

Proof of a Conjecture on the Seidel Energy of Graphs

S. Akbari^{a,*}, M. Einollahzadeh, M. M. Karkhaneei^a and M. A. Nematollahi^a

^aDepartment of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

Abstract

Let G be a graph with the vertex set $\{v_1, \dots, v_n\}$. The Seidel matrix of G is an $n \times n$ matrix whose diagonal entries are zero, ij -th entry is -1 if v_i and v_j are adjacent and otherwise is 1 . The Seidel energy of G is defined to be the sum of absolute values of all eigenvalues of the Seidel matrix of G . Haemers conjectured that the Seidel energy of any graph of order n is at least $2n - 2$ and, up to Seidel equivalence, the equality holds for K_n . We establish the validity of Haemers' Conjecture in general.

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1 Introduction and Terminology

Throughout this paper all graphs we consider are simple and finite. For a graph G , we denote the set of vertices and edges of G by $V(G)$ and $E(G)$, respectively. The complement of G denoted by \overline{G} and the complete graph of order n is denoted by K_n . In this paper, for $v \in V(G)$, $N_G(v)$ and $N_G[v]$ denote the open neighborhood and the close neighborhood of v in G , respectively.

For every Hermitian matrix A and any real number $p > 0$, the p -energy of A , $\mathcal{E}_p(A)$, is defined to be sum of the p -th power of absolute values of the eigenvalues of A . The well-known concept of energy of a graph G denoted by $\mathcal{E}(G)$ is $\mathcal{E}_1(A)$, where A is the adjacency matrix of G .

Let G be a graph and $V(G) = \{v_1, \dots, v_n\}$. The *Seidel matrix* of G , denoted by $\mathcal{S}(G)$, is an $n \times n$ matrix whose diagonal entries are zero, ij -th entry is -1 if v_i and v_j are adjacent and otherwise is 1 (It is noteworthy that at first, van Lint and Seidel introduced the concept of

*E-mail addresses: s_akbari@sharif.edu (S. Akbari), eynollahzadehsamadi@gmail.com (M. Einollahzadeh), karkhaneei@gmail.com (M. M. Karkhaneei) mohammadali.nematollahi69@student.sharif.edu (M.A. Nematollahi).

Seidel matrix for the study of equiangular lines in [7]). The p -Seidel energy of G is defined to be $\mathcal{E}_p(\mathcal{S}(G))$. By the *Seidel energy* of G , we mean 1-Seidel energy of G and denote by $\mathcal{E}(\mathcal{S}(G))$. The *Seidel switching* of G is defined as follows: Partition $V(G)$ into two subsets V_1 and V_2 , delete the edges between V_1 and V_2 and join all vertices $v_1 \in V_1$ and $v_2 \in V_2$ which are not adjacent. Therefore, if we call the new graph by G' , then we have $\mathcal{S}(G') = D\mathcal{S}(G)D$, where D is a diagonal matrix with entries 1 (resp. -1) corresponding to the vertices of V_1 (resp. V_2) ([3]). Hence, $\mathcal{S}(G)$ and $\mathcal{S}(G')$ are similar and they have the same Seidel energy. Note, that if one of the V_1 or V_2 is empty, then G remains unchanged. Two graphs G_1 and G_2 are called *S-equivalent* (resp. *SC-equivalent*) if G_2 is obtained from G_1 (resp. G_1 or $\overline{G_1}$) by a Seidel switching. Note that in either cases, $\mathcal{S}(G_2)$ is similar to $\mathcal{S}(G_1)$ or $-\mathcal{S}(G_1)$, hence $\mathcal{E}(\mathcal{S}(G_1)) = \mathcal{E}(\mathcal{S}(G_2))$.

For every square matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, by $S_k(A)$ we mean $S_k(\lambda_1, \dots, \lambda_n)$, where $S_k(x_1, \dots, x_n)$ is the k -th elementary symmetric polynomial in n variables; i.e.

$$S_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad (S_0(x_1, \dots, x_n) := 1).$$

Also, for every $m \times n$ matrix R and the index sets $I \subseteq \{1, \dots, n\}$ and $J \subseteq \{1, \dots, m\}$, $R_{I,J}$ is the submatrix of R obtained by the restriction of R to the rows I and the columns J . By R^* , we mean conjugate transpose of R .

Haemers in [3] introduced the concept of Seidel energy of a graph and he proposed the following conjecture:

Conjecture. For every graph G of order n , $\mathcal{E}(\mathcal{S}(G)) \geq \mathcal{E}(\mathcal{S}(K_n)) = 2n - 2$.

This conjecture was first investigated by Haemers for $n \leq 10$ and then was settled for $n \leq 12$ in [2]. Ghorbani in [1] proved Haemers' Conjecture for the graphs G of order n such that $n - 1 \leq |\det(\mathcal{S}(G))|$. Also, Oboudi in [6] proved Haemers' Conjecture for every k -regular graph G of order n such that $k \neq \frac{n-1}{2}$ and G has no eigenvalue in $(-1, 0)$. Here, we establish two following theorems which are the main results of the paper:

Theorem 1. Let G be a graph of order n . Then, for every real number $p \in (0, 2)$

$$\mathcal{E}_p(\mathcal{S}(G)) > (n - 1)^p + (n - 2).$$

Theorem 2. For every graph G of order n , $\mathcal{E}(\mathcal{S}(G)) \geq \mathcal{E}(\mathcal{S}(K_n)) = 2n - 2$. Moreover, if G and K_n are not SC-equivalent, then the inequality is strict.

Hence, Theorem 2 proves Haemers' Conjecture. Note that if one restricts the attention to the circulant graphs, then the nature of the Haemers' Conjecture resembles the sharp Littlewood Conjecture on the minimum of the L^1 -norm of polynomials (on the unit circle in the complex

plane) whose absolute values of coefficients are equal to 1.¹ For the special class of polynomials with ± 1 coefficients, Klemeš proved the sharp Littlewood Conjecture [4] and in the procedure of his proof, he used the following equality:

$$\mathcal{E}_p(A) := \sum_{i=1}^n |\lambda_i|^p = C_{\frac{p}{2}} \int_0^\infty \ln \left(\sum_{k=0}^n S_k(A^2) t^k \right) t^{-\frac{p}{2}-1} dt, \quad \text{for every } p \in (0, 2), \quad (1)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix A .

The Equation (1) comes from the Equation (2) which can be checked by a change of variable:

$$\alpha^p = C_p \int_0^\infty \ln(1 + \alpha t) t^{-p-1} dt, \quad C_p = \left(\int_0^\infty \ln(1 + t) t^{-p-1} dt \right)^{-1}, \quad (2)$$

where $p \in (0, 1)$ and $\alpha = r e^{i\theta}$ is a complex number which is not a negative real number, $r > 0$ and $-\pi < \theta < \pi$. Indeed, if for every natural number n and complex numbers $\alpha_1, \dots, \alpha_n$ (no α_i is negative real number) we define $f(t) = \prod_{i=1}^n (1 + t\alpha_i)$, then by Equation (2) we have

$$\sum_{i=1}^n \alpha_i^p = C_p \int_0^\infty \ln(f(t)) t^{-p-1} dt, \quad \text{for every } p \in (0, 1). \quad (3)$$

Now, one can expand $f(t)$ as $\sum_{k=0}^n S_k(\alpha_1, \dots, \alpha_n) t^k$. Hence, if for every i , $\alpha_i = \lambda_i^2$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A , then Equation (1) is obtained.

Throughout this paper, we consider the branch $r^p e^{ip\theta}$ for α^p . We use Equation (1) to establish a lower bound for $S_k(A^2)$, where A is the Seidel matrix of a graph, and then we prove Haemers' Conjecture in general.

2 Main Theorems

In this section, we prove Theorems 1 and 2. So Haemers' Conjecture holds. First, we need the following well-known identity, by the Cauchy-Binet Theorem (See [5, p.776]).

Theorem 3 (Cauchy-Binet). If B is an $n \times n$ matrix of the form $B = RR^*$ for some matrix R , then

$$S_k(B) = \sum_{I, J} (\det R_{I, J})^2,$$

where the summation is taken over all k -subsets $I, J \subset \{1, \dots, n\}$.

¹Thanks to T. Tao's comment in <https://mathoverflow.net/q/302424/53059>

Lemma 4. Let G be a graph of order n and $A = \mathcal{S}(G)$. Then, for $k = 1, \dots, n$, we have

$$S_k(A^2) \geq n(n-1) \binom{n-2}{k-1}.$$

Proof. By Theorem 3, $S_k(A^2) = \sum_{I,J} (\det(A_{I,J}))^2$, where the summation is taken over all k -subsets I and J . Now, we prove the following claim.

Claim. For every two subsets I and J such that $|I \cap J| = k-1$, $|\det(A_{I,J})| \geq 1$.

To prove the claim, note that for every (i, j) , if $i \neq j$, then $a_{i,j} = \pm 1$ and otherwise $a_{i,j} = 0$. So, if $|I \cap J| = k-1$, then after applying permutations on rows and columns of $A_{I,J}$, modulo 2, $A_{I,J}$ has the following form:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}.$$

In the above matrix, subtract the first column from the other columns and get the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

whose determinant is 1. So $\det(A_{I,J})$ is an odd number and the claim is proved. On the other hand, the number of pairs (I, J) with the above property is $n(n-1) \binom{n-2}{k-1}$ and the proof is complete. \square

Proof of Theorem 1. Let $A = \mathcal{S}(G)$ and $\lambda_1, \dots, \lambda_n$ be its eigenvalues. By Equation (1), for every $p \in (0, 2)$, we have

$$\mathcal{E}_p(A) = \sum_{i=1}^n |\lambda_i|^p = C_{\frac{p}{2}} \int_0^\infty \ln\left(\sum_{k=0}^n S_k(A^2) t^k\right) t^{-\frac{p}{2}-1} dt.$$

Since $\prod_{i=1}^n (1 + t\lambda_i^2) = \sum_{k=0}^n S_k(A^2) t^k$, by Lemma 4, for every $t \geq 0$, we have

$$\prod_{i=1}^n (1 + t\lambda_i^2) = 1 + \sum_{k=1}^n S_k(A^2) t^k \geq 1 + \sum_{k=1}^{n-1} n(n-1) \binom{n-2}{k-1} t^k = 1 + n(n-1)t(1+t)^{n-2}. \quad (4)$$

Note that for every $t > 0$, we have

$$\begin{aligned}(1+t)^{n-2} - 1 &= t((1+t) + \cdots + (1+t)^{n-3}) \\ &\leq t(n-3)(1+t)^{n-3} \\ &< t(n-2)(1+t)^{n-3}.\end{aligned}$$

Hence, Inequality (4) can be written as follows:

$$\begin{aligned}\prod_{i=1}^n (1+t\lambda_i^2) &\geq 1 + n(n-1)t(1+t)^{n-2} \\ &= 1 + (n-2)t(1+t)^{n-2} + (n^2 - 2n + 2)t(1+t)^{n-2} \\ &\geq 1 + (n-2)t(1+t)^{n-3} + (n^2 - 2n + 2)t(1+t)^{n-2} \\ &> (1+t)^{n-2} + (n^2 - 2n + 2)t(1+t)^{n-2} \\ &= (1+t)^{n-2}(1 + (n^2 - 2n + 2)t).\end{aligned}$$

So, by Equation (1), for every $0 < p < 2$,

$$\begin{aligned}\mathcal{E}_p(A) &= \sum_{i=1}^n |\lambda_i|^p = C_{\frac{p}{2}} \int_0^\infty \ln\left(\prod_{i=1}^n (1+t\lambda_i^2)\right) t^{-\frac{p}{2}-1} dt \\ &> C_{\frac{p}{2}} \left((n-2) \int_0^\infty \ln(1+t) t^{-\frac{p}{2}-1} dt + \int_0^\infty \ln(1 + (n^2 - 2n + 2)t) t^{-\frac{p}{2}-1} dt \right) \\ &= (n-2) + (n^2 - 2n + 2)^{\frac{p}{2}} > (n-2) + (n-1)^p.\end{aligned}$$

The proof is complete. □

Now, if we apply Theorem 1 for $p = 1$, then we have the following corollary

Corollary 5. For every graph G of order n , $\mathcal{E}(\mathcal{S}(G)) > 2n - 3$.

In order to prove Theorem 2, we strengthen the inequality of Lemma 4. For this purpose, we pay attention to the 2×2 submatrices of $\mathcal{S}(G)$ with an odd number of -1 , which have a key rule in the value of $S_k(A^2)$ and hence in $\mathcal{E}(\mathcal{S}(G))$. We state the following definition.

Definition 6. Let G be a graph. An ordered pair (X, Y) of disjoint subsets of $V(G)$ with $|X| = |Y| = 2$, is called an *odd pair* if the number of edges with one endpoint in X and another in Y is odd. We denote the number of odd pairs in G by $s(G)$.

Let (X, Y) be an odd pair. Consider the 2×2 submatrix of $A = \mathcal{S}(G)$ whose rows and

columns are corresponding to the vertices of X and Y . This submatrix has the form

$$\begin{pmatrix} \pm 1 & \pm 1 \\ \pm 1 & \pm 1 \end{pmatrix},$$

which contains one or three -1 . It is easily seen that the determinant of this matrix is ± 2 .

Lemma 7. Let G be a graph of order n and $A = \mathcal{S}(G)$. Then, for $k = 1, \dots, n-2$, we have

$$S_k(A^2) \geq n(n-1) \binom{n-2}{k-1} + 4s(G) \binom{n-4}{k-2}. \quad (5)$$

Proof. The term $n(n-1) \binom{n-2}{k-1}$ on the right hand side of (5) is deduced by the proof of Lemma 4. Now, suppose that (X, Y) be an odd pair. If $I \subset G$, $|I| = k-2$ and $I \cap X = I \cap Y = \emptyset$, we claim that the absolute value of determinant of $k \times k$ submatrix of $\mathcal{S}(G)$ corresponding to $\mathcal{S}(G)_{I \cup X, I \cup Y}$ is at least 2. To prove the claim, note that after applying some permutations to its rows and columns of $\mathcal{S}(G)_{I \cup X, I \cup Y}$ one can see that $\mathcal{S}(G)_{I \cup X, I \cup Y}$ is changed to U , where

$$U = \left(\begin{array}{cc|cccc} & & \pm 1 & \pm 1 & \cdots & \pm 1 \\ & & \pm 1 & \pm 1 & \cdots & \pm 1 \\ \hline \pm 1 & \pm 1 & 0 & \pm 1 & \cdots & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 & \cdots & \pm 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm 1 & \pm 1 & \pm 1 & \pm 1 & \cdots & 0 \end{array} \right),$$

and T has an odd number of -1 . After multiplying the rows and columns of T by -1 if necessary, T can be changed to the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now, by applying some elementary column operations on U , one can obtain the following matrix:

$$\left(\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ \hline * & & R & & & \end{array} \right).$$

Clearly, $R = I_{k-2}$ and $\det R = 1$ modulo 2. This implies that

$$|\det U| \geq 2, \quad (\text{over real numbers}).$$

We have $\binom{n-4}{k-2}$ number of subsets I with the desired property. Note that we have $s(G)\binom{n-4}{k-2}$ number of $k \times k$ submatrices which introduced. Because, if $\mathcal{S}(G)_{I \cup X, I \cup Y} = \mathcal{S}(G)_{I' \cup X', I' \cup Y'}$, where (X, Y) and (X', Y') are two odd pairs, then

$$I = (I \cup X) \cap (I \cup Y) = (I' \cup X') \cap (I' \cup Y') = I',$$

and therefore, $X = X'$ and $Y = Y'$. Now, by Theorem 3, these submatrices have $4s(G)\binom{n-4}{k-2}$ contribution in $S_k(A^2)$. To complete the proof, note that the $k \times k$ submatrices of $\mathcal{S}(G)$ which considered here and those in the proof of Lemma 4 (which gives us the first term on the right hand side of (5)) do not have any intersection. \square

Lemma 8. Let G be a graph of order n which is not SC-equivalent to K_n . Then G has at least one odd pair.

Proof. One can see that if $n \leq 3$, then G is SC-equivalent to K_n . So, assume that $n \geq 4$. Let $v \in V(G)$ be an arbitrary vertex. By applying a Seidel switching on G with respect to $N_G[v]$ and $\overline{N_G[v]}$, one can suppose that v is adjacent to all vertices of G . Let v_i, v_j and v_k be vertices of $G \setminus \{v\}$ such that v_i and v_j are adjacent but v_i and v_k are not adjacent. Then, $(\{v, v_i\}, \{v_j, v_k\})$ is an odd pair in G .

If such v_i, v_j and v_k do not exist, then one can deduce that $G \setminus \{v\}$ has no induced subgraph isomorphic to $K_1 \cup K_2$ or $\overline{K_1 \cup K_2}$. Therefore, $G \setminus \{v\}$ is either complete graph or empty graph. Hence $G = K_n$ or using a Seidel switching with respect to $\{v\}$ and $G \setminus \{v\}$, we can delete all edges of G and so $G = \overline{K_n}$. \square

Lemma 9. If G is a graph of order n and $s(G) \geq 1$, then $s(G) \geq 2(n-3)^2$.

Proof. Let $V(G) = \{v_1, \dots, v_n\}$. Without loss of generality, suppose that $(\{v_1, v_2\}, \{v_3, v_4\})$ is an odd pair. Now, for every $i, 4 \leq i \leq n$, the parity of the number of edges between v_i and $\{v_1, v_2\}$ is different from the parity of the number of edges between v_3 and $\{v_1, v_2\}$ or v_4 and $\{v_1, v_2\}$. Hence, exactly one of the $(\{v_1, v_2\}, \{v_3, v_i\})$ and $(\{v_1, v_2\}, \{v_4, v_i\})$ is an odd pair. So, if we show the second element of this odd pair by V_i ($v_i \in V_i$), then $(\{v_1, v_2\}, V_i)$ is an odd pair. Similarly, if $i \geq 4$ and $v_j \notin V_i \cup \{v_1\}$ ($1 \leq j \leq n$), then exactly one of the $(\{v_1, v_j\}, V_i)$ and $(\{v_2, v_j\}, V_i)$ is an odd pair. Denote the first element of this odd pair by W_j and hence, (W_j, V_i) is an odd pair. So, we have $(n-3)^2$ odd pairs. Note that for every i and j , (V_i, W_j) is an odd pair, too and is not equal to an odd pair $(W_{j'}, V_{i'})$. Therefore we have at least $2(n-3)^2$ odd pairs in total, as desired. \square

Now, we state a technical lemma for the cubic polynomials with positive coefficients.

Lemma 10. If $f(t) = 1 + at + bt^2 + ct^3$ is a cubic polynomial with positive coefficients, then

$$C_{\frac{1}{2}} \int_0^\infty \ln(f(t)) t^{-\frac{3}{2}} dt \geq \sqrt{a + 2\sqrt{b + 2\sqrt{ac}}}.$$

Proof. Since the constant term of f is 1, one can consider the following factorization

$$f(t) = (1 + \alpha_1 t)(1 + \alpha_2 t)(1 + \alpha_3 t),$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Since f is positive over $\mathbb{R}^{\geq 0}$, no α_i is a negative real number, because otherwise f has a positive root, a contradiction. On the other hand, f has a real root, so at least one of α_i , say α_1 , is a positive real number. Hence, α_2 and α_3 are either positive real numbers or conjugate complex numbers. Note that by our convention about the arguments of complex numbers, if α and β are conjugate, then $\sqrt{\alpha}\sqrt{\beta} = \sqrt{\alpha\beta}$. Now by (3),

$$\sum_{i=1}^3 \sqrt{\alpha_i} = C_{\frac{1}{2}} \int_0^\infty \ln(f(t)) t^{-\frac{3}{2}} dt.$$

Define $X = \sum_{i=1}^3 \sqrt{\alpha_i}$. We have

$$X^2 = \sum_{i=1}^3 \alpha_i + 2 \sum_{i < j} \sqrt{\alpha_i \alpha_j} = a + 2 \sum_{i < j} \sqrt{\alpha_i \alpha_j} \quad (6)$$

$$\left(\sum_{i < j} \sqrt{\alpha_i \alpha_j} \right)^2 = \sum_{i < j} \alpha_i \alpha_j + 2 \sqrt{\alpha_1 \alpha_2 \alpha_3} \sum_{i=1}^3 \sqrt{\alpha_i} = b + 2(\sqrt{c})X. \quad (7)$$

In both cases, either α_i is positive or $\alpha_2 = \overline{\alpha_3}$, the values $\sum_{i < j} \sqrt{\alpha_i \alpha_j}$ and \sqrt{c} are positive numbers. So, Equation (6) implies $X \geq \sqrt{a}$. Now, (6) and (7) imply that,

$$X = \sqrt{a + 2 \sum_{i < j} \sqrt{\alpha_i \alpha_j}} = \sqrt{a + 2\sqrt{b + 2\sqrt{c}X}} \geq \sqrt{a + 2\sqrt{b + 2\sqrt{ac}}}.$$

□

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. As stated in [3], using a computer search one can see that $\mathcal{E}(S(G)) \geq$

$2n - 2$ for $n \leq 10$. So, we assume that $n > 10$. Now, by Lemma 7, for $k = 1, \dots, n$, we have

$$S_k(A^2) \geq n(n-1) \binom{n-2}{k-1} + 4s(G) \binom{n-4}{k-2}, \quad (8)$$

where $A = \mathcal{S}(G)$. Hence, for the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and $t > 0$, we have

$$\begin{aligned} \prod_{i=1}^n (1 + t\lambda_i^2) &= 1 + \sum_{k=1}^n S_k(A^2) t^k \\ &\geq 1 + \sum_{k=1}^n (n(n-1) \binom{n-2}{k-1} + 4s(G) \binom{n-4}{k-2}) t^k \\ &= 1 + n(n-1)t(1+t)^{n-2} + 4s(G)t^2(1+t)^{n-4} \\ &= 1 + (n-4)t(1+t)^{n-2} + (n^2 - 2n + 4)t(1+t)^{n-2} + 4s(G)t^2(1+t)^{n-4} \\ &> (1+t)^{n-4}(1 + (n^2 - 2n + 4)t(1+t)^2 + 4s(G)t^2), \end{aligned}$$

where the last inequality follows from

$$(1+t)^{n-4} < 1 + (n-4)t(1+t)^{n-5} < 1 + (n-4)t(1+t)^{n-2}.$$

Therefore we have

$$\begin{aligned} \mathcal{E}(\mathcal{S}(G)) &= \sum_{i=1}^n |\lambda_i| = C_{\frac{1}{2}} \int_0^\infty \ln \left(\prod_{i=1}^n (1 + t\lambda_i^2) \right) t^{-\frac{3}{2}} dt \\ &> C_{\frac{1}{2}} \int_0^\infty \ln((1+t)^{n-4}(1 + (n^2 - 2n + 4)t(1+t)^2 + 4s(G)t^2)) t^{-\frac{3}{2}} dt \\ &= C_{\frac{1}{2}} \int_0^\infty \ln(1+t)^{n-4} t^{-\frac{3}{2}} dt + C_{\frac{1}{2}} \int_0^\infty \ln(1 + (n^2 - 2n + 4)t(1+t)^2 + 4s(G)t^2) t^{-\frac{3}{2}} dt \\ &= (n-4) + C_{\frac{1}{2}} \int_0^\infty \ln(1 + (n^2 - 2n + 4)t(1+t)^2 + 4s(G)t^2) t^{-\frac{3}{2}} dt. \quad (9) \end{aligned}$$

Let $g(t) = 1 + (n^2 - 2n + 4)t(1+t)^2 + 4s(G)t^2 = 1 + \alpha t + (2\alpha + 4s(G))t^2 + \alpha t^3$, where $\alpha = n^2 - 2n + 4$. Hence by Lemma 10, we have

$$\begin{aligned} C_{\frac{1}{2}} \int_0^\infty \ln(g(t)) t^{-\frac{3}{2}} dt &\geq \sqrt{\alpha + 2\sqrt{(2\alpha + 4s(G)) + 2\alpha}} \\ &= \sqrt{\alpha + 4\sqrt{\alpha + s(G)}}. \end{aligned}$$

Note that for every natural number n , $\alpha = n^2 - 2n + 4 \geq \frac{3}{4}n^2$ which implies that

$$C_{\frac{1}{2}} \int_0^\infty \ln(g(t))t^{-\frac{3}{2}} \geq \sqrt{n^2 - 2n + 4 + 4\sqrt{\frac{3}{4}n^2 + s(G)}}. \quad (10)$$

Therefore if $s(G) \geq \frac{3}{2}n^2$, we have

$$C_{\frac{1}{2}} \int_0^\infty \ln(g(t))t^{-\frac{3}{2}} \geq \sqrt{n^2 - 2n + 4 + 6n} = n + 2,$$

which Equation (9) yields that, $\mathcal{E}(\mathcal{S}(G)) > 2n - 2$. Notice that for any graph G , which is SC-equivalent to K_n , $\mathcal{E}(\mathcal{S}(G)) = 2n - 2$ holds. Hence we assume that G is not SC-equivalent to K_n . In this case, by Lemma 9, $s(G) \geq 2(n - 3)^2$ and so for $n \geq 23$, we have $s(G) \geq \frac{3}{2}n^2$, as desired. Also, if G is not SC-equivalent to K_n , then by Lemma 8, there is an odd pair in G , say $(\{v_1, v_2\}, \{v_3, v_4\})$, and as the proof of Lemma 9 shows, all $2(n - 3)^2$ odd pairs of G contain v_1 or v_2 . Hence if $G \setminus \{v_1, v_2\}$ is not SC-equivalent to K_{n-2} , G has $2(n - 5)^2$ other odd pairs. Notice that for every natural number $n \geq 11$, $2(n - 3)^2 + 2(n - 5)^2 \geq \frac{3}{2}n^2$, as desired. To complete the proof, we consider the graph G with $10 < n = |V(G)| < 23$ and $G \setminus \{v_1, v_2\}$ is SC-equivalent to K_{n-2} . Now, by applying the Seidel switching on G or its complement, one can assume that $G \setminus \{v_1, v_2\} \simeq K_{n-2}$. The number of these graphs, up to isomorphism, is at most $2n^2$ and so by a computer search, it can be easily checked that the assertion holds for these graphs. The proof is complete. \square

Now, by [6, Theorem 13], we close the paper with the following corollary.

Corollary 11. For every graph G , $\mathcal{E}(\mathcal{S}(G)) \geq \mathcal{E}(G)$.

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