac{c}{20}n^{1/m}$. Note that any m-path in G'_{γ} has at most $\lfloor m/2 \rfloor$ internal vertices in T_{γ} , therefore the number of m-paths in G'_{γ} is at least

$$\frac{1}{2}\nu\left(\frac{c}{20}n^{1/m}\right)^{\lfloor m/2 \rfloor}\left(\gamma n^{1/m}\right)^{\lceil m/2 \rceil} \geq \frac{1}{2}\left(\frac{9c}{2000m}\right)^{m/(m+1)}\left(\frac{c\gamma}{20}\right)^{\lfloor m/2 \rfloor}n^2.$$

But the number of *m*-paths in *G* cannot be larger than n^2 , since otherwise there is a pair of vertices being the endpoints of two *m*-paths, contradicting the girth of *G* is larger than 2m. Hence,

$$\frac{1}{2} \left(\frac{9c}{2000}\right)^{m/(m+1)} \left(\frac{c\gamma}{20}\right)^{\lfloor m/2 \rfloor} < 1,$$

or

$$\gamma < \frac{20}{c} \left(2 \left(\frac{2000}{9c} \right)^{m/(m+1)} \right)^{1/\lfloor m/2 \rfloor} < \frac{40}{c} \left(\frac{2000}{9c} \right)^{3/(m+1)}.$$

Therefore, if

$$\gamma > \frac{40}{c} \left(\frac{2000}{9c}\right)^{3/(m+1)}$$

then $e(S_{\gamma}, T_{\gamma}) < \frac{c}{2}n^{1+1/m} \leq \frac{e(G)}{2}$. Finally, since $|S_{\gamma}| \leq \frac{200m}{\gamma}n$, the third condition is fulfilled by (1). This finishes the proof.

3 Counting short cycles

From now on, we suppose that G is a bipartite *n*-vertex graph having at least $cn^{1+1/m}$ edges with girth larger than 2m, such that for some constants $c_1, c_2 > 0$, there holds $c_1 n^{1/m} \leq \deg_G(x) \leq c_2 n^{1/m}$ for any vertex $x \in V(G)$.

In this section, we give a lower bound on the number of the 2ℓ -cycles in G, for each $m + 1 \leq \ell \leq 2m$.

We first sketch the idea. Let x be a vertex in G. Suppose we have a path of odd length k in $\Gamma_G^m(x) \cup \Gamma_G^{m+1}(x)$ with endpoints $w_0 \in \Gamma_G^m(x)$ and $w_k \in \Gamma_G^{m+1}(x)$. For each neighbor y in $\Gamma_G^m(x)$ of w_k , the four paths joining x to w_0 , w_0 to w_k , w_k to y, and y to x form a closed walk, which contains a cycle, as shown in Figure 1. We show in Section 3.1 that generically these paths are internally disjoint, i.e. the length of the cycle is 2m+k+1. Then we count in Section 3.2 the number of such paths and the number of neighbours of w_k . Finally, we obtain the desired lower bound in Section 3.3.

3.1Internally disjoint closed walk

Note that for distinct $i, j \leq m+1$, we know that $\Gamma_G^i(x) \cap \Gamma_G^j(x)$ is empty because G is bipartite and has girth larger than 2m. In particular, the subgraph G_x of G induced by the vertices $\Gamma_G^m(x) \cup \Gamma_G^{m+1}(x)$ is bipartite with bipartition $\{\Gamma_G^m(x), \Gamma_G^{m+1}(x)\}$. Hence, any path of odd length in G_x has one endpoint in $\Gamma_G^m(x)$ and the other endpoint in $\Gamma_G^{m+1}(x)$.

For $1 \leq i \leq m$ and any vertex $w \in \Gamma_G^i(x)$, there is a unique (x, w)path P_w^x of length *i* in *G*. Note that for any two vertices $y_1, y_2 \in \Gamma_G^m(x)$, the intersection of the paths $P_{y_1}^x$ and $P_{y_2}^x$ must be a path, of which x is an



Figure 1: The four paths form a closed walk, which contains a cycle

endpoint. The following lemma guarantees that if $w_0 \in \Gamma_G^m(x)$ and $w_k \in \Gamma_G^{m+1}(x)$, there is at most one neighbour $u \in \Gamma_{G_x}^1(w_k)$ so that $P_{w_0}^x$ and P_u^x intersect internally, see Figure 2.



Figure 2: Other neighbours of w_k give internally disjoint paths

Lemma 6. Suppose two vertices $y_1, y_2 \in \Gamma_G^m(x)$ share a common neighbour $w \in \Gamma_G^{m+1}(x)$, then the paths P_{y_1} and P_{y_2} are internally disjoint.

Proof. Suppose the paths $P_{y_1}^x$ and $P_{y_2}^x$ intersects internally, then their intersection must be a path of length $L \ge 1$, with endpoints x and v, for some



Figure 3: A cycle of length at most 2m formed

 $v \in \Gamma_G^L(x)$. Thus, the union of the paths $P_{y_1} \setminus P_v$, $P_{y_2} \setminus P_v$ and the edges (y_2, w) , (w, y_1) is a cycle of length $2(m - L) + 2 \leq 2m$ in G, as in Figure 3, contradiction.

Now, given a path $P = (w_0, w_1, \ldots, w_k)$ of odd length $k \leq 2m - 1$ in G_x with $w_0 \in \Gamma_G^m(x)$ and $w_k \in \Gamma_G^{m+1}(x)$. Note that $V(P) \cap \Gamma_{G_x}^1(w_k) = \{w_{k-1}\}$ as girth $(G_n) > 2m$. Let $y \in \Gamma_{G_x}^1(w_k)$. As shown in Figure 4, the four paths $P_{w_0}^x$, P, (w_k, y) and P_y^x contain a cycle of length 2m + k + 1, with at most two exceptions, namely $y = w_{k-1}$ and y = u.



Figure 4: Most neighbours of w_k give cycles of length 2m + k + 1

3.2 Number of paths in G_x

Note that in G_x , the minimum degree can be as small as 1. Instead of counting the number of paths of a given length in G_x , we work with a subgraph of G_x having large minimum degree. We adopt the result from Section 2.

It is easy to see that $v(G_x) \leq n$ and

$$e(G_x) \ge \left|\Gamma_G^m(x)\right| c_1 n^{1/m} \ge \left(c_1 n^{1/m}\right)^{m+1} = c_1^{m+1} n^{1+1/m}.$$

Using Lemma 4, we obtain a bipartite subgraph H_x of G_x having order at least $\left(\frac{9c_1^{m+1}}{1000m}\right)^{1/(1+m)}$ n, size at least $\frac{9c_1^{m+1}}{10}n^{1+1/m}$, and $\frac{c_1^{m+1}}{10}n^{1/m} \leq \deg_{H_x}(u) \leq c_2n^{1/m}$, for any vertex $u \in V(H_x)$, with bipartition $\{A_x, B_x\}$, where $A_x = \Gamma_G^m(x) \cap H_x$, and $B_x = \Gamma_G^{m+1}(x) \cap H_x$.

Lemma 7. Let k be an odd number satisfying $1 \le k \le 2m - 1$. The number of k-paths in G_x is at least

$$\frac{9c_1^{(m+1)(k+1)}}{10^{k+1}c_2}n^{1+k/m}$$

Proof. The result follows from

$$|B_x| \ge \frac{e(H_x)}{c_2 n^{1/m}} \ge \frac{9c_1^{m+1}}{10c_2}n$$

and that the number of k-paths in G_x is at least

$$|B_x| \left(\frac{c_1^{m+1}}{10} n^{1/m}\right)^k$$

3.3 Lower bound on the number of short cycles

The work in the preceding sections allows us to find a lot cycles in G. It is clear that a 2ℓ -cycle can be counted by at most 2ℓ times as each vertex of the cycle can play the role of x once.

We are ready to give a lower bound on the number of short even cycles, up to length 4m in G.

Proposition 8. Let m be a positive integer. Let G be a bipartite n-vertex graph having girth larger than 2m and $c_1n^{1/m} \leq \deg_G(v) \leq c_2n^{1/m}$ for any vertex $v \in V(G)$, for some $c_1, c_2 > 0$. Then for $m + 1 \leq \ell \leq 2m$, the number of 2ℓ -cycles in G is at least $\alpha_\ell n^{2\ell/m}$, where

$$\alpha_{\ell} = \frac{9}{2\ell c_2} \left(\frac{c_1^{m+1}}{10}\right)^{2\ell - 2m+1} > 0.$$

Proof. Using Lemma 7 with $k = 2\ell - 2m - 1$ and the observation above, the number of 2ℓ -cycles in G is at least

$$\frac{v(G)}{2\ell} \left(\min_{x \in V(G)} \deg(H_x) \right) \left(\min_{x \in V(G)} \text{ number of } (2\ell - 2m - 1) \text{-paths in } H_x \right)$$

$$\geq \frac{n}{2\ell} \left(\frac{c_1^{m+1}}{10} n^{1/m} \right) \left(\frac{9c_1^{(2\ell - 2m)(m+1)}}{10^{2\ell - 2m}c_2} n^{(2\ell - m - 1)/m} \right)$$

$$= \frac{9}{2\ell c_2} \left(\frac{c_1^{m+1}}{10} \right)^{2\ell - 2m + 1} n^{2\ell/m}.$$

4 Proof of main theorem

To count the number of longer cycles, we observe that H_x has all the nice properties we wanted, namely bipartite, almost regular and large girth, we can apply Proposition 8 to H_x and get many short cycles in H_x . From them, we obtain a lot of paths in H_x , and each of them corresponds to many longer cycles in G as in Section 3. These longer cycles give many longer paths in H_x and again, each of these paths corresponds to many even longer cycles in G_x . Eventually, we have Theorem 2.

For simplicity, we will assume that m is even from now on. For odd m, one can proceed similarly.

Changing the parameters in Proposition 8, the number of 2ℓ -cycles in H_x is at least

$$\frac{9}{2\ell c_2} \left(\frac{c_1^{(m+1)^2}}{10^{m+2}}\right)^{2\ell-2m+1} n^{2\ell/m},$$

for $2\ell \in L_0 := \{3m, 3m+2, 3m+4, \dots, 4m\}$, and so the number of paths of length $2\ell - m \in \{2m, 2m+2, 2m+4, \dots, 3m\}$ in H_x is at least

$$\frac{9}{c_2} \left(\frac{c_1^{(m+1)^2}}{10^{m+2}} \right)^{2\ell - 2m+1} n^{2\ell/m}.$$

Then, for $2\ell \in L_0$, the number of $((2\ell - m) + 2m + 2)$ -cycles in G is at least

$$\frac{n}{2\ell+m+2} \frac{c_1^{2(m+1)}}{100} n^{2/m} \frac{9}{c_2} \left(\frac{c_1^{(m+1)^2}}{10^{m+2}}\right)^{2\ell-2m+1} n^{2\ell/m}$$
$$= \frac{9}{(2\ell+m+2)c_2} \frac{c_1^{2(m+1)}}{100} \left(\frac{c_1^{(m+1)^2}}{10^{m+2}}\right)^{2\ell-2m+1} n^{(2\ell+m+2)/m},$$

or for $2\ell \in L_1 := \{4m+2, 4m+4, \ldots, 5m+2\}$, the number of 2ℓ -cycles in G is at least $\alpha_{\ell} n^{2\ell/m}$, where α_{ℓ} is a positive constant depending on m, c_1, c_2, ℓ only. Repeating the same argument with the sets $L_j := \{3m+j(m+2), 3m+j(m+2)+2, \ldots, 4m+j(m+2)\}$, the proof of Theorem 2 is completed.

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