# Counterexamples to a conjecture of Las Vergnas 

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Abstract<br>We present counterexamples to a 30 -year-old conjecture of Las Vergnas [J. Combin. Theory Ser. B, 1988] regarding the Tutte polynomial of binary matroids.

Based on an evaluation established for the Tutte polynomial of plane graphs on (3,3), Michel Las Vergnas made three conjectures in LV88, in increasing strength, regarding the Tutte polynomial of binary matroids. The first and weakest of these [LV88, Conjecture 4.1] was proved in [Jae89] and in a more general setting in Bou91.
Theorem 1 ( Jae89, Bou91). For every binary matroid $M$, the value $T_{M}(3,3) / T_{M}(-1,-1)$ is an odd integer.

We remark that, for a binary matroid $M, T_{M}(-1,-1)=(-1)^{|E(M)|}(-2)^{b(M)}$, where $b(M)$ is the dimension of the bicycle space of $M$ [RR78, Theorem 9.1]. The other two conjectures remained open for a long time (and were recalled again in 2004 in [ELV04]). It was shown by Gordon Royle Roy13 that $M\left(K_{8}\right)$ is a counterexample to the third and strongest of the three conjectures LV88, Conjecture 4.3]. In fact, an exhaustive search using the dataset of binary matroids with at most 15 elements of [FW11] reveals several more counterexamples.

We now state the second conjecture [LV88, Conjecture 4.2], which is stronger than the first and weaker than the third conjecture.
Conjecture $2([\boxed{\mathrm{LV} 88}])$. For every binary matroid $M$ and every integer $z$, the value $T_{M}(-1+$ $4 z,-1+4 z) / T_{M}(-1,-1)$ is an odd integer.

Thus Theorem 1 corresponds to the value $z=1$ in Conjecture 2, It turns out that $M\left(K_{8}\right)$ is not a counterexample to this conjecture. Also, an exhaustive search using the above-mentioned dataset of FW11] reveals no counterexample to this conjecture. Consequently, any counterexample has at least 16 elements. In fact, for each binary matroid $M$ with less than 16 elements, $Q_{M}(z):=T_{N}(-1+4 z,-1+4 z) / T_{N}(-1,-1)$ turns out to have only integer coefficients. This, together with the fact that both $Q_{M}(0)=1$ and $Q_{M}(1)$ are odd (the latter by Theorem 11), implies that $Q_{M}(z)$ is an odd integer for all integers $z$.

Using SageMath Sage we found that the binary matroid $G$ with 24 elements corresponding to the extended binary Golay code (see, e.g., the appendix of Oxl11 for a definition) is a counterexample to Conjecture 2 Moreover, the rank-6 minor $N$ of $G$ with 18 elements having the following reduced representation over GF (2)

$$
\left(\begin{array}{llllllllllll}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right)
$$

is another counterexample. Indeed, $N$ has the following Tutte polynomial $T_{N}(x, y)$

$$
\begin{array}{r}
y^{12}+6 y^{11}+21 y^{10}+56 y^{9}+126 y^{8}+252 y^{7}+x^{6}+45 x y^{5}+462 y^{6}+12 x^{5}+6 x^{4} y+225 x y^{4}+ \\
747 y^{5}+72 x^{4}+111 x^{3} y+240 x^{2} y^{2}+675 x y^{3}+1017 y^{4}+247 x^{3}+591 x^{2} y+1095 x y^{2}+1057 y^{3}+ \\
417 x^{2}+909 x y+723 y^{2}+231 x+231 y
\end{array}
$$

We have $T_{N}(-1,-1)=2^{6}$ and $Q_{N}(z)=T_{N}(-1+4 z,-1+4 z) / T_{N}(-1,-1)$ is equal to

$$
\begin{array}{r}
262144 z^{12}-393216 z^{11}+344064 z^{10}-180224 z^{9}+73728 z^{8}-18432 z^{7}+8320 z^{6}-1248 z^{5}+ \\
2616 z^{4}-1012 z^{3}+\frac{195}{2} z^{2}-\frac{15}{2} z+1
\end{array}
$$

Consequently, $Q_{N}(z)$ is even for $z \in\{-2,-1,2\}$, contradicting Conjecture 2 ,
Finally, the self-dual (but not identically self-dual) rank-9 minor $N^{\prime}$ of $G$ having the following reduced representation over GF(2)

$$
\left(\begin{array}{lllllllll}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0
\end{array}\right)
$$

is yet another counterexample with 18 elements.
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