On Sequences of Polynomials Arising from Graph Invariants

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Abstract

Graph polynomials are deemed useful if they give rise to algebraic characterizations of various graph properties, and their evaluations encode many other graph invariants. Algebraic: The complete graphs K_n and the complete bipartite graphs $K_{n,n}$ can be characterized as those graphs whose matching polynomials satisfy a certain recurrence relations and are related to the Hermite and Laguerre polynomials. An encoded graph invariant: The absolute value of the chromatic polynomial $\chi(G, X)$ of a graph G evaluated at -1 counts the number of acyclic orientations of G.

In this paper we prove a general theorem on graph families which are characterized by families of polynomials satisfying linear recurrence relations. This gives infinitely many instances similar to the characterization of $K_{n,n}$. We also show where to use, instead of the Hermite and Laguerre polynomials, linear recurrence relations where the coefficients do not depend on n.

Finally, we discuss the distinctive power of graph polynomials in specific form.

Keywords: Graph polynomials, Chromatic Polynomial, Orthogonal polynomials

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1. Introduction and background

1.1. Wilf's Recognition problem

H. Wilf asked in [Wil73] to characterize and recognize the instances of the chromatic polynomial. C.D. Godsil and I. Gutman [GG81] gave a characterization of the instances of the defect matching polynomial $\mu(G; X)$ for paths P_n , cycles C_n , complete graphs K_n and bipartite complete graphs $K_{n,n}$ in terms of orthogonal polynomials. We want to put Wilf's question and C.D. Godsil and I. Gutman's observation into a larger perspective. First we have to fix some terminology. Let \mathcal{G} denote the class of all finite graphs with no multiple edges. A graph property is a class of graphs $\mathcal{C} \subseteq \mathcal{G}$ closed under graph isomorphism. A graph polynomial with r indeterminates⁴ $\overline{X} = (X_1, \ldots, X_r)$ is a function \mathbf{P} from all finite graphs into the polynomial ring $\mathbb{Z}[\overline{X}]$ which is invariant under graph isomorphism. We write $\mathbf{P}(G; \overline{X})$ for the polynomial associated with the graph G.

Definition 1. A graph polynomial \mathbf{P} is computable if

- (i) **P** is a Turing computable function, and additionally,
- (ii) the range of \mathbf{P} , the set

 $\{p(\bar{X}) \in \mathbb{Z}[\bar{X}]: \text{ there is a graph } G \text{ with } \mathbf{P}(G; \bar{X}) = p(\bar{X})\}$

is Turing decidable.

In this paper we give a general formulation to Wilf's question.

Problem 1 (Recognition and Characterization Problem:). Given a graph polynomial $\mathbf{P}(G; \bar{X})$ and a graph property C, define

 $\mathcal{Y}_{\mathbf{P},\mathcal{C}} = \{ p(\bar{X}) \in \mathbb{Z}[\bar{X}] : \exists G \in \mathcal{C} \text{ with } \mathbf{P}(G;\bar{X}) = p(\bar{X}) \}$

- (i) The recognition problem asks for an algebraic method to decide membership in $\mathcal{Y}_{\mathbf{P},\mathcal{C}}$.
- (ii) The characterization problem asks for an algebraic characterization of $\mathcal{Y}_{\mathbf{P},\mathcal{C}}$, i.e., an algebraic characterization of the coefficients of $p(\bar{X})$.

Both the recognition and the characterization problem were stated explicitly for the chromatic polynomial $\chi(G; X)$ and \mathcal{C} the class of all finite graphs by H. Wilf, [Wil73], and he deemed them to be very difficult.

When H. Wilf asked the question about the chromatic polynomial he had an algebraic and descriptive answer in mind. Something like, a polynomial

⁴If the polynomial is univariate, we write X instead of \bar{X}

p(X) is a chromatic polynomial of a some graph G iff the coefficients satisfy some relations. The conjecture, that the absolute values of the coefficients of the chromatic polynomial form a unimodal sequence, only recently proved by J. Huh, [Huh15] has its origin in Wilf's question. H. Wilf was not concerned about algorithmic complexity.

From a complexity point of view, we note that deciding whether a given polynomial p(X) is a chromatic polynomial of a some graph G can be decided by brute force in exponential time as follows:

- (i) Use the degree d_p of p(X) to determine the upper bound on the size of the candidate graph G. In the case of the chromatic polynomial we have |V(G)| = d_p.
- (ii) Let I(n) be the number of graphs, up to isomorphism, of order n. Listing all graphs, up to isomorphism, of order n, is exponential in n.
- (iii) For $i \leq I(d_p)$ compute the chromatic polynomial $\chi(G_i; X)$ and test if $p(X) = \chi(G_i; X)$. Evaluating $\chi(G_i; X)$ for X = a and $a \in \mathbb{N}$ is in $\sharp \mathbf{P}$.

The same argument works for many other graph polynomials.

Problem 2 (Algorithmic version of Wilf's problem:). Given a graph polynomial **P** and a graph property C, determine the complexity of the recognition problem for $\mathcal{Y}_{\mathbf{P},C}$.

One can view the result of C.D. Godsil and I. Gutman, [GG81] as a solution of a very special case of Wilf's Characterization Problem, where $\mathbf{P} = \mu(G; X)$ is the matching polynomial, and \mathcal{C} is the the indexed family of P_n , cycles C_n , complete graphs K_n and bipartite complete graphs $K_{n,n}$. The solution to the Recognition Problem is then given by verifying that the polynomial p(X) in question satisfies a recurrence relation. We shall discuss this and generalizations thereof in Section 4.

In this paper we are interested in two questions:

- (A): How can we get solutions to Wilf's Characterization Problem for a general class of graph polynomials?
- (B): Given such a solution, what does it say about the underlying graphs?

Using classical results and a general theorem from [FM08], this paper gives solutions to (**A**) similar to the characterization of C.D. Godsil and I. Gutman, [GG81], for a large class of graph polynomials and indexed families of graphs G_n by replacing, in many cases, the orthogonal polynomials by polynomials given by other linear recurrence relations with constant coefficients. We shall see in Section 4.3 how to formulate a meta-theorem which captures many cases for special classes of graphs.

1.2. Algebraic vs graph theoretic properties of graph polynomials

As for (\mathbf{B}) , the answer depends on the particular way the graph polynomial is represented. The situation is comparable to linear algebra, with matrices

and linear maps $f: V \to W$ between vector spaces V and W. If we choose bases in V and W, we can associate with f a matrix M_f representing f. Two similar matrices represent the same linear map in terms of different choices of bases. As every matrix is similar to a triangular matrix, triangularity is a property of the matrix M_f and not of the linear map f. However, $\det(M_f) = 0$ is both a property of M_f and of the linear map: f is singular iff $\det(M_f) = 0$ iff $\det(M) = 0$ for every matrix M similar to M_f .

Let $g_n(X)$ be a polynomial basis of $\mathbb{Z}[X]$. A graph polynomial is always written in the form

$$P(G;X) = \sum_{i} a_i(G) \cdot g_i(X)$$

where the coefficients are graph parameters. The graph polynomial P(G; X) defines an equivalence relation on the the class of finite graphs: Two graphs G_1, G_2 are P-equivalent iff $P(G_1; X) = P(G_2; X)$. The various equivalence relations induced by a graph polynomial P(G; X) are partially ordered by the refinement relation. In analogy to similarity of matrices, we say that two graph polynomials have the same distinctive power, or are d.p.-equivalent, if they induce the same equivalence relation on graphs with the same number of vertices, edges and connected components. A property of a graph polynomial is a *semantic (aka* graph theoretic) property if it is invariant under d.p.-equivalence. Otherwise it is a property of the representation, i.e., the choice of the polynomial bases, and we speak of syntactic (aka algebraic) properties of the graph polynomial. A more detailed treatment is given in Section 2. Semantic properties of a graph polynomial **P** cannot be expressed in terms of algebraic properties of $\mathcal{Y}_{P,\mathcal{C}}$ alone without relating to the particular form of **P**. For a thorough discussion of this, cf. [MRB14]. There we argued that determining the location of the roots of a graph polynomial is not a semantic property of graph polynomials. We showed that every graph polynomial can be transformed with mild transformations into a d.p.-equivalent graph polynomial with roots almost wherever we want them to be.

Here, in contrast to [MRB14], we will focus on several classes of very naturally defined graph polynomials, the generalized chromatic polynomials and polynomials defined as generating functions of induced or spanning subgraphs, and determinant polynomials. Restricting the form of the graph polynomials means restricting the coefficients of the graph polynomial in a way that allows a natural combinatorial interpretation. We show that for \mathcal{P} either the generalized chromatic polynomials or the graph polynomials defined as generating functions of induced or spanning subgraphs, the algebraic properties of the resulting graph polynomials are semantic properties in the sense that for every graph polynomial $\mathbf{P} \in \mathcal{P}$ there is exactly one different graph polynomial $\mathbf{Q} \in \mathcal{P}$ which is d.p.-equivalent to \mathbf{P} , (Theorem 5.7). In other words, we give a characterization of d.p.-equivalence for graph polynomials in a particular simple form.

1.3. Main results

Our main contributions in this paper are more conceptual than technical. We put Wilf's recognition and characterization problem, originally formulated for the chromatic polynomial, into the general framework of the systematic study of graph polynomials. To do this, we reinterpret diverse results from the literature into this general framework. This leads us to the following results:

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a family of graphs and **P** a graph polynomial.

- (i) We use a general criterion from [FM08] to show that the sequence of polynomials $\mathbf{P}(G_n; \bar{X})$ satisfying a linear recurrence relation with constant coefficients is C-finite⁵, (Theorem 4.9).
- (ii) Let **P** be a graph polynomial. A graph G is **P**-unique if whenever for a graph H we have $\mathbf{P}(H; X) = \mathbf{P}(G; X)$ then H is isomorphic to G. If every G_n is **P**-unique we use this recurrence relation to give characterizations of $\mathcal{Y}_{P,C}$ for many graph polynomials, (Theorem 4.8).
- (iii) Graph polynomials are compared by their respective distinctive power (d.p.- and s.d.p.-equivalence). We characterize d.p.- and s.d.p.-equivalence, Proposition 2.2. For d.p.-equivalent graph properties C, D this gives: C and D are d.p.-equivalent iff C = D or C = G D, (Proposition 5.1)
- (iv) In Section 5 we study d.p.- and s.d.p.-equivalence of graph polynomials $\mathbf{P}_{\mathcal{C}}^{ind}(G;X)$ and $\mathbf{P}_{\mathcal{D}}^{span}(G;X)$ obtained as generating functions for induced and spanning subgraphs in more detail. They refine the d.p.-equivalence relation of the respective graph properties \mathcal{C} and \mathcal{D} . Theorems 5.13 and 5.14 show that there are infinitely many mutually d.p.-incomparable graph polynomials of this form.
- (v) Theorem 5.8 shows that C-finiteness is a semantic property of graph polynomials obtained as generating functions for induced and spanning subgraphs.
- (vi) Also in Section 5 we study d.p.-equivalence of generalized chromatic polynomials $\chi_{\mathcal{C}}(G; X)$. They also refine the d.p.- and s.d.p.-equivalence relation of graph property \mathcal{C} . Theorem 5.16 states that there are infinitely many mutually s.d.p.-incomparable graph polynomials of this form.
- (vii) Finally, we consider graph polynomials which are generating functions of relations $\mathbf{P}_{\Phi(A)}(G; X)$, and we show that not every graph polynomial of this form can be written as a generating function of induced (spanning) subgraphs or as a generalized chromatic polynomial $\chi_{\mathcal{C}}(G; X)$, Theorems 5.17 and 5.18.

2. How to define and compare graph polynomials?

2.1. Typical forms of graph polynomials

In this paper we look at five types of graph polynomials: generalized chromatic polynomials and polynomials defined as generating functions of induced or spanning subgraphs, and determinant polynomials, and contrast this to graph polynomials arising from generating functions of relations.

More precisely, let \mathcal{C} be a graph property.

⁵ The terminology C-finite is usually used for sequences of natural numbers, and we adopt it here for polynomials, cf. [PWZ96].

- **Generalized chromatic:** Let $\chi_{\mathcal{C}}(G; k)$ denote the number of colorings of G with at most k colors such that each color class induces a graph in \mathcal{C} . It was shown in [KMZ08, KMZ11] that $\chi_{\mathcal{C}}(G; k)$ is a polynomial in k for any graph property \mathcal{C} . Generalized chromatic polynomials are further studied in [GHK⁺17].
- **Generating functions:** Let $A \subseteq V(G)$ and $B \subseteq E(G)$. We denote by G[A] the induced subgraph of G with vertices in A, and by $G\langle B \rangle$ the spanning subgraph of G with edges in B.
 - (i) Let \mathcal{C} a graph property.

$$P_{\mathcal{C}}^{ind}(G;X) = \sum_{A \subseteq V: G[A] \in \mathcal{C}} X^{|A|}$$

(ii) Let \mathcal{D} a graph property which is closed under adding isolated vertices, i.e., if $G \in \mathcal{D}$ then $G \sqcup K_1 \in \mathcal{D}$.

$$P_{\mathcal{D}}^{span}(G;X) = \sum_{B \subseteq E: G < B > \in \mathcal{D}} X^{|B|}$$

Generalized Generating functions: Let $X_i : i \leq r$ be indeterminates and $f_i : i \leq r$ be graph parameters. We also consider graph polynomials of the form

$$P^{ind}_{\mathcal{C},f_1,\ldots,f_r}(G;X) = \sum_{A \subseteq V:G[A] \in \mathcal{C}} \prod_{i=1}^r X_i^{f_i(G[A])}$$

and

$$P^{span}_{\mathcal{C},f_1,\ldots,f_r}(G;X) = \sum_{B \subseteq E: G < B > \in \mathcal{D}} \prod_{i=1}^r X_i^{f_i(G < B >)}$$

Determinants: Let M_G be a matrix associated with a graph G, such as the adjacency matrix, the Laplacian, etc. Then we can form the polynomial $\det(\mathbf{1} \cdot X - M_G)$.

Special cases are the chromatic polynomial $\chi(G; X)$, the independence polynomial I(G; X), the Tutte polynomial T(G; X, Y) and the characteristic polynomial of a graph $p_{char}(G; X)$. Note that, in the sense of the following subsection, $\chi(G; X)$, I(G; X) and $p_{char}(G; X)$ are mutually d.p.-incomparable, and $\chi(G; X)$ has strictly less distinctive power than T(G; X, Y).

In Section 5.6 we shall see that there are graph polynomials defined in the literature which seemingly do not fit the above frameworks. This is the case for the usual definition of the *generating matching polynomial*:

$$\sum_{M \subseteq E(G): \mathrm{match}(M)} X^{|M|}$$

where match(M) says that V(G), M is a matching. However, we shall see in Section 5.6 that there is another definition of the same polynomial which is an

generating function. In stark contrast to this, we shall prove there, that the dominating polynomial

$$DOM(G;X) = \sum_{A \subseteq V(G): \Phi_{dom}(A)} X^{|A|}$$

where $\Phi_{dom}(A)$ says that A is a dominating set of G, cannot be written as a generating function, (Theorem 5.17). This motivates the next definition, see also Section 5.6.

Generating functions of a relation Let Φ be a property of pairs (G, A) where G is a graph and $A \subseteq V(G)^r$ is an r-ary relation on G. Then the generating function of Φ is defined by

$$\mathbf{P}_{\Phi}(G;X) = \sum_{A \subseteq V(G)^r : \Phi(G,A)} X^{|A|}$$

The most general graph polynomials Further generalizations of chromatic polynomials were studied in [MZ06, KMZ11, Kot12] and in [GGN13, GNdM16]. In [MZ06, KMZ11] it was shown that the most general graph polynomials can be obtained using model theory as developed in [Zil93, CH03]. A similar approach was used in [GGN13, GNdM16] based on ideas from [dlHJ95]. However, for our presentation here, the graph polynomials we have defined so far suffice.

2.2. Comparing graph polynomials

We denote by n(G), m(G), k(G) the number of vertices, edges and connected components of G. Let \mathbf{P} be a graph polynomial. A graph G is \mathbf{P} -unique if every graph H with $\mathbf{P}(H; \bar{X}) = \mathbf{P}(G; \bar{X})$ is isomorphic to G. We say that two graphs G, H are similar if the have the same number of vertices, edges and connected components. Two graphs G, H are \mathbf{P} -equivalent if $\mathbf{P}(H; \bar{X}) = \mathbf{P}(G; \bar{X})$. \mathbf{P} distinguishes between G and H if G and H are not \mathbf{P} -equivalent.

Two graph polynomials $\mathbf{P}(G; \bar{X})$ and $\mathbf{Q}(G; \bar{Y})$ with r and s indeterminates respectively can be compared by their *distinctive power* on similar graphs: \mathbf{P} is at most as distinctive as $\mathbf{Q}, \mathbf{P} \leq_{s.d.p} \mathbf{Q}$ if any two similar graphs G, H which are \mathbf{Q} -equivalent are also \mathbf{P} -equivalent. \mathbf{P} and \mathbf{Q} are s.d.p.-equivalent, $\mathbf{P} \sim_{s.d.p} \mathbf{Q}$ if for any two similar graphs G, H \mathbf{P} -equivalence and \mathbf{Q} -equivalence coincide. We can also compare graph polynomials on graphs without requiring similarity. In this case we say that a graph polynomial \mathbf{P} is at most as distinctive as \mathbf{Q} , $\mathbf{P} \leq_{d.p.} \mathbf{Q}$, if for all graphs G_1 and G_2 we have that

$$\mathbf{Q}(G_1) = \mathbf{Q}(G_2)$$
 implies $\mathbf{P}(G_1) = \mathbf{P}(G_2)$

P and **Q** are d.p.-equivalent iff both $\mathbf{P} \leq_{d.p.} \mathbf{Q}$ and $\mathbf{Q} \leq_{d.p.} \mathbf{P}$. D.p.-equivalence is stronger that s.d.p.-equivalence:

Lemma 2.1. For any two graph polynomials \mathbf{P} and \mathbf{Q} we have: $\mathbf{P} \leq_{d.p.} \mathbf{Q}$ implies $\mathbf{P} \leq_{s.d.p.} \mathbf{Q}$.

In this paper we concentrate on d.p.-equivalence and speak of s.d.p.-equivalence only when it is needed. In a sequel to this paper we will investigate in detail what can be said of both notions.

Part (ii) of the following Proposition was shown in [MRB14], and (iii) follows from Definition 1(i) and (ii).

Proposition 2.2. (i) **P** is at most as distinctive as **Q**, **P** $\leq_{d.p}$ **Q**, iff there is a function $F : \mathbb{Z}[\bar{Y}] \to \mathbb{Z}[\bar{X}]$ such that for every graph G we have

$$\mathbf{P}(G;\bar{X}) = F(\mathbf{Q}(G;\bar{Y}))$$

(ii) **P** is at most as distinctive as **Q** on similar graphs, $\mathbf{P} \leq_{s.d.p} \mathbf{Q}$, iff there is a function $F : \mathbb{Z}[\bar{Y}] \times \mathbb{Z}^3 \to \mathbb{Z}[\bar{X}]$ such that for every graph G we have

$$\mathbf{P}(G; \bar{X}) = F(\mathbf{Q}(G; \bar{Y}), n(G), m(G), k(G))$$

(iii) Furthermore, both for d.p. and s.d.p., if both \mathbf{P} and \mathbf{Q} are computable, then F is computable, too.

Proof of (iii): The function F from (i) or (ii) is not unique. However, because the range of polynomials given by \mathbf{Q} is assumed to be decidable, we can choose F such that F(p) = 0 for all $p \in \mathbb{Z}[\bar{X}]$ such that there is no graph G with $\mathbf{Q}(G; \bar{X}) = p$. F chosen in this way now is computable.

Remark 2.3. In the literature [MN09, Sok05] on the Tutte polynomial s.d.p.equivalence is implicitly used to compare the various forms of the Tutte polynomial and the Potts model. The various forms of the Tutte polynomial are not d.p.-equivalent. The same is true for the various forms of the matching polynomial as discussed in, say, [LP86].

Graph polynomials are supposed to give information about graphs. The algebraic characterization of $\mathcal{Y}_{\mathbf{P},\mathcal{C}}$ uses the coefficients of the polynomial $\mathbf{P}(G; \bar{X})$ to characterize \mathcal{C} . However, such a characterization depends on the presentation of $\mathbf{P}(G; \bar{X})$ with respect to the basic polynomials chosen to write $\mathbf{P}(G; \bar{X})$. In the univariate case, the basic polynomials are usually, but not always, X^n . Sometimes one uses $\binom{X}{n}$ instead, or the falling factorial $X_{(n)} = X \cdot (X-1) \cdot \dots \cdot (X-n+1)$. On the other side, the notion of d.p.-equivalence captures properties of graphs independently of the presentations of the particular graph polynomials. A statement involving a graph polynomial \mathbf{P} is a proper statement about graphs, if it is invariant under d.p.-equivalence. Otherwise, it is merely a statement about graphs via the particular presentation of the graph polynomial.

Problem 3 (Invariance under d.p.-equivalence). Are there algebraic characterizations of $\mathcal{Y}_{\mathbf{P},C}$ which are invariant under d.p.- or s.d.p.-equivalence?

In [MRB14], this question was studied concerning the location of the roots of $\mathbf{P}(G; \bar{X})$. The answer was negative even if the class of graph polynomials considered is closed under substitutions, and prefactors. However, the various versions of the Tutte polynomials and matching polynomials can be obtained from each other in this way.

In the light of the discussion in [MRB14], we now look at the following problem:

Problem 4 (Distinctive power). Given two graph polynomials \mathbf{P}, \mathbf{Q} in a specific form such as generalized chromatic polynomials, polynomials defined as generating functions of induced or spanning subgraphs, or determinant polynomials, characterize when they are d.p-equivalent.

In Section 5, we will discuss this problem for the case of polynomials defined as generating functions of induced subgraphs and generalized chromatic polynomials, Theorem 5.7 and Proposition 5.10. These theorems do not hold for generating functions of relations, see Section 5.6.

3. The recognition and characterization problems

Let $\mathbf{P}(G; \bar{X})$ be a computable graph polynomial, and let \mathcal{C} be a graph property. Recall from Section 1, that the *Recognition Problem for* $\mathbf{P}(G; \bar{X})$ and \mathcal{C} is the question, whether, given a polynomial $s(\bar{X}) \in \mathbb{Z}[\bar{X}]$, there is a graph $G_s \in \mathcal{C}$ such that $\mathbf{P}(G_s; \bar{X}) = s(\bar{X})$? The *Characterization Problem for* $\mathbf{P}(G; \bar{X})$ and \mathcal{C} asks for a description of the set of polynomials

$$\mathcal{Y}_{\mathbf{P},\mathcal{C}} = \{ p(\bar{X}) \in \mathbb{Z}[\bar{X}] : \exists G \in \mathcal{C} \text{ with } \mathbf{P}(G;\bar{X}) = p(\bar{X}) \}.$$

If C is the class of all finite graphs, we write $\mathcal{Y}_{\mathbf{P}}$. We also noted in Section 1, that there is a brute force solution for the Recognition Problem as follows:

Observation 3.1. Assume that $\mathbf{P}(G; \bar{X})$ is a computable graph polynomial for which we can give a bound $\beta(\mathbf{P}(G; \bar{X}))$ for the size of G. Then, checking whether a polynomial $p(\bar{X})$ is in $\mathcal{Y}_{\mathbf{P}}$ can be done by computing $\mathbf{P}(G; \bar{X})$ for all graphs smaller than $\beta(\mathbf{P}(G; \bar{X}))$.

What we are looking for should be better than that.

The problem may be easier for certain graph properties C in the relative version. There are many such characterization in the literature, we just give one here for the sake of illustration.

Example 3.1 (Taken from [DKT05]). Let C be the class of finite connected graphs. Then $\mathcal{Y}^r_{\chi,C}$ consists of all instances of the chromatic polynomial which have 0 as a root with multiplicity one.

It is easy to define graph polynomials P(G; X) with a *trivial recognition*, i.e., where for every polynomial

$$s(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{N}[X]$$

there is a graph G_s with $P(G_s; X) = s(X)$.

Proposition 3.2. Let $MaxCl(G; X) = \sum_i mcl_i(G)X_i$ be the graph polynomial where $mcl_i(G)$ denotes the number of maximal cliques of size *i*. MaxCl(G; X) has a trivial recognition.

Proof. Let $s(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{N}[X]$ and let G_s be the graph which is the disjoint union of a_i -many cliques of size *i*. Then $MaxCl(G_s; X) = s(X)$.

Problem 5. Find more naturally defined graph polynomials with trivial recognition.

To show that *not all polynomials* are chromatic polynomials, one can use various properties of the coefficients. One sufficient condition is that the coefficients are alternating in sign for connected graphs. For a characterization, more properties of the coefficients are needed. One such property is the fact that its coefficients (or their absolute values) are unimodal or logconcave, cf. [Huh15], which was suggested also for the independence polynomial, and other graph polynomials. However, showing unimodality of the coefficients is notoriously hard, [Sta89, Bre92, Bra15, Huh15].

4. Characterizations using recurrence relations

Let P_n , C_n and K_n denote, respectively, the path, the cycle and the complete graph on n vertices, and $K_{n,m}$ denote the complete bipartite graph on n + m vertices. We let Path be the family of paths P_n , and Cycle, Clique and CBipartite the families of C_n , K_n and $K_{n,n}$ respectively. For a class of graphs Cclosed under isomorphisms, we denote the class of graphs consisting of disjoint unions of graphs in C by DU(C).

Let $\mathbf{P}(G; \bar{X})$ be a graph polynomial and G_n be a sequence of graphs. The sequence of polynomials $\mathbf{P}_n(\bar{X}) = \mathbf{P}(G_n; \bar{X})$ is *C*-finite if there is $q \in \mathbb{N}$ and there are polynomials $f_i(\bar{X}) \in \mathbb{Z}[\bar{X}], i \in [q]$ such that

$$\mathbf{P}_{n+q}(X) = \sum_{i=0}^{q-1} f_i(\bar{X}) \mathbf{P}_i(\bar{X})$$

4.1. The characteristic polynomial $p_{char}(G; X)$

Let G be an undirected graph and A_G is symmetric adjacency matrix. The characteristic polynomial is defined as

$$p_{char}(G; X) = \det(\mathbf{1} \cdot X - A_G)$$

We note that $p_{char}(G; X)$ is *multiplicative*, i.e., if H is the disjoint union of G_1 and G_2 then $p_{char}(H; X) = p_{char}(G_1; X) \cdot p_{char}(G_2; X)$.

Proposition 4.1 (Taken from [BH12, Chapter 14.4.2]). The graphs P_n , C_n , K_n and $K_{n,n}$ are p_{char} -unique.

Proposition 4.2 (A.J. Schwenk [Sch74]).

The sequences of polynomials $p_{char}(P_n)$, $p_{char}(C_n)$, $p_{char}(K_n)$ and $p_{char}(K_{n,n})$ are all C-finite.

Proposition 4.1 gives us:

Theorem 4.3.

- (i) G is isomorphic to P_n iff $p_{char}(G; X) = p_{char}(P_n; X)$.
- (ii) G is isomorphic to C_n iff $p_{char}(G; X) = p_{char}(C_n; X)$.
- (iii) G is isomorphic to K_n iff $p_{char}(G; X) = p_{char}(K_n; X)$.
- (iv) G is isomorphic to $K_{n,n}$ iff $p_{char}(G; X) = p_{char}(K_{n,n}; X)$.

Proposition 4.2 gives us similar characterizations of $\mathcal{Y}_{p_{char},\text{Path}}, \mathcal{Y}_{p_{char},\text{Cycle}}, \mathcal{Y}_{p_{char},\text{Clique}}$ and $\mathcal{Y}_{p_{char},\text{CBipartite}}$ using C-finiteness.

4.2. The matching polynomials

Let $m_k(G)$ denote the number of k-matchings of a graph G on n vertices. Let $\mu(G; X)$ be the defect matching polynomial (aka acyclic polynomial)

$$\mu(G;X) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k(G)(-1)^k X^{n-2k}$$

Theorem 4.4 (C.D. Godsil and I. Gutman [GG81]). On forests F we have $\mu(F; X) = p_{char}(F; X)$.

We look for characterizations of $\mathcal{Y}_{\mu,\text{Path}}, \mathcal{Y}_{\mu,\text{Cycle}}, \mathcal{Y}_{\mu,\text{Clique}}$ and $\mathcal{Y}_{\mu,\text{CBipartite}}$.

We need the recursive definitions of the orthogonal polynomials of Chebyshev, Hermite and Laguerre, cf. [Chi11]: The Chebyshev polynomials $T_n(X)$ and $U_n(X)$ are defined recursively as follows:

$$T_0(X) = 1, T_1(X) = X$$
 and $T_{n+1}(X) = 2X \cdot T_n(X) - T_{n-1}(X)$

and

$$U_0(X) = 1, U_1(X) = 2X$$
 and $U_{n+1}(X) = 2X \cdot U_n(X) - U_{n-1}(X)$

These two recurrence relations are linear in $U_n(X)$ and their coefficients are elements of $\mathbb{N}[X]$ and do not depend on n.

The Hermite polynomials $He_n(X)$ are defined recursively as follows:

$$He_0(X) = 1, He_1(X) = X$$
 and $He_{n+1}(X) = X \cdot He_n(X) - n \cdot He_{n-1}(X)$

The Laguerre polynomials $L_n(X)$ are defined recursively as follows:

$$L_0(X) = 1, L_1(X) = 1 - X$$
 and $L_{n+1}(X) = \frac{2n + 1 - x}{n+1} \cdot L_n(X) - \frac{n}{n+1} L_{n-1}(X)$

These recurrence relation are linear in $He_n(X)$ and $L_n(X)$ and their coefficients are elements of $\mathbb{N}[X]$, respectively $\mathbb{Q}[X]$ and do depend on n. Theorem 4.5 (C.D. Godsil and I. Gutman [GG81]).

- (i) $\mu(C_n; 2X) = 2 \cdot T_n(X)$
- (ii) $\mu(P_n; 2X) = U_n(X)$
- (iii) $\mu(K_n; X) = He_n(X)$
- (iv) $\mu(K_{n,n};X) = (-1)^n \cdot L_n(X^2)$

For related theorems, cf. also [God81, Ges89, NR04, DG04].

Theorem 4.6.

- (i) ([BF95]) The C_n 's are μ -unique.
- (ii) ([Noy03]) K_n and $K_{n,n}$ are μ -unique.
- (iii) ([BH12, Proposition 14.4.6]) P_n is p_{char}-unique, and, as it is a tree, also μ-unique.

Putting all this together we get:

Theorem 4.7.

- (i) A graph G is isomorphic to a cycle C_n iff $\mu(G; X) = 2 \cdot T_n(X)$. In other words, $\mathcal{Y}_{\mu,\text{Cycle}}$ can be characterized using a linear recurrence relation with constant coefficients in $\mathbb{Z}[X]$.
- (ii) A graph G is isomorphic to a path P_n iff $\mu(G; 2X) = U_n(X)$. In other words, $\mathcal{Y}_{\mu,\text{Path}}$ can be characterized using a linear recurrence relation with constant coefficients in $\mathbb{Z}[X]$.
- (iii) A graph G is isomorphic to a complete graph K_n iff $\mu(G; X) = He_n(X)$. In other words, $\mathcal{Y}_{\mu,\text{Clique}}$ can be characterized using a recurrence relation where the coefficients depend on n.
- (iv) A graph G is isomorphic to a complete bipartite graph $K_{n,n}$ iff $\mu(G; X) = (-1)^n \cdot L_n(X^2)$. In other words, $\mathcal{Y}_{\mu,\text{CBipartite}}$ can be characterized using a recurrence relation where the coefficients depend on n.

4.3. An abstract theorem

Our discussion of the characteristic polynomial can be formulated abstractly. We state the following observation as a theorem.

Theorem 4.8. Let **P** be a graph polynomial and $C = \{G_n : n \in \mathbb{N}\}$ be given as a sequence of graphs. Assume the following:

(i) The sequence of polynomials P(G_n; X̄) satisfies some recurrence relation.
(ii) Each G_n is P-unique.

Then $\mathcal{Y}_{P,\mathcal{C}}$ is characterized algebraically by the property: H is isomorphic to G_n iff $\mathbf{P}(H; \bar{X}) = \mathbf{P}(G_n; \bar{X})$. This can be checked using the recurrence relations.

One can also formulate an analogue of this theorem for families of graphs G_{n_1,\ldots,n_k} depending on k indices.

We look also at the following indexed families of graphs:

- W_n : The wheels $C_n \bowtie K_1$.
- L_n : The ladders $L_n = C_n \times K_2$.
- M_n : The Möbius ladders M_n are obtained from C_{2n} by connecting any pair of opposite vertices.
- C_n^2 : The square of the cycle C_n obtained by connecting any two vertices of distance two.
- $Grid_{n,m}$: The square grids of size $(n \times m)$.

For a graph polynomial \mathbf{P} , an indexed family G_n of graphs is \mathbf{P} -recursive if the sequence of polynomials $\mathbf{P}(G_n; \bar{X})$ is C-finite. Using the main theorem from [FM08] one can prove the following for indexed families of graphs G_n of bounded tree-width.

Theorem 4.9. Let m_0 be fixed. The families $P_n, C_n, Grid_{n,m_0}, W_n, L_n, M_n, C_n^2$ are all C-finite for the graph polynomials p_{char}, μ, χ , and T.

- **Remark 4.10.** 1. Actually, the sequences from Theorem 4.9 are C-finite for every graph polynomial definable in Monadic Second Order Logic (MSOL), such as the independence polynomial, [LM05], and the edge elimination polynomial $\xi(G; X, Y, Z)$, [TAM11]. However, we do not want in this paper to get involved with definability theory or the formalisms of (Monadic) Second Order Logic, i.e., we want to keep it logic-free.
 - 2. The way Theorem 4.9 is stated, it is non-constructive, because it does not say anything about the form of the recurrence relation. It only asserts C-finiteness, without giving the coefficients or the depth of the recursion.
 - 3. The results of [FM08] cannot be applied to $K_n, K_{n,n}$ because the sequences of graphs $K_n, K_{n,n}$ have unbounded tree-width. In fact the resulting families of chromatic and Tutte polynomials are not C-finite, [BDS72]. In general, linear recurrence relations for a sequence of polynomials where the coefficients depend on n are not C-finite, because the coefficients may grow too fast.

The following is folklore for the chromatic polynomial and due to I. Gessel [Ges95] for the Tutte polynomial.

Theorem 4.11. The families $K_n, K_{n,n}$ satisfy recurrence relations with the coefficients depending on n for the chromatic and Tutte polynomials:

$$\chi(K_n; X) = (X - n + 1) \cdot \chi(K_{n-1}; X)$$

 $T(K_n; X, Y) = \sum_{k=1}^n \binom{n-1}{k-1} (X+Y+Y^2+\ldots+Y^{k-1}) \cdot T(K_n; 1, Y) \cdot T(K_{n-k}: X, Y)$

and similar for $K_{n,n}$.

4.4. The chromatic and the Tutte polynomials

To give further applications of Theorem 4.8 we collect some results from [DKT05, dMN04] on χ -unique and T-unique graphs.

Theorem 4.12. 1. Let t_m be a tree on m edges. Then $T(t_m; X) = X^m$. Hence the paths P_n are neither χ -unique nor T-unique.

- 2. $C_n, K_n, K_{n,m}$ are all χ -unique, hence T-unique.
- 3. W_n, L_n, M_n and C_n^2 are T-unique but not χ -unique.

Now, Theorem 4.12 allows us to give more algebraic characterizations using recurrence relations for these sequences via the chromatic and the Tutte polynomial.

5. Distinctive power

5.1. s.d.p.-equivalence and d.p-equivalence of graph properties

A class of graphs S which consists of all graphs having the same number of vertices, edges and connected components is called a *similarity class*.

Let C be a graph property. Two graphs G, H are C-equivalent if either both are in C or both are not in C. We denote by \overline{C} the graph property $\mathcal{G} - C$.

Therefore we have:

- **Proposition 5.1.** (i) Two graph properties C_1 and C_2 are d.p.-equivalent iff either $C_1 = C_2$ or $C_1 = \overline{C_2}$.
- (ii) Two graph properties C_1 and C_2 are s.d.p.-equivalent iff for every similarity class S either $C_1 \cap S = C_2 \cap S$ or $C_1 \cap S = \overline{C_2} \cap S$.

Proof. (i): It is straightforward that if $C_2 = C_1$ or $C_2 = \overline{C_1}$ then C_1 and C_2 are d.p.-equivalent.

For the other direction, we prove first that $C_1 \subseteq C_2$ or $C_1 \subseteq \overline{C_2}$.

By a symmetrical argument, we then prove also $C_2 \subseteq C_1$ or $C_2 \subseteq \overline{C_1}$, $\overline{C_1} \subseteq C_2$ or $\overline{C_1} \subseteq \overline{C_2}$ and $\overline{C_2} \subseteq C_1$ or $\overline{C_2} \subseteq \overline{C_1}$. Now the result follows.

(ii): Fix \mathcal{S} . The proof is the same but relativized to \mathcal{S} .

Remark 5.2. If C_1 and C_2 are s.d.p.-equivalent it is possible that for a similarity class S we have $C_1 \cap S = C_2 \cap S$ but for another similarity class S' we have $C_1 \cap S' = \overline{C_2} \cap S'$.

- **Proposition 5.3.** (i) Let C_1 and C_2 be two graph properties. Assume that both C_1 and C_2 are not empty and do not contain all finite graphs, and that $C_1 \neq C_2$ and $C_1 \neq \overline{C_2}$. Then C_1 and C_2 are d.p.-incomparable, i.e., $C_1 \not\leq_{d.p.} C_2$ and $C_2 \not\leq_{d.p.} C_1$.
- (ii) Let C₁ and C₂ be two graph properties. Assume there is a similarity class S such that both C₁ ∩ S and C₂ ∩ S are not empty and do not contain all finite graphs in S, and that C₁ ∩ S ≠ C₂ ∩ S and C₁ ∩ S ≠ C₂ ∩ S. Then C₁ and C₂ are s.d.p.-incomparable, i.e., C₁ ≤ s.d.p. C₂ and C₂ ≤ s.d.p. C₁.

Proof. We prove only (i) and leave the proof of (ii) to the reader. Assume $G_1 \in (\mathcal{C}_1 - \mathcal{C}_2), G_2 \in (\mathcal{C}_2 - \mathcal{C}_1)$ and $G_3 \in \mathcal{C}_1 \cap \mathcal{C}_2$, the other cases being similar. Then $G_2, G_3 \in \mathcal{C}_2$. If $\mathcal{C}_1 \leq_{d.p.} \mathcal{C}_2$, we would have that both $G_2, G_3 \in \mathcal{C}_1$, or both $G_2, G_3 \notin \mathcal{C}_1$, a contradiction.

In the next two subsections we look at graph polynomials, which are either generating functions, or count colorings which, in both cases, solely depend on a graph property \mathcal{C} .

5.2. Graph polynomials as generating functions

Let \mathcal{C} be a graph property, and \mathcal{D} be a graph property closed under adding and removal isolated vertices. Recall from Section 2 the definitions

$$\mathbf{P}_{\mathcal{C}}^{ind}(G;X) = \sum_{A \subseteq V: G[A] \in \mathcal{C}} X^{|A|} \quad \text{and} \quad \mathbf{P}_{\mathcal{D}}^{span}(G;X) = \sum_{B \subseteq E: G \langle B \rangle \in \mathcal{D}} X^{|B|}.$$

Let |V(G)| = n(G) and |E(G)| = m(G).

Proposition 5.4. (i) $\mathcal{C} \leq_{d.p.} \mathbf{P}_{\mathcal{C}}^{ind}(G; X)$ and (ii) $\mathcal{D} \leq_{d.p.} \mathbf{P}_{\mathcal{D}}^{span}(G; X)$.

Proof. (i) follows from the fact that $G \in \mathcal{C}$ iff the coefficient of $X^{n(G)}$ in $\mathbf{P}_{\mathcal{C}}^{ind}(G;X)$ does not vanish.

Similarly, (ii) follows from the fact that $G \in \mathcal{C}$ iff the coefficient of $X^{m(G)}$ in $\mathbf{P}_{\mathcal{C}}^{span}(G;X)$ does not vanish.

From Lemma 2.1 we get immediately:

Corollary 5.5. (i) $\mathcal{C} \leq_{s.d.p.} \mathbf{P}_{\mathcal{C}}^{ind}(G; X)$ and (ii) $\mathcal{D} \leq_{s.d.p.} \mathbf{P}_{\mathcal{D}}^{span}(G; X)$.

Proposition 5.6. With |V(G)| = n(G) and |E(G)| = m(G) we have:

(i) $\mathbf{P}_{\mathcal{C}}^{ind}(G; X) + \mathbf{P}_{\bar{\mathcal{C}}}^{ind}(G; X) = (1+X)^{n(G)}$ (ii) $\mathbf{P}_{\mathcal{D}}^{span}(G; X) + \mathbf{P}_{\bar{\mathcal{D}}}^{span}(G; X) = (1+X)^{m(G)}$

Proof. (i): Put

$$c_i(G) = |\{A \subseteq V(G) : |A| = i, G[A] \in \mathcal{C}\}|$$

and

$$\bar{c}_i(G) = |\{A \subseteq V(G) : |A| = i, G[A] \notin \mathcal{C}\}|.$$

Clearly,

$$c_i(G) + \bar{c}_i(G) = \binom{n(G)}{i},$$

hence

$$\sum_{i=0}^{n(G)} \left(c_i(G) + \bar{c}_i(G) \right) X^i = (1+X)^{n(G)}.$$

(ii) is similar, but we need that for a set of edges $A \subseteq E(G)$ the spanning subgraph $G(A) = (V(G), A) \in \mathcal{D}$ iff $V(A), A) \in \mathcal{D}$, where $V(A) = \{v \in V(G) :$ there is $u \in V(G)$ with $(u, v) \in A$. **Proposition 5.7.** Let C_1 and C_2 , D_1 and D_2 be graph properties such that C_1 and C_2 and D_1 and D_2 are pairwise d.p.-equivalent,

- (i) $\mathbf{P}_{\mathcal{C}_1}^{ind}(G; X)$ and $\mathbf{P}_{\mathcal{C}_2}^{ind}(G; X)$ are s.d.p.-equivalent;
- (ii) If, additionally, D₁ and D₂ are closed under the addition and removal of isolated vertices, then P^{span}_{D₁}(G; X) and P^{span}_{D₂}(G; X) are s.d.p.-equivalent;

Proof. We prove only (i), (ii) is proved analogously.

(i): We use Proposition 5.1. If $C_1 = C_2$, clearly, $\mathbf{P}_{C_1}^{ind}(G; X) = \mathbf{P}_{C_2}^{ind}(G; X)$, hence they are d.p.-equivalent. If $C_1 = \overline{C_2}$, we use Proposition 5.6 together with Proposition 2.2. But Proposition 5.6 depends on the n(G), hence we get only that $\mathbf{P}_{C_1}^{ind}(G; X)$ and $\mathbf{P}_{C_2}^{ind}(G; X)$ are s.d.p.-equivalent.

Let G_n be an indexed sequence of graphs such that the sequence of polynomials $X^{|V(G_n)|}$ is C-finite. This assumption is true for all the examples from Section 4.3, and in particular for Theorem 4.5, provided the function $|V(G_n)|$ is linear in n. We shall now show that C-finiteness of the sequences of polynomials $\mathbf{P}_{\mathcal{C}}^{ind}(G_n; X)$ of Theorem 4.5 is a semantic property graph polynomials as generating functions. However, the particular form of the recurrence relation is not.

Theorem 5.8. Let C_1 and C_2 , D_1 and D_2 be graph properties such that C_1 and C_2 and D_1 and D_2 are pairwise d.p.-equivalent, and let G_n be an indexed sequence of graphs. Furthermore, assume that the sequence of polynomials $X^{|V(G_n)|}$ is *C*-finite. Then

(i)
$$\mathbf{P}_{\mathcal{C}_1}^{ind}(G_n; \bar{X})$$
 is C-finite iff $\mathbf{P}_{\mathcal{C}_2}^{ind}(G_n; \bar{X})$ is C-finite.
(ii) $\mathbf{P}_{\mathcal{D}_1}^{span}(G_n; \bar{X})$ is C-finite iff $\mathbf{P}_{\mathcal{D}_2}^{span}(G_n; \bar{X})$ is C-finite;

Proof. This follows in both cases from the fact that the sum and difference of two C-finite sequences is again C-finite together with Proposition 5.6. \Box

5.3. Generalized chromatic polynomials

Recall from the introduction the definition of $\chi_{\mathcal{C}}(G;k)$ as the number of colorings of G with at most k colors such that each color class induces a graph in \mathcal{C} .

Theorem 5.9 (J. Makowsky and B. Zilber, cf. [KMZ11]). $\chi_{\mathcal{C}}(G;k)$ is a polynomial in k for any graph property \mathcal{C} .

In contrast to Proposition 5.6 the relationship between $\chi_{\mathcal{C}}(G;k)$ and $\chi_{\bar{\mathcal{C}}}(G;k)$ is not at all obvious.

Problem 6. What can we say about $\chi_{\bar{C}}(G;k)$ in terms of $\chi_{\mathcal{C}}(G;k)$?

Proposition 5.10. There are two classes C_1 and C_2 which are d.p.-equivalent but such that χ_{C_1} and χ_{C_2} are not d.p.-equivalent.

Proof. Let C_1 be all the disconnected graphs and Let C_2 be all the connected graphs. As they are complements of each other, they are d.p.-equivalent. We compute for K_i :

$$\chi_{\mathcal{C}_1}(K_i; j) = 0, j \in \mathbb{N}^+$$

because there is no way to partition K_i into any number of disconnected parts. Hence $\chi_{\mathcal{C}_1}(K_i; X) = 0.$

$$\chi_{\mathcal{C}_2}(K_i; 2) = 2^i - 2$$

because every partial of K_i into two nonempty parts gives two connected graphs. Therefore $\chi_{\mathcal{C}_2}$ distinguishes between cliques of different size, whereas $\chi_{\mathcal{C}_1}$ does not.

We note, however, that the analogue of Proposition 5.7 for generalized chromatic polynomials remains open.

5.4. d.p.-equivalence of graph polynomials

The converse of Theorem 5.7(i) and (ii) is not true:

Proposition 5.11. There are graph properties C_1 and C_2 which are not d.p.equivalent, but such that

- (i) **P**^{ind}_{C1}(G; X) and **P**^{ind}_{C2}(G; X) are d.p.-equivalent.
 (ii) χ_{C1}(G; X) and χ_{C2}(G; X) are d.p.-equivalent.

Proof. For (i) Let $C_1 = \{K_1\}$ and $C_2 = \{K_2, E_2\}$ where E_n is the graph on nvertices and no edges.

We compute:

$$\mathbf{P}_{\mathcal{C}_1}^{ind}(G;X) = n(G) \cdot X$$
$$\mathbf{P}_{\mathcal{C}_2}^{ind}(G;X) = \binom{n(G)}{2} \cdot X^2$$

For (ii) we choose $C_1 = \{K_1\}$ as before, but $C_2 = \{K_1, K_2, E_2\}$. Claim 1: $\chi_{\mathcal{C}_2}(G, X) \leq_{d.p.} n(G)$

Proof of Claim 1: Let G_1 and G_2 be two graphs with the same number of vertices. W.l.o.g. assume they have the same vertex set $V(G_1) = V(G_2) = V$. Now notice for every $f: V \to [k], f$ is a \mathcal{C}_2 -coloring of G_1 iff it is a f is a \mathcal{C}_2 -coloring of G_2 . Hence $\chi_{\mathcal{C}_2}(G_1, X) = \chi_{\mathcal{C}_2}(G_2, X)$ whenever G_1 and G_2 have the same number of vertices.

Claim 2: $n(G) \leq_{d.p.} \chi_{\mathcal{C}_2}(G, X)$

Proof of Claim 2: First denote for every m, $n_{even}(m) = \prod_{i=0}^{m-1} {\binom{2(m-i)}{2}}$ and $n_{odd}(m) = \prod_{i=0}^{m-1} {2 \choose i}$ For every graph G, there is a natural number m(G) such that n(G) = 2m(G) or n(G) = 2m(G) + 1. If n(G) = 2m(G), $\chi_{\mathcal{C}_2}(G, m(G)) = n_{even}(m(G)).$ If $n(G) = 2m(G) + 1, \chi_{\mathcal{C}_2}(G, m(G)) = n_{odd}(m(G)).$ Note $n_{odd}(r) > n_{even}(r)$ for every natural number r. The minimal natural number r such that $\chi(G,r) > 0$ is equal to m(G). We get that the minimal r such that $\chi_{\mathcal{C}_2}(G, r) > 0$ determines n(G). Hence $\chi_{\mathcal{C}_1}$ and $\chi_{\mathcal{C}_2}$ are d.p.-equivalent. \square

We leave it to the reader to construct the corresponding counterexample for $\mathbf{P}_{\mathcal{D}}^{span}(G;X).$

We cannot use Proposition 5.3 to show that there infinitely many d.p.incomparable graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{ind}(G; X)$. However, we can construct explicitly infinitely many d.p.-incomparable graph polynomials of this form.

5.5. Many d.p.-inequivalent graph polynomials

For the rest of this section, let C_i be the undirected circle on *i* vertices, and C_i^* the graph which consists of a copy of C_{i-1} together with a new vertex v which is connected to exactly one of the vertices of C_{i-1} . Clearly, C_i and C_i^* are similar. Furthermore, let $C_i = \{C_i\}$, and let G_i^k consist of the disjoint union of k-many copies of C_i , and let \hat{G}_i^k consist of the disjoint union of k-1 copies of C_i^* together with one copy of C_i . Again, \hat{G}_i^k and G_i^k are similar.

We compute:

Lemma 5.12.

$$\mathbf{P}_{\mathcal{C}_{j}}^{ind}(G_{i}^{k};X) = \mathbf{P}_{\mathcal{C}_{j}}^{ind}(\hat{G}_{i}^{k};X) = 0 \text{ for } i \neq j, i \neq j+1,$$
(i)
$$\mathbf{P}_{\mathcal{C}_{i}}^{ind}(G_{i}^{k};X) = k \cdot X^{i}$$
(ii)

$$\mathbf{P}_{\mathcal{C}_i}^{ind}(G_i^k;X) = k \cdot X^i \tag{ii}$$

$$\mathbf{P}_{\mathcal{C}_i}^{ind}(\hat{G}_i^k; X) = X^i \tag{iii}$$

Theorem 5.13. For all i, j with $i \neq j$ and $i \neq j+1$ the polynomials $\mathbf{P}_{\mathcal{C}_i}^{ind}$ and $\mathbf{P}_{\mathcal{C}_{i}}^{ind}$ are d.p.-incomparable, hence there are infinitely many d.p.-inequivalent graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{ind}(G; X)$.

Proof. Assume $i, j \geq 3$ with $i \neq j$ and $i \neq j+1$. We first prove $\mathbf{P}_{\mathcal{C}_i}^{ind} \not\leq_{d.p.} \mathbf{P}_{\mathcal{C}_j}^{ind}$ for $i \neq j$ and $i \neq j + 1$.

We look at the graphs G_j^2 and \hat{G}_j^2 . $\mathbf{P}_{\mathcal{C}_j}^{ind}(G_j^2; X) = 2 \cdot X^i$ by Lemma 5.12(ii). $\mathbf{P}_{\mathcal{C}_i}^{ind}(\hat{G}_j^2;X) = X^i$ by Lemma 5.12(iii). Hence, $\mathbf{P}_{\mathcal{C}_i}^{ind}$ distinguishes between the two graphs G_j^2 and \hat{G}_j^2 . However, $\mathbf{P}_{\mathcal{C}_i}^{ind}(G_j^2; X) = \mathbf{P}_{\mathcal{C}_i}^{ind}(\hat{G}_j^2; X) = 0$, by Lemma 5.12(i). Hence, $\mathbf{P}_{\mathcal{C}_i}^{ind}$ does not distinguish between the two graphs.

To prove $\mathbf{P}_{\mathcal{C}_j}^{ind} \not\leq_{d.p.} \mathbf{P}_{\mathcal{C}_i}^{ind}$ for $j \neq i$ and $j \neq i+1$. we look at the graphs G_i^2 and \hat{G}_i^2 . In this case $\mathbf{P}_{\mathcal{C}_j}^{ind}$ does not distinguish between the two graphs G_i^2 and \hat{G}_i^2 , but $\mathbf{P}_{\mathcal{C}_i}^{ind}$ does.

Theorem 5.14. There are infinitely many d.p.-inequivalent graph polynomials of the form $\mathbf{P}_{\mathcal{C}}^{span}(G; X)$.

Proof. The proof mimics the proof of Theorem 5.13 with following changes:

Instead of C_i we use $\mathcal{D}_i = \{C_i \sqcup E_j : j \in \mathbb{N}\}$ and

$$\begin{split} \mathbf{P}_{\mathcal{C}_{j}}^{span}(G_{i}^{k};X) &= 0 \text{ for } i \neq j, i \neq j+1 \\ \mathbf{P}_{\mathcal{C}_{i}}^{span}(G_{i}^{k};X) &= k \cdot X^{i}. \\ \mathbf{P}_{\mathcal{C}_{j}}^{span}(\hat{G}_{i}^{k};X) &= \begin{cases} 0 & i \neq j, i \neq j+1 \\ (k-1) \cdot X^{j} & i = j+1 \end{cases} \\ \mathbf{P}_{\mathcal{C}_{i}}^{span}(\hat{G}_{i}^{k};X) &= X^{i}. \end{split}$$

Next we look at chromatic polynomials $\chi_i(G; X) = \chi_{\mathcal{C}_i}(G; X)$. We use the following obvious lemma:

Lemma 5.15. (i) For $X = \lambda \in \mathbb{N}$:

$$\chi_i(G_i^k;\lambda) = \begin{cases} \lambda_{(k)} & \lambda \ge k\\ 0 & else \end{cases}$$

(ii)

$$\chi_j(G_i^k,\lambda) = 0$$

provided that $i \neq j$.

(iii)

$$\chi_j(\hat{G}_i^k, \lambda) = 0$$

provided that $k \geq 2$ or $k = 1, i \neq j$.

Theorem 5.16. For all $i \neq j$ the polynomials χ_i and χ_j are d.p.-incomparable, hence there are infinitely many d.p.-incomparable graph polynomials of the form χ_c .

Proof. $\chi_i \not\leq_{d.p.} \chi_j$:

We look at the graphs G_i^2 and \hat{G}_i^2 . By Lemma 5.15 χ_j does not distinguish between G_i^2 and \hat{G}_i^2 . However, χ_i distinguishes between them.

To show that $\chi_j \not\leq_{d.p.} \chi_i$, we look at the graphs G_j^2 and \hat{G}_j^2 . By Lemma 5.15 χ_i does not distinguish between G_j^1 and G_j^2 . However, χ_j does distinguish between them.

5.6. Generating functions of a relation

If, instead of counting induced (spanning) subgraphs with a certain graph property $\mathcal{C}(\mathcal{D})$, we count *r*-ary relations with a property $\Phi(A)$, we get a generalization of both the generating functions of induced (spanning) subgraphs. Here the summation is defined by

$$\mathbf{P}_{\Phi}(G;X) = \sum_{A \subseteq E(G): \Phi(A)} X^{|A|}.$$

For example, the generating matching polynomial, defined as

$$m(G;X) = \sum_{A \subseteq E(G): \Phi_{match}(A)} X^{|A|}.$$

can be written as

$$m(G; X) = \sum_{A \subseteq E(G): G\langle A \rangle \in \mathcal{D}_{match}} X^{|A|}$$

with \mathcal{D}_{match} being the disjoint union of isolated vertices and isolated edges.

However, not every graph polynomial $\mathbf{P}_{\Phi}(G; X)$ can be written as a generating function of induced (spanning) subgraphs.

Consider the graph polynomial

$$DOM(G; X) = \sum_{A \subseteq V(G): \Phi_{dom}(A)} X^{|A|}$$

where $\Phi_{dom}(A)$ says that A is a dominating set of G. We compute:

$$DOM(K_2; X) = 2X + X^2$$
 (1)

$$DOM(E_2,;X) = X^2 \tag{2}$$

Theorem 5.17. (i) There is no graph property C such that

$$DOM(G; X) = \mathbf{P}_{\mathcal{C}}^{ind}(G; X).$$

(ii) There is no graph property \mathcal{D} such that

$$DOM(G; X) = \mathbf{P}_{\mathcal{D}}^{span}(G; X).$$

Proof. (i): Assume, for contradiction, there is such a \mathcal{C} , and that $K_1 \in \mathcal{C}$. The coefficient of X in $\mathbf{P}_{\mathcal{C}}^{ind}(E_2; X)$ is 2 because $K_1 \in \mathcal{C}$. However, the coefficient of X in $DOM(E_2; X)$ is 0, by equation (2), a contradiction.

Now, assume $K_1 \notin C$. The coefficient of X in $\mathbf{P}_{\mathcal{C}}^{ind}(K_2; X)$ is 0, because $K_1 \notin C$. However, the coefficient of X in $DOM(K_2; X)$ is 2, by equation (1), another contradiction.

(ii): Assume, for contradiction, there is such a \mathcal{D} . The coefficient of X in $\mathbf{P}_{\mathcal{D}}^{span}(K_2; X)$ is ≤ 1 , because K_2 has only one edge. However, the coefficient of X in $DOM(K_2; X)$ is 2, by equation (1), a contradiction.

We can use Equation (1) also to show the following:

Theorem 5.18. There is no graph property C such that

$$DOM(G; X) = \chi_{\mathcal{C}}(G; X).$$

Proof. First we note that $\chi_{\mathcal{C}}(G;1) = 1$ iff $\chi_{\mathcal{C}}(G;1) \neq 0$ iff $G \in \mathcal{C}$. Assume that $K_2 \in \mathcal{C}$. Then we have, using Equation (1),

$$\chi_{\mathcal{C}}(K_2; 1) = 1 = DOM(K_2, 1) = 3,$$

a contradiction.

Assume that $K_2 \notin \mathcal{C}$. then we have, using Equation (1),

$$\chi_{\mathcal{C}}(K_2; 1) = 0 = DOM(K_2, 1) = 3,$$

another contradiction.

5.7. Determinant polynomials

There are only two matrices associated with graphs which have been used to define graph polynomials: the adjacency matrix and the Laplacian. The two resulting determinant polynomials are d.p.-incomparable. It is conceivable to to define other matrix presentations of graphs, and ask when they give rise to d.p.-equivalent determinant polynomials. The characterization and recognition problem in this case amounts to the question when the characteristic polynomial of a matrix is the the characteristic polynomial arising from a graph. However, in this paper we do not pursue this further.

5.8. Characterizing d.p.-equivalence for special classes of graph polynomials

Theorems 5.7 and Proposition 5.10 and Proposition 5.11 show that d.p.equivalence of C and C_1 , respectively D and D_1 , is not enough to characterize d.p.-equivalence of generating functions or generalized chromatic polynomials defined by C and D. Sometimes d.p.-equivalence of graph properties only implies and s.d.p.-equivalence of the corresponding graph polynomials.

Problem 7. Characterize d.p.-equivalence of graph polynomials arising from C and D as

- 1. generalized chromatic polynomials;
- 2. generating functions of induced are spanning subgraphs;
- 3. generating functions of relations.

6. Conclusions and open problems

In the light of our general framework to study Wilf's characterization and recognition problems for graph graph polynomials, we have shown how to characterize the instances of a graph polynomial $\mathbf{P}(G; \bar{X})$ of various indexed sequences of graphs G_i or $G_{i,j}$ using C-finite sequences of polynomials on $\mathbb{Z}[\bar{X}]$. Our method works for many graph polynomials and indexed sequences of graphs as described in the general framework of [FM08], provided that each graph in the indexed sequence is **P**-unique. This improves the characterization of the instances of the defect matching polynomial given in [GG81], and generalizes it to infinitely many other graph polynomials and indexed sequences of graphs.

It also shows that for graph polynomials given as generating functions of induced or spanning subgraphs with a given property, C-finiteness is a semantic property. It remains unclear, whether this also applies to generalized chromatic polynomials.

However, this approach to the algebraic characterization of graph properties, as envisaged by the late Herbert Wilf in [Wil73], is just a very small step forward. The characterization of the polynomials which are instances of the prominent graph polynomials, the matching, chromatic or characteristic polynomials, remains wide open.

In the final section we also briefly discussed whether and when such a characterization found for a graph polynomial **P** sheds light on other graph polynomials which are d.p.- or s.d.p-equivalent to **P**. In forthcoming paper we shall discuss d.p.- or s.d.p-equivalence of graph polynomials from a logical point of view, [KMR17].

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References

References

- [BDS72] N.L. Biggs, R.M. Damerell, and D.A. Sand. Recursive families of graphs. J. Combin. Theory Ser. B, 12:123–131, 1972.
- [BF95] R.A. Beezer and E.J. Farrell. The matching polynomial of a regular graph. Discrete Mathematics, 137(1):7–18, 1995.
- [BH12] A.E. Brouwer and W.H. Haemers. Spectra of Graphs. Springer Universitext. Springer, 2012.
- [Bra15] P. Branden. Unimodality, log-concavity, real-rootedness and beyond. Handbook of Enumerative Combinatorics, 87:437, 2015.
- [Bre92] F. Brenti. Expansions of chromatic polynomials and log-concavity. Transactions of the American Mathematical Society, 332(2):729–756, 1992.
- [CH03] G. Cherlin and E. Hrushovski. Finite structures with few types, volume 152 of Annals of Mathematics Studies. Princeton University Press, 2003.
- [Chi11] T.S. Chihara. An introduction to orthogonal polynomials. Courier Corporation, 2011.
- [DG04] P. Diaconis and A. Gamburd. Random matrices, magic squares and matching polynomials. JOURNAL OF COMBINATORICS, 11(4):R2, 2004.
- [DKT05] F.M. Dong, K.M. Koh, and K.L. Teo. Chromatic Polynomials and Chromaticity of Graphs. World Scientific, 2005.

- [dlHJ95] P. de la Harpe and F. Jaeger. Chromatic invariants for finite graphs: Theme and polynomial variations. *Linear Algebra and its Applications*, 226-228:687–722, 1995.
- [dMN04] A. de Mier and M. Noy. On graphs determined by their Tutte polynomials. Graphs and Combinatorics, 20.1:105–119, 2004.
- [FM08] E. Fischer and J.A. Makowsky. Linear recurrence relations for graph polynomials. In A. Avron, N. Dershowitz, and A. Rabinowitz, editors, *Boris* (*Boaz*) A. Trakhtenbrot on the occasion of his 85th birthday, volume 4800 of LNCS, pages 266–279. Springer, 2008.
- [Ges89] I. Gessel. Generalized rook polynomials and orthogonal polynomials. In q-Series and Partitions, pages 159–176. Springer, 1989.
- [Ges95] I. Gessel. Enumerative applications of a decomposition for graphs and digraphs. Discrete mathematics, 139(1):257–271, 1995.
- [GG81] C.D. Godsil and I. Gutman. On the theory of the matching polynomial. Journal of Graph Theory, 5(2):137–144, 1981.
- [GGN13] D. Garijo, A. Goodall, and J. Nešetřil. Polynomial graph invariants from homomorphism numbers. arXiv:1308.3999 [math.CO], 2013.
- [GHK⁺17] A. Goodall, M. Hermann, T. Kotek, J.A. Makowsky, and S.D. Noble. On the complexity of generalized chromatic polynomials. arXiv http://arxiv.org/abs/1701.06639, 2017.
- [GNdM16] A.J. Goodall, J. Nešetřil, and P. Ossona de Mendez. Strongly polynomial sequences as interpretations of trivial structures. J. Appl. Logic, xx(x):xxyy, 2016. arXiv:1405.2449.
 - [God81] C.D. Godsil. Hermite polynomials and a duality relation for matchings polynomials. *Combinatorica*, 1(3):257–262, 1981.
 - [Huh15] J. Huh. h-vectors of matroids and logarithmic concavity. Advances in Mathematics, 270:49–59, 2015.
- [KMR17] T. Kotek, J.A. Makowsky, and E.V. Ravve. A logician's view of graph polynomials. Preprint, 2017.
- [KMZ08] T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In Computer Science Logic, CSL'08, volume 5213 of Lecture Notes in Computer Science, pages 339–353, 2008.
- [KMZ11] T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In M. Grohe and J.A. Makowsky, editors, *Model Theoretic Methods* in Finite Combinatorics, volume 558 of Contemporary Mathematics, pages 207–242. American Mathematical Society, 2011.
- [Kot12] T. Kotek. *Definability of combinatorial functions*. PhD thesis, Technion Israel Institute of Technology, Haifa, Israel, March 2012.

- [LM05] V.E. Levit and E. Mandrescu. The independence polynomial of a graph - a survey. In S. Bozapalidis, A. Kalampakas, and G. Rahonis, editors, *Proceedings of the 1st International Conference on Algebraic Informatics*, pages 233–254. Aristotle University of Thessaloniki, Department of Mathematics, Thessaloniki, 2005.
- [LP86] L. Lovász and M.D. Plummer. Matching Theory, volume 29 of Annals of Discrete Mathematics. North Holland, 1986.
- [MN09] C. Merino and S. D. Noble. The equivalence of two graph polynomials and a symmetric function. Combinatorics, Probability & Computing, 18(4):601– 615, 2009.
- [MRB14] J.A. Makowsky, E.V. Ravve, and N.K. Blanchard. On the location of roots of graph polynomials. *European Journal of Combinatorics*, 41:1–19, 2014.
 - [MZ06] J.A. Makowsky and B. Zilber. Polynomial invariants of graphs and totally categorical theories. MODNET Preprint No. 21, http://www.logique.jussieu.fr/ modnet/Publications/Preprint%20 server, 2006.
- [Noy03] M. Noy. On graphs determined by polynomial invariants. Theoretical Computer Science, 307:365–384, 2003.
- [NR04] M. Noy and A. Ribó. Recursively constructible families of graphs. Advances in Applied Mathematics, 32:350–363, 2004.
- [PWZ96] M. Petkovsek, H. Wilf, and D. Zeilberger. A=B. AK Peters, 1996.
- [Sch74] A.J. Schwenk. Computing the characteristic polynomial of a graph. In *Graphs and Combinatorics*, pages 153–172. Springer, 1974.
- [Sok05] A.D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *arXiv*, CO:0503607, 2005.
- [Sta89] R.P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometrya. Annals of the New York Academy of Sciences, 576(1):500-535, 1989.
- [TAM11] P. Tittmann, I. Averbouch, and J.A. Makowsky. The enumeration of vertex induced subgraphs with respect to the number of components. *European Journal of Combinatorics*, 32(7):954–974, 2011.
 - [Wil73] H.S. Wilf. Which polynomials are chromatic. In Proc. Colloq. Combinatorial Theory, Rome, 1973.
 - [Zil93] B. Zilber. Uncountably Categorical Theories, volume 117 of Translations of Mathematical Monographs. American Mathematical Society, 1993.