An isoperimetric inequality for antipodal subsets of the discrete cube

David Ellis and Imre Leader

14th September 2016

Abstract

We say a family of subsets of $\{1, 2, \ldots, n\}$ is *antipodal* if it is closed under taking complements. We prove a best-possible isoperimetric inequality for antipodal families of subsets of $\{1, 2, \ldots, n\}$ (of any size). Our inequality implies that for any $k \in \mathbb{N}$, among all such families of size 2^k, a family consisting of the union of two antipodal $(k-1)$-dimensional subcubes has the smallest possible edge boundary.

1 Introduction

Isoperimetric questions are classical objects of study in mathematics. In general, they ask for the minimum possible 'boundary-size' of a set of a given 'size', where the exact meaning of these words varies according to the problem.

The classical isoperimetric problem in the plane asks for the minimum possible perimeter of a shape in the plane with area 1. The answer, that it is best to take a circle, was 'known' to the ancient Greeks, but it was not until the 19th century that this was proved rigorously, by Weierstrass in a series of lectures in the 1870s in Berlin.

The isoperimetric problem has been solved for n-dimensional Euclidean space $\mathbb{E}^n$, for the n-dimensional unit sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$, and for n-dimensional hyperbolic space $\mathbb{H}^n$ (for all n), with the natural notion of boundary in each case, corresponding to surface area for sufficiently 'nice' sets. (For background on isoperimetric problems, we refer the reader to the book of Burago and Zalgaller [2], the surveys of Osserman [6] and of Ros [8], and the references therein.) One of the most well-known open problems in the area is to solve the isoperimetric problem for n-dimensional real projective space $\mathbb{RP}^n$, or equivalently for antipodal subsets of the n-dimensional sphere $\mathbb{S}^n$. (We say a subset $\mathcal{A} \subseteq \mathbb{S}^n$ is *antipodal* if $\mathcal{A} = -\mathcal{A}$.) The conjecture can be stated as follows.

Conjecture 1. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let μ denote the n-dimensional Hausdorff measure on $\mathbb{S}^n$. Let $\mathcal{A} \subseteq \mathbb{S}^n$ be open and antipodal. Then there exists* *a set* $\mathcal{B} \subseteq \mathbb{S}^n$ *such that* $\mu(\mathcal{B}) = \mu(\mathcal{A})$, $\sigma(\mathcal{B}) \leq \sigma(\mathcal{A})$, *and*

$$\mathcal{B} = \{x \in \mathbb{S}^n : \sum_{i=1}^{r} x_i^2 > a\}$$

for some $r \in [n]$ *and some* $a \in \mathbb{R}$.

Here, if $\mathcal{A} \subseteq \mathbb{S}^n$ is an open set, then $\sigma(\mathcal{A})$ denotes the surface area of $\mathcal{A}$, i.e. the $(n-1)$-dimensional Hausdorff measure of the topological boundary of $\mathcal{A}$.

Only the cases $n = 2$ and $n = 3$ of Conjecture 1 are known, the former being 'folklore' and the latter being due to Ritoré and Ros [7]. In this paper, we prove a discrete analogue of Conjecture 1.

First for some definitions and notation. If X is a set, we write $\mathcal{P}(X)$ for the power-set of X. For $n \in \mathbb{N}$, we write $[n] := \{1, 2, \ldots, n\}$, and we let Q_n denote the graph of the n-dimensional discrete cube, i.e. the graph with vertex-set $\mathcal{P}([n])$, where x and y are joined by an edge if $|x \Delta y| = 1$. If $\mathcal{A} \subseteq \mathcal{P}([n])$, we write $\partial\mathcal{A}$ for the *edge-boundary* of $\mathcal{A}$ in the discrete cube Q_n, i.e. $\partial\mathcal{A}$ is the set of edges of Q_n which join a vertex in $\mathcal{A}$ to a vertex outside $\mathcal{A}$. We write $e(\mathcal{A})$ for the number of edges of Q_n which have both end-vertices in $\mathcal{A}$. We say that two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}([n])$ are *isomorphic* if there exists an automorphism σ of Q_n such that $\mathcal{B} = \sigma(\mathcal{A})$. Clearly, if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, then $|\partial\mathcal{A}| = |\partial\mathcal{B}|$.

The *binary ordering* on $\mathcal{P}([n])$ is defined by $x < y$ iff $\max(x \Delta y) \in y$. An *initial segment of the binary ordering on* $\mathcal{P}([n])$ is the set of the first k (smallest) elements of $\mathcal{P}([n])$ in the binary ordering, for some $k \leq 2^n$. For any $k \leq 2^n$, we write $\mathcal{I}_{n,k}$ for the initial segment of the binary ordering on $\mathcal{P}([n])$ with size k.

Harper [3], Lindsay [5], Bernstein [1] and Hart [4] solved the edge isoperimetric problem for Q_n, showing that among all subsets of $\mathcal{P}([n])$ of given size, initial segments of the binary ordering on $\mathcal{P}([n])$ have the smallest possible edge-boundary.

In this paper, we consider the edge isoperimetric problem for antipodal sets in Q_n. If $x \subseteq [n]$, we define $\overline{x} := [n] \setminus x$, and if $\mathcal{A} \subseteq \mathcal{P}([n])$, we define $\overline{\mathcal{A}} := \{\overline{x} : x \in \mathcal{A}\}$. We say a family $\mathcal{A} \subseteq \mathcal{P}([n])$ is *antipodal* if $\mathcal{A} = \overline{\mathcal{A}}$. This notion is of course the natural analogue in the discrete cube of antipodality in $\mathbb{S}^n$; indeed, identifying $\mathcal{P}([n])$ with $\{-1, 1\}^n \subseteq \sqrt{n} \cdot \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ in the natural way, $x \mapsto \overline{x}$ corresponds to the antipodal map $\mathbf{v} \mapsto -\mathbf{v}$.

We prove the following best-possible edge isoperimetric inequality for antipodal families.

Theorem 2. *Let* $\mathcal{A} \subseteq \mathcal{P}([n])$ *be antipodal. Then*

$$|\partial\mathcal{A}| \geq |\partial(\mathcal{I}_{n,|\mathcal{A}|/2} \cup \overline{\mathcal{I}_{n,|\mathcal{A}|/2}})|.$$

We remark that Theorem 2 implies that if $\mathcal{A} \subseteq \mathcal{P}([n])$ is antipodal with $|\mathcal{A}| = 2^k$ for some $k \in [n-1]$, then $|\partial\mathcal{A}| \geq |\partial(\mathcal{S}_{k-1} \cup \overline{\mathcal{S}_{k-1}})|$, where $\mathcal{S}_{k-1} := \mathcal{I}_{n,2^{k-1}} = \{x \subseteq [n] : x \subseteq [k-1]\}$ is a $(k-1)$-dimensional subcube. In other words, a union of two antipodal subcubes has the smallest possible edge-boundary, over all antipodal sets of the same size.

To prove Theorem 2, it will be helpful for us to rephrase it slightly. Firstly, observe that for any $\mathcal{A} \subseteq \mathcal{P}([n])$, we have $\partial(\mathcal{A}^c) = \partial\mathcal{A}$, and that for any $k \leq 2^{n-1}$, the family $(\mathcal{I}_{n,k} \cup \overline{\mathcal{I}_{n,k}})^c$ is isomorphic to the family $\mathcal{I}_{n,2^{n-1}-k} \cup \overline{\mathcal{I}_{n,2^{n-1}-k}}$, via the isomorphism $x \mapsto x\Delta\{n\}$. Hence, by taking complements, it suffices to prove Theorem 2 in the case $|\mathcal{A}| \leq 2^{n-1}$.

Secondly, for any family $\mathcal{A} \subseteq \mathcal{P}([n])$, we have

$$2e(\mathcal{A}) + |\partial\mathcal{A}| = n|\mathcal{A}|, \tag{1}$$

so Theorem 2 is equivalent to the statement that if $\mathcal{A} \subseteq \mathcal{P}([n])$ is antipodal, then

$$e(\mathcal{A}) \leq e(\mathcal{I}_{n,|\mathcal{A}|/2} \cup \overline{\mathcal{I}_{n,|\mathcal{A}|/2}}).$$

Note also that if $\mathcal{B}$ is an initial segment of the binary ordering on $\mathcal{P}([n])$ with $|\mathcal{B}| \leq 2^{n-2}$, then $\mathcal{B} \subseteq \{x \subseteq [n] : \ x \cap \{n-1,n\} = \emptyset\}$ and $\overline{\mathcal{B}} \subseteq \{x \subseteq [n] : \{n-1,n\} \subseteq x\}$, so $\mathcal{B} \cap \overline{\mathcal{B}} = \emptyset$ and $e(\mathcal{B},\overline{\mathcal{B}}) = 0$. Moreover, it is easy to see that $\overline{\mathcal{B}}$ is isomorphic to $\mathcal{B}$, and therefore $e(\overline{\mathcal{B}}) = e(\mathcal{B})$. Hence,

$$e(\mathcal{B} \cup \overline{\mathcal{B}}) = e(\mathcal{B}) + e(\overline{\mathcal{B}}) = 2e(\mathcal{B}).$$

If $k, n \in \mathbb{N}$ with $k \leq 2^n$, we write $F(k) := e(\mathcal{I}_{n,k})$. (It is easy to see that $F(k)$ is independent of n.) Putting all this together, we see that Theorem 2 is equivalent to the following:

$$e(\mathcal{A}) \leq 2F(|\mathcal{A}|/2) \quad \forall \mathcal{A} \subseteq \mathcal{P}([n]) : \ |\mathcal{A}| \leq 2^{n-1}, \ \mathcal{A} \text{ is antipodal.} \tag{2}$$

Now for a few words about our proof. In the special cases of $|\mathcal{A}| = 2^{n-1}$ and $|\mathcal{A}| = 2^{n-2}$, Theorem 2 can be proved by an easy Fourier-analytic argument, but it is fairly obvious that this argument has no hope of proving the theorem for general set-sizes. Our proof of Theorem 2 is purely combinatorial; we prove a stronger statement by induction on n. Our aim is to do induction on n in the usual way: namely, by choosing some $i \in [n]$ and considering the upper and lower i-sections of $\mathcal{A}$, defined respectively by

$$\mathcal{A}_i^+ := \{x \in \mathcal{P}([n] \setminus \{i\}) : \ x \cup \{i\} \in \mathcal{A}\}, \quad \mathcal{A}_i^- := \{x \in \mathcal{P}([n] \setminus \{i\}) : \ x \in \mathcal{A}\}.$$

However, a moment's thought shows that an i-section of an antipodal family need not be antipodal. For example, if $\mathcal{A} = \mathcal{S}_{k-1} \cup \overline{\mathcal{S}_{k-1}}$ (a union of two antipodal $(k-1)$-dimensional subcubes), then for any $i \geq k$, the i-section $\mathcal{A}_i^-$ consists of a single $(k-1)$-dimensional subcube, which is not an antipodal family. This rules out an inductive hypothesis involving antipodal families.

Hence, we seek a stronger statement, about arbitrary subsets of $\mathcal{P}([n])$; one which we can prove by induction on n, and which will imply Theorem 2. It turns out that the right statement is as follows. For any $\mathcal{A} \subseteq \mathcal{P}([n])$ (not necessarily antipodal), we define

$$f(\mathcal{A}) := 2e(\mathcal{A}) + |\mathcal{A} \cap \overline{\mathcal{A}}|.$$

To prove Theorem 2, it suffices to prove the following.

Theorem 3. *For any $n \in \mathbb{N}$ and any $\mathcal{A} \subseteq \mathcal{P}([n])$ with $|\mathcal{A}| \leq 2^{n-1}$, we have*

$$f(\mathcal{A}) \leq 2F(|\mathcal{A}|). \tag{3}$$

Indeed, assume that Theorem 3 holds. Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be antipodal with $|\mathcal{A}| \leq 2^{n-1}$. We have $\mathcal{A}_n^- = \overline{\mathcal{A}_n^+}$ and $|\mathcal{A}_n^+| = |\mathcal{A}|/2 \leq 2^{n-2}$, and so

$$\begin{aligned} e(\mathcal{A}) &= e(\mathcal{A}_n^+) + e(\mathcal{A}_n^-) + |\mathcal{A}_n^+ \cap \mathcal{A}_n^-| = e(\mathcal{A}_n^+) + e(\overline{\mathcal{A}_n^+}) + |\mathcal{A}_n^+ \cap \overline{\mathcal{A}_n^+}| \\ &= 2e(\mathcal{A}_n^+) + |\mathcal{A}_n^+ \cap \overline{\mathcal{A}_n^+}| = f(\mathcal{A}_n^+) \leq 2F(|\mathcal{A}_n^+|) = 2F(|\mathcal{A}|/2), \end{aligned}$$

implying (2) and so proving Theorem 2.

Note that the function f takes the same value (namely, $k2^k$) when $\mathcal{A}$ is a k-dimensional subcube, as when $\mathcal{A}$ is the union of two antipodal $(k-1)$-dimensional subcubes. This is certainly needed in order for our inductive approach to work, by our above remark about the i-sections of the union of two antipodal subcubes.

We prove Theorem 3 in the next section; in the rest of this section, we gather some additional facts we will use in our proof.

We will use the following lemma of Hart from [4].

Lemma 4. *For any $x, y \in \mathbb{N} \cup \{0\}$, we have*

$$F(x+y) - F(x) - F(y) \geq \min\{x, y\}.$$

Equality holds if y is a power of 2 and $x \leq y$.

We will also use the following easy consequence of Lemma 4.

Lemma 5. *Let $x, y \in \mathbb{N} \cup \{0\}$ and let $n \in \mathbb{N}$ such that $x + y \leq 2^n$, $y \geq 2^{n-1}$ and $y \leq 2^{n-1} + x$. Then*

$$F(x+y) - F(y) - F(x) - y + 2^{n-1} \geq x.$$

Proof. Write $z := y - 2^{n-1}$; then $z \leq x$ and $x + z \leq 2^{n-1}$. We therefore have

$$\begin{aligned} F(x+y) - F(y) - F(x) - y + 2^{n-1} &= F(2^{n-1} + x + z) - F(2^{n-1} + z) - F(x) - z \\ &= F(2^{n-1}) + F(x+z) + x + z \\ &\quad - F(2^{n-1}) - F(z) - z - F(x) - z \\ &= F(x+z) - F(x) - F(z) + x - z \\ &\geq \min\{x, z\} + x - z \\ &= x, \end{aligned}$$

where the last inequality uses Lemma 4. □

We also need the following lemma, which says that for any family $\mathcal{A} \subseteq \mathcal{P}([n])$, one coordinate from every pair of coordinates is such that the upper and lower sections of $\mathcal{A}$ corresponding to that coordinate are 'somewhat' close in size.

Lemma 6. *Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\mathcal{A} \subseteq \mathcal{P}([n])$. Then for any $1 \leq i < j \leq n$, we have*

$$\min\{\big||\mathcal{A}_i^+| - |\mathcal{A}_i^-|\big|, \big||\mathcal{A}_j^+| - |\mathcal{A}_j^-|\big|\} \leq 2^{n-2}.$$

Proof. Without loss of generality, by considering $\{A\Delta S : A \in \mathcal{A}\}$ for some $S \subseteq \{i,j\}$, we may assume that $|\mathcal{A}_i^+| \leq |\mathcal{A}_i^-|$ and that $|\mathcal{A}_j^+| \leq |\mathcal{A}_j^-|$. Interchanging i and j if necessary, we may assume that $|(\mathcal{A}_i^-)_j^+| \geq |(\mathcal{A}_i^+)_j^-|$. Then we have

$$\begin{aligned}0 \leq |\mathcal{A}_j^-| - |\mathcal{A}_j^+| &= |(\mathcal{A}_i^-)_j^-| + |(\mathcal{A}_i^+)_j^-| - |(\mathcal{A}_i^-)_j^+| - |(\mathcal{A}_i^+)_j^+| \\ &\leq |(\mathcal{A}_i^-)_j^-| - |(\mathcal{A}_i)^+)_j^+| \\ &\leq |(\mathcal{A}_i^-)_j^-| \\ &\leq 2^{n-2},\end{aligned}$$

proving the lemma. □

We also need the following.

Lemma 7. *Let $n \in \mathbb{N}$ and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}([n])$. Then*

$$2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| \leq |\mathcal{C} \cap \overline{\mathcal{C}}| + |\mathcal{D} \cap \overline{\mathcal{D}}| + 2\min\{|\mathcal{C}|, |\mathcal{D}|\}. \tag{4}$$

Proof. Note that both sides of the above inequality are invariant under interchanging $\mathcal{C}$ and $\mathcal{D}$, so it suffices to prove the lemma in the case $|\mathcal{C}| \leq |\mathcal{D}|$. By inclusion-exclusion, we have

$$2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| = 2|\mathcal{C} \cap (\mathcal{D} \cup \overline{\mathcal{D}})| + 2|\mathcal{C} \cap (\mathcal{D} \cap \overline{\mathcal{D}})| \leq 2|\mathcal{C}| + 2|\mathcal{C} \cap (\mathcal{D} \cap \overline{\mathcal{D}})|,$$

so it suffices to prove that

$$2|\mathcal{C} \cap (\mathcal{D} \cap \overline{\mathcal{D}})| \leq |\mathcal{C} \cap \overline{\mathcal{C}}| + |\mathcal{D} \cap \overline{\mathcal{D}}|.$$

Writing $\mathcal{E} = \mathcal{D} \cap \overline{\mathcal{D}}$, it suffices to prove that for any antipodal set $\mathcal{E} \subseteq \mathcal{P}([n])$, and any set $\mathcal{C} \subseteq \mathcal{P}([n])$, we have

$$2|\mathcal{C} \cap \mathcal{E}| \leq |\mathcal{C} \cap \overline{\mathcal{C}}| + |\mathcal{E}|.$$

This follows immediately from inclusion-exclusion again; indeed, we have

$$2|\mathcal{C}\cap\mathcal{E}| = |\mathcal{C}\cap\mathcal{E}| + |\mathcal{C}\cap\overline{\mathcal{E}}| = |\mathcal{C}\cap\mathcal{E}| + |\overline{\mathcal{C}}\cap\mathcal{E}| = |(\mathcal{C}\cap\overline{\mathcal{C}})\cap\mathcal{E}| + |(\mathcal{C}\cup\overline{\mathcal{C}})\cap\mathcal{E}| \leq |\mathcal{C}\cap\overline{\mathcal{C}}| + |\mathcal{E}|,$$

whenever $\mathcal{E}$ is antipodal. □

Finally, we note that for any $\mathcal{A} \subseteq \mathcal{P}([n])$, we have

$$\begin{aligned}f(\mathcal{A}^c) &= 2e(\mathcal{A}^c) + |\mathcal{A}^c \cap \overline{\mathcal{A}^c}| = n|\mathcal{A}^c| - |\partial(\mathcal{A}^c)| + |\mathcal{A}^c \cap \bar{\mathcal{A}}^c| \\ &= n|\mathcal{A}| + n(|\mathcal{A}^c| - |\mathcal{A}|) - |\partial\mathcal{A}| + |(\mathcal{A} \cup \overline{\mathcal{A}})^c| \\ &= n|\mathcal{A}| - |\partial\mathcal{A}| + n(2^n - 2|\mathcal{A}|) + 2^n - |\mathcal{A} \cup \overline{\mathcal{A}}| \\ &= 2e(\mathcal{A}) + n(2^n - 2|\mathcal{A}|) + 2^n - 2|\mathcal{A}| + |\mathcal{A} \cap \overline{\mathcal{A}}| \\ &= f(\mathcal{A}) + 2(n+1)(2^{n-1} - |\mathcal{A}|).\end{aligned} \tag{5}$$

Moreover, using (1) and the fact that $(\mathcal{I}_{n,k})^c$ is isomorphic to $\mathcal{I}_{n,2^n-k}$, we have

$$\begin{aligned} 2F(k) - 2F(2^n - k) &= kn - \partial(\mathcal{I}_{n,k}) - ((2^n - k)n - \partial((\mathcal{I}_{n,k})^c)) \\ &= kn - \partial(\mathcal{I}_{n,k}) - ((2^n - k)n - \partial(\mathcal{I}_{n,k})) \\ &= (2k - 2^n)n \qquad (6) \end{aligned}$$

for any $k \leq 2^n$. It follows from (5) and (6), by taking complements, that Theorem 3 is equivalent to the inequality

$$f(\mathcal{A}) \leq 2F(|\mathcal{A}|) + 2|\mathcal{A}| - 2^n \quad \forall \mathcal{A} \subseteq \mathcal{P}([n]) : \ |\mathcal{A}| \geq 2^{n-1}. \qquad (7)$$

2 Proof of Theorem 3

Our proof is by induction on n. The base case $n = 1$ of Theorem 3 is easily checked. We turn to the induction step. Let $n \geq 2$, and assume that Theorem 3 holds when n is replaced by $n - 1$. Let $\mathcal{A} \subseteq \mathcal{P}([n])$ with $|\mathcal{A}| \leq 2^{n-1}$. Observe that for any $i \in [n]$, we have

$$\begin{aligned} f(\mathcal{A}) &= 2e(\mathcal{A}) + |\mathcal{A} \cap \overline{\mathcal{A}}| \\ &= 2e(\mathcal{A}_i^+) + 2e(\mathcal{A}_i^-) + 2|\mathcal{A}_i^+ \cap \mathcal{A}_i^-| + |\mathcal{A}_i^+ \cap \overline{\mathcal{A}_i^-}| + |\mathcal{A}_i^- \cap \overline{\mathcal{A}_i^+}| \\ &= 2e(\mathcal{A}_i^+) + 2e(\mathcal{A}_i^-) + 2|\mathcal{A}_i^+ \cap \mathcal{A}_i^-| + 2|\mathcal{A}_i^+ \cap \overline{\mathcal{A}_i^-}|. \qquad (8) \end{aligned}$$

We now split into two cases.

Case 1. Firstly, suppose that there exists $i \in [n]$ such that $\max\{|\mathcal{A}_i^+|, |\mathcal{A}_i^-|\} \leq 2^{n-2}$. Without loss of generality, we may assume that this holds for $i = n$, i.e. that $\max\{|\mathcal{A}_n^+|, |\mathcal{A}_n^-|\} \leq 2^{n-2}$. We may also assume that $|\mathcal{A}_n^+| \leq |\mathcal{A}_n^-|$. Then, defining $\mathcal{C} := \mathcal{A}_n^+ \subseteq \mathcal{P}([n-1])$ and $\mathcal{D} := \mathcal{A}_n^- \subseteq \mathcal{P}([n-1])$, and invoking (8) with $i = n$, we have

$$\begin{aligned} f(\mathcal{A}) &= 2e(\mathcal{C}) + 2e(\mathcal{D}) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| \\ &= f(\mathcal{C}) + f(\mathcal{D}) - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}|. \qquad (9) \end{aligned}$$

We now apply the induction hypothesis to $\mathcal{C}$ and $\mathcal{D}$. Since $|\mathcal{C}| \leq |\mathcal{D}| \leq 2^{n-2}$, we may apply (3), obtaining $f(\mathcal{C}) \leq 2F(|\mathcal{C}|)$ and $f(\mathcal{D}) \leq 2F(|\mathcal{D}|)$. Substituting the last two inequalities into (9), we obtain

$$\begin{aligned} f(\mathcal{A}) &\leq 2F(|\mathcal{C}|) + 2F(|\mathcal{D}|) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| \\ &= 2F(|\mathcal{C}| + |\mathcal{D}|) - \big(2F(|\mathcal{C}| + |\mathcal{D}|) - 2F(|\mathcal{C}|) - 2F(|\mathcal{D}|)\big) \\ &\quad + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| \\ &\leq 2F(|\mathcal{A}|) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| - 2\min\{|\mathcal{C}|, |\mathcal{D}|\} \\ &= 2F(|\mathcal{A}|) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| - 2|\mathcal{C}| \\ &\leq 2F(|\mathcal{A}|), \end{aligned}$$

where the second inequality follows from Lemma 4, and the third inequality follows from Lemma 7. This completes the induction step in Case 1.

Case 2. Secondly, suppose that Case 1 does not occur, i.e. that $\max\{|\mathcal{A}_j^+|, |\mathcal{A}_j^-|\} > 2^{n-2}$ for all $j \in [n]$. By Lemma 6, there exists $i \in [n]$ such that $\big||\mathcal{A}_i^+| - |\mathcal{A}_i^-|\big| \leq 2^{n-2}$, and therefore

$$2^{n-2} < \max\{|\mathcal{A}_i^+|, |\mathcal{A}_i^-|\} \leq \min\{|\mathcal{A}_i^+|, |\mathcal{A}_i^-|\} + 2^{n-2}.$$

Without loss of generality, we may assume that this holds for $i = n$, and that $|\mathcal{A}_n^+| \leq |\mathcal{A}_n^-|$, so that

$$2^{n-2} < |\mathcal{A}_n^-| \leq 2^{n-2} + |\mathcal{A}_n^+|.$$

Defining $\mathcal{C} := \mathcal{A}_n^+ \subseteq \mathcal{P}([n-1])$ and $\mathcal{D} := \mathcal{A}_n^- \subseteq \mathcal{P}([n-1])$ as before, and invoking (8) with $i = n$, we have

$$\begin{aligned} f(\mathcal{A}) &= 2e(\mathcal{C}) + 2e(\mathcal{D}) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| \\ &= f(\mathcal{C}) + f(\mathcal{D}) - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}|. \end{aligned} \tag{10}$$

Now, since $|\mathcal{C}| \leq |\mathcal{D}|$, we have $2|\mathcal{C}| \leq |\mathcal{C}| + |\mathcal{D}| = |\mathcal{A}| \leq 2^{n-1}$ and therefore $|\mathcal{C}| \leq 2^{n-2}$. On the other hand, we have $|\mathcal{D}| > 2^{n-2}$. Applying the induction hypothesis to $\mathcal{C}$ and $\mathcal{D}$ (using (3) for $\mathcal{C}$ and (7) for $\mathcal{D}$), we obtain $f(\mathcal{C}) \leq 2F(|\mathcal{C}|)$ and $f(\mathcal{D}) \leq 2F(|\mathcal{D}|) + 2|\mathcal{D}| - 2^{n-1}$; substituting these two inequalities into (10) yields

$$\begin{aligned} f(\mathcal{A}) &\leq 2F(|\mathcal{C}|) + 2F(|\mathcal{D}|) + 2|\mathcal{D}| - 2^{n-1} + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| \\ &= 2F(|\mathcal{C}| + |\mathcal{D}|) - \big(2F(|\mathcal{C}| + |\mathcal{D}|) - 2F(|\mathcal{C}|) - 2F(|\mathcal{D}|) - 2|\mathcal{D}| + 2 \cdot 2^{n-2}\big) \\ &\quad + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| \\ &\leq 2F(|\mathcal{A}|) + 2|\mathcal{C} \cap \mathcal{D}| + 2|\mathcal{C} \cap \overline{\mathcal{D}}| - |\mathcal{C} \cap \overline{\mathcal{C}}| - |\mathcal{D} \cap \overline{\mathcal{D}}| - 2|\mathcal{C}| \\ &\leq 2F(|\mathcal{A}|), \end{aligned}$$

where the second inequality uses Lemma 5, applied with $x = |\mathcal{C}|$ and $y = |\mathcal{D}|$, and with $n - 1$ in place of n, and the third inequality uses Lemma 7. This completes the induction step in Case 2, proving the theorem.

3 Conclusion

We feel that our proof of Theorem 3 (and therefore of Theorem 2) is somewhat delicate, as it relies on the fact that, in the inductive step, the terms involving F can be dealt with using the fortunate properties of the function F (in Lemmas 4 and 5), and the other terms can be dealt with using the elementary inequality in Lemma 7. We also note that there is a nested sequence of families (with one family of every possible size), each of which is extremal for Theorem 2. In contrast, the (conjectural) extremal families in Conjecture 1 do not have this 'nested' property. Hence, perhaps unfortunately, we feel that Theorem 2, and our proof thereof, may shed only a limited amount of light on Conjecture 1.

Acknowledgements

We would like to thank an anonymous referee for suggesting the above proof of Lemma 6, which is simpler than our original argument.

References

[1] A. J. Bernstein, Maximally connected arrays on the n-cube, *SIAM J. Appl. Math.* **15** (1967), 1485–1489.

[2] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Springer-Verlag, Berlin (1988).

[3] L. H. Harper, Optimal assignment of numbers to vertices, *SIAM J. Appl. Math.*, **12** (1964), 131–135.

[4] S. Hart, A note on the edges of the n-cube, *Discrete Math.* **14** (1976), pp 157–161.

[5] J. H. Lindsey II, Assignment of numbers to vertices, *Amer. Math. Monthly* **71** (1964), 508–516.

[6] R. Osserman, The isoperimetric inequality, *Bull. Amer. Math. Soc.* **84** (1978), 1182–1238.

[7] M. Ritoré and A. Ros, Stable mean constant curvature tori and the isoperimetric problem in three space forms, *Comment. Math. Helvet.* **67** (1992), 293–305.

[8] A. Ros, *The Isoperimetric Problem*, Lecture notes from the Clay Mathematics Institute Summer School on the Global Theory of Minimal Surfaces, June–July 2001, Mathematical Sciences Research Institute, Berkeley, California. Available at `http://www.ugr.es/ aros/isoper.pdf`.